SEMI-ANALYTIC FLUX FORMULAS
FOR SHIELDING CALCULATIONS

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ABSTRACT

A special coordinate system based on the work of H. Ono and A. Tsuro has been used to derive exact semi-analytic formulas for the flux from cylindrical, spherical, toroidal, rectangular, annular and truncated cone volume sources; from cylindrical, spherical, truncated cone, disk and rectangular surface sources; and from curved and tilted line sources. In most of the cases where the source is curved, shields of the same curvature are allowed in addition to the standard slab shields; cylindrical shields are also allowed in the rectangular volume source flux formula. An especially complete treatment of a cylindrical volume source is given, in which dose points may be arbitrarily located both within and outside the source, and a finite cylindrical shield may be considered. Detector points may also be specified as lying within spherical and annular source volumes. The integral functions encountered in these formulas require at most two-dimensional numeric integration in order to evaluate the flux values. The classic flux formulas involving only slab shields and slab, disk, line, sphere and truncated cone sources become some of the many special cases which are given in addition to the more general formulas mentioned above.

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SEMI-ANALYTIC FLUX FORMULAS FOR SHIELDING CALCULATIONS

O. J. Wallace

I. INTRODUCTION

The formulas for calculating fluxes from radioactive source volumes in idealized shield configurations which are given in Chapter 9 of Rockwell (Ref. 1) were drawn almost entirely from the work of Foderaro and Obenshain (Ref. 2). Since that time a great deal of work has been done in this field, and much of this work is summarized in Ref. 3. There are, however, many errors in Ref. 3; and some important results (notably those reported by Ono and Tsuro and by Rhodes, Stallwood and Mott in Refs. 4, 5, and 6, and recent work by Dr. Kalman Shure of Bettis) are not included. Moreover, the methods used in Ref. 4, 5, and 6 may be applied to many different source and shield geometries to obtain useful results.

The present work will attempt to summarize the flux calculation formulas given in Ref. 1-6, and will also present new material, including exact semi-analytic formulas for the flux at the center of a toroidal source, the flux from a rectangular volume source with slab and cylindrical shields, the flux from cylindrical, spherical, truncated cone and rectangular surface sources, and the flux from an annular source. In all cases where a source is curved, curved shields are allowed as well as slab shields, and in no case does the evaluation of the resulting integral functions require more than two-dimensional numeric integration. A. Tsuro, in Ref. 5, gives the only treatment of curved shields available in the open literature.

The derivations are given in some detail because the methods, once understood, may be applied to situations not covered here.

All formulas given in this report are based on the point kernel

$$\phi_{un}(\vec{r}') = \frac{S(\vec{r})e^{-\mu t}}{4\pi\rho^2}$$

where $S(\vec{r})$ is the strength of a point isotropic source at $\vec{r}$, $\phi_{un}(\vec{r}')$ is the uncollided flux arriving at a detector point located at $\vec{r}'$, $\rho = |\vec{r}' - \vec{r}|$, and μt is the optical distance through an intervening shield. The integration of this point kernel over regular source volumes is the basic problem we encounter in finding fluxes due to extended sources, and our basic equation becomes

$$\phi_{un}(\vec{r}') = \int_v \frac{S_v(\vec{r})e^{-\bar{\mu}\rho}dv}{4\pi\rho^2}$$

where S_v is the source strength per unit volume, $\bar{\mu}$ is an attenuation coefficient homogenized over the source and shield(s) and $\bar{\mu}\rho$, the optical distance, is now a function of both external shield thickness and the self-attenuation of the source volume.

The flux formulas which result from the formal integration of the point-kernel equation over various regular geometric shapes require the numeric evaluation of at most two-dimensional integral functions. The integrands are non-singular, and all ray-tracing is semi-analytic, that is, implicit in the flux formulas themselves. These characteristics make these formulas easy to program, and a number of the formulas given here have been implemented in the SPAR1 computer program described in Ref. 7. The numeric integration of these two-dimensional integral functions is treated in detail in Ref. 7.

All the flux formulas presented here assume that the source strength S_v is constant throughout the source volume. There are two general ways in which this restriction may be removed:

The first way requires that the source strength be expressed in functional form such that it may be included in the integrand before formal integration takes place. Some examples of this technique are given in Refs. 1, 2, and 3. Similar modifications of the flux formulas given here may be derived by the reader for his specific problems.

A second and more general way in which a variable source distribution may be represented is that used in the SPAN4 program (Ref. 8), which does a general three-dimensional numeric integration of the point kernel over a volume, with explicit ray-tracing being performed. This permits the source distribution to be variable in one, two, or three dimensions.

II. FLUX FROM CYLINDRICAL AND ANNULAR SOURCES

This section presents formulas for the flux from cylindrical and annular sources for the following source and shield geometries:

- A. Cylindrical Source with Lateral Detector Point, Slab and Cylindrical Shields
- B. Annular Source with Lateral Detector Point, Slab and Cylindrical Shields
- C. Cylindrical Self-Source
- D. Annular Source with Interior Detector Point
- E. Annular Self-Source
- F. Cylindrical Source with Detector Point Off End
- G. Cylindrical Surface Source with Lateral Detector Point, Slab and Cylindrical Shields
- H. Cylindrical Surface Source with Interior Detector Point
- I. Cylindrical Surface Source with Axial Detector Point and Slab Shield

Reference 3 summarizes much of the work which has been done with regard to cylindrical and other volume and area sources, but some errors are present in the material which is presented. Reference 3 also gives a useful bibliography. References 4, 5, and 6 are basic to the results presented here. References 9, 10, and 11 present flux formulas and derivations involving only slab shields. All these references are valuable in that they furnish supplementary information on methods and formulas which may be cross checked. The reader should consult as many of these references as possible.

A. Cylindrical Source with Lateral Detector Point, Slab and Cylindrical Shields

Adopt a coordinate system which is logical for calculating fluxes using the point kernel, as shown in Figure 1. This approach comes from Refs. 4 and 5 and allows us to consider cylindrical as well as slab shields.

From Figure 1

$$\varphi_0 = \sin^{-1} \frac{R}{a+R}, \quad (1)$$

$$dV = (r \sec \varphi \sec^2 \theta \, d\theta)(r \sec^2 \varphi \, d\varphi)dr^*, \quad (2)$$

and we may define

- t = slab shield thickness
- μ = slab shield macroscopic attenuation coefficient
- a = radial distance from edge of source to detector point P
- R = source cylinder radius
- h = source cylinder height
- μ_s = macroscopic self-attenuation coefficient of source material
- $\vec{r}$ = vector lying along $\overline{OP}$; r is the magnitude of $\vec{r}$.
- S_v = source strength per unit volume
(assumed to be constant)
- μ_c = macroscopic attenuation coefficient of the cylindrical shield
- c = cylindrical shield thickness

*Figures 28 and 29 may aid in visualizing this volume element.

The total attenuation factor is then

$$A = \frac{e^{-[\mu t + \mu_c (r_1 - r_3) + \mu_s (r_2 - r_1)] \sec \varphi \sec \theta}}{4\pi (r \sec \varphi \sec \theta)^2} \quad (3)$$

where

$$r_1 = R \left\{ \left(1 + \frac{a}{R}\right) \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi} \right\} \quad (4)$$

$$r_2 = R \left\{ \left(1 + \frac{a}{R}\right) \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi} \right\} \quad (5)$$

$$r_3 = (R + c) \left\{ \left(1 + \frac{a-c}{R+c}\right) \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(1 + \frac{a-c}{R+c}\right)^2 \sin^2 \varphi} \right\} \quad (6)$$

r_1 and r_2 are the projection on $\overline{OP}$ of the intersections of $\overline{PQ}$ and the base of the source cylinder, as shown in Figure 1. r_3 is the projection on $\overline{OP}$ of the intersection of $\overline{PQ}$ and the outer surface of the cylindrical shield.

We define ϕ_{un} as the uncollided flux at the point P in Figure 1 emitted by the volumetric source. Then

$$d\phi_{un} = S_v \cdot A \cdot dV \quad (7)$$

FIGURE 1
CYLINDRICAL SOURCE WITH LATERAL
DETECTOR POINT, SLAB AND
CYLINDRICAL SHIELDS

Now determine limits of integration. As shown in Fig. 2, the cylinder may be divided into two regions, labeled I and II, and

$$\tan \theta_1 = \frac{\ell}{r_1 \sec \varphi} \quad (8)$$

$$\tan \theta_2 = \frac{\ell}{r_2 \sec \varphi} \quad (9)$$

As shown in Figures 2 and 3, in Region II,

$$r \sec \varphi = \ell \cot \theta$$

so that we may define

$$r = \ell \cot \theta \cos \varphi. \quad (10)$$

Therefore the limits are, in Region I,

$$\varphi = (0, \varphi_0)$$

$$r = (r_1, r_2), \text{ and}$$

$$\theta = \left(0, \tan^{-1} \frac{\ell}{r_2 \sec \varphi} \right).$$

while in Region II

$$\varphi = (0, \varphi_0),$$

$$r = (r_1, \ell \cot \theta \cos \varphi), \text{ and}$$

$$\theta = \left(\tan^{-1} \frac{\ell}{r_2 \sec \varphi}, \tan^{-1} \frac{\ell}{r_1 \sec \varphi} \right).$$

FIGURE 2
DIVISION OF A CYLINDRICAL SOURCE
INTO REGIONS I AND II

FIGURE 3
DETAIL OF DIVISION OF CYLINDRICAL SOURCE
INTO REGIONS I AND II

Formal integration of (7) gives, after substitution from (2) and (3)

$$\begin{aligned} \Phi_{un} = & \frac{S_V}{2\pi} \int_0^\varphi \int_{r_1}^{r_2} \int_0^{\theta_2} \exp \left\{ -[\mu t + \mu_c(r_1 - r_3) + \mu_s(r - r_1)] \sec \varphi \sec \theta \right\} \sec \varphi \, dr \, d\varphi \, d\theta \\ & + \frac{S_V}{2\pi} \int_0^\varphi \int_{r_1}^{r_\ell} \int_{\theta_2}^{\theta_1} \exp \left\{ -[\mu t + \mu_c(r_1 - r_3) + \mu_s(r - r_1)] \sec \varphi \sec \theta \right\} \sec \varphi \, dr \, d\varphi \, d\theta \end{aligned} \quad (11)$$

where r_1 , r_2 , r_3 , θ_1 , θ_2 , and r_ℓ are given by Eqs. (4), (5), (7), (8), (9), and (10), respectively.

The integration over r may be accomplished immediately, giving

$$\begin{aligned} \Phi_{un} = & \frac{S_V}{2\pi\mu_s} \int_0^\varphi \int_0^{\theta_2} \cos \theta \left[1 - \exp \left\{ -\mu_s(r_2 - r_1) \sec \varphi \sec \theta \right\} \right] \\ & \exp \left\{ -[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta \right\} \, d\varphi \, d\theta \\ & + \frac{S_V}{2\pi\mu_s} \int_0^\varphi \int_{\theta_2}^{\theta_1} \cos \theta \left[1 - \exp \left\{ -\mu_s(\ell \cot \theta \cos \varphi - r_1) \sec \varphi \sec \theta \right\} \right] \\ & \exp \left\{ -[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta \right\} \, d\varphi \, d\theta \end{aligned} \quad (12)$$

where

$$(r_2 - r_1) = 2R \cos \varphi \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}$$

A number of special cases should be noted.

Special Case #1: $l \rightarrow \infty$, giving an infinite cylindrical source

From (12),

$$\begin{aligned} \phi_{un} = \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_0} \int_0^{\pi/2} \cos \theta \left[1 - \exp \left\{ -\mu_s(r_2 - r_1) \sec \varphi \sec \theta \right\} \right] \\ \exp \left\{ -(\mu_t + \mu_c(r_1 - r_3)) \sec \theta \sec \varphi \right\} d\varphi d\theta \end{aligned} \quad (13)$$

Special Case #2: partial cylindrical source

As shown in Fig. 4, the flux from various portions of a cylindrical source may be found by simply changing the φ -limits to φ_1 and φ_2 , where

$$0 \leq \varphi_1 \leq \varphi_2 \leq \varphi_0 = \sin^{-1} \frac{R}{a+R}.$$

This generalizes (12) to

$$\begin{aligned} \phi_{un} = \frac{S_V}{4\pi\mu_s} \int_{\varphi_1}^{\varphi_2} \int_0^{\theta_2} \cos \theta \left[1 - \exp \left\{ -\mu_s(r_2 - r_1) \sec \varphi \sec \theta \right\} \right] \\ \exp \left\{ -[\mu_t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta \right\} d\varphi d\theta \\ + \frac{S_V}{4\pi\mu_s} \int_{\varphi_1}^{\varphi_2} \int_{\theta_2}^{\theta_1} \cos \theta \left[1 - \exp \left\{ -\mu_s(l \cot \theta \cos \varphi - r_1) \sec \varphi \sec \theta \right\} \right] \\ \exp \left\{ -[\mu_t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta \right\} d\varphi d\theta \end{aligned} \quad (14)$$

Special Case #3: $\mu_s = 0$, giving a non-attenuating cylindrical source

From (11),

$$\begin{aligned} \Phi_{un} = & \frac{S_v R}{\pi} \int_0^{\varphi_0} \int_0^{\theta_2} \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi} e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta} d\varphi d\theta \\ & + \frac{S_v}{2\pi} \int_0^{\varphi_0} \int_{\theta_2}^{\theta_1} [\ell \cot \theta - r_1 \sec \varphi] e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta} d\varphi d\theta \end{aligned} \quad (15)$$

and if $\ell \rightarrow \infty$,

$$\Phi_{un} = \frac{S_v R}{\pi} \int_0^{\varphi_0} \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi} e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta} d\varphi d\theta \quad (16)$$

The generalizations of (14) apply to (15) and (16) also.

Special Case #4: $\mu_s = \mu = \mu_c = 0$, giving an unshielded, non-attenuated source

From (15)

$$\begin{aligned} \Phi_{un} = & \frac{S_V R}{\pi} \int_0^{\varphi_0} \tan^{-1} \left(\frac{l}{r_2 \sec \varphi} \right) \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi} d\varphi \\ & + \frac{S_V l}{2\pi} \int_0^{\varphi_0} \log \sqrt{\frac{l^2 + r_2^2 \sec^2 \varphi}{l^2 + r_1^2 \sec^2 \varphi}} d\varphi \\ & - \frac{S_V}{2\pi} \int_0^{\varphi_0} \left\{ \tan^{-1} \left(\frac{l}{r_1 \sec \varphi} \right) - \tan^{-1} \left(\frac{l}{r_2 \sec \varphi} \right) \right\} r_1 \sec \varphi d\varphi \end{aligned} \quad (17)$$

Ref. 10 expresses this equation in terms of elliptic functions.

And if $l \rightarrow \infty$, from (16)

$$\Phi_{un} = \frac{S_V R}{2} \int_0^{\varphi_0} \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi} d\varphi. \quad (18)$$

The generalizations of (14) also apply to (17) and (18).

Special Case #5: $a = 0.0$, giving the flux at a point on the cylinder surface

From Figure 5,

$$r_1 = 0$$

$$r_2 = 2R \cos^2 \varphi$$

while in region II of Figure 5, (10) still applies,

$$\theta_1 = \frac{\pi}{2} \quad \text{and}$$

$$\theta_2 = \tan^{-1} \frac{l}{2R \cos \varphi}.$$

Then from (12), the uncollided flux at the point P in Figure 5 is

$$\begin{aligned} \Phi_{un} = & \frac{S_v}{2\pi\mu_s} \int_0^{\varphi = \frac{\pi}{2}} \int_0^{\tan^{-1} \frac{l}{2R \cos \varphi}} \cos \theta \left[1 - e^{-2\mu_s R \cos \varphi \sec \theta} \right] d\varphi d\theta \\ & + \frac{S_v}{2\pi\mu_s} \int_0^{\varphi = \frac{\pi}{2}} \int_{\tan^{-1} \frac{l}{2R \cos \varphi}}^{\frac{\pi}{2}} \cos \theta \left[1 - e^{-\mu_s l \csc \theta} \right] d\varphi d\theta \end{aligned} \quad (19)$$

or, rearranging,

$$\begin{aligned} \Phi_{un} = & \frac{S_v}{4\mu_s} - \frac{S_v}{2\pi\mu_s} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos \theta \exp \left\{ -2\mu_s R \cos \varphi \sec \theta \right\} d\varphi d\theta \\ & - \frac{S_v}{2\pi\mu_s} \int_0^{\frac{\pi}{2}} \int_{\theta_2}^{\frac{\pi}{2}} \cos \theta \exp \left\{ -\mu_s l \csc \theta \right\} d\varphi d\theta \end{aligned} \quad (20)$$

If $\ell \rightarrow \infty$, this becomes

$$\Phi_{un} = \frac{S_v}{4\mu_s} - \frac{S_v}{2\pi\mu_s} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos \theta e^{-2\mu_s R \cos \varphi \sec \theta} d\varphi d\theta \quad (21)$$

The generalizations of (14) apply to (20) and (21)

Special Case #6: $a = 0.0$ and $\mu_s = 0$, giving the non-attenuated flux on the cylinder surface

From (11), with $r_1 = 0$ and $r_2 = 2R \cos^2 \varphi$

$$\Phi_{un} = \frac{S_v R}{\pi} \int_0^{\frac{\pi}{2}} \tan^{-1} \left(\frac{\ell}{2R \cos \varphi} \right) \cos \varphi d\varphi + \frac{S_v \ell}{2\pi} \int_0^{\frac{\pi}{2}} \log \sqrt{1 + \left(1 + \frac{2R \cos \varphi}{\ell} \right)^2} d\varphi \quad (22)$$

and the generalization of (14) applies.

B. Annular Source with Lateral Detector Point, Slab and Cylindrical Shields

Adopting the same coordinate system as is used in section II.A, we may write (see Figures 6 and 7)

$$\varphi_1 = \sin^{-1} \frac{R_1}{R_0 + a} \quad (23)$$

$$\varphi_2 = \sin^{-1} \frac{R_0}{R_0 + a} \quad (24)$$

where R_1 and R_0 are the inner and outer radii of the annulus respectively. dV , t , μ , ℓ , μ_s , S_v , and r are defined as under Section II.A. μ_h is the macroscopic attenuation coefficient of the material in the central hole of the annulus.

FIGURE 6
ANNULAR SOURCE WITH LATERAL DETECTOR POINT
AND CYLINDRICAL AND SLAB SHIELDS

FIGURE 7
THE THREE REGIONS OF INTEGRATION
WHICH DEPEND ON THE RADII
OF THE ANNULUS

Now generalizing the development given in Section II.A, we may write equations analogous to equations (4), (5) and (3):

$$r_{01} = R_0 \left\{ \left(1 + \frac{a}{R_0} \right) \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(1 + \frac{a}{R_0} \right)^2 \sin^2 \varphi} \right\} \quad (25)$$

$$r_{02} = R_0 \left\{ \left(1 + \frac{a}{R_0} \right) \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(1 + \frac{a}{R_0} \right)^2 \sin^2 \varphi} \right\} \quad (26)$$

and if we set $a_1 = a + R_0 - R_1$, then

$$r_{11} = R_1 \left\{ \left(1 + \frac{a_1}{R_1} \right) \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(1 + \frac{a_1}{R_1} \right)^2 \sin^2 \varphi} \right\} \quad (27)$$

$$r_{12} = R_1 \left\{ \left(1 + \frac{a_1}{R_1} \right) \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(1 + \frac{a_1}{R_1} \right)^2 \sin^2 \varphi} \right\} \quad (28)$$

r_3 is defined by (6) with R replaced by R_0 .

Now consider the attenuation expressions in the three regions of figure 7. For region I in Fig. 7, defined by $\varphi_1 \leq \varphi \leq \varphi_2$, equations (3) and (14) apply, with r_1 replaced by r_{01} , r_2 replaced by r_{02} , R replaced by R_0 , and φ_1 and φ_2 defined by (23) and (24) respectively.

