Similarity Solutions for Homogeneous and Nonhomogeneous Aerosol Balance Equations

by
Ronald L. Drake

July 1976

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SIMILARITY SOLUTIONS FOR HOMOGENEOUS AND
NONHOMOGENEOUS AEROSOL BALANCE EQUATIONS

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July 1976

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ABSTRACT

This report is concerned with similarity solutions for homogeneous and nonhomogeneous aerosol balance equations. The systems describe the time evolution of an aerosol population in fluid media under the influences of volumetric sources and sinks, coagulation of particles, condensational growth, a vertical wind field, gravitational settling, and vertical diffusion. Applying transformation equations, the physical space is mapped into a one-dimensional similarity space. Direct and indirect methods are used to solve the governing equations for the number densities in the similarity space. The forms of the solutions vary with the changes in the system parameters. Hence, several relationships between the parameters and the similarity spectra and the integral moments of the spectra are obtained. For example, the heavier-tailed spectra correspond to the unbounded coagulation parameters and the lighter-tailed spectra correspond to the bounded kernels.

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1. INTRODUCTION

This report is concerned with the similarity transformation of homogeneous and nonhomogeneous aerosol balance equations. The independent variables in the nonhomogeneous system are as follows: the volume of a particle u or v , the time t , and the vertical coordinate z . In addition, we assume that the z -axis is positive in a direction opposite to that of gravity and has its origin at the earth's surface. The independent variables in the homogeneous system are u or v and t . The only dependent variable in the systems is the number density $n=n(z,v,t)$ defined in the following way: ndv is the total number of particles in the interval $v \pm \frac{1}{2} dv$ at time t and location z per unit volume of the medium in which the particles are embedded.

The parameters that govern the system are the coagulation parameter $\beta(z,u,v,t)$, the condensational growth rate $I(z,v,t)$, the volumetric source and sink distribution $S(z,v,t)$, the vertical velocity field in the medium $W(z,t)$, the particle settling velocity $W_s(z,v,t)$ and the eddy diffusivity of the medium $D(z,t)$. For the homogeneous system, the z -dependency is suppressed and W , W_s and D are dropped. The physical units of these quantities are as follows: $\{n\}=L^{-6}$, $\{I\}=\{\beta\}=L^3T^{-1}$, $\{S\}=L^{-6}T^{-1}$, $\{W\}=\{W_s\}=LT^{-1}$ and $\{D\}=L^2T^{-1}$, where A is the dimension of A , L the characteristic length and T the characteristic time.

In the six parameters given above, we have assumed variations in time and space. It is reasonable to allow W and D to vary with z and t since they are properties of the ambient flow field of the medium. In addition, the ambient properties and composition of the medium influence various coefficients in the other parameters; thus, β , I , S and W_s also depend on z and t . The diffusivity D should depend upon v ; however, we were unable to obtain a similarity transformation when $D=D(z,v,t)$, hence, we assume that $D=D(z,t)$.

There have been several examples of time dependent parameters reported in the literature. For example, Heisler and Friendlander [1974] investigated the following growth mechanism in their study of urban aerosols:

$$I(v,t) = C(v,t)v^p ,$$

where $C(v,t)$ is a slowly-varying function of v and "plays the role" of a reaction rate. When the exponent p is equal to unity, the growth of particles is dominated by reactions occurring within the particles. When $p = 2/3$, the growth is dominated by a reaction which occurs on the surface of the particle. When $p = 1/3$, the mechanism is the well-known Maxwellian condensation, see Pich et al [1970]. Usually, the Maxwellian condensation assumes that $C(v,t) = C(t)$, where the time-variation is due to the supersaturation of the gaseous phase of the condensing medium.

The coagulation parameter $\beta(u,v,t)$ can be time-dependent if the particles are embedded in a medium with a time-dependent turbulent structure, see Stewart and Townsend [1951]. If the coagulating particles are influenced by electrical phenomena, then β will vary with time for environments around and in thunderstorms, see Zebel [1966]. Another example of a time-varying β is given in Nathans et al [1970]. They found that the absolute temperature and the rate of change of temperatures in an expanding cloud of debris from a nuclear air blast are so great that the viscosity, density and mean-free-path of the air changes with time. Hence, any kernel β that contains these quantities must also vary with time as the debris cloud expands and cools.

In this report we consider both homogeneous and non-homogeneous systems. Homogeneous systems have been considered for the past several years; however, there is little theoretical information on nonhomogeneous systems except for certain numerical studies. Some of the recent literature dealing with homogeneous systems are as follows. A literature survey on pure coagulation is given in Drake [1972a]; pure condensation processes are discussed in Pich et al 1970 , Brock [1973], and Heisler and Friedlander [1974]; Melzak [1973] extends the binary coagulation theory to the coagulation of three or more particles and to particle breakup; and Klett [1975] considers steady state, source-enhanced, coagulating systems.

The objective of this report is to determine the system parameters, initial conditions and boundary conditions that will allow n to evolve such that a similarity solution exists. In other words, we are searching for the conditions that will allow $n(z,v,t)$ to be replaced by a function of a single variable $\psi(\eta)$. Hence, we want the independent variables z , v and t to coalesce into a single variable η , and n to be replaced by the similarity solution, or spectrum, $\psi(\eta)$. In Figure 1, we show a schematic of this process for the homogeneous case.

There are several reasons why the search for similarity solutions is a meaningful activity. For example:

1. Similarity solutions, or "self-preserving" spectra, have been observed both in the laboratory and in the field (see Swift and Friedlander [1964], Clark and Whitby [1967], and Nathans et al [1970]).
2. The qualitative analysis of the system for $\psi(\eta)$ is usually more fruitful than the analysis of the system for $n(z,v,t)$; and the results obtained for $\psi(\eta)$ are applicable to $n(z,v,t)$.
3. The governing equation for $\psi(\eta)$ is an ordinary integro-differential equation as opposed to a partial integro-differential equation for $n(z,v,t)$. Therefore, if the transformed system represents a physical system of interest and numerical methods are required for obtaining solutions, the system for $\psi(\eta)$ requires considerably less

computer time than the original system.

4. Any exact solution of the transformed system can be used as a test case for numerical techniques that have been devised to solve for the $n(z,v,t)$ in more complex physical systems.
5. Similarity solutions can be used to approximate the solutions for nonsimilar systems if the parameters producing the nonsimilarity solutions are "reasonably close" to the ones producing the similarity solutions. The idea of "reasonably close" is considered in Drake [1972b].
6. Even though a set of transformed systems may not exactly describe the physical system of interest, the systems may be sufficiently close to the physical system that the resultant $\psi(\eta)$'s will give meaningful upper and lower bounds for the $n(z,v,t)$ in question.
7. From a set of similarity solutions we may be able to formulate a series or an asymptotic expansion for $n(z,v,t)$ in terms of these solutions. These formulas will then be useful in solving more complex systems.

In Section 2 we discuss the physical systems considered in this report and give the defining equations. The transformed system for the homogeneous case is given in Section 3. In addition, in Section 4 we derive several properties of

the transformed system. Section 5 is concerned with the evaluation of important integral power moments of n in terms of the similarity spectrum $\psi(\eta)$. Direct methods of obtaining $\psi(\eta)$ for the homogeneous system are discussed in Section 6 and indirect methods are given in Section 7. Sections 8 and 9 are concerned with transforming the nonhomogeneous system into a similarity system and in Section 10 we derive an indirect solution technique for the transformed system. Section 11 gives a detailed example of this technique. Finally, Section 12 lists the conclusions of the study.

2. THE PHYSICAL SYSTEMS

The defining equation for the evolution of an aerosol in a nonhomogeneous medium is given by

$$\boxed{\frac{\partial n}{\partial t} + \frac{\partial In}{\partial v} + \frac{\partial Wn}{\partial z} - \frac{\partial W_s n}{\partial z} = \frac{\partial \left[D \frac{\partial n}{\partial z} \right]}{\partial z} + \frac{1}{2} \tilde{N} + S} \quad , \quad (1)$$

where

$$\tilde{N}(z, v, t) = \int_0^v \tilde{F}(z, u, v-u, t) du - 2 \int_0^\infty \tilde{F}(z, u, v, t) du \quad , \quad (2)$$

and

$$\begin{aligned} \tilde{F}(z, u, v, t) &= \beta(z, u, v, t) n(z, u, t) n(z, v, t) \\ &= \beta(z, v, u, t) n(z, v, t) n(z, u, t) \\ &= \tilde{F}(z, v, u, t) \geq 0 \quad . \end{aligned} \quad (3)$$

The second term on the left in Equation (1) is the condensational growth term; the third is the advection term; the fourth is the gravitational settling term; the first on the right is the turbulent diffusion term; the second is the coagulation term; and the third and last term is a source and sink distribution. In Equation (2), the first integral accounts for the production of v-particles due to the coagulation of smaller particles and the second integral accounts for the loss of v-particles due to v-particles coagulating with other particles.

The source term S in Equation (1) can be incorporated into the growth term I by replacing

$$\frac{\partial I n}{\partial v} - S$$

by

$$\frac{\partial}{\partial v} \left[n \left(\tilde{I} - \frac{\int_0^v S(z, u, t) du}{n} \right) \right] \equiv \frac{\partial}{\partial v} [\tilde{I} n] ,$$

where $\tilde{I}$ is the new "growth" term. However, we will continue to consider Equation (1) where S and I are treated separately.

There are three integral power moments of $n(z, v, t)$ that are important both from physical and mathematical points of view. They are

$$N(z, t) = \int_0^{\infty} n(z, v, t) dv , \quad (4)$$

$$\sigma(z,t) = \sqrt[3]{36\pi} \int_0^{\infty} v^{2/3} n(z,v,t) dv , \quad (5)$$

$$\phi(z,t) = \int_0^{\infty} v n(z,v,t) dv , \quad (6)$$

where $N(z,t)$ is the total number density, $\sigma(z,t)$ the total surface area density, and $\phi(z,t)$ the total volume concentration. The physical units of these quantities are given by: $\{N\}=L^{-3}$, $\{\sigma\}=L^{-1}$, and $\{\phi\}=\text{unitless}$.

For the homogeneous system we require the governing equations for $N(t)$ and $\phi(t)$. These are obtained from Equation (1) by multiplying by 1 and v , respectively, and then integrating with respect to v from 0 to ∞ . The results are

$$\begin{aligned} \frac{d N(t)}{dt} = & - \frac{1}{2} \int_0^{\infty} \int_0^{\infty} \tilde{F}(u,v,t) du dv + \lim_{v \rightarrow +0} \left[I(v,t) n(v,t) \right] + \\ & + \int_0^{\infty} S(v,t) dv , \end{aligned} \quad (7)$$

and

$$\frac{d \phi(t)}{dt} = \int_0^{\infty} I(v,t) n(v,t) dv + \int_0^{\infty} v S(v,t) dv . \quad (8)$$

The existence of the first integral in Equation (8) implies that

$$\begin{aligned} \lim_{v \rightarrow +0} [v I(v,t) n(v,t)] &= \lim_{v \rightarrow +\infty} [v I(v,t) n(v,t)] = \\ &= \lim_{v \rightarrow +\infty} [I(v,t) n(v,t)] = 0 . \end{aligned} \quad (9)$$

If Equation (7) is meaningful, then the limit on the right-hand side must exist. However, it need not be zero as some investigators have tacitly assumed.

Equation (7) states that the total number density decreases due to coagulation, increases due to sources, decreases due to sinks, and increases due to gas-to-particle conversion. The gas-to-particle mechanism is represented by the positive values of the limit term in Equation (7). Similarly, Equation (8) states that the total volume density increases due to condensational growth and the source distribution, and decreases due to the sink distribution.

3. THE TRANSFORMED, HOMOGENEOUS SYSTEM

The most successful set of similarity variables for the homogeneous system corresponding to Equation (1) is the set introduced by Friedlander [1964]. The variables are

$$\eta = v/\langle v \rangle \quad , \quad \xi = u/\langle v \rangle \quad , \quad (10)$$

and

$$\psi(\eta) = f(t) n(v, t) \quad , \quad (11)$$

where $\langle v \rangle$ is a mean volume defined by

$$\langle v \rangle = \frac{\phi(t)}{N(t)} \quad , \quad (12)$$

and where $f(t)$ is defined by

$$f(t) = \frac{\phi(t)}{N^2(t)} \quad . \quad (13)$$

The choice for $f(t)$ was such that

$$M_0 = M_1 = 1 \quad , \quad (14)$$

where

$$M_v(\eta) = \int_{\eta}^{\infty} \xi^v \psi(\xi) d\xi \quad , \quad (15)$$

are integral power moments of $\psi(\eta)$ and where

$$M_v(0) = M_v \quad , \quad M_v(\infty) = 0 \quad , \quad (16)$$

for all v . The derivative of $M_v(\eta)$ with respect to η is given by

$$\frac{dM_v(\eta)}{d\eta} = -\eta^v \psi(\eta) \quad . \quad (17)$$

For the existence of $\psi(\eta)$, we require that the initial spectrum $n(v,0)$ satisfy Equation (11), namely,

$$n(v,0) = \frac{\psi(\eta)}{f(0)} \quad . \quad (18)$$

In addition, the parameters β , I and S must be restricted. As candidates (see Ruckenstein and Chi, [1975]), let us consider the following:

$$\beta(u,v,t) = \alpha(t)\beta(\xi,\eta) = \alpha(t)\beta(\eta,\xi) \quad , \quad (19)$$

$$I(v,t) = a(t)I(\eta) \quad , \quad (20)$$

$$S(v,t) = b(t)S(\eta) \quad . \quad (21)$$

The time-dependent terms in (19) to (21) enable us to consider rather complex functions of u and v for our choices of β , I and S . However, the time-dependent coefficients in

β , I and S are restricted by Equations (19) to (21) and by the equations given in the next paragraph.