For region II, which is defined by $0 \leq \varphi \leq \varphi_1$, $r_{01} \leq r \leq r_{11}$, the attenuation is

$$A_{II} = \frac{e^{-[\mu_t + \mu_c(r_{01}-r_3) + \mu_s(r-r_{01})] \sec \varphi \sec \theta}}{4\pi r^2 \sec^2 \varphi \sec^2 \theta} \quad (29)$$

and for region III, defined by $0 \leq \varphi \leq \varphi_1$, $r_{12} \leq r \leq r_{02}$,

$$A_{III} = \frac{e^{-[\mu t + \mu_c (r_{01}-r_3) + \mu_s (r_{11}-r_{01}) + \mu_s (r-r_{12}) + \mu_h (r_{12}-r_{11})]} \sec \varphi \sec \theta}{4\pi r^2 \sec^2 \varphi \sec^2 \theta} \quad (30)$$

The uncollided fluxes at the point P in Figure 6 from the three regions I, II, and III will be called Φ_1 , Φ_2 , and Φ_3 . Φ_1 is found immediately from (14).

$$\begin{aligned} \Phi_1 = & \frac{S_v}{4\pi\mu_s} \int_{\varphi_1}^{\varphi_2} \int_0^{\theta_{02}} \cos \theta \left[1 - \exp \left\{ -\mu_s (r_{02}-r_{01}) \sec \varphi \sec \theta \right\} \right] \\ & \exp \left\{ -[\mu t + \mu_c (r_{01}-r_3)] \sec \varphi \sec \theta \right\} d\varphi d\theta \\ & + \frac{S_v}{4\pi\mu_s} \int_{\varphi_1}^{\varphi_2} \int_{\theta_{02}}^{\theta_{01}} \cos \theta \left[1 - \exp \left\{ -\mu_s (\ell \cot \theta \cos \varphi - r_{01}) \sec \varphi \sec \theta \right\} \right] \\ & \exp \left\{ -[\mu t + \mu_c (r_{01}-r_3)] \sec \varphi \sec \theta \right\} d\varphi d\theta \end{aligned}$$

$$\text{where } \theta_{02} = \tan^{-1} \frac{\ell}{r_{02} \sec \varphi} \quad \text{and } \theta_{01} = \tan^{-1} \frac{\ell}{r_{01} \sec \varphi}, \quad (31)$$

and by similar reasoning,

$$\begin{aligned} \Phi_2 = & \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_0^{\theta_{11}} \cos \theta \left[1 - \exp \left\{ -\mu_s (r_{11}-r_{01}) \sec \varphi \sec \theta \right\} \right] \\ & \exp \left\{ -[\mu t + \mu_c (r_{01}-r_3)] \sec \varphi \sec \theta \right\} d\varphi d\theta \\ & + \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_{\theta_{11}}^{\theta_{01}} \cos \theta \left[1 - \exp \left\{ -\mu_s (\ell \cot \theta \cos \varphi - r_{01}) \sec \varphi \sec \theta \right\} \right] \\ & \exp \left\{ -[\mu t + \mu_c (r_{01}-r_3)] \sec \varphi \sec \theta \right\} d\varphi d\theta \end{aligned}$$

$$\text{where } \theta_{11} = \tan^{-1} \frac{\ell}{r_{11} \sec \varphi}. \quad (32)$$

$$\begin{aligned}
\Phi_3 = & \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_0^{\theta_{02}} \cos \theta \left[1 - \exp\left\{-\mu_s(r_{02}-r_{i2})\sec \varphi \sec \theta\right\} \right] \\
& \exp\left\{-\left[\mu t + \mu_c(r_{01}-r_3) + \mu_s(r_{i1}-r_{01}) + \mu_h(r_{i2}-r_{i1})\right]\sec \varphi \sec \theta\right\} d\varphi d\theta \\
+ & \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_{\theta_{02}}^{\theta_{i2}} \cos \theta \left[1 - \exp\left\{-\mu_s(l \cot \theta \cos \varphi - r_{i2})\sec \varphi \sec \theta\right\} \right] \\
& \exp\left\{-\left[\mu t + \mu_c(r_{01}-r_3) + \mu_s(r_{i1}-r_{01}) + \mu_h(r_{i2}-r_{i1})\right]\sec \varphi \sec \theta\right\} d\varphi d\theta
\end{aligned}$$

where $\theta_{i2} = \tan^{-1} \frac{l}{r_{i2} \sec \varphi}$ (33)

Finally, the flux from the full annulus is

$$\Phi_{un} = 2(\Phi_1 + \Phi_2 + \Phi_3) \quad (34)$$

Consider one special case, that of a detector point on the surface of the annulus at the base. We have $a = 0$, the terms involving μt and μ_c do not appear, and

$$\begin{aligned}
\alpha_1 &= \sin^{-1} \frac{R_i}{R_0} \\
\alpha_2 &= \frac{\pi}{2} \\
r_{01} &= 0 \\
r_{02} &= 2R_0 \cos^2 \alpha
\end{aligned} \quad (35)$$

$$r_{i1} = R_i \left\{ \frac{R_0}{R_i} \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(\frac{R_0}{R_i}\right)^2 \sin^2 \varphi} \right\} \quad (36)$$

$$r_{i2} = R_i \left\{ \frac{R_0}{R_i} \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(\frac{R_0}{R_i}\right)^2 \sin^2 \varphi} \right\} \quad (37)$$

The fluxes from the three regions I, II, and III of Fig. 7 are then

$$\begin{aligned} \Phi_1' &= \frac{S_v}{4\pi\mu_s} \int_{\varphi_1}^{\frac{\pi}{2}} \int_0^{\Theta_2} \cos \theta \left[1 - \exp\{-2\mu_s R_0 \cos \varphi \sec \theta\} \right] d\varphi d\theta \\ &+ \frac{S_v}{4\pi\mu_s} \int_{\varphi_1}^{\frac{\pi}{2}} \int_{\Theta_2}^{\frac{\pi}{2}} \cos \theta \left[1 - \exp\{-\mu_s l \csc \theta\} \right] d\varphi d\theta, \end{aligned} \quad (38)$$

where $\Theta_2 = \tan^{-1} \frac{l}{2 R_0 \cos \varphi}$;

$$\begin{aligned} \Phi_2' &= \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_0^{\theta_{i1}} \cos \theta \left[1 - \exp\{-\mu_s r_{i1} \sec \varphi \sec \theta\} \right] d\varphi d\theta \\ &+ \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_{\theta_{i1}}^{\frac{\pi}{2}} \cos \theta \left[1 - \exp\{-\mu_s l \csc \theta\} \right] d\varphi d\theta; \end{aligned} \quad (39)$$

$$\begin{aligned}
\Phi'_3 = & \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_0^{\Theta_2} \cos \theta \left[1 - \exp \left\{ -\mu_s (2R_0 \cos^2 \varphi - r_{i2} \sec \varphi \sec \theta) \right\} \right] \\
& \exp \left\{ -\left[\mu_s r_{i1} + \mu_h (r_{i2} - r_{i1}) \right] \sec \varphi \sec \theta \right\} d\varphi d\theta \\
+ & \frac{S_v}{4\pi\mu_s} \int_0^{\varphi_1} \int_0^{\Theta_{i2}} \cos \theta \left[1 - \exp \left\{ \left(\cot \theta \cos \varphi - r_{i2} \right) \sec \varphi \sec \theta \right\} \right] \\
& \exp \left\{ -\left[\mu_s r_{i2} + \mu_h (r_{i2} - r_{i1}) \right] \sec \varphi \sec \theta \right\} d\varphi d\theta ; \quad (40)
\end{aligned}$$

and the flux from the full annulus at a surface detector point at the base is

$$\Phi_{un} = 2(\Phi'_1 + \Phi'_2 + \Phi'_3) \quad (41)$$

C. Cylindrical Self-Source

The length of $\overline{PM}$ is given by $\overline{PM} = a \cos \varphi + \sqrt{R^2 - a^2 \sin^2 \varphi}$ and the projection of PM on PQ in Figure 8 is

$$r_1 = a \cos^2 \varphi + \cos \varphi \sqrt{R^2 - a^2 \sin^2 \varphi} \quad (42)$$

The differential volume is

$$dV = (r \sec^2 \varphi d\varphi) (r \sec \varphi \sec^2 \theta d\theta) dr \quad (43)$$

and the uncollided flux at the point P in Figure 8 is given by

$$\begin{aligned} \Phi_{un} = \frac{S_v}{2\pi} \int_0^\pi \int_0^{r_1} \int_0^{\tan^{-1} \frac{l}{r_1 \sec \varphi}} \frac{l}{r_1 \sec \varphi} \frac{1}{\sec \varphi e^{\mu_s r \sec \varphi \sec \theta}} dr d\varphi d\theta \\ + \frac{S_v}{2\pi} \int_0^\pi \int_0^l \cot \theta \cos \varphi \int_{\tan^{-1} \frac{l}{r_1 \sec \varphi}}^{\frac{\pi}{2}} \frac{1}{\sec \varphi e^{\mu_s r \sec \varphi \sec \theta}} dr d\varphi d\theta \end{aligned} \quad (44)$$

The integration over r yields

$$\begin{aligned} \Phi_{un} = & \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_0^{\tan^{-1} \frac{l}{r_1 \sec \varphi}} \frac{\cos \theta}{\cos \theta} \left[1 - e^{-\mu_s r_1 \sec \varphi \sec \theta} \right] d\varphi d\theta \\ & + \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_{\tan^{-1} \frac{l}{r_1 \sec \varphi}}^{\frac{\pi}{2}} \cos \theta \left[1 - e^{-\mu_s l \csc \theta} \right] d\varphi d\theta. \end{aligned} \quad (45)$$

Rearranging gives

$$\begin{aligned} \Phi_{un} = & \frac{S_v}{2\mu_s} - \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_0^{\tan^{-1} \frac{l}{r_1 \sec \varphi}} \frac{\cos \theta}{\cos \theta} e^{-\mu_s r_1 \sec \varphi \sec \theta} d\varphi d\theta \\ & - \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_{\tan^{-1} \frac{l}{r_1 \sec \varphi}}^{\frac{\pi}{2}} \cos \theta e^{-\mu_s l \csc \theta} d\varphi d\theta. \end{aligned} \quad (46)$$

Consider five special cases:

Special Case #1: $\ell \rightarrow \infty$

From (46) ,

$$\Phi_{un} = \frac{S_v}{2\mu_s} - \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_0^{\frac{\pi}{2}} \cos \theta e^{-\mu_s r_1 \sec \varphi \sec \theta} d\varphi d\theta . \quad (46a)$$

Special case #2: $\mu_s = 0$

from (44)

$$\Phi_{un} = \frac{S_v}{2\pi} \int_0^\pi r_1 \sec \varphi \tan^{-1} \left(\frac{\ell}{r_1 \sec \varphi} \right) d\varphi \quad (46b)$$

$$+ \frac{S_v}{2\pi} \ell \int_0^\pi \log \sqrt{1 + \left(\frac{r_1 \sec \varphi}{\ell} \right)^2} d\varphi .$$

Special case #3: $a = 0$

Several equivalent expressions may be given for this case. We first change the coordinate system to that of Figure 8a and proceed in order to get a result in terms of the E_1 function as directly as possible:

$$dV = (r \tan \theta d\varphi) (r \sec^2 \theta d\theta) dr$$

$$A = \frac{e^{-\mu_s r \sec \theta}}{4\pi r^2 \sec^2 \theta} , \text{ and} \quad (47)$$

$$d\Phi_{un} = S_v \cdot A \cdot dV \quad (48)$$

$$\Phi_{un} = \frac{S_v}{4\pi} \int_0^{2\pi} \int_0^{\tan^{-1} \frac{R}{r}} \int_0^{\ell} \tan \theta e^{-\mu_s r \sec \theta} dr d\varphi d\theta, \text{ or}$$

$$\Phi_{un} = \frac{S_v}{2} \int_{r=0}^{\ell} \int_{\theta=0}^{\tan^{-1} \frac{R}{r}} \tan \theta e^{-\mu_s r \sec \theta} d\theta dr \quad \text{which is equivalent to}$$

$$\Phi_{un} = \frac{S_v}{2} \int_0^{\ell} \left[E_1(\mu_s r) - E_1\left(\mu_s r \sqrt{1 + (R/r)^2}\right) \right] dr. \quad (49a)$$

(See the Appendix for a discussion of the exponential functions of the first and second kinds, commonly called the E_1 and E_2 functions. These functions are also discussed in Refs. 1, 2, and 3.)

Let $y = \mu_s r$. Then from (49a),

$$\Phi_{un} = \frac{S_v}{2\mu_s} \int_0^{\mu_s \ell} \left[E_1(y) - E_1\left(y \sqrt{1 + \left(\frac{\mu_s R}{y}\right)^2}\right) \right] dy$$

or

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[1 - E_2(\mu_s \ell) - \int_0^{\mu_s \ell} E_1\left(\sqrt{y^2 + (\mu_s R)^2}\right) dy \right]. \quad (50)$$

The corresponding equation obtained from (46) with $a = 0$ and $r_1 = R \cos \varphi$ is

$$\Phi_{un} = \frac{S_v}{2\mu_s} - \frac{S_v}{2\mu_s} \int_0^{\tan^{-1} \frac{l}{R}} \cos \theta e^{-\mu_s R \sec \theta} d\theta - \frac{S_v}{2\mu_s} \int_{\tan^{-1} \frac{l}{R}}^{\frac{\pi}{2}} \cos \theta e^{-\mu_s l \csc \theta} d\theta$$

or

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[1 - G(\tan^{-1} \frac{l}{R}, \mu_s R) - E_2(\mu_s l) + \frac{l}{\sqrt{R^2 + l^2}} E_2(\mu_s \sqrt{R^2 + l^2}) \right] \quad (51)$$

Still another derivation may be given, based on a line source (line sources are discussed in Section III). Setting up the integrand in terms of the variables x and ψ marked in Figure 8a, we have

$$\Phi_{un} = \frac{S_v}{2} \int_0^R \int_0^{\tan^{-1} \frac{l}{x}} \frac{e^{-\mu_s x \sec \psi} x^2 d\psi}{x^2} = \frac{S_v}{2} \int_0^R F(\tan^{-1} \frac{l}{x}, \mu_s x) dx,$$

$$\Phi_{un} = \frac{S_v}{2\mu_s} \int_0^{\mu_s R} F(\tan^{-1} \frac{\mu_s l}{\mu_s x}, \mu_s x) d(\mu_s x) = \frac{S_v}{2\mu_s} G(\mu_s l, \mu_s R). \quad (51a)$$

The function $G(\tan^{-1} \frac{l}{R}, \mu_s R)$ which appears in (51) is the G-function of Ono and Tsuro (Ref. 4) and the function $F(\tan^{-1} \frac{l}{x}, \mu_s x)$ is the classic Sievert Integral of the first kind, which is discussed in Refs. 1, 2, 3, 12 and 13. The function $G(\mu_s l, \mu_s R)$ is the G-function of Federaro and Obenshein which is documented in Refs. 1 and 2. All these functions are defined in the Appendix of this report.

FIGURE 8
CYLINDRICAL SOURCE WITH DETECTOR POINT
ON DIAMETER

FIGURE 8a
CYLINDRICAL SOURCE WITH
DETECTOR POINT AT
END OF AXIS

Special Case #4: $\mu_s = 0, a = 0$

From (49),

$$\Phi_{un} = \frac{S_v}{2} \int_0^{\ell} \int_0^{\tan^{-1} \frac{R}{r}} \tan \theta \, d\theta \, dr = \frac{S_v}{2} \int_0^{\ell} \log \sec \left(\tan^{-1} \frac{R}{r} \right) dr$$

let $\theta = \tan^{-1} \frac{R}{r}$, $r = R \cot \theta$, and

$$dr = -R \csc^2 \theta d\theta.$$

Make substitutions and integrate by parts. The result is

$$\Phi_{un} = \frac{S_v}{2} \ell \left[\frac{1}{2} \log \left(1 + \frac{R}{\ell} \right)^2 + \frac{R}{\ell} \tan^{-1} \left(\frac{\ell}{R} \right) \right] \quad (53)$$

D. Annular Source with Interior Detector Point

The intersections of PQ and the circles of radii R_1 and R_0 , (in Fig. 9) are, after projection on $\overline{PM}$ (which is parallel to the vector $\vec{r}$)

$$r_1 = a \cos^2 \varphi - \cos \varphi \sqrt{R_1^2 - a^2 \sin^2 \varphi} \quad \text{and} \quad (54)$$

$$r_2 = a \cos^2 \varphi + \cos \varphi \sqrt{R_0^2 - a^2 \sin^2 \varphi},$$

and we have

$$dV = (r \sec^2 \varphi \, d\varphi) (r \sec \varphi \sec^2 \theta \, d\theta) dr,$$

μ_h = macroscopic attenuation coefficient of the material in the center of the annulus,

and μ_s , S_v , and ℓ are defined as under Section II.A.

Then the uncollided flux at the point P in Figure 9 is given by

$$\begin{aligned} \Phi_{un} = & \frac{S_v}{2\pi} \int_0^\pi \int_{r_1}^{r_2} \int_0^{\tan^{-1} \frac{l}{r_2 \sec \varphi}} \frac{1}{\sec \varphi} e^{-[\mu_s(r-r_1) + \mu_h r_1] \sec \varphi \sec \theta} d\varphi d\theta \\ & + \frac{S_v}{2\pi} \int_0^\pi \int_{r_1}^l \cot \theta \cos \varphi \int_{\tan^{-1} \frac{l}{r_2 \sec \varphi}}^{\tan^{-1} \frac{l}{r_1 \sec \varphi}} \frac{1}{\sec \varphi} e^{-[\mu_s(r-r_1) + \mu_h r_1] \sec \varphi \sec \theta} d\varphi d\theta. \end{aligned} \quad (55)$$

Integration over r gives

$$\begin{aligned} \Phi_{un} = & \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_0^{\tan^{-1} \frac{l}{r_2 \sec \varphi}} \frac{1}{\cos \theta} \left[1 - e^{-\mu_s(r_2-r_1) \sec \varphi \sec \theta} \right] e^{-\mu_h r_1 \sec \varphi \sec \theta} d\varphi d\theta \\ & + \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_{\tan^{-1} \frac{l}{r_2 \sec \varphi}}^{\tan^{-1} \frac{l}{r_1 \sec \varphi}} \frac{1}{\cos \theta} \left[1 - e^{-\mu_s[l \cot \theta \cos \varphi - r_1] \sec \varphi \sec \theta} \right] \\ & e^{-\mu_h r_1 \sec \varphi \sec \theta} d\varphi d\theta. \end{aligned} \quad (56)$$

Consider two special cases:

Special Case #1: $a = 0$

From (54) and (56)

$$r_1 = R_1 \cos \varphi,$$

$$r_2 = R_0 \cos \varphi,$$

and the flux at the center of the annulus is

$$\Phi_{un} = \frac{S_v}{2\mu_s} \int_0^{\tan^{-1} \frac{l}{R_0}} \cos \theta \left[1 - e^{-\mu_s [R_0 - R_1] \sec \theta} \right] e^{-\mu_h R_1 \sec \theta} d\theta$$

$$+ \frac{S_v}{2\mu_s} \int_{\tan^{-1} \frac{l}{R_0}}^{\tan^{-1} \frac{l}{R_1}} \cos \theta \left[1 - e^{-\mu_s [l \cot \theta - R_1] \sec \theta} \right] e^{-\mu_h R_1 \sec \theta} d\theta. \quad (57)$$

This may also be written as

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left\{ G \left(\tan^{-1} \frac{l}{R_1}, \mu_h R_1 \right) - G \left(\tan^{-1} \frac{l}{R_0}, [\mu_s (R_0 - R_1) + \mu_h R_1] \right) \right.$$

$$\left. - \int_{\tan^{-1} \frac{l}{R_0}}^{\tan^{-1} \frac{l}{R_1}} e^{-[\mu_s l \csc \theta + R_1 (\mu_h - \mu_s) \sec \theta]} \cos \theta d\theta \right\}. \quad (57a)$$

If we further assume that $\mu_s = \mu_h$, this reduces to the difference of two evaluations of Eq. (51), one made with $R = R_0$ and the other with $R = R_1$. The G-function used here is the same as that used in Eq. (51).

Special Case #2: $a = R_1$

From (54) and (56),

$$r_1 = R_1 \cos^2 \varphi,$$

$$r_2 = R_1 \cos^2 \varphi + \cos \varphi \sqrt{R_0^2 - R_1^2 \sin^2 \varphi},$$

$$r_2 - r_1 = \cos \varphi \sqrt{R_0^2 - R_1^2 \sin^2 \varphi},$$

and

$$\begin{aligned}
\Phi_{un} = & \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_0^{\tan^{-1} \frac{l}{r_2 \sec \varphi}} \frac{1}{\cos \theta} \left[1 - e^{-\mu_s (r_2 - r_1) \sec \varphi \sec \theta} \right] e^{-\mu_h R_1 \cos \varphi \sec \theta} d\varphi d\theta \\
& + \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_{\tan^{-1} \frac{l}{r_2 \cos \varphi}}^{\tan^{-1} \frac{l}{R_1 \cos \varphi}} \frac{1}{\cos \theta} \left[1 - e^{-\mu_s [l \csc \theta - R_1 \cos \varphi \sec \theta]} \right] \\
& \quad e^{-\mu_h R_1 \cos \varphi \sec \theta} d\varphi d\theta. \quad (58)
\end{aligned}$$

(58) may be rewritten in a form similar to that of Eq. (57a) if the reader desires to do so.

E. Annular Self-Source

See Figure 9 and consider the detector point P_1 . The flux at this point may be found by dividing the annulus at the location of P_1 and handling the outer portion using Eq. (58) and the inner portion using Eq. (41).

F. Cylindrical Source with Detector Point Off End

First consider a cylindrical source with an axial detector point and slab shield (See Figure 10). Refs. 3 and 11 also treat this problem. The derivation given here follows closely that given by Kruger and Atze in Ref. 14. The volume element and attenuation function are respectively

$$dV = (r \tan \theta d\phi)(r \sec^2 \theta d\theta)dr \quad \text{and} \quad A = \frac{e^{-[\mu_s(r-a) + \mu t] \sec \theta}}{4\pi r^2 \sec^2 \theta}$$

and the uncollided flux at the point P in Figure 10 is given by

$$\Phi_{un} = \frac{S_v}{2} \int_a^{a+l} \int_0^{\tan^{-1} \frac{R}{r}} \tan \theta e^{-[\mu_s(r-a) + \mu t] \sec \theta} d\theta dr. \quad (59)$$

Using the definitions of, and the relationships between, the E_1 and E_2 functions given in the Appendix, we have, from (59)

$$\Phi_{un} = \frac{S_v}{2} \int_a^{a+l} \left[E_1[\mu_s(r-a) + \mu t] - E_1\left[\left\{\mu_s(r-a) + \mu t\right\} \sqrt{1 + \left(\frac{R}{r}\right)^2}\right] \right] dr. \quad (60)$$

Now let $y = \mu_s(r-a)$, giving

$$\Phi_{un} = \frac{S_v}{2\mu_s} \int_0^{\mu_s l} E_1(y + \mu t) dy - \frac{S_v}{2\mu_s} \int_0^{\mu_s l} E_1\left[(y + \mu t) \sqrt{1 + \left(\frac{\mu_s R}{y + \mu_s a}\right)^2}\right] dy, \quad (61)$$

which integrates to give

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[E_2(\mu t) - E_2(\mu t + \mu_s \ell) - \int_0^{\mu_s \ell} E_1 \left\{ (y + \mu t) \sqrt{1 + \left(\frac{\mu_s R}{y + \mu_s a} \right)^2} \right\} dy \right]. \quad (62)$$

Compare this to the truncated cone, Eq. (186).

Three special cases will be considered:

Special Case #1: $\mu_s = 0$

From (59),

$$\Phi_{un} = \frac{S_v}{2} \int_a^{a+\ell} \int_0^{\tan^{-1} \frac{R}{r}} \tan \theta e^{-\mu t \sec \theta} d\theta dr$$

or

$$\Phi_{un} = \frac{S_v}{2} \ell E_1(\mu t) - \frac{S_v}{2} \int_a^{a+\ell} E_1 \left(\mu t \sqrt{1 + \left(\frac{R}{r} \right)^2} \right) dr. \quad (63)$$

Let $r = x + a$, $\tan \theta = \frac{R}{r} = \frac{R}{x + a}$, from which

$$dx = R \csc^2 \theta d\theta.$$

Make these substitutions and integrate by parts; we obtain

$$\Phi_{un} = \frac{S_v}{2} \ell E_1(\mu t) - \frac{S_v}{2} R \left[-\frac{a}{R} E_1\left(\mu t \sqrt{1 + \left(\frac{R}{a}\right)^2}\right) + \frac{a + \ell}{R} E_1\left(\mu t \sqrt{1 + \left(\frac{R}{a + \ell}\right)^2}\right) + F\left(\tan^{-1} \frac{R}{a + \ell}, \mu t\right) - F\left(\tan^{-1} \frac{R}{a}, \mu t\right) \right] \quad (64)$$

where $E_1(\mu t)$ is the exponential function of the first kind and $F(\tan^{-1} \frac{R}{a}, \mu t)$ is the Sievert integral of the first kind. These functions are defined in the Appendix

Special Case #2: $\mu_s = \mu t = 0$.

From (59)

$$\begin{aligned} \Phi_{un} &= \frac{S_v}{2} \int_a^{a+\ell} \int_0^{\tan^{-1} \frac{R}{r}} \tan \theta \, d\theta \, dv \\ \Phi_{un} &= \frac{S_v}{2} (a + \ell) \left[\frac{1}{2} \log \left(1 + \left(\frac{R}{a + \ell} \right)^2 \right) - \frac{R}{a + \ell} \tan^{-1} \frac{R}{a + \ell} \right] \\ &\quad - \frac{S_v}{2} a \left[\frac{1}{2} \log \left(1 + \left(\frac{R}{a} \right)^2 \right) - \frac{R}{a} \tan^{-1} \frac{R}{a} \right], \end{aligned} \quad (65)$$

and if $a = 0$ also,

$$\Phi_{un} = \frac{S_v}{2} \left[\frac{\ell}{2} \log \left(1 + \left(\frac{R}{\ell} \right)^2 \right) + R \tan^{-1} \frac{\ell}{R} \right]. \quad (65a)$$

Special Case #3: $l \rightarrow \infty$

From (60)

$$\Phi_{un} = \frac{S_v}{2} \int_a^{\infty} \left\{ E_1[\mu_s(r-a) + \mu t] - E_1\left[\mu_s(r-a) + \mu t \sqrt{1 + \left(\frac{R}{r}\right)^2}\right] \right\} dr.$$

Let $y = \mu_s(r-a)$ and integrate (see (61)) assuming the whole contribution comes from the lower limit. This gives the approximation

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[E_2(\mu t) - \frac{1}{\sqrt{1 + \left(\frac{R}{a}\right)^2}} E_2\left(\mu t \sqrt{1 + \left(\frac{R}{a}\right)^2}\right) \right] \quad (66)$$

which is an upper limit. The expression was first published by Krüger and Seeligman in Ref. 15. (66) is the same expression as that for an infinitely long truncated cone source with detector point at the apex, as given in Eq. (191).

Now consider the same cylindrical source but move the detector point off the axis as shown in Figure 10a. Let the displacement h of the point P off the axis be less than the cylinder radius R . In terms of the coordinates θ , φ , and ρ , the volume element is given by

$$dV = (\rho \sin \theta d\varphi)(\rho d\theta)d\rho = \rho^2 \sin \theta d\rho d\varphi d\theta \quad (66a)$$

and the attenuation is given by

$$T = \frac{e^{-\mu_s(\rho - a \sec \theta) - \mu t \sec \theta}}{4\pi \rho^2} \quad (66b)$$

The length of $\rho_2 = \overline{AD} = \overline{BC}$ in Figure 10a is

$$\rho_2 = h \cos \varphi + \sqrt{R^2 - h^2 \sin^2 \varphi} \quad (66c)$$

1

and the flux integral breaks naturally into two parts

$$\begin{aligned}
 \Phi_{un} &= \int S_v T dV \\
 &= \frac{S_v}{2\pi} \int_0^\pi \int_0^{\theta_1} \int_{a \sec \theta}^{(a+l) \sec \theta} e^{-\mu_s \rho} d\rho \cdot e^{(\mu_s a \sec \theta - \mu t \sec \theta)} \sin \theta d\varphi d\theta \\
 &+ \frac{S_v}{2\pi} \int_0^\pi \int_{\theta_1}^{\theta_2} \int_{a \sec \theta}^{\rho_2 \csc \theta} e^{-\mu_s \rho} d\rho \cdot e^{(\mu_s a \sec \theta - \mu t \sec \theta)} \sin \theta d\varphi d\theta \quad (66d)
 \end{aligned}$$

where

$$\theta_1 = \tan^{-1} \frac{\rho_2}{a+l}$$

and

$$\theta_2 = \tan^{-1} \frac{\rho_2}{a}.$$

Integration with respect to ρ gives the flux at the point P in Figure 10a as

$$\begin{aligned}
 \Phi_{un} &= \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_0^{\theta_1} \left[1 - e^{-\mu_s l \sec \theta} \right] e^{-\mu t \sec \theta} \sin \theta d\varphi d\theta \\
 &+ \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_{\theta_1}^{\theta_2} \left[1 - e^{-\mu_s [\rho_2 \csc \theta - a \sec \theta]} \right] e^{-\mu t \sec \theta} \sin \theta d\varphi d\theta. \quad (66e)
 \end{aligned}$$

Using relations from the Appendix, (66e) may be rearranged and integrated into a more convenient form:

$$\begin{aligned}
\Phi_{un} &= \frac{S_v}{2\mu_s} \left[E_2(\mu t) - E_2(\mu_s \ell + \mu t) \right] \\
&- \frac{S_v}{2\pi\mu_s} \int_0^\pi \left[\cos\theta_2 E_2(\mu t \sec\theta_2) - \cos\theta_1 E_2((\mu t + \mu_s \ell) \sec\theta_1) \right] d\varphi \\
&- \frac{S_v}{2\pi\mu_s} \int_0^\pi \int_{\theta_1}^{\theta_2} e^{-[\mu_s (\rho \csc\theta - a \sec\theta) + \mu t \sec\theta]} \sin\theta \, d\varphi \, d\theta.
\end{aligned}$$

(66f)

In the special case where $\mu_s = 0$, (66d) gives

$$\begin{aligned}
\Phi_{un} &= \frac{S_v \ell}{2\pi} \int_0^\pi \int_0^{\theta_1} \tan\theta \, e^{-\mu t \sec\theta} \, d\varphi \, d\theta \\
&+ \frac{S_v h}{2\pi} \int_0^\pi \int_{\theta_1}^{\theta_2} \sin\theta \cos\varphi \, e^{-\mu t \sec\theta} \, d\varphi \, d\theta + \frac{S_v}{2\pi} \int_0^\pi \int_{\theta_1}^{\theta_2} \sin\theta \sqrt{R^2 - h^2 \sin^2\varphi} \, e^{-\mu t \sec\theta} \, d\varphi \, d\theta \\
&- \frac{S_v a}{2\pi} \int_0^\pi \int_{\theta_1}^{\theta_2} \tan\theta \, e^{-\mu t \sec\theta} \, d\varphi \, d\theta.
\end{aligned}$$

(66g)

Integrating (66g) and using various relations from the Appendix, we have, after combining terms,

$$\begin{aligned}\Phi_{un} = & \frac{S_v \ell}{2} E_1(\mu t) - \frac{S_v(\ell+a)}{2\pi} \int_0^\pi E_1(\mu t \sec \theta_1) d\varphi \\ & + \frac{S_v a}{2\pi} \int_0^\pi E_1(\mu t \sec \theta_2) d\varphi \\ & + \frac{S_v}{2\pi} \int_0^\pi [\cos \theta_1 E_2(\mu t \sec \theta_1) - \cos \theta_2 (\mu t \sec \theta_2)] \rho_2 d\varphi.\end{aligned}\quad (66h)$$

Here, ρ_2 is defined by (66c) and θ_1 and θ_2 are defined following (66d).

G. Cylindrical Surface Source with Lateral Detector Point, Slab and Cylindrical Shields

(See Figure 11)

r_1 , r_2 , and r_3 are given by (4), (5), and (6), respectively. The total differential area for any pair of angle values (φ, θ) is the sum of two terms:

$$dA = (r_1^2 + r_2^2) \sec^2 \theta \sec^2 \varphi \sec \gamma d\varphi d\theta. \quad (67)$$

$$\sec \gamma = \frac{R}{\sqrt{R^2 - (R+a)^2 \sin^2 \varphi}} = \frac{1}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} \quad (68)$$

as shown in Figure 12, where the differential arc dL

FIGURE II
CYLINDRICAL SURFACE SOURCE WITH SLAB
AND CYLINDRICAL SHIELDS

is given by

$$dL = r_2 \sec \varphi \sec \gamma \, d\varphi = r_2 \sec \varphi \frac{R}{\sqrt{R^2 - (R+a)^2 \sin^2 \varphi}} d\varphi . \quad (69)$$

The limit φ_0 of the angle variable φ is defined by (1).

Now for the front portion of the arc defined by $0 \leq \varphi \leq \varphi_0$, the attenuation A_1 is given by

$$A_1 = \frac{e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta}}{4\pi r_1^2 \sec^2 \varphi \sec^2 \theta} , \quad (70)$$

and for the rest of the arc, the attenuation μ_h of the cylinder material enters, so that the attenuation is

$$A_2 = \frac{e^{-[\mu t + \mu_c(r_1 - r_3) + \mu_h(r_2 - r_1)] \sec \varphi \sec \theta}}{4\pi r_2^2 \sec^2 \varphi \sec^2 \theta} . \quad (71)$$

The uncollided flux at the point P in Figure 11 is then

$$\begin{aligned} \Phi_{un} = & \frac{S_a}{2\pi} \int_0^{\varphi_0} \int_0^{\tan^{-1} \frac{l}{r_1 \sec \varphi}} \frac{e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta}}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} d\varphi d\theta \\ & + \frac{S_a}{2\pi} \int_0^{\varphi_0} \int_0^{\tan^{-1} \frac{l}{r_2 \sec \varphi}} \frac{e^{-[\mu t + \mu_c(r_1 - r_3) + \mu_h(r_2 - r_1)] \sec \varphi \sec \theta}}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} d\varphi d\theta . \end{aligned} \quad (72)$$

Two special cases may be noted:

Special Case #1: $\mu_t = \mu_c = \mu_h = 0$

$$\Phi_{un} = \frac{S_a R}{2(a+2R)} \int_0^{\tan^{-1} \frac{l}{a}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (73)$$

where
$$k = \frac{2\sqrt{R(a+R)}}{2R+a} . \quad (74)$$

This is an elliptic function of the first kind. A similar derivation is given in detail in Section II.H, Eqs. (84) thru (89).

Special Case #2: $a = 0$

If $a = 0$, then $r_1 = 0$, $r_1 - r_3 = 0$, $\mu_t = 0$, $\varphi_0 = \frac{\pi}{2}$ and $r_2 = 2R \cos^2 \varphi$. Making these substitutions in (72) gives

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sec \varphi \, d\varphi \, d\theta + \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\tan^{-1} \frac{l}{2R \cos \varphi}} \frac{l}{2R \cos \varphi} \frac{d\theta}{\sec \varphi e^{-2\mu_h R \cos \varphi \sec \theta}} \, d\varphi \, d\theta \quad (75)$$

which may be simplified, but which is undefined at the upper limit on φ . This situation is therefore not amenable to analysis by this technique.

H. Cylindrical Surface Source with Interior Detector Point

In Figure 13,

$$\begin{aligned} r_1 &= a \cos \varphi + \sqrt{R^2 - a^2 \sin^2 \varphi} \\ r_2 &= -a \cos \varphi + \sqrt{R^2 - a^2 \sin^2 \varphi} \\ r_1 + r_2 &= 2\sqrt{R^2 - a^2 \sin^2 \varphi} \end{aligned} \quad (76)$$

Now see Figure 12. By a similar construction and argument we may write, in analogy to (69), the following expressions for the differential arc length dL , noting that we have changed the definitions of r_1 and r_2 as given by (76):

$$\begin{aligned} dL &= r_1 \sec \gamma \, d\varphi \\ \text{and} \\ dL &= r_2 \sec \gamma \, d\varphi \end{aligned} \quad (77)$$

$$\begin{aligned} \text{where } \sec \gamma &= \frac{R}{\sqrt{R^2 - a^2 \sin^2 \varphi}} \\ &= \frac{1}{\sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi}} . \end{aligned} \quad (78)$$

and the differential area is given by

$$\begin{aligned} \text{and } dA &= (r_1 \sec \gamma \, d\varphi)(r_1 \sec^2 \theta \, d\theta) \quad \text{or by} \\ dA &= (r_2 \sec \gamma \, d\varphi)(r_2 \sec^2 \theta \, d\theta) . \end{aligned} \quad (79)$$

Then the uncollided flux at the point P in Figure 13, is given by

$$\begin{aligned}\Phi_{un} = & \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\tan^{-1} \frac{l}{r_1}} \frac{r_1^2 \sec \gamma \sec^2 \theta}{(r_1 \sec \theta)^2} e^{-\mu_h r_1 \sec \theta} d\varphi d\theta \\ & + \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\tan^{-1} \frac{l}{r_2}} \frac{r_2^2 \sec \gamma \sec^2 \theta}{(r_2 \sec \theta)^2} e^{-\mu_h r_2 \sec \theta} d\varphi d\theta\end{aligned}$$

which may be simplified to

$$\begin{aligned}\Phi_{un} = & \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\tan^{-1} \frac{l}{r_1}} \frac{e^{-\mu_h r_1 \sec \theta} d\theta d\varphi}{\sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi}} \\ & + \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\tan^{-1} \frac{l}{r_2}} \frac{e^{-\mu_h r_2 \sec \theta} d\theta d\varphi}{\sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi}}\end{aligned}\tag{80}$$

A number of special cases should be noted.

Special Case #1: $a = 0$

In this case $dA = R^2 \sec^2 \theta \, d\varphi \, d\theta$

$$\text{and} \quad \Phi_{\text{un}} = \frac{S}{2} \frac{a}{R} \int_0^{\tan^{-1} \frac{l}{R}} e^{-\mu_h R \sec \theta} d\theta = \frac{S}{2} F\left(\tan^{-1} \frac{l}{R}, \mu_h R\right) \quad (81)$$

Special Case #2: $l \rightarrow \infty$

In this case $\tan^{-1} \frac{l}{r_1} = \tan^{-1} \frac{l}{r_2} = \frac{\pi}{2}$ and we may combine the integrals of (80) to get

$$\Phi_{\text{un}} = \frac{S}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{e^{-\mu_h R \sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi} \sec \theta} (e^{-\mu_h a} + e^{\mu_h a})}{\sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi}} d\varphi \, d\theta \quad (82)$$

Special Case #3: $l \rightarrow \infty, a = 0$

$$\Phi_{\text{un}} = \frac{S}{2} F\left(\frac{\pi}{2}, \mu_h R\right) \quad (83)$$

$F(\theta, b)$ is the Sievert Integral of the first kind. See the Appendix.

Special Case #4: $\mu_h = 0$

This derivation will proceed using a coordinate system centered at the origin 0 rather than at the detector point P. See Figure 13a. We have

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^\pi \int_0^l \frac{R d\alpha dz}{R^2 + a^2 + z^2 - 2Ra \cos \alpha} \quad (84)$$

Integration with respect to α using (136a) gives

$$\Phi_{un} = \frac{S_a R}{2} \int_0^l \frac{dz}{\sqrt{(R^2 + z^2 + a^2)^2 - 4R^2 a^2}} \quad (85)$$

Let

$$z = (R - a) \tan \theta$$

$$dz = (R - a) \sec^2 \theta d\theta$$

$$\chi = \sqrt{(R^2 + z^2 + a^2)^2 - 4R^2 a^2}$$

and substitute in (85). We get

$$\Phi_{un} = \frac{S_a}{2} R \int_0^{\tan^{-1} \frac{l}{R-a}} \frac{\frac{l}{R-a}}{\chi} \frac{(R-a) \sec^2 \theta d\theta}{\chi} \quad (86)$$

Now
$$\chi^2 = [R^2 + (R^2 - 2aR + a^2) \tan^2 \theta + a^2]^2 - 4R^2 a^2.$$

Substitute $1 - \sec^2 \theta = \tan^2 \theta$ and perform the indicated squaring and collect terms, which gives

$$\chi^2 = (R - a)^2 \sec^2 \theta [(R - a)^2 \sec^2 \theta + 4aR] ,$$

whence

$$\chi = (R - a) \sec^2 \theta [(R - a)^2 + 4aR \cos^2 \theta]^{\frac{1}{2}} ,$$

$$\chi = (R - a) \sec^2 \theta [(R + a)^2 - 4aR \sin^2 \theta]^{\frac{1}{2}}$$

and (86) becomes

$$\Phi_{un} = \frac{S_a R}{2} \int_0^{\tan^{-1} \frac{l}{R-a}} \frac{(R-a) \sec^2 \theta d\theta}{(R-a) \sec^2 \theta \sqrt{(R+a)^2 - 4aR \sin^2 \theta}}$$

or

$$\Phi_{un} = \frac{S_a R}{2(R+a)} \int_0^{\tan^{-1} \frac{l}{R-a}} \frac{d\theta}{\sqrt{1 - \frac{4aR}{(R+a)^2} \sin^2 \theta}} . \quad (87)$$

Now let

$$k = \frac{2\sqrt{aR}}{R+a} . \quad (88)$$

Clearly, $k^2 < 1$, and (87) is the Legendre normal form of the elliptic integral of the first kind. The uncollided flux at the point P in Figure 13a is then

$$\Phi_{un} = \frac{S_a}{2} \frac{R}{(R+a)} \int_0^{\tan^{-1} \frac{l}{R-a}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} . \quad (89)$$

FIGURE 13
CYLINDRICAL SURFACE SOURCE
WITH INTERIOR DETECTOR POINT

FIGURE 13a
CYLINDRICAL SURFACE SOURCE
WITH INTERIOR DETECTOR POINT
NO ATTENUATION

Equation (73) may be found from (88) and (89) by replacing the distance a in the numerator of (88) by $(a+R)$ and by replacing $(R+a)$ by $2R+a$ in the denominators of (88) and (89). The upper limit of (89) becomes $\tan^{-1} \frac{L}{a}$ in this case.