For the existence of $\psi(\eta)$, the quantities $\alpha(t)$, $a(t)$ and $b(t)$ must be chosen such that

$$A_1 = \lim_{\eta \rightarrow +0} [I(\eta)\psi(\eta)] = \text{finite constant}, \quad (22)$$

$$A_2 = \frac{a(t)}{\alpha(t)\phi(t)} = \text{constant}, \quad (23)$$

$$A_3 = \frac{b(t)\phi(t)}{\alpha(t)N^3(t)} = \text{constant}. \quad (24)$$

Without loss of generality, we can further assume that $\alpha(t)$, $a(t)$ and $b(t)$ satisfy the following equations:

$$\int_0^\infty I(\eta)\psi(\eta)d\eta = 1, \quad (25)$$

$$S_1 = \int_0^\infty \eta S(\eta)d\eta = 1, \quad (26)$$

$$\int_0^\infty \int_0^\infty F(\xi, \eta)d\xi d\eta = 1, \quad (27)$$

where

$$F(\xi, \eta) = \beta(\xi, \eta)\psi(\xi)\psi(\eta) = F(\eta, \xi), \quad (28)$$

$$S_v(\eta) = \int_\eta^\infty \xi^v S(\xi)d\xi, \quad (29)$$

and where $S_v(0) = S_v$ and $S_v(\infty) = 0$.

Using Equations (19) to (24), we obtain the following classifications: $A_2=0$ for no condensation, $A_2=\infty$ for no coagulation, $0 < A_2 < \infty$ for both condensation and coagulation, $A_3=0$ for no sources and sinks, $A_3=\infty$ for no coagulation, and $0 < A_3 < \infty$ for both sources and sinks and coagulation.

Combining Equations (7) to (11) and (22) to (29) give

$$\frac{dN(t)}{dt} = - \frac{\alpha(t)}{2} [1 - 2A_1A_2 - 2S_0A_3]N^2(t) , \quad (30)$$

and

$$\frac{d\phi(t)}{dt} = \alpha(t) [A_2 + A_3]N(t)\phi(t) . \quad (31)$$

Assuming the existence of $\psi(\eta)$ and given the forms of β , I and S , we can easily solve (30) and (31) for $\phi(t)$ and $N(t)$.

Given the quantities $\phi(t)$ and $N(t)$ and assuming the existence of $\psi(\eta)$, we can obtain $\sigma(t)$ from the following:

$$\sigma(t) = \sqrt[3]{36\pi} M_{2/3} N^{1/3}(t) \phi^{2/3}(t) . \quad (32)$$

Using Equations (10), (11), (30) and (31), the transformation of the homogeneous system corresponding to Equation (1) becomes

$$\begin{aligned} & 2A_2 [I(\eta)\psi(\eta) - (1-A_1)\eta\psi(\eta) - A_1M_0(\eta)] + \\ & + 2A_3 [S_0(\eta) - (1-S_0)\eta\psi(\eta) - S_0M_0(\eta)] = \\ & = \eta\psi(\eta) - M_0(\eta) - \int_{\eta}^{\infty} N(\xi)d\xi , \end{aligned} \quad (33)$$

where

$$N(\xi) = \int_0^{\xi} F(\theta, \xi - \theta) d\theta - 2 \int_0^{\infty} F(\theta, \xi) d\theta . \quad (33a)$$

In summary, the homogeneous system possesses a similarity solution if the following conditions are all satisfied:

1. $n(v,0)$ satisfies Equation (18);
2. β , I and S satisfy Equations (19) to (29);
3. Equation (33) has a solution satisfying Equation (14).

Given $\psi(\eta)$; $n(v,t)$ is obtained from (10) and (11) and $N(t)$, $\phi(t)$ and $\sigma(t)$ are derived from (30) to (32). However, we do not require the actual expression for $\psi(\eta)$, but only its existence, in order to determine certain properties of the transformed system and to obtain the quantities $N(t)$, $\phi(t)$ and $\sigma(t)$ in terms of $\psi(\eta)$ and the system parameters.

4. PROPERTIES OF THE TRANSFORMED SYSTEM

The existence of the moments in (14) implies that

$$\lim_{\eta \rightarrow +0} [\eta \psi(\eta)] = \lim_{\eta \rightarrow +\infty} [\eta^2 \psi(\eta)] = 0 . \quad (34)$$

The second limit in (34) indicates that ψ must decrease faster than η^{-2} as $\eta \rightarrow +\infty$. It is interesting to note that the famous r^{-4} power law of Junge [1963] corresponds to $\psi(\eta) \rightarrow \eta^{-2}$ as $\eta \rightarrow +\infty$.

Equation (27) restricts both $\psi(\eta)$ and $\beta(\xi, \eta)$ through the following limits on $F(\xi, \eta)$:

$$\lim_{\substack{\eta \rightarrow +0 \\ \xi \rightarrow +0}} [\eta \xi F(\xi, \eta)] = \lim_{\substack{\eta \rightarrow +\infty \\ \xi \rightarrow +\infty}} [\eta \xi F(\xi, \eta)] = 0 . \quad (35)$$

Similarly, Equations (25) and (26) and a finite S_0 restrict $I(\eta)$ and $S(\eta)$, namely,

$$\lim_{\eta \rightarrow +0} [\eta I(\eta) \psi(\eta)] = \lim_{\eta \rightarrow +\infty} [\eta I(\eta) \psi(\eta)] = 0, \quad (36)$$

and

$$\lim_{\eta \rightarrow +0} [\eta S(\eta)] = \lim_{\eta \rightarrow +\infty} [\eta^2 S(\eta)] = 0. \quad (37)$$

The results in Equations (34) to (36) are similar to the results in Ruckenstein and Chi [1975]; however, β , I and S need not be restricted to algebraic functions.

The quantities A_1 , A_2 and A_3 are arbitrary except for the restrictions placed on them by Equations (19) to (27). In fact, we can show that Equation (33) places no further restrictions on them. For example, if we take the limit of Equation (33) as $\eta \rightarrow +0$ and if we integrate (33) with respect to η from 0 to $+\infty$, only $F(\xi, \eta)$ is further restricted, namely,

$$\lim_{\substack{\xi \rightarrow +\infty \\ \eta \rightarrow +\infty}} [\eta^2 \xi^2 F(\xi, \eta)] = 0. \quad (38)$$

Suppose $\psi(\eta)$ decreases like η^{-2-p} as $\eta \rightarrow +\infty$, where p is an arbitrarily small positive number. Then (38) implies that $\beta(\xi, \eta)$ must be nonincreasing as ξ and η approach infinity. One interpretation of this result is that the coagulation parameter that gives the power law of Junge must approach a constant or

must decrease as ξ and η approach infinity. A complementary result by Drake [1972b] showed that ψ did not exist in the ordinary sense when β was equal to $(\xi+\eta)$ or to $(\xi\eta)$. Also, Wang and Friedlander [1967] encountered problems with the laminar shear and turbulent shear kernels for large values of η . In these cases β varies as $[\eta^{1/3} + \xi^{1/3}]^3$.

Equation (27) eliminates certain candidates for $F(\xi, \eta)$. For example, $F(\xi, \eta)$ can not be of the following forms:

$$\begin{aligned} F(\xi, \eta) &= h(\xi) + h(\eta) \quad , \\ F(\xi, \eta) &= h(\xi\eta) \quad . \end{aligned} \tag{39}$$

For if we combine (27) and (39), contradictions are obtained, namely,

$$\begin{aligned} \int_0^\infty \int_0^\infty [h(\xi) + h(\eta)] d\xi d\eta &= 2 \int_0^\infty \int_0^\infty h(\xi) d\xi d\eta \\ &= 2C \int_0^\infty d\eta = +\infty, \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \int_0^\infty h(\xi\eta) d\xi d\eta &= \int_0^\infty \int_0^\infty \frac{1}{\tau} h(\xi) d\xi d\tau \\ &= C \int_0^\infty \frac{d\tau}{\tau} = +\infty, \end{aligned}$$

where C is a positive constant.

A misinterpretation of similarity solutions that should be corrected is the one that occurred in Lui and Whitby [1968].

They stated that a system in dynamic equilibrium and possessing a self-preserving spectrum must satisfy the following equation:

$$2\psi + \eta \, d\psi/d\eta = 0 \quad . \quad (40)$$

They further stated that for sufficiently large η , the solution of Equation (40) is the famous Junge power law, namely,

$$\psi(\eta) = (\text{constant}) \, \eta^{-2} \quad .$$

However, Equation (40), as well as the concept that equilibrium and similarity are mutually satisfied, is invalid. Suppose we rewrite (40) as

$$d[\eta\psi(\eta)] + \psi(\eta)d\eta = 0 \quad ,$$

and then integrate from $\eta = 0$ to $\eta = +\infty$. The result is clearly contradictory, namely, " $1=0$ ". The reason a contradiction results is due to the fact that similarity solutions are not steady-state solutions but involve time implicitly through Equations (10) and (11).

The form of Equation (33) can be simplified by employing the following substitutions:

$$I(\eta) = (1-A_1)\eta + f(\eta) \quad , \quad (41)$$

$$S_o(\eta) = [(1-S_o)\eta + g(\eta)]\psi(\eta) \quad , \quad (42)$$

where $f(\eta)$ and $g(\eta)$ are new system parameters. Equations (22)

to (27) restrict $f(\eta)$ and $g(\eta)$ in the following ways:

$$\lim_{\eta \rightarrow +\infty} [\eta f(\eta) \psi(\eta)] = \lim_{\eta \rightarrow +\infty} [\eta g(\eta) \psi(\eta)] = 0 \quad (43)$$

$$\lim_{\eta \rightarrow +0} [f(\eta) \psi(\eta)] = A_1, \quad \lim_{\eta \rightarrow +0} [g(\eta) \psi(\eta)] = S_0, \quad (44)$$

$$\int_0^\infty f(\eta) \psi(\eta) d\eta = A_1, \quad \int_0^\infty g(\eta) \psi(\eta) d\eta = S_0. \quad (45)$$

Further, let us introduce the parameter $h(\eta)$ defined by

$$h(\eta) = f(\eta) + A_5 g(\eta), \quad (46)$$

where $A_5 = A_3/A_2$. The restrictions placed on $h(\eta)$ are given by

$$\lim_{\eta \rightarrow +0} [h(\eta) \psi(\eta)] = \int_0^\infty h(\eta) \psi(\eta) d\eta = A_6, \quad (47)$$

$$\lim_{\eta \rightarrow +\infty} [\eta h(\eta) \psi(\eta)] = 0, \quad (48)$$

where $A_6 = A_1 + A_5 S_0$.

If $A_6 = 0$ in Equation (47), then $h(\eta)$ is either identically zero or changes sign since ψ must be nonnegative and not identically zero. If $h(\eta) \equiv 0$, the condensation term must be

$$I(\eta) = (1 + A_5) \eta - A_5 \frac{S_0(\eta)}{\psi(\eta)}. \quad (49)$$

If there are no sources or sinks ($A_1 = A_5 = A_6 = 0$), then

$$I(\eta) = \eta. \quad (49a)$$

Equation (49a) implies that the condensation mechanism is volume dependent whenever A_1 is zero and there are no sources and sinks. This contradicts the results in Pich et al [1970] where they assume that I can be Maxwellian when $A_1=0$ and β is a Brownian kernel.

If $A_6 \neq 0$ in (47), then the limits in Equations (34) and (47) imply that

$$\lim_{\eta \rightarrow +0} [\eta^{-1} h(\eta)] = +\infty,$$

or

$$h(\eta) \rightarrow \eta^p, \quad p < 1, \quad \text{as } \eta \rightarrow +0. \quad (50)$$

Combining Equations (33), (41), (42) and (46) gives the general form of the transformed system, namely,

$$2A_2 [h(\eta)\psi(\eta) - A_6 M_0(\eta)] = \eta\psi(\eta) - M_0(\eta) - \int_{\eta}^{\infty} N(\xi) d\xi. \quad (51)$$

Equation (51) defines at least seven different systems depending upon the values of the constants A_1 , A_2 and A_3 (or A_2 , A_5 and A_6). However, there are only three forms of the equation that must be considered in this study. The first form is Equation (51); and the three cases that it represents are as follows:

I. Coagulation-Condensation-Source

$$0 < A_2 < +\infty, \quad -\infty < A_3 < 0, \quad 0 < A_3 < +\infty.$$

II. Coagulation-Condensation

$$0 < A_2 < +\infty, \quad A_3 = A_5 = 0.$$

III. Coagulation-Source

$$A_2 = 0, \quad -\infty < A_3 < 0, \quad 0 < A_3 < +\infty,$$

$$2A_2h(\eta) = 2A_3g(\eta), \quad A_2A_6 = A_3S_0.$$

The second form of the similarity equation is given by

$$\boxed{\eta\psi(\eta) - M_0(\eta) = \int_{\eta}^{\infty} N(\xi) d\xi} \quad (52)$$

Equation (52) represents the following physical system:

IV. Coagulation

$$A_2 = A_2A_6 = A_3 = 0.$$

The final form of the similarity equation is given by

$$\boxed{h(\eta)\psi(\eta) = A_6M_0(\eta)} \quad (53)$$

Equation (53) represents the following systems:

V. Condensation-Source

$$A_2 = +\infty, \quad A_3 = \pm\infty, \quad A_3/A_2 = \text{finite}.$$

VI. Source

$$A_2 = \text{finite}, \quad A_3 = \pm\infty.$$

VII. Condensation

$$A_2 = +\infty, \quad A_3 = \text{finite}.$$

5. EVALUATION OF $N(t)$, $\sigma(t)$ AND $\phi(t)$

If the homogeneous system possesses a similarity solution, then Equations (30) and (31) can be rewritten as

$$\frac{d\phi}{dN} = p \frac{\phi}{N}, \quad (54)$$

and

$$\frac{dN}{dt} = A(t)N^{2-q}, \quad (55)$$

where $A(t)$ and the constants p and q will be specified later.