Comparison of (80) with $\mu_h = 0$ and (89) gives the relation

$$\int_0^{\frac{\pi}{2}} \frac{\left(\tan^{-1} \frac{L}{r_1} + \tan^{-1} \frac{L}{r_2} \right) d\varphi}{\sqrt{1 - \left(\frac{a}{R} \right)^2 \sin^2 \varphi}} = \frac{\pi R}{(a+R)} \int_0^{\tan^{-1} \frac{L}{R-a}} \frac{d\theta}{\sqrt{1 - \frac{4aR}{(a+R)^2} \sin^2 \theta}} \quad (89a)$$

I. Cylindrical Surface Source with Axial Detector Point and Slab Shield

In Figure 14,

$$dA = (Rd\varphi)(R \csc^2 \theta d\theta), \quad (90)$$

and the attenuation is given by

$$A_1 = \frac{e^{-[\mu_h(r-a) + \mu t] \sec \theta}}{4\pi R^2 \csc^2 \theta}, \quad (91)$$

where

$$r = R \cot \theta.$$

Then

$$d\Phi_{un} = S_a \cdot A_1 \cdot dA \quad (92)$$

and the uncollided flux at the point P in Figure 14 is given by

$$\Phi_{un} = \frac{S_a}{2} \int_{\tan^{-1} \frac{R}{a+\sqrt{a^2+R^2}}}^{\tan^{-1} \frac{R}{a}} e^{-[\mu_h(R \cot \theta - a) + \mu t] \sec \theta} d\theta, \quad (93)$$

or

$$\Phi_{un} = \frac{S_a}{2} \int_{\tan^{-1} \frac{R}{a+\sqrt{a^2+R^2}}}^{\tan^{-1} \frac{R}{a}} e^{-[\mu_h R \csc \theta + (\mu t - \mu_h a) \sec \theta]} d\theta. \quad (94)$$

Note the similarity of this equation and Equation (98).

One special case may be mentioned:

if $\mu_h = 0$, we have

$$\Phi_{un} = \frac{S_a}{2} \int_{\tan^{-1} \frac{R}{a+\sqrt{a^2+R^2}}}^{\tan^{-1} \frac{R}{a}} e^{-\mu t \sec \theta} d\theta = \frac{S_a}{2} \left[F\left(\tan^{-1} \frac{R}{a}, \mu t\right) - F\left(\tan^{-1} \frac{R}{a+\sqrt{a^2+R^2}}, \mu t\right) \right] \quad (95)$$

For a discussion of the $F(\theta_0, b)$ function (Sieverts Integral of the first kind) see the Appendix.

FIGURE 14
CYLINDRICAL SURFACE SOURCE WITH AXIAL
POINT AND SLAB SHIELD

III. FLUX FROM LINE SOURCES AND FLAT SURFACE SOURCES

This section presents formulas for the flux from line and flat surface sources for the following combinations of sources, shields and detector points:

- A. Line Source in Absorbing Medium with Perpendicular Slab Shield, Lateral Detector Point
- B. Line Source with Tilted Slab Shield, Lateral Detector Point
- C. Parallel Line Source with Slab Shield
- D. Circular Line Source with Exterior Detector Point
- E. Circular Line Source with Interior Detector Point
- F. Disk Source with Axial Detector Point
- G. Disk Source with Off-set Axial Detector Point
- H. Annular Disk Source with Interior Detector Point
- I. Disk Source with Detector Point at Center of Circular Notch
- J. Disk Source with Exterior Lateral Detector Point
- K. Rectangular Surface Source with Detector Point on Perpendicular
- L. Rectangular Surface Source with Lateral Detector Point

Linear and disk sources are treated in References 1, 2, and 3. Ref. 16 also discusses linear sources and plane and disk sources are discussed in Refs. 17, 18, 19, and 20.

A. Line Source in Absorbing Medium with Perpendicular Slab Shield, Lateral Detector Point

The absorbing medium of attenuation $\bar{\mu}$ in Figure 15 could represent a slab shield parallel to the line source and homogenized over the distance a' , where

$$a' = \frac{a\ell}{h + \ell}.$$

If this slab shield parallel to the line source has thickness t and macroscopic attenuation coefficient μ , then

$$\bar{\mu} = \frac{\mu t}{a'} = \frac{\mu t(h + \ell)}{a\ell}.$$

Then $\rho = a \sec \theta$ in Figure 15,
the attenuation factor A is

$$A = \frac{e^{-[\bar{\mu}(a \sec \theta - h \csc \theta) + \pi \tau \csc \theta]}}{4\pi(a \sec \theta)^2}, \quad (96)$$

and the differential line length and angle limits are

$$d\ell = a \sec^2 \theta d\theta,$$

$$\theta_1 = \tan^{-1} \frac{h}{a}, \text{ and}$$

$$\theta_2 = \tan^{-1} \frac{h+\ell}{a}.$$

The differential flux is

$$d\Phi_{un} = S_L \cdot A \cdot d\ell$$

and the uncollided flux at the point P in Figure 15 is

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_{\theta_1}^{\theta_2} e^{-[\bar{\mu}(a \sec \theta - h \csc \theta) + m\tau \csc \theta]} d\theta. \quad (97)$$

If we further homogenize the perpendicular shield over the distance h , we have

$$\mu_h = \frac{m\tau}{h}$$

and

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_{\theta_1}^{\theta_2} e^{-[\bar{\mu} a \sec \theta + (\mu_h - \bar{\mu})h \csc \theta]} d\theta. \quad (98)$$

Note the similarity of Equations (98) and (94).

Several special cases are of interest. Refer to (98).

Special Case #1: $h = 0$ and $a' = a$, $\bar{\mu}a = \mu t$

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_0^{\tan^{-1} \frac{\ell}{a}} e^{-\bar{\mu}a \sec \theta} d\theta = \frac{S_L}{4\pi a} F(\tan^{-1} \frac{\ell}{a}, \bar{\mu}a) \quad (99)$$

This is the classic line source equation, involving only a parallel uniform slab shield as shown in Figure 15a. The function $F(\tan^{-1} \frac{\ell}{a}, \bar{\mu}a)$ is discussed in the Appendix. Note that if $\bar{\mu} = 0$, this reduces to

$$\Phi_{un} = \frac{S_L}{4\pi a} \tan^{-1} \frac{\ell}{a}. \quad (100)$$

An interesting property of this classic line source case is that the same analysis and flux formula applies to both the straight line source ℓ and the circular arc line source ℓ' in Figure 15a. The length of ℓ' is $a \cdot \theta_0 = a \tan^{-1} \frac{\ell}{a}$. The differential arc is

$$d\ell' = a d\theta \quad (100a)$$

and the attenuation is given by

$$A = \frac{e^{-\mu t \sec \theta}}{4\pi a^2}. \quad (100b)$$

Then

$$d\Phi_{un} = S_L \cdot A \cdot d\ell'$$

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_0^{\tan^{-1} \frac{\ell}{a}} e^{-\mu t \sec \theta} d\theta = \frac{S_L}{4\pi a} F(\tan^{-1} \frac{\ell}{a}, \mu t) \quad (100c)$$

which is the same as (99).

Special Case #2: $\mu_h = 0$

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_{\theta_1}^{\theta_2} e^{-\bar{\mu}(a \sec \theta - h \csc \theta)} d\theta \quad (101)$$

This gives the flux from a line source with a parallel slab shield which does not extend beyond the end of the line source.

Special Case #3: $\bar{\mu} = 0, a \neq 0$

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_{\theta_1}^{\theta_2} e^{-\mu_h h \csc \theta} d\theta \quad (102)$$

This gives the flux from a line source with a perpendicular slab shield. If we change the variable we can express this in terms of the $F(\theta_0, b)$ function.

Let

$$\psi = \frac{\pi}{2} - \theta,$$

$$\psi_1 = \frac{\pi}{2} - \theta_1, \text{ and} \quad (103)$$

$$\psi_2 = \frac{\pi}{2} - \theta_2.$$

Then

$$\Phi_{un} = \frac{S_L}{4\pi a} \int_{\psi_2}^{\psi_1} e^{-\mu_h h \sec \psi} d\psi = \frac{S_L}{4\pi a} [F(\psi_1, \mu_h h) - F(\psi_2, \mu_h h)] \quad (104)$$

Special case #4: $l \rightarrow \infty$ and $h = 0$

This implies $\theta_2 \rightarrow \frac{\pi}{2}$ and (99) gives

$$\phi_{un} = \frac{S_L}{4\pi a} F\left(\frac{\pi}{2}, \bar{\mu}a\right) = \frac{S_L}{4\pi a} G(\bar{\mu}a) \quad (105)$$

for a parallel uniform shield only.

If $\bar{\mu}a = 0$ also, we get

$$\phi_{un} = \frac{S_L}{8a} \quad (106)$$

Finally, if $l \rightarrow \infty$, (104) becomes

$$\phi_{un} = \frac{S_L}{4\pi a} F(\psi_1, \mu_h h) \quad (107)$$

for the case with perpendicular slab shield only.

FIGURE 15. LINE SOURCE IN ABSORBING MEDIUM WITH PERPENDICULAR SLAB SHIELD

FIGURE 15a. CLASSIC LINE SOURCE WITH SLAB SHIELD, AND EQUIVALENT CIRCULAR ARC SOURCE

B. Line Source with Tilted Slab Shield,
Lateral Detector Point

In Figure 16a, the equivalent line source parallel to the shield subtends the angle θ_0 and has length

$$l' = a \tan(\alpha + \theta_0) - a \tan \alpha$$

where

$$a \geq (l + t \csc \alpha) \tan \alpha .$$

The uncollided flux at the point P in Figure 16a is given by

$$\Phi_{un} = \frac{S_L}{4\pi a} [F(\alpha + \theta_0, b) - F(\alpha, b)] \quad (108)$$

where $b = \mu t$. The function $F(\theta_0, b)$ is discussed in the Appendix.

In Figure 16b, where the shield is tilted in the opposite direction, the equivalent parallel line source again subtends the angle designated as θ_0 . The length of the equivalent line source is, in this case

$$l' = a \tan \alpha + a \tan(\theta_0 - \alpha)$$

where

$$a \geq t \sec \alpha$$

and the uncollided flux at the point P in Figure 16b is

$$\Phi_{un} = \frac{S_L}{4\pi a} [F(\alpha, b) + F(\theta_0 - \alpha, b)] . \quad (109)$$

FIGURE 16A. LINE SOURCE
WITH TILTED SLAB SHIELD

FIGURE 16B. LINE SOURCE
WITH TILTED SLAB SHIELD

C. Line Source with Perpendicular Slab Shield.

See Figure 17. The uncollided flux at the point P is

$$\phi_{un} = \frac{S_L}{4\pi} \int_a^{a+l} \frac{e^{-[\mu t + \mu_s(r-a)]}}{r^2} dr. \quad (110)$$

Let

$$t = \mu_s r \quad \text{and} \\ dt = \mu_s dr.$$

Then

$$\phi_{un} = \frac{S_L}{4\pi} \mu_s e^{(\mu_s a - \mu t)} \int_{\mu_s a}^{\mu_s(a+l)} \frac{e^{-t}}{t^2} dt \quad (111)$$

and

$$\phi_{un} = \frac{S_L}{4\pi} e^{(\mu_s a - \mu t)} \left[\frac{E_2(\mu_s a)}{a} - \frac{E_2(\mu_s(a+l))}{a+l} \right]. \quad (112)$$

If $\mu_s = 0$, we have, from (110)

$$\phi_{un} = \frac{S_L}{4\pi} e^{-\mu t} \left[\frac{1}{a} - \frac{1}{a+l} \right]. \quad (113)$$

D. Circular Line Source with Exterior Detector Point

Refer to Figure 12 and Equations (68) and (69), which define $\sec \gamma$ and the arc length dL . φ_0 , r_1 , r_2 , and r_3 are defined by (1), (4), (5), and (6), respectively. μ , μ_s , t , and μ_c are defined under Section II.A.

The attenuation factor is (see Figure 18)

$$A = \frac{e^{-[\mu t + \mu_c(r_1 - r_3) + \mu_s(r_2 - r_1)] \sec \varphi}}{4\pi (r \sec \varphi)^2} \quad (114)$$

for the arc MK and the total uncollided flux at the point P in Figure 18 is

$$\begin{aligned} \Phi_{un} = & \frac{S_L}{4\pi} \cdot 2 \int_0^{\varphi_0} \frac{e^{-[\mu t + \mu_c(r_1 - r_3) + \mu_s(r_2 - r_1)] \sec \varphi} r_2 \sec \gamma \sec \varphi d\varphi}{(r_2 \sec \varphi)^2} \\ & + \frac{S_L}{4\pi} \cdot 2 \int_0^{\varphi_0} \frac{e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi} r_1 \sec \gamma \sec \varphi d\varphi}{(r_1 \sec \varphi)^2} . \end{aligned} \quad (115)$$

$$\Phi_{un} = \frac{S_L}{2\pi} \int_0^{\varphi_0} \frac{e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi}}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} \left[\frac{e^{-\mu_s(r_2 - r_1) \sec \varphi}}{r_2} + \frac{1}{r_1} \right] \cos \varphi d\varphi \quad (116)$$

$$= \frac{S_L}{2\pi} \int_0^{\varphi_0} \frac{e^{-[\mu t + \mu_c(r_1 - r_3)] \sec \varphi}}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} \left[\frac{r_1 e^{-2\mu_s R \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} + r_2}{(a^2 + 2aR)} \right] \sec \varphi d\varphi \quad (117)$$

This can be regrouped and manipulated in various ways at the user's convenience.

If there is no attenuation ($\mu = \mu_c = \mu_s = 0$),

$$\Phi_{un} = \frac{S_L}{2\pi a(a+2R)} \int_0^{\varphi_0} \frac{(r_1+r_2)\sec \varphi \, d\varphi}{\sqrt{1 - \left(1+\frac{a}{R}\right)^2 \sin^2 \varphi}}.$$

If we substitute $r_1 + r_2 = 2(a+R)\cos^2 \varphi$, we get

$$\Phi_{un} = \frac{S_L}{2\pi} \cdot \frac{2(a+R)}{a(a+2R)} \int_0^{\varphi_0} \frac{\cos \varphi \, d\varphi}{\left(\frac{a+R}{R}\right) \sqrt{\left(\frac{R}{a+R}\right)^2 - \sin^2 \varphi}}$$

which may be integrated to give

$$\Phi_{un} = \frac{S_L}{2} \frac{R}{a(2R+a)}, \quad (118)$$

and this is valid so long as $a > 0$.

FIGURE 17. LINE SOURCE WITH PERPENDICULAR
SLAB SHIELD

FIGURE 18. CIRCULAR LINE SOURCE
WITH SLAB AND CIRCULAR SHIELDS
AND EXTERIOR DETECTOR POINT

E. Circular Line Source with Interior Detector Point

We shall consider the simpler cases first.

Case 1 (See Figure 19)

If the detector point is at the center, and we assume attenuation, the uncollided flux at the point P in Figure 19 is

$$\Phi_{un} = \frac{S_L}{4\pi} \int_0^{2\pi} \frac{e^{-\mu_s R}}{R^2} d\varphi, \text{ or}$$
$$\Phi_{un} = \frac{S_L}{2R} e^{-\mu_s R}. \quad (119)$$

If the point P lies a distance h above the plane of the circular line source, we have

$$\Phi_{un} = \frac{S_L}{4\pi} \int_0^{2\pi} \frac{e^{-\mu_s \sqrt{R^2+h^2}}}{R^2+h^2} d\varphi, \text{ or}$$
$$\Phi_{un} = \frac{S_L R}{2(R^2+h^2)} e^{-\mu_s \sqrt{R^2+h^2}}. \quad (120)$$

Case 2 (See Figure 20)

In Figure 20, the detector point P lies on the end of a diameter, and the vector $r = \overline{PM}$ is related to the radius vector and the diameter by the equation

$$r = 2R \cos^2 \varphi$$

or
$$r \sec \varphi = 2R \cos \varphi .$$

The attenuation factor is

$$A = \frac{e^{-\mu_s r \sec \varphi}}{4\pi (r \sec \varphi)^2} ,$$

and the differential arc length is

$$d\ell = r \sec^2 \varphi d\varphi .$$

Then

$$d\Phi_{un} = S_L \cdot A \cdot d\ell ; \text{ and after simplifying, we have}$$

$$\Phi_{un} = \frac{S_L}{4\pi R} \int_0^{\varphi_0} e^{-2\mu_s R \cos \varphi} \sec^2 \varphi d\varphi \quad (121)$$

This gives an answer approaching infinity when φ_0 approaches $\frac{\pi}{2}$. If $\mu_s = 0$, we have

$$\Phi_{un} = \frac{S_L}{4\pi R} \tan \varphi_0 . \quad (122)$$

Case 3 (See Figure 21)

The detector point is interior to the circular line source but arbitrarily located. r_1 and r_2 are given by (76), $d\ell$ is given by (77), and $\sec \gamma$ is given by (78). Then the uncollided flux at the point P in Figure 21 is given by

$$\Phi_{un} = 2 \cdot \frac{S_L}{4\pi} \int_0^{\frac{\pi}{2}} \frac{e^{-\mu_s r_1} r_1 \sec \gamma d\varphi}{r_1^2} + \frac{2 \cdot S_L}{4\pi} \int_0^{\frac{\pi}{2}} \frac{e^{-\mu_s r_2} r_2 \sec \gamma d\varphi}{r_2^2},$$

$$\Phi_{un} = \frac{S_L}{2\pi} \int_0^{\frac{\pi}{2}} \frac{e^{-\mu_s r_1} d\varphi}{r_1 \sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi}} + \frac{S_L}{2\pi} \int_0^{\frac{\pi}{2}} \frac{e^{-\mu_s r_2} d\varphi}{r_2 \sqrt{1 - \left(\frac{a}{R}\right)^2 \sin^2 \varphi}}. \quad (123)$$

This relation may be manipulated by the user at his convenience.

If $\mu_s = 0$, (122) becomes

$$\Phi_{un} = \frac{S_L}{2\pi} \int_0^{\frac{\pi}{2}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \sec \gamma d\varphi = \frac{S_L R}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{R^2 - a^2}$$

$$\Phi_{un} = \frac{S_L R}{2(R^2 - a^2)} \quad (124)$$

which is valid so long as $a < R$.

FIGURE 19. CIRCULAR LINE SOURCE WITH DETECTOR POINT IN CENTER

FIGURE 20. CIRCULAR LINE SOURCE WITH DETECTOR POINT AT END OF A DIAMETER

FIGURE 21. CIRCULAR LINE SOURCE WITH
ARBITRARY INTERIOR DETECTOR POINT

F. Disk Source with Axial Detector Point

In Figure 22,

$$dA = (a \tan \theta d\varphi)(a \sec^2 \theta d\theta), \quad (125)$$

$$\theta_0 = \tan^{-1} \frac{R}{a}, \quad (126)$$

and the uncollided flux at point P in Figure 22 is given by

$$\Phi_{un} = \frac{S_a}{4\pi} \int_0^{2\pi} \int_0^{\theta_0} \frac{e^{-\mu t \sec \theta} a^2 \tan \theta \sec^2 \theta d\theta d\varphi}{(a \sec \theta)^2}$$

or

$$\Phi_{un} = \frac{S_a}{2} \int_0^{\theta_0} \tan \theta e^{-\mu t \sec \theta} d\theta. \quad (127)$$

Using relations from the Appendix, this may be written

$$\Phi_{un} = \frac{S_a}{2} [E_1(b) - E_1(b \sec \theta_0)]. \quad (128)$$

If $\mu = 0$,

$$\Phi_{un} = \frac{S_a}{2} \int_0^{\theta_0} \tan \theta d\theta = \frac{S_a}{2} \log (\sec \theta_0). \quad (129)$$

If $R \rightarrow \infty$, we have, from (128)

$$\Phi_{un} = \frac{S_a}{2} E_1(b), \quad (130)$$

which gives the flux from an infinite plane source with slab shield.

This is the classic disk source which is given in References 1, 2, and 3. Smith and Storm give an analysis of the disk source with detector point not on the axis in Reference 19. We shall consider this problem in the next section.

G. Disk Source with Off-Set Axial Detector Point

In Figure 23,

$$\rho^2 = r^2 + h^2 + a^2 - 2rh \cos \alpha \quad (131)$$

Homogenize the slab shield over the distance a . The equivalent attenuation coefficient is then

$$\bar{\mu} = \mu \frac{t}{a} \quad (132)$$

and the flux at point P in Figure 23 is given by

$$\phi_{un} = \frac{2 \cdot S}{4\pi} \int_0^\pi \int_0^R \frac{e^{-\bar{\mu}\rho} r \, dr \, d\alpha}{\rho^2} \quad (133)$$

We may define the function

$$D(R, a, h, \bar{\mu}) = 2 \int_0^\pi \int_0^R \frac{e^{-\bar{\mu}\rho} r \, dr \, d\alpha}{\rho^2} \quad (134)$$

If we make the substitutions

$$\begin{aligned} X &= \bar{\mu}\rho \quad \text{and} \\ x &= \bar{\mu}r, \end{aligned}$$

we obtain

$$D(R, a, h, \bar{\mu}) = D(\bar{\mu}R, \bar{\mu}a, \bar{\mu}h) = 2 \int_0^\pi \int_0^{\bar{\mu}R} \frac{e^{-X} x \, dx \, d\alpha}{X^2} \quad (135)$$

which is a function of only three variables.