The solutions of (54) and (55) are given by

$$N(t) = N(0) \left[1 + B(t) \right]^{\frac{1}{q-1}}, \quad (56)$$

and

$$\phi(t) = \phi(0) \left[1 + B(t) \right]^{\frac{p}{q-1}}, \quad (57)$$

where

$$B(t) = (q-1) \left[N(0) \right]^{1-q} \int_0^t A(u) du. \quad (58)$$

Combining (32), (56) and (57) gives

$$\sigma(t) = \sigma(0) \left[1 + B(t) \right]^{\frac{2p+1}{3(q-1)}}, \quad (59)$$

where

$$\sigma(o) = \sqrt[3]{36\pi} M_{2/3} [N(o)]^{1/3} [\phi(o)]^{2/3} \quad (60)$$

The existence of a similarity solution $\psi(\eta)$ not only allows us to express σ in terms of N and ϕ but also allows us to express all integral power moments of $n(v,t)$ in terms of $N(t)$ and $\phi(t)$, or, in this case, in terms of $[1 + B(t)]$. Suppose we denote the power moments of $n(v,t)$ by

$$m_v(t) = \int_0^\infty v^v n(v,t) dv \quad (61)$$

Then, combining Equations (61), (10), (11), and (16) gives

$$m_v(t) = M_v [\phi(t)]^v [N(t)]^{1-v} \quad (62)$$

Substituting Equations (56) and (57) into (62) gives

$$m_v(t) = m_v(o) [1 + B(t)]^{\frac{1-v+vp}{q-1}} \quad (63)$$

where

$$m_v(o) = M_v [\phi(o)]^v [N(o)]^{1-v} \quad (63a)$$

The relationships between $a(t)$, $b(t)$ and $\alpha(t)$ can be obtained from (23), (24), (56) and (57); they are

$$a(t) = \frac{a(o)}{\alpha(o)} [1 + B(t)]^{\frac{p}{q-1}} \alpha(t) \quad (64)$$

$$b(t) = \frac{b(o)}{\alpha(o)} [1 + B(t)]^{\frac{3-p}{q-1}} \alpha(t) , \quad (65)$$

$$a(t) = \frac{a(o)}{b(o)} [1 + B(t)]^{\frac{2p-3}{q-1}} b(t) . \quad (66)$$

Before we consider special cases of Equations (56) and (57), let us discuss some of the limitations of the coefficients and the time-dependent parameters. For physical reasons $\alpha(t)$ and $a(t)$ are assumed to be nonnegative. The source term, Equation (21), includes both sources and sinks; thus $b(t)$ can take on both signs.

The quantity $\psi(\eta)$ is nonnegative and the quantity $I(\eta)$ could be positive, negative or zero in Equation (22). Hence, A_1 could be positive, negative or zero. A negative A_1 can be interpreted as a loss of particles due to evaporation while a positive A_1 can be interpreted as a supply of particles from a secondary source (a gas-to-particle conversion process). Equation (49a) gives the result for $A_1=0$.

Constant A_2 , Equation (23), must be nonnegative; but A_3 and A_5 may be positive, negative or zero since $b(o)$ can be positive, negative or zero. Obviously, $A_6 = A_1 + A_5 S_0$ depends upon the choices for A_1 , A_5 and S_0 .

The quantity S_0 can be positive, negative or zero as long as $S(\eta)$ is such that $S_1=1$. For example, suppose that

$$S(\eta) = \frac{\eta^{a-1} q(\eta)}{h(a+1)} , \quad (67)$$

where

$$q(\eta) = e^{-\eta} \ln(\eta), \quad h(a) = \Gamma(a) \Psi(a) \quad .$$

The function $\Gamma(a)$ is the well-known gamma function, see Figure 2; and $\Psi(a) = \frac{d}{da} \ln[\Gamma(a)]$ is the psi function, see Figure 3. The corresponding expression for S_0 is given by

$$S_0 = \frac{h(a)}{h(a+1)} \quad .$$

Thus, S_0 is zero at a zero of $\Psi(a)$, for example, at $a_0 \approx 1.46163$. If $1 < a < a_0$, then $S_0 < 0$ and $h(a+1) > 0$. If $a > a_0$, then $S_0 > 0$. Figure 4 gives a representative graph of $S(\eta)$ for any $a > 1$. If $b(t) > 0$ in Equation (21), then we have a particle sink for $0 < \eta < 1$ and a source for $\eta > 1$. A more realistic case may be obtained if $b(t) < 0$; thus, we have a small particle source and a large particle sink.

In Table I we have listed several special cases of Equations (56) and (57). The relationships between the system parameters and the total volume and total number concentrations of the aerosol populations can be determined from Equations (19) to (24).

6. DIRECT SOLUTION METHODS FOR THE HOMOGENEOUS SYSTEM

The only direct solution technique for solving the homogeneous system known to this author is to use some type of integral transform. From past experience, the most successful transform has been that due to Laplace. However,

this solution method imposes added restrictions on $h(\eta)$ and $\beta(\xi, \eta)$, namely,

$$h(\eta) = a + b\eta + e\eta^2 + f\eta^3, \quad (68)$$

and

$$\beta(\xi, \eta) = A + B(\xi + \eta) + C(\xi\eta), \quad (69)$$

where A, B, C, a, b, e and f are constants.

The constants in (68) and (69) must also satisfy certain restrictions. For example, combining (68) and (43) to (45) give the following:

$$A_6 = a\psi(0) = a + b + eM_2 + fM_3, \quad \lim_{\eta \rightarrow +\infty} [\eta^4 \psi(\eta)] = 0. \quad (70)$$

Combining (27) and (69) gives

$$A + 2B + C = 1, \quad (71)$$

where A, B and C are all nonnegative.

Substituting (68) and (69) into Equation (51) gives

$$\begin{aligned} & 2A_2[a\psi' + b(\eta\psi)' + e(\eta^2\psi)' + f(\eta^3\psi)' + A_6\psi] \\ &= (\eta\psi)' + \psi + A(\psi*\psi) + B\eta(\psi*\psi) + \\ &+ C[(\eta\psi)*(\eta\psi)] - 2A\psi - 2B\psi \\ &- 2B\eta\psi - 2C\eta\psi, \end{aligned}$$

(72)

where $\psi = \psi(\eta)$, the "prime" indicates differentiation with respect to η , and $F * G$ is the convolution of $F(\eta)$ and $G(\eta)$.

If the Laplace transform of $\psi(\eta)$ is given by

$$L[\psi(\eta)] \equiv \int_0^{\infty} \psi(\eta) e^{-s\eta} d\eta = \omega(s) \quad , \quad (73)$$

then the transform of (72) is as follows:

$$\begin{aligned} & [2A_2 f s] \omega''' - [2A_2 e s] \omega'' + \\ & + [2A_2 b s - s + 2B + 2C] \omega' + \\ & + [1 - 2A - 2B - 2A_2 A_6 - 2A_2 a s] \omega + \\ & + A \omega^2 - B[\omega^2]' + C[\omega']^2 + 2A_2 A_6 = 0 \quad , \end{aligned} \quad (74)$$

where $\omega = \omega(s)$ and the prime indicates differentiation with respect to s .

Equation (74) is a third-order, nonlinear ordinary differential equation. In general, it has no "closed form" solutions. However, there are several special cases of (74) for which solutions exist or are known not to exist in the ordinary sense. Some of the special cases are given in the following list. This list includes solutions for (52) as well as for (51).

$$\text{I. } A = 1, B = C = a = b = e = f = A_2 A_6 = 0.$$

Equation (74) becomes

$$\omega(\omega-1) = \delta \omega' .$$

The solutions of this equation are

$$\omega = 0, 1, \frac{1}{1+\delta} .$$

The inverse transform of 0 is a trivial solution; the inverse transform of 1 is a generalized function; and the inverse of $(1+\delta)^{-1}$ is $\exp(-\eta)$. We ignore the first two solutions and only consider the last one, which is a well-known result that has been obtained by several authors.

$$\text{II. } B = 1/2, A = C = a = b = e = f = A_2 A_6 = 0.$$

Equation (74) becomes

$$\omega'[\omega + \delta - 1] = 0 .$$

The solutions are

$$\omega = \text{Constant}, \quad 1 - \delta .$$

Hence, the inverse of ω is either zero or a generalized function. Thus, there are no nontrivial solutions in the ordinary sense, that possess Laplace transforms, also see Drake [1972b].

$$\text{III. } C = 1, A = B = a = b = e = f = A_2 A_6 = 0.$$

In this case, (74) becomes

$$[\omega']^2 + (2 - \delta)\omega' + \omega = 0 .$$

This equation is a Clairaut type whose general solution is

$$\omega(\delta) = E(2-\delta) - E^2 ,$$

and whose singular solution is

$$\omega(s) = \frac{1}{4}(s-2)^2 ,$$

where E is a constant. When $E=0$, we have the trivial solution. When $E \neq 0$, we get only generalized solutions for $\psi(\eta)$. Thus, we have the same conclusion as in Example II, also see Drake [1972b].

IV. Trial Solution: $\psi = \exp[-\eta]$.

Since we have four free parameters in (68), a question arises, "Can we find values of a , b , e and f such that there are ordinary solutions of (51) for the values of A , B and C in Examples II and III?" As an example, suppose we choose ψ to be $\exp[-\eta]$. Then, the transform of ψ is given by $\omega = (1+s)^{-1}$; the derivatives of ω are given by

$$\omega^{(\eta)} = (-1)^\eta \eta! (1+s)^{-\eta-1} ,$$

where η is the η^{th} derivative of ω with respect to s .

Substituting $\omega^{(\eta)}$ into (74) results in the following relationships:

$$\begin{aligned} A_6 &= a , \\ A + 2B + C &= 1 , \\ b &= -2e - 6f \\ C &= -12A_2f , \\ 2B &= -C - 4A_2e , \\ A &= 1 + C - 2A_2b , \end{aligned}$$

where a and A_2 are assumed to be arbitrary.

If the degree of β is greater than zero in (69), then $h(\eta)$ can not be identically zero if $\psi = \exp [-\eta]$. That is, we require some type of condensational growth or source and sink distribution to get the exponential solution for these β 's. The following examples satisfy the above relationships among the constants A, B, C, a, b, e and f .

$$(a) \quad A = C = 0, \quad B = \frac{1}{2}$$

$$\therefore f = 0, \quad e = -\frac{1}{4A_2}, \quad b = \frac{1}{2A_2};$$

$$(b) \quad A = B = 0, \quad C = 1$$

$$\therefore f = -\frac{1}{12A_2}, \quad e = -\frac{1}{4A_2}, \quad b = \frac{1}{A_2};$$

$$(c) \quad A = B = C = \frac{1}{4}$$

$$\therefore f = -\frac{1}{48A_2}, \quad e = -\frac{3}{16A_2}, \quad b = \frac{1}{2A_2}.$$

Examples (a) and (b) have the same kernels β as Examples II and III, respectively. Hence, by imposing a condensational growth mechanism and/or a source and sink distribution upon the system, we can change the character of the solution $\psi(\eta)$ from generalized functions to exponential functions.

Now, let us consider Equation (53). The general solution of (53) is given by

$$\psi(\eta) = A_6 [h(\eta)]^{-1} \exp \left[-A_6 \int \{h(\eta)\}^{-1} d\eta \right]. \quad (75)$$

By choosing the form of $h(\eta)$, we can easily derive an explicit form for ψ from (75). However, the choice of $h(\eta)$ must be such that (14) is satisfied.

If

$$h(\eta) = b(\eta+a)^{p+1}, \quad p > 0,$$

then

$$\int_0^{\infty} h(\eta) \psi(\eta) d\eta \text{ diverges.}$$

If

$$h(\eta) = A_6 (1+a)^{-1} (a+\eta), \quad a > 0,$$

then

$$\psi(\eta) = (1+a) a^{1+a} (a+\eta)^{-a-2}.$$

Thus, for small a , this last result is the Junge power law. Combining these last two results we note that $h(\eta)$ cannot increase faster than η as $\eta \rightarrow +\infty$, and the limiting form of $\psi(\eta)$ is the Junge power-law.

Another power law is given when

$$h(\eta) = A_6 \frac{\pi(4\eta^4+1)}{8\eta} \left[1 - \frac{2}{\pi} \tan^{-1}(2\eta^2) \right],$$

namely

$$\psi(\eta) = \frac{8\eta}{(4\eta^4 + 1)}.$$

There are several forms of $h(\eta)$ that result in $\psi(\eta)$'s of the following form:

$$\psi(\eta) = p(\eta) \exp [-q(\eta)].$$

Table II gives a list of the $h(\eta)$'s and the corresponding functions $p(\eta)$ and $q(\eta)$. Example 1 in Table II has several interesting special cases, see Table III. If $h(\eta)$ only represents condensational growth, then Solution 3 in Table III is the well-known Maxwellian growth, and Solution 4 is a surface dominated condensational growth mechanism.

7. INDIRECT SOLUTION METHODS FOR THE HOMOGENEOUS SYSTEM

The first step in deriving our indirect solution techniques is to assume that $N(\xi)$ is known in Equation (51), see Wang [1966]. The solution of the resulting linear, integral equation is given by

$$\psi(\eta) = \frac{\exp [K(\eta)]}{2A_2 h(\eta) - \eta} \left[2A_2 A_6 + \int_0^\eta N(\xi) \exp \{-K(\xi)\} d\xi \right], \quad (76)$$

where

$$K(\eta) = \int \frac{1 - 2A_2 A_6}{2A_2 h(\eta) - \eta} d\eta. \quad (77)$$

A special case of (76) occurs when $2A_2 A_6 = 1$. Thus,

$K(\eta) \equiv 0$ and (76) reduces to

$$\psi(\eta) = \frac{1 + \int_0^\eta N(\xi) d\xi}{2A_2 h(\eta) - \eta}. \quad (78)$$

When $A_2 = 0$ and A_6 is finite in (76), we have the "pure coagulation" process. For this case we have

$$K(\eta) = \ln(1/\eta),$$

and

$$\psi(\eta) = \eta^{-2} \int_{\eta}^{\infty} \xi N(\xi) d\xi, \quad (79)$$

where

$$\int_0^{\infty} \xi N(\xi) d\xi = 0.$$

Equation (79) is an alternate form of Equation (52).

Equation (51) can also be written as

$$h(\eta) = A_6 \frac{M_0(\eta)}{\psi(\eta)} + \frac{1}{2A_2} \left[\eta - \frac{M_0(\eta)}{\psi(\eta)} - \frac{1}{\psi(\eta)} \int_{\eta}^{\infty} N(\xi) d\xi \right]. \quad (80)$$

The equations in (76) to (80) are the bases for our indirect methods. These methods are given below:

Method I.

- (a) Choose $h(\eta)$.
- (b) Compute $K(\eta)$ from Equation (77).
- (c) Choose $g(\xi, \eta; a)$ in

$$F(\xi, \eta) = \frac{g(\xi, \eta; a)}{\int_0^{\infty} \int_0^{\infty} g(\xi, \eta; a) d\xi d\eta}.$$

- (d) Compute $N(\xi)$ from Equation (33a).
- (e) Compute $\psi(\eta)$ from Equation (76).
- (f) Evaluate the parameter "a" from

$$M_0 = M_1 = 1.$$

- (g) Compute $\beta(\xi, \eta)$ from Equation (28).

Method II.

- (a) Choose $g(\xi, \eta; a)$.
- (b) Compute $N(\xi)$ from Equation (33a).
- (c) Compute $\psi(\eta)$ from Equation (79).
- (d) Evaluate the parameter "a" from
$$M_0 = M_1 = 1 .$$
- (e) Compute $\beta(\xi, \eta)$ from Equation (28).

Method III.

- (a) Assume $h(\eta) \neq 0$.
- (b) Assume that the coagulation process and the condensation-source processes in Equation (80) are uncoupled.
- (c) Let $\psi(\eta)$ be given by Method II.
- (d) Using this $\psi(\eta)$, evaluate $h(\eta)$ from
$$h(\eta) = \frac{A_6 M_0(\eta)}{\psi(\eta)} .$$
- (e) The coagulation and condensation-source processes are coupled in the untransformed system through the quantities $\phi(t)$ and $N(t)$.

Method IV.

- (a) Choose $\psi(\eta)$ such that $M_0 = M_1 = 1$, see Equation 14.
- (b) Choose $\beta(\xi, \eta)$ such that Equation (27) is satisfied.
- (c) Compute $N(\xi)$ from Equation (33a).
- (d) Compute $h(\eta)$ from Equation (80).

A convenient way to satisfy Item (a) in Method IV is to use the table of Mellin transforms found in Erdélyi et al

[1954]. These can be applied in the following way. Suppose we write $\psi(\eta)$ as

$$\psi(\eta; \delta) = A \eta^{\delta-1} q(\eta) \quad , \quad (81)$$

where the Mellin transform of $q(\eta)$ is given by

$$h(\delta) = \int_0^{\infty} q(\eta) \eta^{\delta-1} d\eta = \frac{1}{A} \int_0^{\infty} \psi(\eta; \delta) d\eta \quad . \quad (82)$$

Combining Equations (14), (81) and (82) gives us a candidate for $\psi(\eta)$, namely

$$\psi(\eta) = \frac{\eta^{\delta-1} q(\eta)}{h(\delta)} \quad , \quad (83)$$

where δ is defined by

$$h(\delta) = h(\delta+1) \quad . \quad (84)$$

A general example of $\psi(\eta)$ that satisfies (14) is given by

$$\psi(\eta) = \frac{A \eta^a}{(\eta^b + m)^c} \quad , \quad (85)$$

where (14) imposes the following restrictions on a , b , c , m and A :

$$m^{\frac{1}{b}} \Gamma\left(\frac{a+2}{b}\right) \Gamma\left(c - \frac{a+2}{b}\right) = \Gamma\left(\frac{a+1}{b}\right) \Gamma\left(c - \frac{a+1}{b}\right) \quad , \quad (86)$$

$$b \Gamma(c) = A \Gamma\left(\frac{a+2}{b}\right) \Gamma\left(c - \frac{a+2}{b}\right) m^{\frac{a+1-bc}{b} + \frac{1}{b}} \quad .$$

In Table IV, we have listed several special cases of (85) and (86). For the $\psi(\eta)$ given in (85), the power moments M_ν only exist for

$$-a-1-\epsilon \leq \nu \leq bc-a-1-\epsilon$$

where $m>0$ and ϵ is an arbitrarily small positive number.

Another example of $\psi(\eta)$ that satisfies (14) is given by

$$\psi(\eta) = A\eta^B e^{-C\eta^D}, \quad (87)$$

where (14) imposes the following restrictions on A, B, C and D:

$$A = \frac{DC^{\frac{B+1}{D}}}{\Gamma\left(\frac{B+1}{D}\right)}, \quad C = \left[\frac{\Gamma\left(\frac{B+2}{D}\right)}{\Gamma\left(\frac{B+1}{D}\right)} \right]^D. \quad (88)$$

Table V gives several special cases of Equations (87) and (88).

Other candidates for $\psi(\eta)$ that satisfy (14) are as follows:

$$\psi(\eta) = \frac{1}{12} \left(\frac{\pi}{\ln 2} \right)^2 \frac{1}{1 + \exp\left[\frac{\pi^2 \eta}{12 \ln 2} \right]},$$

$$\psi(\eta) = \eta^{-3} \exp \left[-\eta^{-1} \right],$$

$$\psi(\eta) = \sqrt{\frac{a}{\pi \eta}} \exp \left[2\sqrt{a} - a\eta - \frac{1}{\eta} \right],$$

where $a=1 \pm 3/2$.

In the remainder of this section, we list several solutions of (51) and (52) obtained by Methods I to IV.

Example 1 (Method 1)

$$\begin{aligned}\text{Let} \quad h(\eta) &= (2A_2)^{-1}(1+\eta) . \\ \text{Then} \quad K(\eta) &= -\eta \text{ and } 2A_2A_6 = 2 . \\ \text{Let} \quad F(\xi, \eta) &= \exp(-\xi-\eta) . \\ \text{Then} \quad N(\xi) &= (\xi-2) \exp(-\xi) , \\ \psi(\eta) &= \frac{1}{2} [\eta-2]^2 \exp[-\eta] , \\ \text{and} \quad \beta(\xi, \eta) &= \frac{4}{[(\eta-2)(\xi-2)]^2} .\end{aligned}\tag{89}$$

In this case the coagulation parameter is unbounded and the function $\psi(\eta)$ is bimodal, see Figure 5.

Example 2 (Method I)

Choose the same quantities $h(\eta)$, $K(\eta)$ and $2A_2A_6$ as in Example 1.

$$\begin{aligned}\text{Let} \quad F(\xi, \eta) &= 4 \exp[-2\xi-2\eta] . \\ \text{Then} \quad N(\xi) &= 4(\xi-1) \exp[-2\xi] , \\ \psi(\eta) &= 2 \exp[-\eta] - 4\eta \exp[-2\eta] , \\ \text{and} \quad \beta(\xi, \eta) &= \frac{\exp[-\xi-\eta]}{[1-2\eta \exp(-\eta)][1-2\xi \exp(-\xi)]} .\end{aligned}\tag{90}$$

In this case β is bounded and $\psi(\eta) > 0$; in addition, $\psi(\eta)$ decreases for increasing η for all $\eta > 0$.

Example 3 (Method II), see Wang [1966]

$$\begin{aligned}\text{Let} \quad F(\xi, \eta) &= A[\exp\{-b(\xi+\eta)\} + \exp\{-a(\xi+\eta)\}] . \\ \text{Then} \quad \psi(\eta) &= \frac{A}{ab} [b \exp(-a\eta) + a \exp(-b\eta)] , \\ \text{where} \quad \frac{a^3b^3}{A} &= ab^3 + a^3b = b^3 + a^3 .\end{aligned}\tag{91}$$

A well-known solution of (52) is obtained when $a = b = 2A = 1$ in Equation (91). For this case we have $\psi(\eta) = \exp[-\eta]$ and $\beta(\xi, \eta) = 1$.

Example 4 (Method II), see Wang [1966]

Let $F(\xi, \eta) = A(\xi + \eta) [\exp\{-b(\xi + \eta)\} + \exp\{-a(\xi + \eta)\}]$.

Then $\psi(\eta) = \frac{A}{a^2 b^2} [b^2(a\eta + 1) \exp(-a\eta) + a^2(b\eta + 1) \exp(-b\eta)]$,
(92)

where $\frac{a^4 b^4}{A} = 2ab^4 + 2a^4 b = 3a^4 + 3b^4$.

A specific example of Equation (92) is given by

$$\psi(\eta) = \frac{3}{8}(3\eta + 2) \exp\left[-\frac{3}{2}\eta\right], \quad (92a)$$

where $a = b = \frac{3}{2}$, $A = \frac{27}{32}$ and

$$\beta(\xi, \eta) = \frac{4}{3} \frac{\xi + \eta}{\left(\eta + \frac{2}{3}\right) \left(\xi + \frac{2}{3}\right)}.$$

Example 5 (Method II)

Let $F(\xi, \eta) = \frac{(a+1)(a+2)a^{1+a}}{[\xi + \eta + a]^{3+a}}$, $a > 0$.

Then $\psi(\eta) = \frac{(a+1)a^{a+1}}{(\eta + a)^{2+a}}$,
(93)

where $\beta(\xi, \eta) = \frac{a+2}{(a+1)a^{a+1}} \frac{(\xi + a)^{2+a} (\eta + a)^{2+a}}{(\xi + \eta + a)^{3+a}}$.

If "a" is sufficiently small, then $\psi(\eta) \approx a\eta^{-2}$ for large η .

Hence for "pure coagulation", the Junge power-law is given by a kernel possessing the features of the above $\beta(\xi, \eta)$.

Example 6 (Method II)

$$\begin{aligned}\text{Let} \quad F(\xi, \eta) &= \frac{25}{3} \exp[-\sqrt{10}(\xi+\eta)] \quad . \\ \text{Then} \quad \psi(\eta) &= \frac{5}{3} [\sqrt{10\eta} + 1] e^{-\sqrt{10\eta}} \quad , \\ \text{where} \quad \beta(\xi, \eta) &= \frac{3 \exp[\sqrt{10\eta} + \sqrt{10\xi} - \sqrt{10(\xi+\eta)}]}{[\sqrt{10\xi} + 1][\sqrt{10\eta} + 1]} \quad .\end{aligned}\tag{94}$$

Since $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$ for $a \geq 0$ and $b \geq 0$, then β is unbounded as ξ and $\eta \rightarrow +\infty$.

Example 7 (Method II)

$$\begin{aligned}\text{Let} \quad F(\xi, \eta) &= a^4 \xi \eta \exp[-a(\xi+\eta)] \quad , \quad a = \frac{7}{3} \quad . \\ \text{Then} \quad \psi(\eta) &= \frac{a^2}{6} [a\eta^2 + 4\eta] e^{-a\eta} \quad , \\ \text{where} \quad \beta(\xi, \eta) &= \frac{36}{[a\eta + 4][a\xi + 4]} \quad .\end{aligned}\tag{95}$$

Example 8 (Method II)

$$\begin{aligned}\text{Let} \quad F(\xi, \eta) &= \frac{\pi}{8} \exp\left[-\left\{\frac{\sqrt{\pi}}{4}(\xi+\eta)\right\}^2\right] \quad . \\ \text{Then} \quad \psi(\eta) &= \frac{\pi}{4} \operatorname{erfc}\left[\frac{\sqrt{\pi}\eta}{4}\right] \quad , \\ \text{where} \quad \beta(\xi, \eta) &= \frac{2}{\pi} \frac{\exp\left[-\frac{\pi}{16}(\xi+\eta)^2\right]}{\operatorname{erfc}\left(\frac{\sqrt{\pi}\eta}{4}\right)\operatorname{erfc}\left(\frac{\sqrt{\pi}\xi}{4}\right)}\end{aligned}\tag{96}$$

and where $\operatorname{erfc}(x)$ is the complementary error function. For large x , we know that

$$\operatorname{erfc}(x) \approx \frac{1}{\sqrt{\pi} x \exp(x^2)} \quad .$$

Thus, for large ξ and η , we have

$$\psi(\eta) \approx \frac{1}{\eta} \exp \left[-\frac{\pi}{16} \eta^2 \right] ,$$

and

$$\beta(\xi, \eta) \approx \frac{\pi}{8} \eta \xi \exp \left[-\frac{\pi}{8} \xi \eta \right] .$$

Hence, β is bounded and approaches zero as ξ and $\eta \rightarrow \infty$.

Example 9 (Method II)

$$\text{Let } F(\xi, \eta) = \frac{2a^2 \exp[-a(\xi+\eta)]}{\sqrt{\pi} \sqrt{a(\xi+\eta)}} .$$

$$\text{Then } \psi(\eta) = 2a \operatorname{erfc}(\sqrt{a\eta}) , \quad (97)$$

$$\text{where } \beta(\xi, \eta) = \frac{\exp[-a(\xi+\eta)]}{2\sqrt{\pi} \sqrt{a(\xi+\eta)} \operatorname{erfc}(\sqrt{a\eta}) \operatorname{erfc}(\sqrt{a\xi})} ,$$

$$\text{and } a = 4 \int_0^\infty \tau^3 \operatorname{erfc}(\tau) d\tau .$$

For small ξ and η , we have

$$\psi(\eta) \approx 2a ,$$

and

$$\beta(\xi, \eta) \approx \frac{1}{2\sqrt{\pi} \sqrt{a(\xi+\eta)}} .$$

For large ξ and η , we have

$$\psi(\eta) \approx 2\sqrt{\frac{a}{\pi\eta}} e^{-a\eta} ,$$

and

$$\beta(\xi, \eta) \approx \frac{\sqrt{a\pi}}{2} \sqrt{\frac{\eta\xi}{\eta+\xi}} .$$

Comparing the $\psi(\eta)$'s in Equations (91) to (97) and the corresponding β 's, we note that the ψ 's with the "lightest tails" correspond to the bounded β 's and the ψ 's with the "heaviest tails" correspond to the unbounded β 's. This seems reasonable since the unbounded kernels (unbounded for large values of ξ and η) promote more growth in the larger particle sizes than the bounded kernels; hence, the resulting spectra possess heavier tails.