If $\bar{\mu} = 0$, from (133) we have

$$\Phi_{un} = \frac{S a}{2\pi} \int_0^\pi \int_0^R \frac{r dr d\alpha}{r^2 + h^2 + a^2 - 2rh \cos \alpha} \quad (136)$$

Then, starting from the integration formula

$$\int \frac{d\alpha}{A + B \cos \alpha} = \frac{2}{\sqrt{A^2 - B^2}} \tan^{-1} \left(\frac{\sqrt{A^2 - B^2} \tan \frac{\alpha}{2}}{A + B} \right) + C \quad (136a)$$

we may obtain after considerable manipulation,

$$\Phi_{un} = \frac{S a}{4} \log \left[\frac{a^2 + R^2 - h^2 + \sqrt{R^4 + 2R^2(a^2 - h^2) + (a^2 + h^2)^2}}{2a^2} \right] \quad (137)$$

(137) reduces to (129) if $h = 0$.

FIGURE 22. DISK SOURCE WITH
SLAB SHIELD, AXIAL POINT

FIGURE 23. DISK SOURCE WITH
OFF-SET AXIAL DETECTOR POINT

We may analyze this situation using a different coordinate system and arrive at equivalent results which are interesting in their own right. Using the coordinate system of Figure 23a, and considering Case I, in which the displacement h of the detector point is less than the radius R of the disk, the area element and the attenuation are given by

$$dA = (a \tan \theta) d(a \tan \theta) = a^2 \tan \theta \sec^2 \theta d\varphi d\theta \quad (137a)$$

and

$$T = \frac{e^{-\mu t \sec \theta}}{4\pi (a \sec \theta)^2} \quad (137b)$$

Also,

$$\rho_1 = h \cos \varphi + \sqrt{R^2 - h^2 \sin^2 \varphi} \quad (\text{See Eq. (66c)}) \quad (137c)$$

and

$$d\Phi_{un} = S_a \cdot T \cdot dA.$$

Then the uncollided flux at the point P in Figure 23a is

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^\pi \int_0^{\tan^{-1} \frac{\rho_1}{a}} e^{-\mu t \sec \theta} \tan \theta d\varphi d\theta, \quad (137d)$$

and using relationships from the Appendix we have

$$\Phi_{un} = \frac{S_a}{2} E_1(\mu t) - \frac{S_a}{2\pi} \int_0^\pi E_1\left(\mu t \sqrt{1 + (\rho_1/a)^2}\right) d\varphi. \quad (137e)$$

In the special case where $h = R$,

$$\rho_1 = 2R \cos \theta,$$

$$\tan^{-1} \frac{\rho_1}{a} = \frac{\pi}{2},$$

and (137d) becomes

$$\Phi_{un} = \frac{S_a}{4} E_1(\mu t) - \frac{S_a}{2\pi} \int_0^{\pi/2} E_1\left(\frac{\mu t}{a} \sqrt{a^2 + 4R^2 \cos^2 \varphi}\right) d\varphi \quad (137f)$$

Now consider Case II, as shown in Figure 23b, where the displacement h is greater than the radius R . The volume element dA and the attenuation T are then given by (137a) and (137b), respectively.

Following the same logic which gave Equations (4) and (5), we may write

$$\rho_1 = h \cos \varphi - \sqrt{R^2 - h^2 \sin^2 \varphi} \quad , \quad (137g)$$

$$\rho_2 = h \cos \varphi + \sqrt{R^2 - h^2 \sin^2 \varphi} \quad ,$$

and the upper limit on φ is now

$$\varphi_0 = \sin^{-1} \frac{R}{h} \quad . \quad (137h)$$

Then the uncollided flux at the point P in Figure 23b is

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a}} \tan \theta e^{-\mu t \sec \theta} d\varphi d\theta \quad (137i)$$

or

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\sin^{-1} \frac{R}{h}} \left[E_1\left(\mu t \sqrt{1 + (\rho_1/a)^2}\right) - E_1\left(\mu t \sqrt{1 + (\rho_2/a)^2}\right) \right] d\varphi \quad (137j)$$

FIGURE 23a. DISK SOURCE WITH SLAB
SHIELD AND OFF-SET AXIAL DETECTOR
POINT.
CASE I: $h \leq R$

FIGURE 23b. DISK SOURCE WITH SLAB
SHIELD AND OFF-SET AXIAL DETECTOR
POINT.
CASE II: $h > R$

H. Annular Disk Source with Interior Detector Point

Consider first a simple case: An annular disk source with detector point at the center. The uncollided flux is given by (See Figure 24)

$$\phi_{un} = \frac{S_a}{4\pi} \cdot 4 \int_0^{\frac{\pi}{2}} \int_{R_0}^R \frac{e^{-\mu_s(r-R_0)} - \mu R_0}{r^2} r dr d\varphi \quad (138)$$

Integrate over the angle and let $t = \mu_s r$. We get

$$\phi_{un} = \frac{S_a}{2} e^{-(\mu-\mu_s)R_0} \int_{\mu_s R_0}^{\mu_s R} \frac{e^{-t} dt}{t} \quad (139)$$

or

$$\phi_{un} = \frac{S_a}{2} e^{-(\mu-\mu_s)R_0} [E_1(\mu_s R_0) - E_1(\mu_s R)] \quad (140)$$

which holds so long as $R_0 > 0$.

If there is no self-attenuation ($\mu_s = 0$) (138) yields the relation

$$\phi_{un} = \frac{S_a}{2} e^{-\mu R_0} \log \frac{R}{R_0} \quad (141)$$

and if $\mu = 0$ also we get

$$\phi_{un} = \frac{S_a}{2} \log \frac{R}{R_0} \cdot \quad (142)$$

Now consider the more difficult case in which the detector point is arbitrarily located inside the annulus. As shown in Figure 25,

$$dA = r \sec^2 \varphi d\varphi \cdot dr$$

$$r_1 = R \left(\frac{a}{R} \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(\frac{a}{R} \right)^2 \sin^2 \varphi} \right) \quad (143)$$

$$r_2 = R \left(-\frac{a}{R} \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(\frac{a}{R} \right)^2 \sin^2 \varphi} \right) \quad (144)$$

Also define r_{01} and r_{02} in terms of the inner radius R_0 :

$$r_{01} = R_0 \left(\frac{a}{R_0} \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(\frac{a}{R_0} \right)^2 \sin^2 \varphi} \right) \quad (145)$$

$$r_{02} = R_0 \left(-\frac{a}{R_0} \cos^2 \varphi + \cos \varphi \sqrt{1 - \left(\frac{a}{R_0} \right)^2 \sin^2 \varphi} \right) \cdot \quad (146)$$

Then the uncollided flux at the point P in Figure 25 is given by

$$\begin{aligned} \phi_{un} = & 2 \cdot \frac{S_a}{4\pi} \int_0^{\frac{\pi}{2}} \int_{r_{01}}^{r_1} \frac{r \sec^2 \varphi e^{-[\mu_s(r-r_{01}) + \mu r_{01}] \sec \varphi}}{(r \sec \varphi)^2} dr d\varphi \\ & + 2 \cdot \frac{S_a}{4\pi} \int_0^{\frac{\pi}{2}} \int_{r_{02}}^{r_2} \frac{r \sec^2 \varphi e^{-[\mu_s(r-r_{02}) + \mu r_{02}] \sec \varphi}}{(r \sec \varphi)^2} dr d\varphi \end{aligned} \quad (147)$$

FIGURE 24. ANNULAR DISK SOURCE
WITH DETECTOR POINT AT CENTER

FIGURE 25. ANNULAR DISK SOURCE
WITH DETECTOR POINT ARBITRARILY
LOCATED INSIDE THE ANNULUS

If we simplify this and let $t = \mu_s r \sec \varphi$, we get

$$\begin{aligned} \Phi_{un} = & \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} [E_1(\mu_s r_{01} \sec \varphi) - E_1(\mu_s r_1 \sec \varphi)] e^{-(\mu - \mu_s) r_{01} \sec \varphi} d\varphi \\ & + \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} [E_1(\mu_s r_{02} \sec \varphi) - E_1(\mu_s r_2 \sec \varphi)] e^{-(\mu - \mu_s) r_{02} \sec \varphi} d\varphi. \end{aligned} \quad (148)$$

Finally, if $\mu_s = 0$, (147) gives

$$\begin{aligned} \Phi_{un} = & \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_{r_{01}}^{r_1} e^{-\mu r_{01} \sec \varphi} \frac{dr}{r} d\varphi + \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \int_{r_{02}}^{r_2} e^{-\mu r_{02} \sec \varphi} \frac{dr}{r} d\varphi, \\ \Phi_{un} = & \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} e^{-\mu r_{01} \sec \varphi} \log \frac{r_1}{r_{01}} d\varphi + \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} e^{-\mu r_{02} \sec \varphi} \log \frac{r_2}{r_{02}} d\varphi; \end{aligned} \quad (149)$$

and if $\mu = 0$ also, we get

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\frac{\pi}{2}} \log \frac{r_1}{r_{01}} \frac{r_2}{r_{02}} d\varphi \quad (150)$$

which may be simplified using (143) and (144) to give

$$\Phi_{un} = \frac{S_a}{4} \log \left(\frac{R^2 - a^2}{R_0^2 - a^2} \right) \text{ for } a < R_0 < R. \quad (151)$$

I. Disk Source with Detector Point at Center of Circular Notch

As shown in Figure 26,

$$r_1 = 2R \cos^2 \varphi \quad (152)$$

and at the intersection of the notch and the disk edge

$$R_0 = r_1 \sec \varphi_0 = 2R \cos \varphi_0, \quad (153)$$

so that
$$\varphi_0 = \cos^{-1} \frac{R_0}{2R}.$$

As above, $dA = r \sec^2 \varphi \, d\varphi \, dr$ and the attenuation $T = \frac{e^{-[\mu_s (r - R_0 \cos \varphi) \sec \varphi + \mu R_0]}}{4\pi (r \sec \varphi)^2}$.
Then the uncollided flux at the point P in Figure 26 is

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\varphi_0} \int_{R_0 \cos \varphi}^{2R \cos^2 \varphi} e^{-[\mu_s (r - R_0 \cos \varphi) \sec \varphi + \mu R_0]} \frac{dr}{r} d\varphi \quad (154)$$

If we separate terms and let $t = \mu_s r \sec \varphi$ we get

$$\Phi_{un} = \frac{S_a}{2\pi} e^{-(\mu - \mu_s)R_0} \left[\cos^{-1} \left(\frac{R_0}{2R} \right) \cdot E_1(\mu_s R_0) - \int_0^{\cos^{-1} \frac{R_0}{2R}} E_1(2\mu_s R \cos \varphi) d\varphi \right]. \quad (155)$$

If $\mu_s = 0$, (154) gives

$$\Phi_{un} = \frac{S_a}{2\pi} e^{-\mu R_0} \left[\cos^{-1} \left(\frac{R_0}{2R} \right) \log \left(\frac{2R}{R_0} \right) + \int_0^{\cos^{-1} \frac{R_0}{2R}} \log \cos \varphi d\varphi \right]. \quad (156)$$

J. Disk Source with Exterior Lateral Detector Point

See Figure 27.

φ_0 , r_1 , r_2 , and r_3 are defined by (1), (4), (5), and (6) respectively.

The area element is given by

$$dA = r \sec^2 \varphi \, d\varphi \, dr ,$$

and

$$d\Phi_{un} = \frac{S_a e^{-[\mu_c(r_1-r_3) + \mu t + \mu_s(r-r_1)]\sec \varphi}}{4\pi(r \sec \varphi)^2} r \sec^2 \varphi \, d\varphi \, dr .$$

Integration gives, for the uncollided flux at the point P in Figure 27,

$$\Phi_{un} = 2 \cdot \frac{S_a}{4\pi} \int_0^{\varphi_0} \int_{r_1}^{r_2} \frac{dr}{r} e^{-\mu_s r \sec \varphi} e^{-[\mu_c(r_1-r_3) - \mu_s r_1 + \mu t]\sec \varphi} d\varphi \quad (157)$$

or

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\varphi_0} [E_1(\mu_s r_1 \sec \varphi) - E_1(\mu_s r_2 \sec \varphi)] e^{-[\mu_c(r_1-r_3) - \mu_s r_1 + \mu t]\sec \varphi} d\varphi . \quad (158)$$

If $\mu_s = 0$, (161) gives

$$\Phi_{un} = \frac{S_a}{2\pi} \int_0^{\varphi_0} \log\left(\frac{r_2}{r_1}\right) e^{-[\mu_c(r_1-r_3) + \mu t]\sec \varphi} d\varphi . \quad (159)$$

From (4) and (5) we can write

$$\log \frac{r_2}{r_1} = \log \left(\frac{(a+R)^2}{a^2 + 2aR} \right) + 2 \log \left[\cos \varphi + \sqrt{\left(\frac{R}{a+R} \right)^2 - \sin^2 \varphi} \right] \quad (160)$$

and substitution of (160) into (159) gives

$$\begin{aligned}\Phi_{un} = & \frac{S_a}{2\pi} \log \left[\frac{(a+R)^2}{a^2 + 2aR} \right] \int_0^{\varphi_0} e^{-[\mu_c(r_1-r_3) + \mu t] \sec \varphi} d\varphi \\ & + \frac{S_a}{\pi} \int_0^{\varphi_0} \log \left[\cos \varphi + \sqrt{\left(\frac{R}{a+R}\right)^2 - \sin^2 \varphi} \right] e^{-[\mu_c(r_1-r_3) + \mu t] \sec \varphi} d\varphi. \quad (161)\end{aligned}$$

If $\mu_c = 0$, this reduces to

$$\begin{aligned}\Phi_{un} = & \frac{S_a}{2\pi} F(\varphi_0, \mu t) \log \left[\frac{(a+R)^2}{a^2 + 2aR} \right] \\ & + \frac{S_a}{\pi} \int_0^{\varphi_0} \log \left[\cos \varphi + \sqrt{\left(\frac{R}{a+R}\right)^2 - \sin^2 \varphi} \right] e^{-\mu t \sec \varphi} d\varphi. \quad (162)\end{aligned}$$

And if $\mu t = 0$ also,

$$\begin{aligned}\Phi_{un} = & \frac{S_a}{2\pi} \varphi_0 \log \left[\frac{(a+R)^2}{a^2 + 2aR} \right] \\ & + \frac{S_a}{\pi} \int_0^{\varphi_0} \log \left[\cos \varphi + \sqrt{\left(\frac{R}{a+R}\right)^2 - \sin^2 \varphi} \right] d\varphi \quad (163)\end{aligned}$$

See the Appendix for information of the Sievert Integral $F(\varphi_0, \mu t)$.

K. Rectangular Surface Source with Detector Point on Perpendicular

For the rectangular surface the element of area is

$$dA = (a \sec^2 \varphi d\varphi)(a \sec \varphi \sec^2 \theta d\theta) \quad (164)$$

as shown in Figures 28 and 29. The upper limits are

$$\varphi_0 = \tan^{-1} \frac{l_1}{a} \quad (165)$$

and

$$\theta_0 = \tan^{-1} \frac{l_2}{a \sec \varphi} . \quad (166)$$

Now assume a slab shield parallel to the rectangular surface source, of thickness t and macroscopic attenuation coefficient μ . The uncollided flux at the point P in Figure 28 is given by

$$\Phi_{un} = \frac{S a}{4\pi} \int_0^{\varphi_0} \int_0^{\theta_0} \frac{dA e^{-\mu t \sec \varphi \sec \theta}}{(a \sec \varphi \sec \theta)^2} ,$$

or

$$\Phi_{un} = \frac{S a}{4\pi} \int_0^{\varphi_0} \int_0^{\theta_0} \sec \varphi e^{-\mu t \sec \varphi \sec \theta} d\varphi d\theta . \quad (167)$$

In terms of the Sievert Integral, (167) becomes

$$\Phi_{un} = \frac{S a}{4\pi} \int_0^{\varphi_0} \sec \varphi F\left(\tan^{-1} \frac{l_2}{a \sec \varphi}, \mu t \sec \varphi\right) d\varphi . \quad (168)$$

If $\mu t = 0$, (167) becomes

$$\Phi_{un} = \frac{S a}{4\pi} \int_0^{\varphi_0} \sec \varphi \tan^{-1} \left(\frac{l_2}{a \sec \varphi} \right) d\varphi . \quad (169)$$

FIGURE 28. RECTANGULAR SURFACE SOURCE
WITH DETECTOR POINT ON PERPENDICULAR

FIGURE 29. DETAIL OF AREA ELEMENT
OF RECTANGULAR SURFACE

To get an upper limit estimate of (169), assume $\theta_0 = \tan^{-1} \frac{l_2}{a}$. This changes the source shape from rectangular to irregular--a large "horn" is added on the upper left portion of the source. Under this assumption, (169) becomes

$$\Phi_{un} = \frac{S_a}{4\pi} \tan^{-1} \frac{l_2}{a} \int_0^{\varphi_0} \sec \varphi \, d\varphi$$

$$\Phi_{un} = \frac{S_a}{4\pi} \tan^{-1} \frac{l_2}{a} \log \left[\sqrt{1 + \left(\frac{l_1}{a} \right)^2} + \frac{l_1}{a} \right]. \quad (170)$$

Compare (170) to (129) noting that (170) may be written

$$\Phi_{un} = \frac{S_a}{4\pi} \varphi_0 \log [\sec \theta_0 + \tan \theta_0] \quad (170a)$$

where $\tan \theta_0 = \frac{l_2}{a}.$

Assuming the irregular source shape mentioned above in deriving (170), and further assuming that the slab shield is replaced by an annular shield curved in the θ -direction and centered on the detector point, we may rewrite (168) as

$$\Phi_{un} = \frac{S_a}{4\pi} \tan^{-1} \frac{l_2}{a} \int_0^{\varphi_0} \sec \varphi \, e^{-\mu t \sec \varphi} \, d\varphi. \quad (170b)$$

This integral is related to the Sievert Integral of the Second Kind $F'(\varphi_0, \mu t)$, which is defined in the Appendix, as follows:

$$\int_0^{\varphi_0} \sec \varphi \, e^{-\mu t \sec \varphi} \, d\varphi = \frac{1}{\mu t} F'(\varphi_0, \mu t) \quad (170c)$$

L. Rectangular Surface Source with Lateral Detector Point

See Figure 30.

$$\theta_1 = \tan^{-1} \frac{l_2}{a+l_1} \quad (171)$$

$$\theta_2 = \tan^{-1} \frac{l_2}{a} \quad (172)$$

$$dA = (r \sec^2 \theta d\theta) dr \quad (173)$$

For $0 \leq \theta \leq \theta_1$, the limits on r are $(a, a + l_1)$ but for $\theta_1 < \theta \leq \theta_2$, the limits on r are $(a, l_2 \cot \theta)$.

Then

$$\begin{aligned} \Phi_{un} = & \frac{S_a}{4\pi} \int_a^{a+l_1} \int_0^{\theta_1} \frac{e^{-[\mu_s r - \mu_s a + \mu t]} \sec \theta}{r} dr d\theta \\ & + \frac{S_a}{4\pi} \int_a^{l_2 \cot \theta} \int_{\theta_1}^{\theta_2} \frac{e^{-[\mu_s r - \mu_s a + \mu t]} \sec \theta}{r} dr d\theta. \end{aligned} \quad (174)$$

Integrating over r and setting $t = \mu_s r \sec \theta$, we get

$$\begin{aligned} \Phi_{un} = & \frac{S_a}{4\pi} \int_0^{\theta_1} [E_1(\mu_s a \sec \theta) - E_1(\mu_s (a+l_1) \sec \theta)] e^{-(\mu t - \mu_s a) \sec \theta} d\theta \\ & + \frac{S_a}{4\pi} \int_{\theta_1}^{\theta_2} [E_1(\mu_s a \sec \theta) - E_1(\mu_s l_2 \csc \theta)] e^{-(\mu t - \mu_s a) \sec \theta} d\theta \end{aligned} \quad (175)$$

FIGURE 30. RECTANGULAR SURFACE
SOURCE WITH LATERAL DETECTOR POINT

If $\mu_s = 0$, integration of (174) yields

$$\begin{aligned} \Phi_{un} = & \frac{S}{4\pi} \log \left(\frac{a+l_1}{a} \right) F(\theta_1, \mu t) \\ & + \frac{S}{4\pi} \int_{\theta_1}^{\theta_2} \log \left(\frac{l_2 \cot \theta}{a} \right) e^{-\mu t \sec \theta} d\theta \end{aligned} \quad (176)$$

The $F(\theta_0, b)$ function is treated in the Appendix.

Finally, if $\mu = \mu_s = 0$, integration of (176) reduces to

$$\Phi_{un} = S_a \frac{\theta_1}{4\pi} \log \left(\frac{a+l_1}{a} \right) + \frac{S}{4\pi} \int_{\theta_1}^{\theta_2} \log \left(\frac{l_2 \cot \theta}{a} \right) d\theta, \quad (177)$$

which may be manipulated into the form

$$\Phi_{un} = \frac{S}{4\pi} \left[\theta_1 \log \frac{a+l_1}{l_2} + \theta_2 \log \frac{l_2}{a} + \int_{\theta_1}^{\theta_2} \log \cot \theta d\theta \right]. \quad (178)$$

IV. FLUX FROM TRUNCATED CONE, TOROIDAL, SPHERICAL, AND RECTANGULAR SOLID SOURCES

This section presents formulas for the flux from truncated cone, toroidal, spherical, and rectangular solid sources for the following combinations of source and detector locations:

- A. Truncated Cone Surface Source, Detector Point at Apex
- B. Truncated Cone Source, Detector Point at Apex
- C. Truncated Cone Source with Axial Detector Point
- D. Toroidal Source with Detector Point at Center
- E. Spherical Source with Interior Detector Point
- F. Spherical Source with Exterior Detector Point
- G. Rectangular Volume Source with Exterior Detector Point
- H. Spherical Surface Source

Truncated cone volume sources are discussed in References 1, 2, and 3, while spherical sources are treated in References 1, 2, 3, 4 and 5. The rectangular volume source with a slab shield only is analyzed in References 3 and 20.