Example 10 (Method II)

A generalization of Example 7 is given by

$$F(\xi, \eta) = g(\xi)g(\eta) .$$

Thus, we have

$$\psi(\eta) = \eta^{-2} [G_1(\eta) * g(\eta) + G_0(\eta) * \eta g(\eta) + G_1 G_0(\eta) - G_1(\eta)] , \quad (98)$$

where $*$ is the convolution operation and

$$G_v(\eta) = \int_{\eta}^{\infty} \eta^v g(\eta) d\eta , \quad G_v = G_v(0) , \quad G_0 = 1 .$$

Equation (98) reduces to (95) when

$$g(\eta) = a^2 \eta \exp[-a\eta] , \quad a = \frac{7}{3} .$$

Example 11 (Method II)

Another general form is given by

$$F(\xi, \eta) = h(\xi+\eta) + \xi\eta g(\xi+\eta) .$$

The corresponding $\psi(\eta)$ is given by

$$\psi(\eta) = H_0(\eta) + \frac{2}{3} \eta G_1(\eta) - \frac{1}{2} \eta^2 G_0(\eta) ,$$

where

$$\begin{aligned} H_v(\eta) &= \int_{\eta}^{\infty} \eta^v h(\eta) d\eta , & H_v &= H_v(0) , \\ G_v(\eta) &= \int_{\eta}^{\infty} \eta^v g(\eta) d\eta , & G_v &= G_v(0) . \end{aligned} \tag{99}$$

and

$$H_1 + \frac{1}{6} G_3 = 1 .$$

A special case of (99) is given by

$$\psi(\eta) = \left[\frac{a}{3} + \frac{5a^2 \eta}{9} + \frac{a^3 \eta^2}{18} \right] \exp[-a\eta] ,$$

where

$$\begin{aligned} g(\tau) &= \frac{a^4}{3} \exp[-a\tau] , \\ h(\tau) &= \frac{a^3 \tau}{3} \exp[-a\tau] , & a &= \frac{16}{9} . \end{aligned}$$

The next set of examples are given in Table VI. These solutions were obtained by Method III. Hence, Table VI lists the quantities $h(\eta)$ that correspond to some of the ψ 's obtained by Method II.

Example 12 (Method IV)

$$\text{Let} \quad F(\xi, \eta) = \sum_{\ell=0}^p (\xi \eta)^{\ell} g_{\ell}(\xi + \eta) .$$

From Equation (80) we have

$$h(\eta) = \frac{A_6 M_0(\eta)}{\psi(\eta)} + \frac{1}{2A_2} \left[\eta - \frac{M_0(\eta)}{\psi(\eta)} + \right. \\ \left. + \frac{1}{\psi(\eta)} \sum_{\ell=0}^p \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \int_{\eta}^{\infty} \frac{\tau^{2\ell+1} - 2\eta^{\ell+j+1} \tau^{\ell-j}}{\ell+j+1} g_{\ell}(\tau) d\tau \right] . \quad (100)$$

A specific example of Equation (100) is given by

$$\beta(\xi, \eta) = A + B \cdot (\xi + \eta) + C\xi\eta ,$$

$$A + 2B + C = 1 ,$$

$$\psi(\eta) = \frac{(m+1)^{m+1}}{m!} \eta^m e^{-(m+1)\eta} .$$

The functions $g_n(\tau)$ in $F(\xi, \eta)$ are defined by

$$g_0(\tau) = \dots = g_{m-1}(\tau) = 0 ,$$

$$g_m(\tau) = \frac{(m+1)^{2m+2} (A+B\tau)}{(m!)^2} e^{-(m+1)\tau} ,$$

$$g_{m+1}(\tau) = \frac{(m+1)^{2m+2} C}{(m!)^2} e^{-(m+1)\tau} .$$

If we let

$$Q_N(\eta; a) e^{-a\eta} = \int_{\eta}^{\infty} \tau^N e^{-a\tau} d\tau ,$$

or

$$Q_N(\eta; a) = \frac{(a\eta)^N + N(a\eta)^{N-1} + N(N-1)(a\eta)^{N-2} + \dots + N!}{a^{N+1}} ,$$

then $h(\eta)$ is given by

$$\begin{aligned}
h(\eta) = & \frac{A_6 Q_m(\eta; m+1)}{\eta^m} + \frac{1}{2A_2} \left[\eta - \frac{Q_m(\eta; m+1)}{\eta^m} + \right. \\
& + \frac{(m+1)^{m+1}}{m! \eta^m} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{1}{m+1+j} \left\{ A Q_{2m+1}(\eta; m+1) + \right. \\
& - 2A \eta^{m+1+j} Q_{m-j}(\eta; m+1) + B Q_{2m+2}(\eta; m+1) + \\
& \left. - 2B \eta^{m+1+j} Q_{m-j+1}(\eta; m+1) \right\} + \\
& + \frac{(m+1)^{m+1}}{m! \eta^m} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \frac{1}{m+2+j} \left\{ C Q_{2m+3}(\eta; m+1) + \right. \\
& \left. \left. - 2C \eta^{m+2+j} Q_{m+1-j}(\eta; m+1) \right\} \right] .
\end{aligned}$$

When $m=0$, $h(\eta)$ reduces to

$$h(\eta) = A_6 + \frac{B}{2A_2} \left[2\eta - \eta^2 \right] + \frac{C}{2A_2} \left[2\eta - \frac{1}{2} \eta^2 - \frac{1}{6} \eta^3 \right] .$$

When $m=1$, $h(\eta)$ reduces to

$$\begin{aligned}
h(\eta) = & A_6 \left[\frac{1}{2} + \frac{1}{4} \eta^{-1} \right] + \frac{1}{2A_2} \left[\frac{1}{2} \eta - \frac{1}{3} \eta^2 \right] + \\
& + \frac{B}{2A_2} \left[\eta - \frac{1}{3} \eta^3 \right] + \\
& + \frac{C}{2A_2} \left[\eta - \frac{1}{6} \eta^3 - \frac{1}{15} \eta^4 \right] .
\end{aligned}$$

Example 13 (Method IV)

Suppose we choose β to be the Brownian kernel, namely

$$\beta(\xi, \eta) = A \left[2 + \sqrt[3]{\xi/\eta} + \sqrt[3]{\eta/\xi} \right] ,$$

where

$$A = \left[2 + 2M_{\frac{1}{3}} M_{-\frac{1}{3}} \right]^{-1} .$$

From Equation (80), we have

$$h(\eta) = \frac{A_6 M_0(\eta)}{\psi(\eta)} + \frac{1}{2A_2} \left[\eta - \frac{M_0(\eta)}{\psi(\eta)} + \right. \quad (101)$$

$$\left. + \frac{M_0(\eta) + M_{\frac{1}{3}} M_{-\frac{1}{3}}(\eta) - M_0(\eta) * \psi(\eta) - \left\{ \eta^{\frac{1}{3}} \psi(\eta) \right\} * M_{-\frac{1}{3}}(\eta)}{\left(1 + M_{-\frac{1}{3}} M_{\frac{1}{3}} \right) \psi(\eta)} \right] .$$

A specific example of Equation (101) is given when $\psi(\eta) = e^{-\eta}$, namely,

$$h(\eta) = A_6 + \frac{1}{2A_2} \left[\eta - 1 + \right.$$

$$\left. + \frac{1 - \eta + \Gamma\left(\frac{4}{3}\right) e^{\eta} M_{-\frac{1}{3}}(\eta) - e^{\eta} \left\{ \eta^{\frac{1}{3}} e^{-\eta} \right\} * M_{-\frac{1}{3}}(\eta)}{1 + \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)} \right] ,$$

where $\Gamma(x)$ is the gamma function and

$$M_{-\frac{1}{3}}(\eta) = \int_{\eta}^{\infty} \xi^{-\frac{1}{3}} e^{-\xi} d\xi$$

is an incomplete gamma function.

8. TRANSFORMATION FROM (z, v, t) TO (η, v)

The equation given in (1) is a partial, integrodifferential equation defined over the (z, v, t) space. The objective of our work is to find similarity transformations that will transform (1) to an ordinary, integrodifferential equation defined over a one dimensional space denoted by θ . The "plan of attack" that will be followed in fulfilling this objective is to first transform the system from the (z, v, t) space to a (η, v) space, and then transform the intermediate system to the θ -space.

The candidate for η is defined by

$$\eta = \eta(z, t) = \frac{z}{\alpha_1(t)} = \frac{z}{\alpha_1} \quad , \quad (102)$$

where $\alpha_1(t)$ is a function that will be specified later in this report. The relationships between the derivatives with respect to η and with respect to z and t are given by

$$\frac{\partial}{\partial z} = \frac{1}{\alpha_1} \frac{\partial}{\partial \eta} \quad , \quad (103)$$

$$\frac{\partial}{\partial t} = - \frac{\alpha_1'}{\alpha_1} \eta \frac{\partial}{\partial \eta} \quad , \quad (104)$$

where the prime indicates differentiation with respect to t .

We assume that n and the system parameters are given by the following:

$$\begin{aligned}
 n &= \alpha_2 \bar{n} = \alpha_2(t) \bar{n}(\eta, v) \quad , \\
 I &= \alpha_3 \bar{I} = \alpha_3(t) \bar{I}(\eta, v) \quad , \\
 W &= \alpha_4 \bar{W} = \alpha_4(t) \bar{W}(\eta) \quad , \\
 W_s &= \alpha_5 \bar{W}_s = \alpha_5(t) \bar{W}_s(\eta, v) \quad , \\
 D &= \alpha_6 \bar{D} = \alpha_6(t) \bar{D}(\eta) \quad , \\
 \beta &= \alpha_7 \bar{\beta} = \alpha_7(t) \bar{\beta}(\eta, u, v) \quad , \\
 S &= \alpha_8 \bar{S} = \alpha_8(t) \bar{S}(\eta, v) \quad .
 \end{aligned} \tag{105}$$

Combining (2), (3) and (105) give the transformed equations for $\tilde{F}$ and $\tilde{N}$, namely,

$$\begin{aligned}
 \tilde{F} &= \alpha_2^2 \alpha_7 \bar{F} = \alpha_2^2(t) \alpha_7(t) \bar{F}(\eta, u, v) \quad , \\
 \tilde{N} &= \alpha_2^2 \alpha_7 \bar{N} = \alpha_2^2(t) \alpha_7(t) \bar{N}(\eta, v) \quad ,
 \end{aligned} \tag{106}$$

where

$$\begin{aligned}
 \bar{F}(\eta, u, v) &= \bar{\beta}(\eta, u, v) \bar{n}(\eta, u) \bar{n}(\eta, v) \\
 &= \bar{\beta}(\eta, v, u) \bar{n}(\eta, v) \bar{n}(\eta, u) \\
 &= \bar{F}(\eta, v, u) \geq 0 \quad ,
 \end{aligned} \tag{107}$$

and

$$\bar{N}(\eta, v) = \int_0^v \bar{F}(\eta, u, v-u) du - 2 \int_0^\infty \bar{F}(\eta, u, v) du. \tag{108}$$

Combining Equations (1) and (102) to (108) gives the defining equation for $\bar{n}$ in the (η, v) - space:

$$\begin{aligned}
 & - A_1 \eta \frac{\partial \bar{n}}{\partial \eta} + A_2 \bar{n} + A_3 \frac{\partial \bar{I} \bar{n}}{\partial v} + A_4 \frac{\partial \bar{W} \bar{n}}{\partial \eta} + \\
 & - A_5 \frac{\partial \bar{W}_s \bar{n}}{\partial \eta} = A_6 \frac{\partial \left[\bar{D} \frac{\partial \bar{n}}{\partial \eta} \right]}{\partial \eta} + \frac{1}{2} A_7 \bar{N} = A_8 \bar{S} \quad ,
 \end{aligned} \tag{109}$$

where the A_i , $i = 1, 2, \dots, 8$, are arbitrary constants.

To guarantee similarity, the $\alpha_i(t)$, $i=1, 2, \dots, 8$, in Equations (102) and (105) must satisfy the following relationships:

$$\begin{aligned}
 \alpha_1(t) &= \alpha_1(o) \exp [A_1 F(t)] \quad , \\
 \alpha_2(t) &= \alpha_2(o) \exp [A_2 F(t)] \quad , \\
 \alpha_3(t) &= A_3 E(t) \quad , \\
 \alpha_4(t) &= \alpha_1(o) A_4 E(t) \exp [A_1 F(t)] \quad , \\
 \alpha_5(t) &= \alpha_1(o) A_5 E(t) \exp [A_1 F(t)] \quad , \\
 \alpha_6(t) &= [\alpha_1(o)]^2 A_6 E(t) \exp [2A_1 F(t)] \quad , \\
 \alpha_7(t) &= [\alpha_2(o)]^{-1} A_7 E(t) \exp [-A_2 F(t)] \quad , \\
 \alpha_8(t) &= \alpha_2(o) A_8 E(t) \exp [A_2 F(t)] \quad ,
 \end{aligned} \tag{110}$$

where $E(t)$ is an arbitrary function of t and $F(t)$ is defined by

$$\frac{dF(t)}{dt} = E(t) \quad . \tag{111}$$

Even though the similarity transformation defined by Equations (102) to (105) imposes restrictions on the α_i 's there can be an infinite number of different forms of the α_i 's since $E(t)$ is an arbitrary function of t . Many of these forms may not be physically meaningful; however, the similarity solutions that result from the various choices of $E(t)$ have several useful purposes. For example, the solutions can be used for checking numerical schemes formulated for more meaningful physical systems and the solutions will contribute to the overall quantitative and qualitative theory of the physical systems defined by Equation (1). Examples of $E(t)$ and the α_i 's are given in Tables VIIa and VIIb.

9. TRANSFORMATION FROM (η, v) TO θ -SPACE

The next step in our search for similarity solutions is to transform the system in Equation (109) into a system involving only one independent variable, θ . The candidate for θ is given by

$$\theta = \frac{v}{\xi_1(\eta)} = \frac{v}{\xi_1} \quad . \quad (112)$$

The derivatives with respect to v and η transform in the following manner:

$$\begin{aligned} \frac{\partial}{\partial v} &= \frac{1}{\xi_1} \frac{d}{d\theta} \quad , \\ \frac{\partial}{\partial \eta} &= - \frac{\xi_1'}{\xi_1} \theta \frac{d}{d\theta} \quad . \end{aligned} \quad (113)$$

In addition, we assume that $\bar{n}$ and the system parameters in Equation (109) transform as

$$\begin{aligned}
\bar{n} &= \xi_2(\eta) \psi(\theta) = \xi_2 \psi & , \\
\bar{I} &= \xi_3(\eta) I(\theta) = \xi_3 I & , \\
\bar{W} &= \xi_4(\eta) = \xi_4 & , \\
\bar{W}_s &= \xi_5(\eta) W_s(\theta) = \xi_5 W_s & , \\
\bar{D} &= \xi_6(\eta) = \xi_6 & , \\
\bar{\beta} &= \xi_7(\eta) \beta(\phi, \theta) = \xi_7 \beta & , \\
\bar{S} &= \xi_8(\eta) S(\theta) = \xi_8 S & , \\
\bar{F} &= \xi_2^2(\eta) \xi_7(\eta) F(\phi, \theta) = \xi_2^2 \xi_7 F & , \\
\bar{N} &= \xi_1(\eta) \xi_2^2(\eta) \xi_7(\eta) N(\theta) = \xi_1 \xi_2^2 \xi_7 N & ,
\end{aligned} \tag{114}$$

where

$$\begin{aligned}
F(\phi, \theta) &= \beta(\phi, \theta) \psi(\phi) \psi(\theta) & , \\
&= \beta(\theta, \phi) \psi(\theta) \psi(\phi) & , \\
&= F(\theta, \phi) \geq 0 & ,
\end{aligned} \tag{115}$$

and

$$N(\theta) = \int_0^\theta F(\phi, \theta - \phi) d\phi - 2 \int_0^\infty F(\phi, \theta) d\phi \quad . \tag{116}$$

Combining Equations (109) and (112) to (116) gives the following:

$$\begin{aligned}
& B_1 \psi + B_2 \theta \psi' + B_3 (I\psi)' - B_4 W_s \psi + \\
& + B_5 \theta (W_s \psi)' = B_6 \theta^2 \psi'' + B_7 N + B_8 S
\end{aligned}
\tag{117}$$

where the B_i 's are arbitrary constants. The system in (117) is an ordinary, integrodifferential equation for the similarity spectrum $\psi(\theta)$. The system parameters are $W_s(\theta)$, $I(\theta)$, $S(\theta)$ and $\beta(\phi, \theta)$.

To guarantee similarity in (117), the quantities ξ_i in Equations (112) and (114) must satisfy the following formulas:

$$\begin{aligned}
\xi_1(\eta) &= G(\eta) \\
\xi_2(\eta) &= C_2 H(\eta) J(\eta) \\
\xi_3(\eta) &= C_3 G(\eta) \\
\xi_4(\eta) &= C_{41} \eta + C_{42} [J(\eta)]^{-1} + C_{43} J'(\eta) [J(\eta)]^{-3} \\
\xi_5(\eta) &= C_5 [J(\eta)]^{-1} \\
\xi_6(\eta) &= C_6 [J(\eta)]^{-2} \\
\xi_7(\eta) &= C_7 [G'(\eta) H(\eta)]^{-1} \\
\xi_8(\eta) &= C_8 H(\eta) J(\eta)
\end{aligned}
\tag{118}$$

where

$$\begin{aligned}
H(\eta) &= [G(\eta)]^{B_4/B_5} \\
J(\eta) &= G'(\eta)/G(\eta)
\end{aligned}
\tag{119}$$

The function $G(\eta)$ is arbitrary. In Tables VIIla and VIIlb

we list examples of $G(\eta)$, $u(\eta)$, $J(\eta)$ and $\xi_i(\eta)$.

The constants C_i in Equation (118) are defined in terms of the constants A_i and B_i . The defining equations are

$$\begin{aligned}
 C_1 &= \frac{B_1 + \frac{B_4}{B_5} \left[B_2 + B_6 - \frac{B_4 B_6}{B_5} \right]}{A_1 + A_2}, \\
 C_2 &= \frac{C_1 A_5}{B_5}, \\
 C_3 &= \frac{B_3}{A_3 C_1}, \\
 C_{41} &= \frac{A_1}{A_4}, \\
 C_{42} &= \frac{1}{A_4 C_1} \left[\frac{2B_4 B_6}{B_5} - B_2 - B_6 \right], \\
 C_{43} &= \frac{B_6}{A_4 C_1}, \\
 C_5 &= \frac{B_5}{A_5 C_1} = C_2^{-1}, \\
 C_6 &= \frac{B_6}{A_6 C_1}, \\
 C_7 &= \frac{2B_5 B_7}{A_5 A_7 C_1^2}, \\
 C_8 &= \frac{B_8 A_5}{A_8 B_5}.
 \end{aligned} \tag{120}$$

10. INDIRECT SOLUTION TECHNIQUES FOR THE NONHOMOGENEOUS SYSTEM

Equation (117) is a nonlinear, ordinary, integrodifferential equation. The "integral" and "nonlinear" character of the system is due to $N(\theta)$. Thus, if $N(\theta) \equiv 0$, Equation (117) reduces to a linear, second order, ordinary differential equation with variable coefficients. This reduced system can be solved in "closed form" for certain values of the parameters, and, in general, it can be solved by series techniques. If $N(\theta) \neq 0$, then indirect solution techniques are required.

The indirect solution techniques that are considered in this section require that $\psi(\theta)$, $F(\phi, \theta)$ or $N(\theta)$ be given. If $N(\theta)$ or $F(\phi, \theta)$ is given, we can solve for $\psi(\theta)$ in terms of the parameters $I(\theta)$, $W_s(\theta)$ and $S(\theta)$. Using this quantity $\psi(\theta)$, we can obtain $\beta(\phi, \theta)$ from $N(\theta)$ or $F(\phi, \theta)$. However, if $\beta(\phi, \theta)$ is obtained from $N(\theta)$ a complex, linear, integral equation must be solved. Hence, it is better to specify $F(\phi, \theta)$ than $N(\theta)$.

If we initially specify $\psi(\theta)$, then Equation (117) reduces to an equation involving the four parameters, $\beta(\phi, \theta)$, $I(\theta)$, $W_s(\theta)$ and $S(\theta)$. Hence, we can solve for one of the parameters in terms of the other three. Since $\beta(\phi, \theta)$ is contained in the integrals of $N(\theta)$, it is best to specify $\beta(\phi, \theta)$ and solve for one of the three remaining parameters in terms of the other two.

From physical considerations, one would probably want to specify $\beta(\phi, \theta)$, $I(\theta)$ and $W_S(\theta)$. Then for any given $\psi(\theta)$, we can solve Equation (117) for $S(\theta)$. Hence, for realistic values of $\beta(\phi, \theta)$, $I(\theta)$ and $W_S(\theta)$, we can derive a correspondence between the similarity spectra $\psi(\theta)$ and the distributions of volumetric sources or sinks $S(\theta)$.

As an example of this correspondence, suppose we let

$$\begin{aligned}\beta(\phi, \theta) &= 1, \\ I(\theta) &= \theta, \\ W_S(\theta) &= \theta,\end{aligned}\tag{121}$$

Thus, Equation (117) becomes

$$\begin{aligned}S(\theta) &= -\frac{B_6}{B_8} \theta^2 \psi'' + \frac{B_2 \theta + B_3 \theta + B_5 \theta^2}{B_8} \psi' + \\ &+ \frac{B_1 + B_3 + 2M_O B_7 + B_5 \theta - B_4 \theta}{B_8} \psi - \frac{B_7}{B_8} \psi * \psi,\end{aligned}\tag{122}$$

where "*" is the convolution operator and M_O is defined by

$$M_O = \int_0^\infty \psi(\theta) d\theta.\tag{123}$$

For simplicity, suppose

$$\begin{aligned}B_1 &= B_2 = B_3 = \frac{1}{2}, \\ B_4 &= B_5 = B_6 = B_7 = B_8 = 1.\end{aligned}\tag{124}$$

Combining Equations (122) and (124) gives

$$\begin{aligned}
S(\theta) = & -\theta^2 \psi'' + (\theta + \theta^2) \psi' + \\
& + (1+2M_O) \psi - \psi * \psi .
\end{aligned}
\tag{125}$$

If

$$\psi_1 = \exp [-\theta] ,$$

then Equation (125) gives

$$S_1 = (3-2\theta-2\theta^2) \exp [-\theta] .$$

If

$$\psi_2 = (1+\theta)^{-2}$$

then we have

$$S_2(\theta) = \left[3-2\theta - \frac{6\theta^2}{(1+\theta)^2} - \frac{2\theta(1+\theta)}{(2+\theta)^2} - \frac{4(1+\theta)^2 \ln(1+\theta)}{(2+\theta)^3} \right] (1+\theta)^{-2} .$$

Finally, if

$$\psi_3 = \frac{1}{2} \theta^2 \exp [-\theta] ,$$

then we have

$$S_3(\theta) = \frac{[90+150\theta-60\theta^2-\theta^3]\theta^2}{60} \exp [-\theta] .$$

In Figure 6 we have plotted the curves for the ψ 's; and in Figure 7 we have the curves for the S 's. Comparing the curves for ψ with those for S shows how the source and sink distributions vary with ψ for fixed values of $\beta(\phi, \theta)$, $W_s(\theta)$ and $I(\theta)$.

Now, suppose we initially specify $F(\phi, \theta)$. Therefore, we can compute $N(\theta)$ from (116). Equation (117) then reduces to a nonhomogeneous, second order, linear, ordinary differential equation given by

$$B_6 \theta^2 \psi'' - \bar{p}(\theta) \psi' - \bar{q}(\theta) \psi = -\bar{r}(\theta) \quad , \quad (126)$$

where

$$\begin{aligned} \bar{p}(\theta) &= B_2 \theta + B_3 I(\theta) + B_5 \theta W_S(\theta) \quad , \\ \bar{q}(\theta) &= B_1 + B_3 I'(\theta) + B_5 \theta W_S'(\theta) - B_4 W_S(\theta) \quad , \\ \bar{r}(\theta) &= B_7 N(\theta) + B_8 S(\theta) \quad . \end{aligned} \quad (127)$$

In Table IX we have given several examples of $F(\phi, \theta)$ along with the corresponding expressions for $N(\theta)$. It is interesting to note the sign changes in Examples 6, 7 and 8 as the exponent of $(\phi + \theta + 1)$ changes.

Suppose $B_6 = 0$ in Equation (126). Then we have a first order linear equation given by

$$\bar{p}(\theta) \psi' + \bar{q}(\theta) \psi = \bar{r}(\theta) \quad . \quad (128)$$

The solution of (128) is given by

$$\begin{aligned} \psi(\theta) &= \psi(0) \exp \left[- \int_0^\theta \frac{\bar{q}(x)}{\bar{p}(x)} dx \right] + \\ &+ \exp \left[- \int_0^\theta \frac{\bar{q}(x)}{\bar{p}(x)} dx \right] \int_0^\theta \frac{\bar{r}(x)}{\bar{p}(x)} \exp \left[+ \int_0^x \frac{\bar{q}(y)}{\bar{p}(y)} dy \right] dx \quad . \end{aligned} \quad (129)$$

Hence, Equation (129) gives the similarity spectrum $\psi(\theta)$ in terms of the assumed quantity $N(\theta)$ and the parameters $I(\theta)$, $W_s(\theta)$ and $S(\theta)$. The parameter $\beta(\phi, \theta)$ is obtained from $\psi(\theta)$ and the quantity $F(\phi, \theta)$. The dimensional results are then obtained from the equations of transformation that map the (z, v, t) system into the θ system. Equation (129) is the exact solution of Equation (117) when $B_6=0$ and the coagulation process is negligible, that is, when $N(\theta) \equiv 0$.

If $B_6 \neq 0$ in Equation (126), we have a second order, differential equation in ψ , namely,

$$\boxed{\psi'' + p(\theta)\psi' + q(\theta)\psi = r(\theta)} \quad (130)$$

where

$$\begin{aligned} p(\theta) &= -\bar{p}(\theta)/B_6\theta^2, \\ q(\theta) &= -\bar{q}(\theta)/B_6\theta^2, \\ r(\theta) &= -\bar{r}(\theta)/B_6\theta^2. \end{aligned} \quad (131)$$

Suppose we let

$$\psi(\theta) = f(\theta)\Gamma(\theta), \quad (132)$$

see Rainville [1964]. Combining (130) and (132) gives

$$f\Gamma'' + (2f' + pf)\Gamma' + (f'' + pf' + qf)\Gamma = r. \quad (133)$$

If we choose an $f(\theta)$ such that

$$f'' + pf' + qf = 0, \quad (134)$$

then Equation (133) reduces to a first order equation in Γ' , namely

$$f\Gamma'' + (2f' + pf)\Gamma' = r \quad (135)$$

The solution of Equation (135) is given by

$$\begin{aligned} \Gamma(\theta) = & \Gamma(0) + \Gamma'(0) \int_0^\theta f^{-2}(x) \exp \left[-\int_0^x p(y) dy \right] dx + \\ & + \int_0^\theta \left\{ f^{-2}(x) \exp \left[-\int_0^x p(y) dy \right] \int_0^x r(y) f(y) \exp \left[\int_0^y p(z) dz \right] dy \right\} dx. \end{aligned} \quad (136)$$

Thus, to determine $\psi(\theta)$, we need only obtain a particular solution of the second order equation in Equation (134).

Rainville [1964] describes techniques for solving systems of the type given in (134) and in Table X we list several specific examples of closed-form solutions of this system.

In summary, we outline a procedure for obtaining similarity solutions by the indirect method. The procedure is as follows:

1. Specify the function $F(\phi, \theta)$;
2. Compute $N(\theta)$ from (116);
3. Realistically choose the quantities $I(\theta)$, $W_s(\theta)$ and $S(\theta)$;
4. Choose a set of values for the constants B_i , $i=1, 2, \dots, 8$;

5. Compute $p(\theta)$, $q(\theta)$ and $r(\theta)$ from (127) and (131);
6. Find a particular solution of Equation (134), see Rainville [1964];
7. Evaluate $\Gamma(\theta)$ from Equation (136);
8. Evaluate $\psi(\theta)$ from Equation (132);
9. Evaluate $\beta(\phi, \theta)$ from Equation (115);
10. Choose a set of values for the constants A_i , $i=1, 2, \dots, 8$;
11. Evaluate the constants C_i , $i=1, 2, \dots, 8$, from Equation (120);
12. Choose a candidate for the function $G(\eta)$;
13. Evaluate the $\xi_i(\eta)$, $i=1, 2, \dots, 8$, from Equation (118);
14. Choose a candidate for the function $E(t)$;
15. Evaluate the $\alpha_i(t)$, $i=1, 2, \dots, 8$, from Equation (110);
16. Equation (112) gives the volume v in terms of θ and ξ_1 ;
17. Equation (102) gives the coordinate z in terms of η and α_1 ;
18. Equations (105) and (114) give the dimensional dependent variable n and the parameters I to S in terms of the α_i 's, ξ_i 's, $\psi(\theta)$ and the transformed parameters I to S . The final form of the dimensional similarity solution is given by

$$n = \alpha_2(t) \xi_2 \left[\frac{z}{\alpha_1(t)} \right] \psi \left[\frac{v}{\xi_1 \left\{ \frac{z}{\alpha_1(t)} \right\}} \right] . \quad (137)$$

Referring to Items 1, 3, 4, 10, 12 and 14, we note that a large (infinite, in fact) variety of similarity solutions can be generated by this procedure. By judicious choices of the constants and parameters, one can generate a useful set of similarity solutions for the system given in Equation (1). Tables VIIa to X can be used in this selection process.