A. Truncated Cone Surface Source, Detector Point at Apex

In Figure 31, the element of area is given by

$$dA = (r \tan \alpha \, d\phi)(\sec \alpha \, dr)$$

We assume that the source lies in an attenuating medium with macroscopic attenuation coefficient μ_s . The uncollided flux at the point P in Figure 31 is given by

$$\Phi_{un} = \frac{S_a}{4\pi} \int_a^{a+l} \int_0^{2\pi} \frac{e^{-[\mu t + \mu_s(r-a)] \sec \alpha} r \tan \alpha \sec \alpha \, d\phi \, dr}{(r \sec \alpha)^2} \quad (179)$$

or

$$\Phi_{un} = \frac{S_a}{2} \int_a^{a+l} \frac{e^{-\mu_s r \sec \alpha} (\sec \alpha \, dr)}{r \sec \alpha} \sin \alpha e^{-[\mu t - \mu_s a] \sec \alpha} \quad (180)$$

Now let

$$x = \mu_s r \sec \alpha$$

$$dx = \mu_s \sec \alpha \, dr$$

and substitute in (180). We get

$$\Phi_{un} = \frac{S_a}{2} e^{-[\mu t - \mu_s a] \sec \alpha} \left[E_1(\mu_s a \sec \alpha) - E_1(\mu_s (a+l) \sec \alpha) \right] \sin \alpha \quad (181)$$

where

$$\sin \alpha = \frac{R_1}{\sqrt{R_1^2 + a^2}}$$

and

$$\sec \alpha = \sqrt{1 + \left(\frac{R_1}{a}\right)^2} \quad .$$

If $\mu_s = 0$, we have, from (180)

$$\Phi_{un} = \frac{S_a}{2} e^{-\mu t \sec \alpha} \sin \alpha \log \left(\frac{a+l}{a} \right), \quad (182)$$

and if $\mu t = 0$ also,

$$\Phi_{un} = \frac{S_a}{2} \frac{R_1}{\sqrt{R_1^2 + a^2}} \log \left(\frac{a+l}{a} \right). \quad (183)$$

FIGURE 31
TRUNCATED CONE SURFACE SOURCE
WITH SLAB SHIELD, DETECTOR
POINT AT APEX

B. Truncated Cone Source, Detector Point at Apex

In Figure 32, the volume element is given by

$$dV = (r \tan \theta d\phi)(r \sec^2 \theta d\theta)dr$$

so the uncollided flux at the point P is given by

$$\Phi_{un} = \frac{S_v}{2} \int_a^{a+l} \int_0^{\theta_0} e^{-[\mu t + \mu_s(r-a)] \sec \theta} \tan \theta d\theta dr \quad (184)$$

where $\theta_0 = \tan^{-1} \frac{R_1}{a}$.

Integrating over r gives

$$\Phi_{un} = \frac{S_v}{2\mu_s} \int_0^{\theta_0} \left[\sin \theta e^{-\mu t \sec \theta} - \sin \theta e^{-(\mu t + \mu_s l) \sec \theta} \right] d\theta \quad (185)$$

Using relations from the Appendix, (185) may be written

$$\begin{aligned} \Phi_{un} = \frac{S_v}{2\mu_s} & \left[E_2(\mu t) - \cos \theta_0 E_2(\mu t \sec \theta_0) \right. \\ & \left. - E_2(\mu t + \mu_s l) + \cos \theta_0 E_2((\mu t + \mu_s l) \sec \theta_0) \right] \end{aligned} \quad (186)$$

This equation may be compared to the cylindrical and spherical results given in Equations (62), (66), and (233), respectively.

This is the classic and useful truncated cone source equation. We may note four special cases: If $\mu t = 0$, (186) reduces to

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[1 - \cos \theta_0 - E_2(\mu_s \ell) + \cos \theta_0 E_2(\mu_s \ell \sec \theta_0) \right]. \quad (187)$$

If $\mu_s = 0$, we have, from (184)

$$\Phi_{un} = \frac{S_v}{2} \ell \int_0^{\theta_0} \tan \theta e^{-\mu t \sec \theta} d\theta \quad (188)$$

or

$$\Phi_{un} = \frac{S_v}{2} \ell \left[E_1(\mu t) - E_1(\mu t \sec \theta_0) \right]. \quad (189)$$

If there is no attenuation at all, $\mu_s = \mu t = 0$, and integration of (188) yields

$$\Phi_{un} = \frac{S_v}{2} \ell \log \sec \theta_0 \quad (190)$$

which may be compared to (129) and (170a).

The formula for an infinitely long truncated cone source may be found by letting $\ell \rightarrow \infty$ in Eq. (186). This equation then becomes

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[E_2(\mu t) - \cos \theta_0 E_2(\mu t \sec \theta_0) \right] \quad (191)$$

which is the same as Equation (66).

C. Truncated Cone Source with Axial Detector Point

Consider the more general case, as shown in Figure 33, where the detector point is arbitrarily located at some point on the axis of a truncated cone. This is a generalization of the case summarized in Eq. (186).

The radius of the truncated cone in terms of r is

$$x = (r + h - a) \tan \alpha ; \quad (192)$$

$$\tan \alpha = \frac{R_1}{h} = \frac{R_2}{h+l} = \frac{R_2 - R_1}{l} . \quad (193)$$

The volume is

$$V = \pi \int_a^{a+l} (r + h - a)^2 \tan^2 \alpha \, dr = \pi \left(\frac{(h+l)^3 - h^3}{3} \right) \frac{R_1^2}{h^2} . \quad (194)$$

The θ -limit in terms of r is

$$\theta_1 = \tan^{-1} \frac{x}{r} = \tan^{-1} \left[\left(\frac{r+h-a}{r} \right) \tan \alpha \right] \quad (195)$$

and another way of expressing the differential volume is

$$dV = 2\pi r \tan \theta \, d\theta \cdot r \sec^2 \theta \, dr = 2\pi r^2 \sec^2 \theta \tan \theta \, dr \, d\theta . \quad (196)$$

The attenuation A is

$$A = \frac{e^{-[\mu_t + \mu_s(r-a)] \sec \theta}}{4\pi r^2 \sec^2 \theta} . \quad (197)$$

Then the differential flux is

$$d\Phi_{un} = S_v \cdot A \cdot dV \quad (198)$$

and the uncollided flux at the point P in Figure 33, is given by

$$\Phi_{un} = \frac{2\pi S_v}{4\pi} \int_a^{a+l} \int_0^{\theta_1} \tan \theta e^{-[\mu t + \mu_s(r-a)] \sec \theta} d\theta dr. \quad (199)$$

Remembering the definition of the E_1 function, we get

$$\Phi_{un} = \frac{S_v}{2} \int_a^{a+l} \left\{ E_1[\mu t + \mu_s(r-a)] - E_1[\mu t + \mu_s(r-a)] \sec \theta_1 \right\} dr \quad (200)$$

and integrating this gives, for the uncollided flux at P,

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[E_2(\mu t) - E_2(\mu t + \mu_s l) \right] - \frac{S_v}{2} \int_a^{a+l} E_1 \left[\mu t + \mu_s(r-a) \right] \sqrt{1 + \left(\frac{r+h-a}{a} \right)^2 \tan^2 \alpha} dr. \quad (201)$$

For a point P' opposite the large end of the truncated cone in Figure 33, a similar formula may be derived. For this case,

$$x = (h' + a' - r') \tan \alpha \quad (202)$$

$$\theta_1' = \tan^{-1} \left[\frac{h' + a' - r'}{r'} \tan \alpha \right] \quad (203)$$

and the uncollided flux at P' is given by

$$\Phi_{un} = \frac{S_v}{2\mu_s} \left[-E_2(\mu t) + E_2(\mu t + \mu_s l) \right] + \frac{S_v}{2} \int_{a'}^{a'+l} E_1 \left[\mu t + \mu_s(r'-a) \right] \sqrt{1 + \left(\frac{h' + a' - r'}{r'} \right)^2 \tan^2 \alpha} dr' \quad (204)$$

where the slab shield is assumed to lie between P' and the large end of the truncated cone source.

FIGURE 32
TRUNCATED CONE SOURCE WITH
DETECTOR POINT AT APEX

FIGURE 33
TRUNCATED CONE SOURCE WITH
AXIAL DETECTOR POINT

D. Toroidal Source with Detector Point at Center

See Figure 34. r_1 , r_2 and r_3 are given by (4), (5), and (6), respectively,

μ = macroscopic attenuation coefficient of the slab shield of thickness t ,
 μ_s = macroscopic self-attenuation coefficient of the toroidal source,
 μ_c = macroscopic attenuation coefficient of the toroidal shield of thickness c ,
and

$$\varphi_0 = \sin^{-1} \frac{R}{a+R} . \quad (205)$$

The differential volume and the attenuation factor A are given by

$$dV = r \sec^2 \varphi \, d\varphi \cdot r \sec \varphi \, d\theta \cdot dr = r^2 \sec^3 \varphi \, dr \, d\varphi \, d\theta \quad (206)$$

and

$$A = \frac{e^{-[\mu_s(r-r_1) + \mu_c(r_1-r_3) + \mu t \sec \theta] \sec \varphi}}{4\pi r^2 \sec^2 \varphi} , \quad (207)$$

respectively. Then the uncollided flux at the point P located at the center of the toroidal segment in Figure 34 is

$$\int d\Phi_{un} = \int_V S_v \cdot A \cdot dV \quad (208)$$

or

$$\Phi_{un} = 2 \cdot \frac{S_v}{4\pi} \int_0^{\varphi_0} \int_0^{\theta} \int_{r_1}^{r_2} e^{-[\mu_s(r-r_1) + \mu_c(r_1-r_3) + \mu t \sec \theta] \sec \varphi} dr \, d\varphi \, d\theta . \quad (209)$$

(209) may be integrated with respect to r to give

$$\Phi_{un} = \frac{S}{2\pi\mu_s} \int_0^{\varphi_0} \int_0^{\theta_0} \cos \varphi \left[1 - e^{-\mu_s(r_2-r_1) \sec \varphi} \right] e^{-[\mu_t \sec \theta + \mu_c(r_1-r_3) \sec \varphi]} d\varphi d\theta. \quad (210)$$

If we assume that there is no toroidal shield and that the slab shield is replaced by an annular shield centered at the detector point P, and curved in the θ -direction, the attenuation of this shield is given by

$$e^{-\mu_t \sec \varphi}.$$

The first term of (210) then becomes

$$\Phi_{\text{partial}} = \frac{S \theta_0}{2\pi\mu_s} \int_0^{\varphi_0} \cos \varphi e^{-\mu_t \sec \varphi} d\varphi \quad (211)$$

and this integral is the same as the function

$$G(\varphi_0, b) = \int_0^{\varphi_0} \cos \varphi e^{-b \sec \varphi} d\varphi \quad (212)$$

defined by Ono and Tsuro in Ref. 4, and which is described in the Appendix.

We may note here that equations (99) and (100c) show that the same flux formula is appropriate to the straight and curved line sources shown in Figures 15 and 15a. A certain similarity of behavior may also be observed by comparing equations (12) and (210). That is, the flux from a toroid with a given radius, shields, and central detector point location will be close to the flux from a cylindrical source with like radius, shields, and detector point location, if the angle subtended by the toroid is related to the height ℓ of the cylinder by the relation

$$\theta_0 = \tan^{-1} \frac{\ell}{a+R}.$$

FIGURE 34

TOROIDAL SEGMENT SOURCE WITH SELF ATTENUATION,
SLAB AND TOROIDAL SHIELDS,
DETECTOR POINT AT CENTER

E. Spherical Source with Interior Detector Point

See Figure 35. We present here the classic derivation from Refs. 1 and 2. The uncollided flux reaching P_3 in Figure 35, will be the sum of the flux from the sphere of radius $(R_0 - d)$ centered at P_3 plus the flux from the region outside this small sphere but inside the large sphere centered at P_1 . The integral over the small sphere yields

$$\begin{aligned}\phi_1 &= S_V \int_0^{R_0-d} 4\pi r^2 dr \frac{e^{-\mu_s r}}{4\pi r^2} \\ &= \frac{S_V}{\mu_s} \left[1 - e^{-\mu_s (R_0-d)} \right].\end{aligned}\quad (213)$$

The volume element used in the integration over the rest of the source is a segment of a spherical shell as shown in Figure 35. Its volume is

$$dV = 2\pi r^2 (1 - \cos \theta) dr \quad (214)$$

Thus

$$\phi_2 = S_V \int_{R_0-d}^{R_0+d} \frac{e^{-\mu_s r}}{4\pi r^2} 2\pi r^2 (1 - \cos \theta) dr. \quad (215)$$

Replacing $\cos \theta$ by its expression in terms of the sides of the triangle

$$\cos \theta = \frac{1}{2} \left[\frac{r}{d} + \frac{1}{r} \left(\frac{d^2 - R_0^2}{d} \right) \right] \quad (216)$$

yields

$$\phi_2 = \phi_{21} + \phi_{22} + \phi_{23} \quad (217)$$

where

$$\begin{aligned}\varphi_{21} &= \frac{S_V}{2} \int_{R_0-d}^{R_0+d} e^{-\mu_s r} dr \\ &= \frac{S_V}{2\mu_s} \left[e^{-\mu_s(R_0-d)} - e^{-\mu_s(R_0+d)} \right],\end{aligned}\quad (218)$$

$$\begin{aligned}\varphi_{22} &= -\frac{S_V}{4} \int_{R_0-d}^{R_0+d} e^{-\mu_s r} \frac{r}{d} dr \\ &= -\frac{S_V}{4\mu_s d} \left\{ e^{-\mu_s(R_0-d)} [\mu_s(R_0-d) + 1] - e^{-\mu_s(R_0+d)} [\mu_s(R_0+d) + 1] \right\},\end{aligned}\quad (219)$$

and

$$\begin{aligned}\varphi_{23} &= -\frac{S_V}{4} \int_{R_0-d}^{R_0+d} \frac{e^{-\mu_s r}}{r} \left(\frac{d^2 - R_0^2}{d} \right) dr \\ &= -\frac{S_V}{4} \left(\frac{d^2 - R_0^2}{d} \right) \left\{ E_1[\mu_s(R_0-d)] - E_1[\mu_s(R_0+d)] \right\}.\end{aligned}\quad (220)$$

After defining $b_4 \equiv \mu_s(R_0-d)$, $b_5 \equiv \mu_s(R_0+d)$, adding $\varphi_1 + \varphi_{21} + \varphi_{22} + \varphi_{23}$ and rearranging terms,

$$\begin{aligned}\phi_{un} &= \frac{S_V}{\mu_s} \left\{ 1 - \frac{e^{-b_4}}{2} - \frac{e^{-b_5}}{2} - \frac{1}{4\mu_s d} \left[e^{-b_4}(1+b_4) - e^{-b_5}(1+b_5) \right] + \frac{b_4 b_5}{4\mu_s d} \left[E_1(b_4) - E_1(b_5) \right] \right\}\end{aligned}\quad (221)$$

When $\mu_s = 0$, (221) reduces to

$$\phi_{un} = \frac{S_V R_0}{2} + \frac{S_V}{4d} (R_0^2 - d^2) \log \left(\frac{R_0+d}{R_0-d} \right). \quad (222)$$

When the point lies on the surface of a spherical source, $d = R_0$, $b_4 = 0$, $b_5 = 2\mu_s R_0$ and the term involving $(R_0^2 - d^2)/4d$ does not appear. Then (221) reduces to

$$\phi_{un} = \frac{S_V}{2\mu_s} \left(1 - \frac{1}{2\mu_s R_0} + \frac{e^{-2\mu_s R_0}}{2\mu_s R_0} \right) \quad (223)$$

and (223) gives the uncollided flux at any point P_2 on the surface of the sphere. When $\mu_s = 0$ in this case, (223) becomes

$$\phi_{un} = \frac{S_V R_0}{2} \cdot \quad (224)$$

Finally, if the detector point is at the point P_1 at the center of the sphere in Figure 35,

$$\phi_{un} = \frac{S_V}{\mu_s} \left(1 - e^{-\mu_s R_0} \right) \quad (225)$$

and if the point is at the center of the sphere with $\mu_s = 0$,

$$\phi_{un} = S_V R_0 \cdot \quad (226)$$

FIGURE 35
SPHERICAL SOURCE
WITH INTERIOR
DETECTOR POINT

F. Spherical Source with Exterior Detector Point

In Figure 36,

$$\varphi_0 = \sin^{-1} \frac{R}{a+R}, \quad (227)$$

$$dV = (r \sec \varphi \sin \varphi d\theta) (r \sec^2 \varphi d\varphi) dr, \quad (228)$$

and

$$A = \frac{e^{-[\mu_t + \mu_c(r_1 - r_3) + \mu_s(r - r_1)] \sec \varphi}}{4\pi (r \sec \varphi)^2}, \quad (229)$$

where r_1 , r_2 and r_3 are given by (4), (5), and (6) respectively. Then

$$d\phi_{un} = S_V \cdot A \cdot dV \quad (230)$$

and the uncollided flux at the point P in Figure 36 is given by

$$\phi_{un} = \frac{S_V}{4\pi} \int_0^{2\pi} \int_0^{\varphi_0} \int_{r_1}^{r_2} \sec \varphi \sin \varphi e^{-[\mu_t + \mu_c(r_1 - r_3) + \mu_s(r - r_1)] \sec \varphi} dr d\varphi d\theta. \quad (231)$$

This integrates to

$$\phi_{un} = \frac{S_V}{2\mu_s} \int_0^{\varphi_0} \left[1 - e^{-\mu_s(r_2 - r_1) \sec \varphi} \right] e^{-[\mu_t + \mu_c(r_1 - r_3)] \sec \varphi} \sin \varphi d\varphi \quad (232)$$

If $\mu_c = 0$, (232) becomes, using relations from the Appendix,

$$\begin{aligned} \Phi_{un} = \frac{S_V}{2\mu_s} & \left[E_2(\mu t) - \cos \varphi_0 E_2(\mu t) \sec \varphi_0 \right] \\ & - \frac{S_V}{2\mu_s} \int_0^{\varphi_0} \sin \varphi e^{-[\mu t + \mu_s(r_2 - r_1)] \sec \varphi} d\varphi. \end{aligned} \quad (233)$$

Compare this to the truncated cone, equations (186) and (191).

If $\mu_t = \mu_c = 0$, (233) reduces to

$$\Phi_{un} = \frac{S_V}{2\mu_s} \left[1 - \cos \varphi_0 - \int_0^{\varphi_0} \sin \varphi e^{-\mu_s(r_2 - r_1) \sec \varphi} d\varphi \right] \quad (234)$$

where

$$r_2 - r_1 = 2R \cos \varphi \sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}. \quad (235)$$

If the self-attenuation μ_s is zero also, we must go back to (231) and integrate the modified equation

$$\Phi_{un} = \frac{S_V}{4\pi} \cdot 2\pi \int_0^{\varphi_0} \int_{r_1}^{r_2} \sec \varphi \sin \varphi dr d\varphi. \quad (236)$$

Using (235), we have

$$\Phi_{un} = S_V \int_0^{\varphi_0} \sin \varphi \sqrt{R^2 - (a+R)^2 \sin^2 \varphi} d\varphi \quad (237)$$

which may be rewritten as follows:

$$\Phi_{un} = S_V \int_0^{\varphi_0} (a+R) \sqrt{\cos^2 \varphi - \frac{a^2 + 2aR}{(a+R)^2}} \sin \varphi d\varphi.$$

Integration of this gives the formula

$$\Phi_{un} = \frac{S_V}{4(a+R)} \left[2R(a+R) - a(a+2R) \log \left(\frac{a+2R}{a} \right) \right]. \quad (238)$$

FIGURE 36
 SPHERE WITH EXTERIOR DETECTOR PT.,
 SLAB SHIELD, SPHERICAL SHIELD

We may also use this formalism to obtain the formula for the flux at a detector point P_2 on the surface of the sphere. In this case, $a = 0$ and there are no exterior shields, and (232) reduces to

$$\phi_{un} = \frac{S_V}{2\mu_s} \int_0^{\varphi_0} \left[1 - e^{-\mu_s(r_2-r_1)\sec\varphi} \right] \sin\varphi \, d\varphi \quad (239)$$

$$\text{with } r_2 - r_1 = 2R \cos^2\varphi \quad (240)$$

$$\text{and } \varphi_0 = \frac{\pi}{2}.$$

This may be readily integrated to give

$$\phi_{un} = \frac{S_V}{2\mu_s} \left[1 - \frac{1}{2\mu_s R} + \frac{e^{-2\mu_s R}}{2\mu_s R} \right] \quad (241)$$

which is the same as (224).