In the next section we give a complete example of the application of this solution procedure.

11. A DETAILED EXAMPLE

Assume that $F(\phi, \theta)$ is given by

$$F(\phi, \theta) = \exp [-\phi - \theta] .$$

Then, from Table IX, Example 1, we have

$$N(\theta) = (\theta - 2) \exp [-\theta] .$$

In addition, let us assume that

$$I(\theta) = \theta ,$$

$$W_s(\theta) = \theta ,$$

$$S(\theta) = (\theta^4 - \theta + 2) \exp [-\theta] .$$

The choice for I can be interpreted as a volume dependent, condensational growth mechanism. The expression for W_s is somewhat unrealistic since Stoke's Law would give $\theta^{2/3}$. However, for simplicity, we will use the above formula for W_s . The expression for $S(\theta)$ is plotted in Figure 8. This relation-

ship could be interpreted as a distribution of sources such that the major contribution to the aerosol is from the "intermediate size ranges", a secondary contribution from the "small size ranges", and an exponentially decaying contribution from the "large size ranges."

A set of B_i 's that is reasonable for this problem is given by

$$\begin{aligned} B_1 &= 2, \quad B_2 = B_4 = 0, \\ B_3 &= B_5 = B_6 = B_7 = B_8 = 1. \end{aligned}$$

Obviously, this is an arbitrary set, and there are an infinite number of other possible sets of B_i 's that one could consider.

From Equations (127) and (131), we have

$$\begin{aligned} p(\theta) &= - [1 + \theta^{-1}] , \\ q(\theta) &= - [\theta^{-1} + 3\theta^{-2}] , \\ r(\theta) &= -\theta^2 \exp [-\theta] . \end{aligned}$$

For these values of p and q , a particular solution of Equation (134) is given by

$$f(\theta) = \theta^{-1} .$$

Therefore, from Equation (136), we have

$$\Gamma(\theta) = 3 - \left[\frac{\theta^3}{2} + \frac{3\theta^2}{2} + 3\theta + 3 \right] \exp [-\theta] .$$

Finally, from (132) we have

$$\psi(\theta) = \frac{3}{\theta} - \left[\frac{\theta^2}{2} + \frac{3\theta}{2} + 3 + \frac{3}{\theta} \right] \exp [-\theta] .$$

In Figure 9, we have plotted the values of $\psi(\theta)$ vs θ for

$0 \leq \theta \leq 10$. Beyond the value of $\theta=10$, $\psi(\theta) \approx 3\theta^{-1}$. For small values of θ , $\psi(\theta) \approx \frac{1}{8} \theta^3 \exp[-\theta]$.

From Equation (115), we obtain the quantity $\beta(\phi, \theta)$, namely,

$$\beta(\phi, \theta) = \frac{4\theta\phi}{[6 \exp(\theta) - \theta^3 - 3\theta^2 - 6\theta - 6][6 \exp(\phi) - \phi^3 - 3\phi^2 - 6\phi - 6]}$$

For small values of ϕ and θ , β reduces to

$$\beta(\phi, \theta) \approx \left[\frac{4}{\phi\theta} \right]^3.$$

For large values of ϕ and θ , we have

$$\beta(\phi, \theta) \approx \frac{\phi\theta}{9} \exp[-\phi-\theta].$$

Choose $E(t) = t^{-1}$ in Equation (110) and Table VIIa, and choose $G(\eta) = \eta$ in Equation (118) and Table VIIIa. From these choices we can obtain the ξ_i and α_i . A reasonable set of values for the A_i 's and C_i 's is the following:

$$\begin{aligned} A_1 &= 1, & A_2 &= -\frac{1}{2} \\ C_1 &= 4, & C_2 &= 4A_5, & A_5 &= \text{arbitrary}, \\ A_3C_3 &= -C_{42}A_4 = C_{43}A_4 = \frac{1}{4}, \\ A_5C_5 &= A_6C_6 = \frac{1}{4}, \\ C_{41}A_4 &= 1, & A_8C_8 &= A_5, & A_7C_7 &= \frac{1}{8A_5}. \end{aligned}$$

The quantity A_5 need not be specified and the individual factors in the A_iC_j products also need not be specified.

Using the transformed results, the values of A_i , B_i and

C_i , and the equations of transformation, we obtain the corresponding dimensional quantities; these are given below:

$$\begin{aligned}\eta &= zt^{-1}, \quad \theta = tvz^{-1}, \\ n(z,v,t) &= 4A_5 t^{1/2} z^{-1} \psi(tvz^{-1}), \\ I(z,v,t) &= \frac{1}{4} vt^{-1}, \\ W_s(z,v,t) &= \frac{1}{4} v, \\ W(z,t) &= \frac{1}{2} zt^{-1}, \\ D(z,t) &= \frac{1}{4} z^2 t^{-1}, \\ \beta(z,u,v,t) &= \frac{1}{8} A_5^{-1} t^{-1/2} \beta(tuz^{-1}, tvz^{-1}), \\ S(z,v,t) &= A_5 t^{-1/2} z^{-1} S(tvz^{-1}).\end{aligned}$$

Even though there was a great deal of arbitrariness in the choice of the constants and parameters in this example, there is still a fair amount of realism in the above parameters. For example, the condensational growth mechanism is volume dependent, which is a possible mechanism of growth. The decrease of I with time to t^{-1} could be interpreted as a decrease in the concentration of the condensing gas or vapor as time increases. The variation of the terminal velocity of the particles with respect to v is reasonable over some size ranges, even though Stoke's Law varies as $v^{2/3}$. The z and t variations of W and D could be interpreted as the conditions in the environment of a converging updraft in a decaying (the t^{-1} variations in W and D) cumulus cloud. Finally, the coagu-

lation parameter β indicates rapid coagulation for the smaller size ranges and at higher elevations (maybe the elevations within the cumulus cloud). Hence, it is felt that the above example is reasonable and outlines a method for the generation of similarity solutions for the system given in Equation (1).

12. CONCLUSIONS

The purpose of this report was to derive similarity solutions for an evolving aerosol population in homogeneous and nonhomogeneous media under the influences of coagulation, condensational growth, a vertical wind field, gravitational settling, turbulent diffusion and a source and sink distribution. The conclusions of this study are given in the following list.

1. Transformation equations were derived that mapped the physical systems into a one-dimensional similarity space.
2. For the homogeneous case the total number and volume concentrations were expressed in terms of the similarity solutions for various formulations of the system parameters.
3. Several limit properties of the similarity solution for the homogeneous case were obtained for arbitrarily small and large values of the similarity variable.

4. Sets of similarity solutions for aerosols evolving in homogeneous media were obtained by direct and indirect methods. The forms of the solutions varied with changes in the system parameters.
5. The homogeneous systems that did not contain coagulation terms were solved directly for every allowable condensation-source process.
6. Certain "pure coagulation" systems, described by $\beta(\xi, \eta)$, did not possess similarity solutions in the ordinary sense; however, if a proper condensation-source process $h(\eta)$ was added to the system, the character of the solutions were changed from generalized functions to exponential functions.
7. The exponential "ramp" function, $\psi(\eta) = \exp [-\eta]$, was found to exist under a variety of different relationships for β and $h(\eta)$ in the homogeneous case.
8. The Junge power law, $\psi(\eta) \sim \text{const. } \eta^{-2}$, also existed under a variety of combinations of β 's and $h(\eta)$'s.
9. The wide "spectrum" of solutions obtained for the pure coagulation process allowed us to study the relationships between the coagulation parameters and the similarity spectrum $\psi(\eta)$. For example, the "heavier tailed" spectra corresponded to the β 's that were unbounded as $(\xi, \eta) \rightarrow (+\infty, +\infty)$, while the "lighter tailed" spectra corresponded to the bounded kernels.

10. An indirect method for the solution of the similarity equation for the nonhomogeneous case was outlined.
11. Finally, an exact solution for the nonhomogeneous case was obtained. This pseudo-realistic solution was derived by the procedure mentioned in Conclusion 10.

Using the same procedure, Conclusions 10 and 11, other similarity solutions can be derived for other sets of parameters. The totality of these similarity solutions can then be used to check numerical techniques derived for "more realistic" systems, to derive qualitative and quantitative results for the system in Equation (1), and can be used to form series solutions or asymptotic expansions for systems other than those for which the similarity solutions were obtained. The important thing to remember in obtaining similarity solutions is that much of the physical structure of Equation (1) is retained even though some of the parameters corresponding to the similarity solutions may not be too realistic. Therefore, if we can get similarity solutions for a wide range of parameters in Equation (1), the resulting set of similarity solutions can be applied to realistic problems in the ways mentioned above.

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APPENDIX A

Figures

FIGURE 1. A schematic of a similarity transformation applied to an evolving aerosol spectrum in a homogeneous medium.

FIGURE 2. The gamma function, $y = \Gamma(x)$.

FIGURE 3. The psi function, $y = \psi(x) = \frac{d}{dx} \ln \Gamma(x)$.

FIGURE 4. An example of a source function $S(\eta)$. The quantities η_1 and η_2 depend upon the value of "a" in Equation (67).

FIGURE 5. The bimodal spectrum defined by Equation (89).

FIGURE 6. Examples of similarity spectra, $\psi(\theta)$.

SOURCE AND SINK DISTRIBUTIONS

FIGURE 7. Examples of source and sink distributions, $S(\theta)$.

FIGURE 8. The source distribution used in the detailed example in Section 11.

FIGURE 9. The similarity spectrum $\psi(\theta)$ used in the detailed example in Section 11.

APPENDIX B

Tables

TABLE I. $N(t)$ and $\phi(t)$ for Various System Parameters

(I) $\alpha(t) \neq 0$ and $2A_2A_6 \neq 1$

$$p = -\frac{2(A_2 + A_3)}{1 - 2A_2A_6}, \quad q = 0, \quad A(t) = -\frac{1}{2}(1 - 2A_2A_6)\alpha(t),$$

$$B(t) = \frac{N(0)}{2}(1 - 2A_2A_6) \int_0^t \alpha(u) du.$$

(Ia) If $2A_2A_6 < 1$, then $B(t) \geq 0$

$N(t) \downarrow$, $\phi(t) \downarrow$ if $A_2 + A_3 < 0$, $\phi(t) = \phi(0)$ if $A_2 = -A_3$,
 $\phi(t) \uparrow$ if $A_2 + A_3 > 0$.

(where $\uparrow$ means increasing and $\downarrow$ means decreasing as t increases)

(Ib) If $2A_2A_6 > 1$, then $B(t) \leq 0$

$N(t) \uparrow$ and may $\rightarrow \infty$ at finite t , $\phi(t)$ same as (Ia).

(II) $\alpha(t) \neq 0$ and $2A_2A_6 = 1$ and $A_2 + A_3 \neq 0$

$$p = \infty, \quad q = 0, \quad A(t) = \beta(t) = 0$$

$$\lim_{\substack{p \rightarrow \infty \\ B(t) \rightarrow 0}} [1 + B(t)]^{\frac{p}{q-1}} = \exp[(A_2 + A_3)N(0) \int_0^t \alpha(u) du],$$

$N(t) = N(0)$, $\phi(t) \uparrow$ if $A_2 + A_3 > 0$, $\phi(t) \downarrow$ if $A_2 + A_3 < 0$.

(IIa) $A_2 + A_3 = 0$

$$N(t) = N(0), \quad \phi(t) = \phi(0), \quad m_v(t) = m_v(0).$$

TABLE I. (continued)

(III) $\alpha(t) \equiv 0$ and $a(t) \neq 0$

$$q = P = \frac{1 + A_5}{A_1 + A_5 S_0}, \quad A(t) = \frac{[N(0)]^P (A_1 + A_5 S_0)}{\phi(0)} a(t),$$

$$B(t) = \left[(1-A_1) + A_5(1 - S_0) \right] \frac{N(0)}{\phi(0)} \int_0^t a(u) du.$$

(IIIa) $A_1 = S_0 = 1$, thus $A_6 = 1 + A_5$

$$N(t) = \frac{N(0)}{\phi(0)} \phi(t) = N(0) \exp \left\{ \frac{(1 + A_5)N(0)}{\phi(0)} \int_0^t a(u) du \right\},$$

$N(t)$ and $\phi(t) \uparrow$ if $1 + A_5 > 0$, $N(t)$ and $\phi(t) \downarrow$ if $A_5 + 1 < 0$,

$N(t) = N(0)$, $\phi(t) = \phi(0)$ if $A_5 = -1$.

(IIIb) $A_6 = 0$ or $A_1 = -A_5 S_0$

$$N(t) = N(0), \quad \phi(t) = \phi(0) + [1 + A_5]N(0) \int_0^t a(u) du,$$

$\phi(t) \uparrow$ if $1 + A_5 > 0$, $\phi(t) = \phi(0)$ if $A_5 = -1$,

$\phi(t) \downarrow$ if $1 + A_5 < 0$.