If there is no attenuation, the flux at P_2 in Figure 36 may be found from (236), and is given by

$$\phi_{un} = \frac{S_V R}{2}. \quad (242)$$

Finally, if we consider only the front half of the sphere, the flux from this hemisphere at the point P is

$$\begin{aligned} \phi_{un} = & \frac{S_V}{2\mu_s} \int_0^{\varphi_1} \left[1 - e^{-\mu_s[a+R-r_1]\sec\varphi} \right] e^{-[\mu_t+\mu_c(r_1-r_3)]\sec\varphi} \sin\varphi \, d\varphi \\ & + \frac{S_V}{2\mu_s} \int_{\varphi_1}^{\varphi_0} \left[1 - e^{-\mu_s[r_2-r_1]\sec\varphi} \right] e^{-[\mu_t+\mu_c(r_1-r_3)]\sec\varphi} \sin\varphi \, d\varphi \end{aligned} \quad (243)$$

$$\text{where } \varphi_1 = \tan^{-1} \frac{R}{a+R}. \quad (244)$$

An interesting case is treated by Korzhov in Ref. 21. Given that the attenuation coefficients of the source and the spherical shield are the same in Figure 36, the uncollided flux at the point P_2 may be found using a

conventional spherical volume coordinate system. Let these coordinates be called ρ , ξ , and α , where α is the polar angle. Then the distance l in Figure 36 is given by

$$l^2 = \rho^2 + (R+c)^2 - 2\rho(R+c) \cos \alpha \quad (245)$$

and since $\mu_c = \mu_s$, we may write

$$\Phi_{un}(P_2) = \frac{S_V}{4\pi} \int_0^{2\pi} d\xi \int_0^R \rho^2 d\rho \int_0^\pi \frac{e^{-\mu_s l} \sin \alpha d\alpha}{l^2} . \quad (246)$$

Now, from (245)

$$\sin \alpha d\alpha = \frac{l dl}{\rho(R+c)} \quad (247)$$

and substituting from (245) and (247) into (246), using the relations between the E_1 , E_2 , and E_3 functions given in the Appendix, and integrating by parts, we obtain the following expression for the uncollided flux at P_2 in Figure 36

$$\Phi_{un} = \frac{S_V R}{2\mu_s (R+c)} \left\{ E_2(\mu_s c) - E_2[\mu_s (2R+c)] - \frac{1}{\mu_s R} E_3(\mu_s c) + \frac{1}{\mu_s R} E_3[\mu_s (2R+c)] \right\} \quad (248)$$

G. Rectangular Volume Source with Exterior Detector Point

See Figures 37a - 37c. We shall analyze this source using an approach somewhat different from that used by Posik in Ref. 22.

We first divide the rectangular source volume into four parts as shown in the figures. The limits for each division are tabulated below. For all divisions of the rectangular solid source, the differential volume is

$$dV = r \sec^2 \varphi d\varphi \cdot r \sec \varphi \sec^2 \theta d\theta \cdot dr$$

or

$$dV = r^2 \sec^2 \theta \sec^3 \varphi dr d\varphi d\theta$$

(249)

Also, in terms of the geometry of Figure 38, the cylindrical shield thickness at angle φ is given by $(r_1 - r_3)$, where

$$r_1 = R \left\{ \left(1 + \frac{\xi - R}{R} \right) \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(1 + \frac{\xi - R}{R} \right)^2 \sin^2 \varphi} \right\} \quad (250)$$

and

$$r_3 = (R+c) \left\{ \left(1 + \frac{\xi - R - c}{R+c} \right) \cos^2 \varphi - \cos \varphi \sqrt{1 - \left(1 + \frac{\xi - R - c}{R+c} \right)^2 \sin^2 \varphi} \right\} \quad (251)$$

and the attenuation A is given by

$$A = \frac{e^{-[\mu_s(r-a) + \mu_t t + \mu_c(r_1 - r_3)] \sec \varphi \sec \theta}}{4\pi r^2 \sec^2 \varphi \sec^2 \theta} \quad (252)$$

Table 1 gives the limits for Divisions I, II, III and IV of the rectangular volume source (see Figures 37a through 37c).

Table 1 - Rectangular Volume Source Limits

<u>Division</u>	<u>r-limits</u>	<u>ϕ-limits</u>	<u>θ-limits</u>
I	$(a, a+l_3)$	$(0, \tan^{-1} \frac{l_1}{a+l_3})$	$(0, \tan^{-1} \frac{l_2}{(a+l_3) \sec \phi})$
II	$(a, l_1 \cot \phi)$	$(\tan^{-1} \frac{l_1}{a+l_3}, \tan^{-1} \frac{l_1}{a})$	$(0, \tan^{-1} \frac{l_2}{(a+l_3) \sec \phi})$
III	$(a, l_2 \cot \theta)$	$(0, \tan^{-1} \frac{l_1}{a+l_3})$	$(\tan^{-1} \frac{l_2}{(a+l_3) \sec \phi}, \tan^{-1} \frac{l_2}{a \sec \phi})$
IV	$(a, l_1 \cot \phi)$	$(\tan^{-1} \frac{l_1}{a+l_3}, \tan^{-1} \frac{l_1}{a})$	$(\tan^{-1} \frac{l_2}{(a+l_3) \sec \phi}, \tan^{-1} \frac{l_2}{l_1 \csc \phi})$

The differential flux is then

$$d\phi_{un} = S_V A \cdot dV$$

and the full equation for the uncollided flux at the point P in Figures 37a - 37c and Figure 38 is

$$\phi_{un} = \frac{S_V}{4\pi} \int_a^{a+l_3} \int_0^{\tan^{-1} \frac{l_1}{a+l_3}} \int_{\tan^{-1} \frac{l_2}{(a+l_3)\sec \varphi}}^{\tan^{-1} \frac{l_2}{(a+l_3)\sec \varphi}} \sec \varphi e^{-[\mu_s(r-a)+\mu_t+\mu_c(r_1-r_3)]} \sec \varphi \sec \theta \, dr \, d\varphi \, d\theta$$

$$+ \frac{S_V}{4\pi} \int_a^{l_1 \cot \varphi} \int_{\tan^{-1} \frac{l_1}{a+l_3}}^{\tan^{-1} \frac{l_1}{a}} \int_0^{\tan^{-1} \frac{l_2}{(a+l_3)\sec \varphi}} \sec \varphi e^{-[\mu_s(r-a)+\mu_t+\mu_c(r_1-r_3)]} \sec \varphi \sec \theta \, dr \, d\varphi \, d\theta$$

$$+ \frac{S_V}{4\pi} \int_a^{l_2 \cot \theta} \int_0^{\tan^{-1} \frac{l_1}{a+l_3}} \int_{\tan^{-1} \frac{l_2}{(a+l_3)\sec \varphi}}^{\tan^{-1} \frac{l_2}{a \sec \varphi}} \sec \varphi e^{-[\mu_s(r-a)+\mu_t+\mu_c(r_1-r_3)]} \sec \varphi \sec \theta \, dr \, d\varphi \, d\theta$$

$$+ \frac{S_V}{4\pi} \int_a^{l_1 \cot \varphi} \int_{\tan^{-1} \frac{l_1}{a+l_3}}^{\tan^{-1} \frac{l_1}{a}} \int_{\tan^{-1} \frac{l_2}{(a+l_3)\sec \varphi}}^{\tan^{-1} \frac{l_2}{l_1 \csc \varphi}} \sec \varphi e^{-[\mu_s(r-a)+\mu_t+\mu_c(r_1-r_3)]} \sec \varphi \sec \theta \, dr \, d\varphi \, d\theta .$$

(253)

Integrating with respect to r gives

$$\begin{aligned}
\phi_{un} = & \frac{S_V}{4\pi\mu_s} \int_0^{\tan^{-1} \frac{l_1}{a+l_3}} \int_0^{\tan^{-1} \frac{l_2}{(a+l_3)\sec\varphi}} \cos\theta \left[1 - e^{-\mu_s l_3 \sec\varphi \sec\theta} \right] e^{-[\mu_t + \mu_c(r_1 - r_3)] \sec\varphi \sec\theta} d\varphi d\theta \\
& + \frac{S_V}{4\pi\mu_s} \int_{\tan^{-1} \frac{l_1}{a+l_3}}^{\tan^{-1} \frac{l_1}{a}} \int_0^{\tan^{-1} \frac{l_2}{(a+l_3)\sec\varphi}} \cos\theta \left[1 - e^{-\mu_s [l_1 \cot\varphi - a] \sec\varphi \sec\theta} \right] e^{-[\mu_t + \mu_c(r_1 - r_3)] \sec\varphi \sec\theta} d\varphi d\theta \\
& + \frac{S_V}{4\pi\mu_s} \int_0^{\tan^{-1} \frac{l_1}{a+l_3}} \int_{\tan^{-1} \frac{l_2}{(a+l_3)\sec\varphi}}^{\tan^{-1} \frac{l_2}{a \sec\varphi}} \cos\theta \left[1 - e^{-\mu_s [l_2 \cot\theta - a] \sec\varphi \sec\theta} \right] e^{-[\mu_t + \mu_c(r_1 - r_3)] \sec\varphi \sec\theta} d\varphi d\theta \\
& + \frac{S_V}{4\pi\mu_s} \int_{\tan^{-1} \frac{l_1}{a+l_3}}^{\tan^{-1} \frac{l_1}{a}} \int_{\tan^{-1} \frac{l_2}{(a+l_3)\sec\varphi}}^{\tan^{-1} \frac{l_2}{l_1 \csc\varphi}} \cos\theta \left[1 - e^{-\mu_s [l_1 \cot\varphi - a] \sec\varphi \sec\theta} \right] e^{-[\mu_t + \mu_c(r_1 - r_3)] \sec\varphi \sec\theta} d\varphi d\theta
\end{aligned}
\tag{254}$$

Two of these integrands may be combined together; they are left separate here because of the interesting shapes of the source volumes. This simplification, and the case in which $\mu_s - \mu_c = \mu = 0$, are left as exercises for the reader.

RECTANGULAR VOLUME SOURCE WITH EXTERIOR DETECTOR POINT

FIGURE 37a

RECTANGULAR VOLUME SOURCE. DETAIL OF DIVISION I, WHICH IS ONE QUARTER OF A RECTANGULAR PRISM

FIGURE 37b

RECTANGULAR VOLUME SOURCE. DETAIL OF DIVISIONS II AND III WHICH ARE WEDGE-SHAPED

FIGURE 37c
RECTANGULAR VOLUME SOURCE. DETAIL OF
DIVISION IV , WHICH IS ONE QUARTER OF
A FOUR-SIDED PYRAMID

FIGURE 38
TOP VIEW OF RECTANGULAR
VOLUME SOURCE WITH SLAB
AND CYLINDRICAL SHIELDS

H. Spherical Surface Source with Exterior Detector Point

See Figure 39.

φ_0 , r_1 , r_2 , and r_3 are given by equations (1), (4), (5), and (6) respectively. The differential surface areas are

$$\begin{aligned} dS_1 &= r_1 \sec \varphi \sin \varphi \, d\theta \cdot r_1 \sec \varphi \sec \gamma \, d\varphi \\ &= r_1^2 \sec^2 \varphi \sin \varphi \sec \gamma \, d\varphi \, d\theta \end{aligned} \quad (255)$$

and

$$\begin{aligned} dS_2 &= r_2 \sec \varphi \sin \varphi \, d\theta \cdot r_1 \sec \varphi \sec \gamma \, d\varphi \\ &= r_2^2 \sec^2 \varphi \sin \varphi \sec \gamma \, d\varphi \, d\theta \end{aligned} \quad (256)$$

where

$$\sec \gamma = \frac{1}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} \quad (257)$$

(See Section II-G and Figure 12).

The attenuation functions are

$$A_1 = \frac{e^{-[\mu t + \mu_c(r_1 - r_3)]} \sec \varphi}{4\pi r_1^2 \sec^2 \varphi} \quad (258)$$

and

$$A_2 = \frac{e^{-[\mu t + \mu_h(r_2 - r_1) + \mu_c(r_1 - r_3)]} \sec \varphi}{4\pi r_2^2 \sec^2 \varphi} \quad (259)$$

and the uncollided flux at the point P in Figure 39 is given by

$$\phi_{un} = \frac{S_a}{2} \int_0^{\varphi_0} \frac{e^{-[\mu_c(r_1-r_3)+\mu t]} \sec \varphi \left[1 + e^{-\mu_h(r_2-r_1) \sec \varphi} \right] \sin \varphi d\varphi}{\sqrt{1 - \left(1 + \frac{a}{R}\right)^2 \sin^2 \varphi}} \quad (260)$$

If $\mu_h = \mu = \mu_c = 0$, (260) becomes

$$\phi_{un} = S_a \frac{R}{a+R} \int_0^{\varphi_0} \frac{\sin \varphi d\varphi}{\sqrt{\cos^2 \varphi - \frac{2aR+a^2}{(a+R)^2}}} \quad (261)$$

This may be integrated to obtain

$$\phi_{un} = \frac{S_a}{2} \frac{R}{a+R} \log \left(\frac{a+2R}{a} \right) \quad (262)$$

To find the uncollided flux at the center of a spherical surface source we observe that

$$d\phi_{un} = S_a A \cdot dS = S_a \frac{e^{-\mu_h R}}{4\pi R^2} \cdot R^2 \sin \theta d\varphi d\theta$$

and

$$\phi_{un} = \frac{8S_a}{4\pi} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \theta e^{-\mu_h R} d\varphi d\theta \quad (263)$$

Then the flux at the center is

$$\phi_{un} = S_a e^{-\mu_h R} \quad (264)$$

If $\mu_h = 0$, this becomes

$$\phi_{un} = S_a \quad (265)$$

FIGURE 39
SPHERICAL SURFACE SOURCE WITH
EXTERIOR DETECTOR POINT

V. GENERALIZATION OF CYLINDRICAL SOURCE FLUX EQUATIONS

This section describes generalizations of cylindrical source flux equations for the following shield and detector point locations:

- A. Cylindrical Source with Arbitrary Lateral Detector Point
- B. Cylindrical Source with Arbitrary Off-End Detector Point and Infinite Cylindrical Shields
- C. Cylindrical Source with Arbitrary Off-End Detector Point and a Finite Cylindrical Shield

The flux formulas given in Section II of this report are not general in several respects. Three major restrictions still exist: i) shields are assumed to be infinite in at least one coordinate, ii) detector points may not be arbitrarily located, and iii) uniform source strength distributions are always assumed. In this section we will relax restrictions i) and ii) somewhat and consider how this attempt to get a little closer to reality complicates the analysis. The special coordinate system discussed in Section II and used throughout the report makes this feasible. Conversely, we will not attempt to treat non-uniform source strength distributions, since this same special coordinate system is not suited to this purpose. See Refs. 1 and 3 for a limited treatment of non-uniform source strength distributions.

We shall first generalize the lateral detector point case, and then consider the off-end detector point case, allowing a finite cylindrical shield.

A. Cylindrical Source with Arbitrary Lateral Detector Point

As shown in Figure 40, this is a generalization of the case given in Figure 1. The detector point is still opposite the source, but we now allow it to be laterally displaced a distance d measured from a line passing through the axis of the cylinder and perpendicular to the slab shield. From equations (1) through (12), and referring to Figures 40 and 41, we may write the following equations which take account of the displacement d :

$$\alpha = \tan^{-1} \frac{d}{a+R} \quad (266)$$

$$a' = \sqrt{(a+R)^2 + d^2} - R \quad (267)$$

$$\varphi_0 = \sin^{-1} \frac{R}{a+R} \quad (268)$$

$$r_1 = (R+a') \cos^2 \varphi - \cos \varphi \sqrt{R^2 - (R+a')^2 \sin^2 \varphi} \quad (269)$$

$$r_2 = (R+a') \cos^2 \varphi + \cos \varphi \sqrt{R^2 - (R+a')^2 \sin^2 \varphi} \quad (270)$$

$$r_3 = (R+a') \cos^2 \varphi - \cos \varphi \sqrt{(R+c)^2 - (R+a')^2 \sin^2 \varphi} \quad (271)$$

$$r_\ell = \ell \cot \theta \cos \varphi \quad (272)$$

$$r_2 - r_1 = 2 \cos \varphi \sqrt{R^2 - (R+a')^2 \sin^2 \varphi} \quad (273)$$

$$\theta_1 = \tan^{-1} \frac{\ell}{r_1 \sec \varphi} \quad (274)$$

$$\theta_2 = \tan^{-1} \frac{\ell}{r_2 \sec \varphi} \quad (275)$$

The uncollided flux at the point P in Figure 40 or Figure 41 is then given by the following generalization of equation (12):

$$\Phi_{un} = \frac{S_V}{4\pi\mu_s} \int_0^{\varphi_0} \int_0^{\theta_2} \cos \theta \left[1 - e^{-\mu_s(r_2-r_1)\sec \varphi \sec \theta} \right] e^{-[\mu_t \sec(\alpha-\varphi) + \mu_c(r_1-r_3)\sec \varphi] \sec \theta} d\varphi d\theta$$

$$+ \frac{S_V}{4\pi\mu_s} \int_0^{\varphi_0} \int_{\theta_2}^{\theta_1} \cos \theta \left[1 - e^{-\mu_s(l \cot \theta \cos \varphi - r_1)\sec \varphi \sec \theta} \right] e^{-[\mu_t \sec(\alpha-\varphi) + \mu_c(r_1-r_3)\sec \varphi] \sec \theta} d\varphi d\theta$$

$$+ \frac{S_V}{4\pi\mu_s} \int_0^{\varphi_0} \int_0^{\theta_2} \cos \theta \left[1 - e^{-\mu_s(r_2-r_1)\sec \varphi \sec \theta} \right] e^{-[\mu_t \sec(\alpha+\varphi) + \mu_c(r_1-r_3)\sec \varphi] \sec \theta} d\varphi d\theta$$

$$+ \frac{S_V}{4\pi\mu_s} \int_0^{\varphi_0} \int_{\theta_2}^{\theta_1} \cos \theta \left[1 - e^{-\mu_s(l \cot \theta \cos \varphi - r_1)\sec \varphi \sec \theta} \right] e^{-[\mu_t \sec(\alpha+\varphi) + \mu_c(r_1-r_3)\sec \varphi] \sec \theta} d\varphi d\theta$$

(276)

We note that this relatively simple extension of eq. (12) requires a new piece of data and the evaluation of four rather than two integrands. The off-end case which we shall consider next is somewhat more complicated.

FIGURE 40

CYLINDRICAL SOURCE WITH SLAB AND
CYLINDRICAL SHIELDS AND LATERAL
DETECTOR POINT. DETECTOR POINT IS
OPPOSITE SOURCE BUT ARBITRARILY
LOCATED WITH RESPECT TO A RADIUS
PERPENDICULAR TO THE SLAB SHIELD

FIGURE 41
CYLINDRICAL SOURCE WITH
LATERAL DETECTOR POINT-
DETAILED TOP VIEW

B. Cylindrical Source Volume with Arbitrary Off-End Detector Point and Infinite Cylindrical Shields

This is an extension of the analysis summed up in equations 66a-66g of Section II. The general situation is shown in Figures 42 and 43. In Section II the case where h (the displacement of the detector point off the axis of the cylinder) is less than the cylinder radius R was considered. Referring to Figures 10 and 43 and to equations 66c through 66e, we see that if

$$h = R, \quad (277)$$

then

$$\phi_2 = 2R \cos \varphi, \quad (278)$$

$$\theta_1 = \tan^{-1} \frac{2R \cos \varphi}{a+l}, \quad (279)$$

$$\theta_2 = \tan^{-1} \frac{2R \cos \varphi}{a}, \quad (280)$$

and from (66e)

$$\begin{aligned} \phi_{un} = & \frac{S_V}{2\pi\mu_s} \int_0^{\frac{\pi}{2}} \int_0^{\theta_1} \left[1 - e^{-\mu_s l \sec \theta} \right] e^{-\mu t \sec \theta} \sin \theta \, d\varphi \, d\theta \\ & + \frac{S_V}{2\pi\mu_s} \int_0^{\frac{\pi}{2}} \int_{\theta_1}^{\theta_2} \left[1 - e^{-\mu_s [2R \cos \varphi \csc \theta - a \sec \theta]} \right] e^{-\mu t \sec \theta} \sin \theta \, d\varphi \, d\theta. \end{aligned} \quad (281)$$

The next step is to consider the case, shown in Figure 43, where

$$h > R$$

and both parallel and perpendicular slab shields are present. A cylindrical shield of thickness c is also present. Assume (for the present) that this cylindrical shield, like the slab shield parallel to the axis of the

FIGURE 42

CYLINDRICAL SOURCE WITH AXIAL OFF-
SET DETECTOR POINT

cylindrical source volume, is of infinite extent, and that the condition on h and R is really

$$h > R + c$$

so that the cylindrical shield is always encountered in any ray-tracing done. Now see Figures 44b, 45b, and 46b. It is easily shown that

$$\rho_1 = h \cos \varphi - \sqrt{R^2 - h^2 \sin^2 \varphi}, \quad (282)$$

$$\rho_2 = h \cos \varphi + \sqrt{R^2 - h^2 \sin^2 \varphi}, \quad (283)$$

$$\rho_3 = h \cos \varphi - \sqrt{(R+c)^2 - h^2 \sin^2 \varphi}, \quad (284)$$

and we may define

$b = \mu t$, the optical distance thru the slab shield parallel to the Z-axis of the cylindrical source ;

$b_p = m\tau$, the optical distance through the slab shield perpendicular to the Z-axis of the cylindrical source ;

$$b_c = \mu_c(\rho_1 - \rho_3) = \mu_c \left(\sqrt{(R+c)^2 - h^2 \sin^2 \varphi} - \sqrt{R^2 - h^2 \sin^2 \varphi} \right), \text{ the} \quad (285)$$

optical distance through the cylindrical shields at angle φ ;

$$\varphi_0 = \sin^{-1} \left(\frac{R}{h} \right), \text{ the limit on } \varphi; \text{ and} \quad (286)$$

$$dV = \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta. \quad (287)$$

Now consider the path of the radius vector ρ through the source and shields of Figure 43 as it sweeps over the cylindrical source volume. The expression for the maximum length of ρ depends on whether ρ terminates on the base or on the side of the cylinder, and the path through the shields

FIGURE 43

CYLINDRICAL SOURCE WITH SLAB SHIELDS PARALLEL AND PERPENDICULAR TO THE SOURCE, CYLINDRICAL SHIELDS, AND AXIAL OFF-CENTER DETECTOR POINT

depends on the position of the detector point. We thus find that the attenuation function changes according to the relationship between the angles

$$\tan^{-1} \frac{\rho_1}{a} \quad \text{and} \quad \tan^{-1} \frac{\rho_2}{a+l}$$

which serve as θ -limits.