(IV) $\alpha(t) \equiv 0$ and $a(t) \equiv 0$

$$P = 1/S_0, \quad q = 3 - P, \quad A(t) = \frac{\phi(0) S_0}{[N(0)]^P} b(t),$$

$$B(t) = \frac{(q - 1) \phi(0) S_0}{[N(0)]^2} \int_0^t b(u) du.$$

TABLE I. (Continued)

(IVa) $S_0 = 1/2$

$$\phi(t) = \phi(o) \exp \left[\frac{\phi(o)}{\{N(o)\}^2} \int_0^t b(u) \, du \right], \quad \frac{N(t)}{N(o)} = \left[\frac{\phi(t)}{\phi(o)} \right]^{1/2},$$

(N(t) and $\phi(t)$ increases or decrease depending upon the sign of $b(t)$..).

TABLE II. Functions $h(\eta)$, $p(\eta)$ and $q(\eta)$

No.	$h(\eta)$	$p(\eta)$	$q(\eta)$
1	$A_6(p+1)^p [\Gamma(1/p+1)]^{-1-p} \eta^{-p}, p > -1$	$\left[\frac{\Gamma(1/p+1)}{(p+1)^p} \right]^{1+p} \eta^p$	$\frac{\left\{ \Gamma(1/p+1) \right\}^{p+1} \eta^{p+1}}{(p+1)^{p+1}}$
2	$\frac{A_6}{3} \left[1 + \sqrt{6\eta} \right]$	3	$\sqrt{6\eta}$
3	$A_6 \left[\eta^3 e^{\frac{1}{\eta}} - \eta^2 - \eta^3 \right]$	η^{-3}	$1/\eta$
4	$\frac{A_6}{20} \left[\left(60\eta \right)^{\frac{2}{3}} + 2 \left(60\eta \right)^{\frac{1}{3}} + 2 \right]$	10	$\sqrt[3]{60\eta}$
5	$\frac{A_6}{4} \left(2 + \eta^{-1} \right)$	4η	2η
6	$\frac{\pi}{2} A_6 \frac{\operatorname{erfc}\left(\frac{\eta}{\sqrt{\pi}}\right)}{\exp\left(-\frac{\eta^2}{\pi}\right)}$	$2/\pi$	η^2/π
7	$\sqrt{2\pi} A_6 \sqrt{\eta} \left[\exp(\eta/2) \operatorname{erfc}(\sqrt{\eta/2}) \right]$	$\frac{1}{\sqrt{2\pi\eta}}$	$\eta/2$

TABLE III. Special Cases of Example 1 in Table II

No.	p	p(n)	q(n)	Notes
1	0	1	n	$\psi(n) = \exp(-n)$
2	1	$\frac{\pi}{2} n$	$\frac{\pi}{4} n^2$	—
3	$-\frac{1}{3}$	$\sqrt[3]{\frac{\pi}{6}} n^{-\frac{1}{3}}$	$\sqrt[3]{\frac{9\pi}{16}} n^{\frac{2}{3}}$	Brock [1971, 1972]
4	$-\frac{2}{3}$	$\sqrt[3]{\frac{2}{9}} n^{-\frac{2}{3}}$	$\sqrt[3]{6} n^{\frac{1}{3}}$	Brock [1971, 1972]

TABLE IV. Special Cases of Equations (85) and (86)

No.	A	a	b	c	m
1	$\frac{2\Gamma(2\delta)}{[\Gamma(\delta)]^2}, 0 < \delta < 1$	$\delta - 1$	1	$1 + 2\delta$	1
2	$\frac{3\sqrt{3}}{2\pi}$	0	3	1	1
3	2	0	1	3	1
4	1	0	2	$3/2$	1

TABLE V. Special Cases of Equations (87) and (88)

No.	A	B	C	D
1	10	0	$\sqrt[3]{60}$	$1/3$
2	$\pi/2$	1	$\pi/4$	2
3	$2/\pi$	0	$1/\pi$	2
4	$\frac{1}{\sqrt{2}\pi}$	$-\frac{1}{2}$	$\frac{1}{2}$	1
5	1	0	1	1
6	4	1	2	1
7	3	0	$\sqrt{6}$	$1/2$

TABLE VI. Solutions Generated by Method III

No.	$\psi(\eta)$	$h(\eta)$
1	$\exp [-\eta]$	A_6
2	Equation (92a)	$\frac{2A_6}{3} \left[1 + \frac{2}{3\eta + 2} \right]$
3	Equation (93)	$\frac{A_6}{(a + 1)} (\eta + a)$
4	Equation (94)	$\frac{A_6}{5} \left[2 + \sqrt{10\eta} + (\sqrt{10\eta} + 1)^{-1} \right]$

TABLE VIIa. Examples of $E(t)$ and $\alpha_i(t)$

Given We Have	$E(t) = 1$	$E(t) = t^{-1}$
$F(t)$	t	$\ln(t)$
$\alpha_1(t)$	$\alpha_1(o) \exp [A_1 t]$	$A_1 t$
$\alpha_2(t)$	$\alpha_2(o) \exp [A_2 t]$	$A_2 t$
$\alpha_3(t)$	A_3	$A_3 t^{-1}$
$\alpha_4(t)$	$\alpha_1(o) A_4 \exp [A_1 t]$	$A_4 t^{A_1-1}$
$\alpha_5(t)$	$\alpha_1(o) A_5 \exp [A_1 t]$	$A_5 t^{A_1-1}$
$\alpha_6(t)$	$\alpha_1^2(o) A_6 \exp [2A_1 t]$	$A_6 t^{2A_1-1}$
$\alpha_7(t)$	$[\alpha_2(o)]^{-1} A_7 \exp [-A_2 t]$	$A_7 t^{-A_2-1}$
$\alpha_8(t)$	$\alpha_2(o) A_8 \exp [A_2 t]$	$A_8 t^{A_2-1}$

TABLE VIIb: Examples of $E(t)$ and $\alpha_i(t)$

Given We Have	$E(t) = 2t$	$E(t) = e^{-t}$
$F(t)$	t^2	$1 - e^{-t}$
$\alpha_1(t)$	$\alpha_1(o) \exp[A_1 t^2]$	$\alpha_1(o) \exp[A_1(1 - e^{-t})]$
$\alpha_2(t)$	$\alpha_2(o) \exp[A_2 t^2]$	$\alpha_2(o) \exp[A_2(1 - e^{-t})]$
$\alpha_3(t)$	$2A_3 t$	$A_3 \exp[-t]$
$\alpha_4(t)$	$2\alpha_1(o) A_4 t \exp[A_1 t^2]$	$\alpha_1(o) A_4 \exp[A_1(1 - e^{-t}) - t]$
$\alpha_5(t)$	$2\alpha_1(o) A_5 t \exp[A_1 t^2]$	$\alpha_1(o) A_5 \exp[A_1(1 - e^{-t}) - t]$
$\alpha_6(t)$	$2\alpha_1^2(o) A_6 t \exp[2A_1 t^2]$	$\alpha_1^2(o) A_6 \exp[2A_1(1 - e^{-t}) - t]$
$\alpha_7(t)$	$2[\alpha_2(o)]^{-1} A_7 t \exp[-A_2 t^2]$	$[\alpha_2(o)]^{-1} A_7 \exp[-A_2(1 - e^{-t}) - t]$
$\alpha_8(t)$	$2\alpha_2(o) A_8 t \exp[A_2 t^2]$	$\alpha_2(o) A_8 \exp[A_2(1 - e^{-t}) - t]$

TABLE VIIIa. Examples of $G(\eta)$ and $\xi_i(\eta)$

<div>Given We Have</div>	$G(\eta) = \eta$	$G(\eta) = \eta^{-1}$
$H(\eta)$	$\frac{B_4}{B_5} \eta$	$\frac{-B_4}{B_5} \eta$
$J(\eta)$	η^{-1}	$-\eta^{-1}$
$\xi_1(\eta)$	η	η^{-1}
$\xi_2(\eta)$	$\frac{B_4}{B_5} \eta - 1$	$-\frac{B_4}{B_5} \eta - 1$
$\xi_3(\eta)$	$C_3 \eta$	$C_3 \eta^{-1}$
$\xi_4(\eta)$	$(C_{41} + C_{42} - C_{43}) \eta$	$(C_{41} - C_{42} - C_{43}) \eta$
$\xi_5(\eta)$	$C_5 \eta$	$-C_5 \eta$
$\xi_6(\eta)$	$C_6 \eta^2$	$+C_6 \eta^2$
$\xi_7(\eta)$	$\frac{-B_4}{B_5} \eta$	$-C_7 \eta^2 + \frac{B_4}{B_5}$
$\xi_8(\eta)$	$\frac{B_4}{B_5} \eta - 1$	$-C_8 \eta^{-1} - \frac{B_4}{B_5}$

TABLE VIIb. Examples of $G(\eta)$ and $\xi_i(\eta)$

We Given Have	$G(\eta) = \exp[\eta]$	$G(\eta) = \exp[-\eta]$
$H(\eta)$	$\exp[B_4/B_5 \eta]$	$\exp[-B_4/B_5 \eta]$
$J(\eta)$	1	-1
$\xi_1(\eta)$	$\exp[\eta]$	$\exp[-\eta]$
$\xi_2(\eta)$	$C_2 \exp[B_4/B_5 \eta]$	$-C_2 \exp[-B_4/B_5 \eta]$
$\xi_3(\eta)$	$C_3 \exp[\eta]$	$C_3 \exp[-\eta]$
$\xi_4(\eta)$	$C_{41} \eta + C_{42}$	$C_{41} \eta - C_{42}$
$\xi_5(\eta)$	C_5	$-C_5$
$\xi_6(\eta)$	C_6	C_6
$\xi_7(\eta)$	$C_7 \exp[-\eta - B_4/B_5 \eta]$	$-C_7 \exp[\eta + B_4/B_5 \eta]$
$\xi_8(\eta)$	$C_8 \exp[B_4/B_5 \eta]$	$-C_8 \exp[-B_4/B_5 \eta]$

TABLE IX. Examples of $F(\phi, \theta)$ and $N(\theta)$

No.	$F(\phi, \theta)$, where $a > 0$	$N(\theta)$
1	$\exp[-a\phi - a\theta]$	$\left(\theta - \frac{2}{a}\right) \exp[-a\theta]$
2	$(\phi + \theta) \exp[-a(\phi + \theta)]$	$\left(\theta^2 - \frac{2\theta}{a} - \frac{2}{a^2}\right) \exp[-a\theta]$
3	$\exp[-a^2(\phi + \theta)^2]$	$\theta \exp[-a^2\theta^2] - \frac{\sqrt{\pi}}{a} \operatorname{erfc}[a\theta]$
4	$\exp\left[-\sqrt{a}(\phi + \theta)\right]$	$\left(\theta - \frac{4\sqrt{\theta}}{\sqrt{a}} - \frac{4}{a}\right) \exp\left[-\sqrt{a}\theta\right]$
5	$(\phi + \theta)^{-1/2} \exp[-a(\phi + \theta)]$	$\sqrt{\theta} \exp[-a\theta] - \frac{2\sqrt{\pi}}{\sqrt{a}} \operatorname{erfc}\left[\sqrt{a}\theta\right]$
6	$(\phi + \theta + 1)^{-2}$	$-(\theta + 2)(\theta + 1)^{-2}$
7	$(\phi + \theta + 1)^{-3}$	$-(\theta + 1)^{-3}$
8	$(\phi + \theta + 1)^{-4}$	$\frac{1}{3}(\theta - 2)(\theta + 1)^{-4}$
9	$g(\phi) g(\theta)$	$g(\theta) * g(\theta) - 2 G_O g(\theta), \quad G_O = \int_0^\infty g(\phi) d\phi$
10	$g(\phi + \theta)$	$\theta g(\theta) - 2 G_O g(\theta), \quad G_O(\theta) = \int_\theta^\infty g(\phi) d\phi$

TABLE X. Specific Examples of Closed-Form Solutions
of Equation 134, From Rainville [1964]

No.	$p(\theta)$	$q(\theta)$	Particular Solution, $f(\theta)$
1	$-\theta H(\theta)$	$H(\theta)$	θ
2	$\theta^{-1} - 1$	$-\theta^{-1}$	$\exp [\theta]$
3	$-\theta^{-1}$	$\theta^{-1} - 1$	$\exp [\theta]$
4	$1 - 2\theta^{-1}$	$-2 - \theta^{-1} + 2\theta^{-2}$	$\theta \exp [\theta]$
5	$1 - \theta^{-1}$	$-\theta^{-1}$	$\exp [-\theta]$
6	θ	1	$\exp \left[-\frac{1}{2} \theta^2 \right]$
7	$1 + \frac{3}{2} \theta^{-1}$	$\frac{3}{2} \theta^{-1} - \frac{1}{2} \theta^{-2}$	$\left[2 + \theta^{-1} \right] \exp [-\theta]$
8	$5 + \frac{5}{2} \theta^{-1}$	$\frac{5}{2} \theta^{-1}$	$\theta^{-3/2} - 10 \theta^{-1/2}$
9	$-1 - \frac{1}{2} \theta^{-1}$	$2 \theta^{-1} + \frac{1}{2} \theta^{-2}$	$\theta - \frac{2}{3} \theta^2$
10	$\frac{3}{2} \theta^{-1}$	$-\frac{1}{2} \theta^{-2}$	$\theta^{1/2}$ <u>or</u> θ^{-1}
11	$-\frac{1}{3} \theta^{-1}$	$-\frac{4}{3} \theta^{-2}$	θ^2 <u>or</u> $\theta^{-2/3}$
12	$-1 - \theta^{-1}$	θ^{-2}	$\theta \exp [\theta]$

TABLE X. (Continued)

No.	$p(\theta)$	$q(\theta)$	Particular Solution, $f(\theta)$
13	$3 + 4 \theta^{-1}$	$3 \theta^{-1}$	$\frac{9}{2} \theta^{-1} - 3 \theta^{-2} + \theta^{-3}$
14	$-1 + \theta^{-1}$	$3 \theta^{-1}$	$1 - 3\theta + \frac{3}{2} \theta^2 - \frac{1}{6} \theta^3$
15	$-\frac{1}{2} + \frac{1}{2} \theta^{-1}$	$-\frac{1}{2} - \frac{1}{2} \theta^{-1}$	$\exp[\theta] \text{ or } e^{\theta} \int_0^{\sqrt{\theta}} \exp\left[-\frac{3}{2} \beta^2\right] d\beta$

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