From (282), (283), (284), and (286), we may write

$$\rho_1(0) = h-R ,$$

$$\rho_2(0) = h+R ,$$

$$\rho_3(0) = h-R-c ,$$

$$\sin \varphi_0 = \frac{R}{h} ,$$

$$\cos \varphi_0 = \frac{\sqrt{h^2 - R^2}}{h} ,$$

$$\rho_1(\varphi_0) = \rho_2(\varphi_0) = \sqrt{h^2 - R^2} ,$$

and $h^2 > R^2$ since $h > R+c$ is assumed.

Refer to Figures 44a and 44b and ignore the cylindrical shield for the moment. In Figure 44a, which we shall call a Case I geometry, we have

$$\frac{\rho_1(0)}{a} < \frac{\rho_2(0)}{a+l}$$

$$\text{or} \quad \frac{h-R}{a} < \frac{h+R}{a+l} .$$

Then as φ increases, $\rho_1(\varphi)$ increases, while $\rho_2(\varphi)$ decreases, but as long as

$$\frac{\rho_1(\varphi)}{a} < \frac{\rho_2(\varphi)}{a+l}$$

we still have a Case I geometry and the source divides into three regions having limits and source exponential attenuation as given in Table 2:

TABLE 2 - Case I Geometry

Region	θ -limits	ρ -limits	Source Exponential Attenuation Function T
i	$\left(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_1}{a} \right)$	$(\rho_1 \csc \theta, (a+l) \sec \theta)$	$e^{-\mu_s(\rho - \rho_1 \csc \theta)}$
ii	$\left(\tan^{-1} \frac{\rho_1}{a}, \tan^{-1} \frac{\rho_2}{a+l} \right)$	$(a \sec \theta, (a+l) \sec \theta)$	$e^{-\mu_s(\rho - a \sec \theta)}$
iii	$\left(\tan^{-1} \frac{\rho_2}{a+l}, \tan^{-1} \frac{\rho_2}{a} \right)$	$(a \sec \theta, \rho_2 \csc \theta)$	$e^{-\mu_s(\rho - a \sec \theta)}$

It may happen, however, that φ_0 may be large enough so that at some angle φ_m , $0 < \varphi_m \leq \varphi_0$, we have

$$\frac{\rho_1(\varphi_m)}{a} = \frac{\rho_2(\varphi_m)}{a+l};$$

substituting from (282) and (283) into this equation we find that

$$\varphi_m = \sin^{-1} \left(\frac{1}{2h} \sqrt{\frac{(2a+l)^2 R^2 - l^2 h^2}{a(a+l)}} \right) \quad (288)$$

and φ_m is always real since $\frac{h-R}{a} < \frac{h+R}{a+l}$ implies that $\frac{h}{R} < \frac{2a+l}{l}$. Thus we pass from the Case I geometry discussed just above to what we call a Case II geometry, as shown in Figures 44b and 44c. Here

$$\frac{\rho_1}{a} \geq \frac{\rho_2}{a+l}.$$

Such a geometry is also shown in Figure 45. Indeed, this shows a case in which

$$\frac{\rho_1(0)}{a} = \frac{h-R}{a} \geq \frac{h+R}{a+l} = \frac{\rho_2(0)}{a+l}$$

and the Case II geometry applies throughout the integration over the source. Here again the source divides into three regions having limits and source

exponential attenuation functions which differ somewhat from those given above for the Case I geometry in Table 2. These are given in Table 3:

TABLE 3 - Case II Geometry

Region	θ -limits	ρ -limits	Source Exponential Attenuation Function T
i	$\left(\tan^{-1} \frac{\rho_1}{a}, \tan^{-1} \frac{\rho_2}{a+l}\right)$	$(\rho_1 \csc \theta, (a+l) \sec \theta)$	$e^{-(\rho - \rho_1 \csc \theta)}$
ii	$\left(\tan^{-1} \frac{\rho_2}{a+l}, \tan^{-1} \frac{\rho_1}{a}\right)$	$(\rho_1 \csc \theta, \rho_2 \csc \theta)$	$e^{-(\rho - \rho_1 \csc \theta)}$
iii	$\left(\tan^{-1} \frac{\rho_1}{a}, \tan^{-1} \frac{\rho_2}{a}\right)$	$(a \sec \theta, \rho_2 \csc \theta)$	$e^{-(\rho - a \sec \theta)}$

Now in all these cases the total attenuation function A is given by (289) below. b , b_p and b_c were defined just after eq. (284).

$$A = \frac{T \left(e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \right)}{4\pi \rho^2} \quad (289)$$

$$\text{and } d\dot{\Phi}_{un} = S_V \cdot A \cdot dV. \quad (290)$$

We may now write the integrals for the uncollided flux for the Case I and Case II geometries, and integrate with respect to ρ . This yields, for Case I

$$\phi_I(\varphi_m) = \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_1}{a+l}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s [(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta$$

$$+ \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s l \sec \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta$$

$$+ \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s (\rho_2 \csc \theta - a \sec \theta)} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta$$

(291)

and for Case II,

$$\Phi_{II}(\varphi_m) \equiv \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s [(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta$$

$$+ \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s (\rho_2 - \rho_1) \csc \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta$$

$$+ \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s (\rho_2 \csc \theta - a \sec \theta)} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta.$$

(292)

In finding the uncollided flux at the point P in Figure 43, the procedure is then as follows:

If $\frac{h-R}{a} < \frac{h+R}{a+l}$, the angle φ_m may be found from (288). φ_m must lie in the range $0 < \varphi_m < \varphi_0$ since (288) may be written

$$\varphi_m = \sin^{-1} \left(\frac{R}{h} \sqrt{\frac{(2a+l)^2 - l^2 h^2 / R^2}{4a(a+l)}} \right)$$

and the quantity under the radical equals unity only if $h = R$, but $h > R + c$ is assumed. Then

$$\phi_{un} = \phi_I(\varphi_m) + \phi_{II}(\varphi_m) \quad (293)$$

where ϕ_I and ϕ_{II} are given by (291) and (292) respectively.

If, however, $\frac{h-R}{a} > \frac{h+R}{a+l}$,

then only the case II geometry applies and

$$\phi_{un} = \phi_{II}(0). \quad (294)$$

Equations (281)-(294) assume that the cylindrical shield(s) are of infinite extent. They also assume, to avoid extra complications, that both the parallel and perpendicular slab shields exist where they intersect - that is, both shields exist in the region where the intersection occurs (see Figure 43) and occupy the same space. Any radiation passing through this region is then attenuated by both shields, which is not physically realistic. The same is true of the region where the cylindrical shield intersects the perpendicular slab shield. In practice this means that the user must be careful in defining the shields in a problem of this kind.

C. Cylindrical Source with Arbitrary Off-End
Detector Point and a Finite Cylindrical Shield

The last step we shall take in generalizing this analysis will be to remove the assumption that the cylindrical shield(s) are infinite and that $h > R + c$. Instead, we shall assume that we have one cylindrical shield which has of the same length as the source, as in Figure 42, and that $h > R$, and obtain flux formulas which take account of this complication. These formulas will be extensions of (291) and (292).

In this configuration, the Case I and Case II geometries still apply and the definition of ϕ_m given in (288) remains the same. The source volume must however be split into more regions because the ray trace does not always pass through the entire finite cylindrical shield (See Figures 43, 44, 45 and 46).

In order to restrict the present analysis to a manageable number of useful cases, we shall require that the cylindrical shield thickness c not exceed a limit which depends on the problem dimensions. This limit is

$$c \leq \frac{2aR}{a+l} \quad \text{or} \quad \frac{c}{a} \leq \frac{2R}{a+l}$$

and if we combine it with the condition for a Case II geometry, namely

$$\frac{h-R}{a} > \frac{h+R}{a+l}$$

we get

$$\frac{c}{2R} \leq \frac{h-R}{h+R},$$

which guarantees that for the limiting configuration of a Case II geometry, the cylindrical shield thickness c will be divided into just two parts for ray-tracing purposes.

That is, we determine our geometry configuration (Case I or Case II) without reference to c , the cylindrical shield thickness, and so we must require that c conform to certain assumptions we make about the ray-trace in Case II geometry.

FIGURE 44a

CASE I, SUBCASE I
SECTION THRU CYLINDER
AT $\phi = 0$

$$\frac{h-R}{a+l} \geq \frac{h-R-c}{a}$$

FIGURE 44b

CASE I FOLLOWED BY CASE II
END VIEW OF CYLINDER
WITH SECTIONS MARKED
(SUBCASE I FOLLOWED BY
SUBCASE 5)

FIGURE 44c

CASE II, SUBCASE 5
SECTION THRU CYLINDER
AT AN ANGLE $\phi > \phi_m$
 $\frac{\rho_1}{a+l} \leq \frac{\rho_2}{a+l} \leq \frac{\rho_1}{a} \leq \frac{\rho_2}{a}$

FIGURE 45a

CASE II, SUBCASE 3
SECTION THRU CYLINDER
AT $\phi = 0$

$$\frac{h+R}{a+l} \geq \frac{h-R-c}{a}$$

FIGURE 45b

CASE II
END VIEW OF CYLINDER
WITH SECTIONS MARKED
(SUBCASE 3 FOLLOWED
BY SUBCASE 4)

FIGURE 45c

CASE II, DIVISION BETWEEN
SUBCASES 3 AND 4
SECTION THRU CYLINDER
AT AN ANGLE ϕ_c SUCH THAT
 $\frac{\rho_3(\phi_c)}{a} = \frac{\rho_2(\phi_c)}{a+l}$

FIGURE 46a
CASE I, SUBCASE 2
SECTION THRU CYLINDER

AT $\phi = 0$

$$\frac{h-R}{a+l} < \frac{h-R-c}{a}$$

FIGURE 46b
CASE I FOLLOWED BY CASE II
END VIEW OF CYLINDER
WITH SECTIONS MARKED
(SUBCASE 2 FOLLOWED BY
SUBCASE 3)

FIGURE 46c
CASE II, SUBCASE 3
SECTION THRU CYLINDER
AT AN ANGLE $\phi > \phi_m$

$$\frac{\rho_1}{a+q} \leq \frac{\rho_3}{a} \leq \frac{\rho_2}{a+q} \leq \frac{\rho_1}{a} \leq \frac{\rho_2}{a}$$

$$\frac{\rho_1}{a+l} \leq \frac{\rho_3}{a} \leq \frac{\rho_2}{a+l} \leq \frac{\rho_1}{a} \leq \frac{\rho_2}{a}$$

Now consider the distance traced through the finite cylindrical shield. There are now five configurations to be considered, two under Case I and three under Case II, and the pertinent limits and optical distances are enumerated in Table 4. Reference to Figures 44, 45 and 46 may be helpful here.

Figure 44a shows a section through a Case I geometry at $\varphi = 0$ and illustrates Subcase 1 of Table 4. Figure 44c shows a section through the same cylinder for $\varphi > \varphi_m$, so that a Case II geometry and Subcase 5 is illustrated. Figure 46a shows subcase 2 of a Case I geometry, which becomes a Case II, Subcase 3 geometry for $\varphi > \varphi_m$, as shown in Figure 46c.

Figure 45a illustrates a Case II geometry at $\varphi = 0$, and so also applies to Subcase 3, while Figure 45c illustrates the situation at $\varphi = \varphi_c$, where we switch from Subcase 3 to Subcase 4.

The initial geometry condition of Subcase 3 is

$$\frac{h+R}{a+l} > \frac{h-R-c}{a} .$$

This is the same as

$$\frac{\rho_2(0)}{a+l} > \frac{\rho_3(0)}{a} ,$$

and from the definitions of ρ_2 and ρ_3 given in (283) and (284), we see that $\rho_2(\varphi)$ decreases with increasing φ , while $\rho_3(\varphi)$ increases, and at some value of φ which we will call φ_c , $0 \leq \varphi_m < \varphi_c < \varphi_0$, and

$$\frac{\rho_2(\varphi_c)}{a+l} = \frac{\rho_3(\varphi_c)}{a} . \quad (294a)$$

Then for $\varphi > \varphi_c$ we must use Subcase 4.

TABLE 4 - Finite Cylindrical Shield Optical Distance

Geometry	Subcase	Relationship of tangents of angles which are θ -limits	The optical distance through the finite cylindrical shield is given by for the θ -limits	
Case I $\frac{h-R}{a} < \frac{h+R}{a+l}$	1	$\frac{\rho_1}{a+l} < \frac{\rho_1}{a} < \frac{\rho_2}{a+l} < \frac{\rho_2}{a}$	$b_a = \mu_c(\rho_1 \csc \theta - a \sec \theta)$	$(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_1}{a})$
	2	$\frac{\rho_1}{a+l} < \frac{\rho_3}{a} < \frac{\rho_1}{a} < \frac{\rho_2}{a+l} < \frac{\rho_2}{a}$	$b_c \csc \theta = \mu_c(\rho_1 - \rho_3) \csc \theta$	$(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_3}{a})$
			$b_a = \mu_c(\rho_1 \csc \theta - a \sec \theta)$	$(\tan^{-1} \frac{\rho_3}{a}, \tan^{-1} \frac{\rho_1}{a})$
	3	$\frac{\rho_1}{a+l} < \frac{\rho_3}{a} < \frac{\rho_2}{a+l} < \frac{\rho_1}{a} < \frac{\rho_2}{a}$	$b_c \csc \theta = \mu_c(\rho_1 - \rho_3) \csc \theta$	$(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_3}{a})$
Case II $\frac{h-R}{a} > \frac{h+R}{a+l}$ or $\frac{\rho_2}{a+l} > \frac{\rho_3}{a}$ $\frac{\rho_1(\varphi)}{a} > \frac{\rho_2(\varphi)}{a+l}$ for $\varphi > \varphi_m$	4	$\frac{\rho_1}{a+l} < \frac{\rho_2}{a+l} < \frac{\rho_3}{a} < \frac{\rho_1}{a} < \frac{\rho_2}{a}$	$b_c \csc \theta = \mu_c(\rho_1 - \rho_3) \csc \theta$	$(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_3}{a})$
	5	$\frac{\rho_1}{a+l} < \frac{\rho_2}{a+l} < \frac{\rho_1}{a} < \frac{\rho_2}{a}$	$b_a = \mu_c(\rho_1 \csc \theta - a \sec \theta)$	$(\tan^{-1} \frac{\rho_3}{a}, \tan^{-1} \frac{\rho_1}{a})$
			$b_c \csc \theta = \mu_c(\rho_1 - \rho_3) \csc \theta$	$(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_3}{a})$
	6	$\frac{\rho_1}{a+l} < \frac{\rho_2}{a+l} < \frac{\rho_1}{a} < \frac{\rho_2}{a}$	$b_a = \mu_c(\rho_1 \csc \theta - a \sec \theta)$	$(\tan^{-1} \frac{\rho_1}{a+l}, \tan^{-1} \frac{\rho_1}{a})$
			$b_c \csc \theta = \mu_c(\rho_1 - \rho_3) \csc \theta$	$(\tan^{-1} \frac{\rho_3}{a}, \tan^{-1} \frac{\rho_1}{a})$

Substitution from (283) and (284) into (294a) does not yield an easy way of computing φ_c . An iterative process may be used to find φ_c , or the reader may attempt to solve (294a) for φ_c as an exercise.

For Subcase 4, the initial geometry condition is

$$\frac{\rho_2(0)}{a+l} < \frac{\rho_3(0)}{a}$$

and this remains true for $\varphi > 0$, so that no value of φ_c is needed for Subcase 4.

Now if we combine the results of Tables 2, 3 and 4 and form flux integrals for the various divisions of the source in the φ - and θ -directions we shall have five formulas, which are given below. The quantities b , b_p , b_a , $b_c \csc \theta$, φ_0 , φ_m and φ_c have all been defined either in Section V.B or in the present section.

Case I, Subcase 1. $\frac{h-R}{a} < \frac{h+R}{a+l}$, φ_m exists, and $\frac{h-R}{a+l} \geq \frac{h-R-c}{a}$; since $\frac{\rho_1(\varphi)}{a+l} \geq \frac{\rho_3(\varphi)}{a}$ for all φ , this case and subcase apply for $0 \leq \varphi \leq \varphi_m$ and Case II, subcase 5 applies for $\varphi_m < \varphi \leq \varphi_0$.

$$\Phi_{I.1}(\varphi_m) = \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_1}{a+l}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s [(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta$$

$$+ \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s l \sec \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta$$

$$+ \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s [\rho_2 \csc \theta - a \sec \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta .$$

Case II, Subcase 5. $\frac{h-R}{a} > \frac{h+R}{a+l}$ or $\frac{\rho_1(\varphi)}{a} > \frac{\rho_2(\varphi)}{a}$ for $\varphi > \varphi_m$ and $\frac{\rho_1}{a+l} > \frac{\rho_3}{a}$ for all φ , so this case and subcase apply for $\varphi_m < \varphi \leq \varphi_0$. $c \leq \frac{2aR}{a+l}$ is assumed.

$$\begin{aligned} \Phi_{II.5}(\varphi_m) &= \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a+l}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s[(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta \\ &+ \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s(\rho_2 - \rho_1) \csc \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta \\ &+ \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s(\rho_2 \csc \theta - a \sec \theta)} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta . \end{aligned}$$

(295a)

Case I, Subcase 2. $\frac{h-R}{a} < \frac{h+R}{a+l}$ and $\frac{h-R}{a+l} < \frac{h-R-c}{a}$; this case and subcase apply for $0 \leq \varphi \leq \varphi_m$. For larger angles $\varphi_m < \varphi \leq \varphi_c$ Case II, Subcase 3 applies. If $\varphi_c < \varphi_0$, Case II, Subcase 4 then is appropriate for $\varphi_c < \varphi \leq \varphi_0$.

$$\begin{aligned}
 \Phi_{I.2}(\varphi_m) = & \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_1}{a+l}}^{\tan^{-1} \frac{\rho_3}{a}} \left[1 - e^{-\mu_s[(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_3}{a}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s[(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s l \sec \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_0^{\varphi_m} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s[\rho_2 \csc \theta - a \sec \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta .
 \end{aligned}$$

Case II, Subcase 3. $\frac{h-R}{a} \geq \frac{h+R}{a+l}$ or $\frac{\rho_1(\varphi)}{a} \geq \frac{\rho_2(\varphi)}{a+l}$ for $\varphi > \varphi_m$; $\frac{h+R}{a+l} \geq \frac{h-R-c}{a}$ or $\frac{\rho_2(\varphi)}{a+l} \geq \frac{\rho_3(\varphi)}{a}$, and $c \leq \frac{2aR}{a+l}$ is assumed. If $\frac{\rho_2(\varphi_c)}{a+l} = \frac{\rho_3(\varphi_c)}{a}$ and $\varphi_m < \varphi < \varphi_c < \varphi_0$ this case and subcase apply; for $\varphi > \varphi_c$, Case II and Subcase 4 are to be used.

$$\begin{aligned}
 \Phi_{II.3}(\varphi_m, \varphi_c) = & \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_c} \int_{\tan^{-1} \frac{\rho_1}{a+l}}^{\tan^{-1} \frac{\rho_3}{a}} \left[1 - e^{-\mu_s [(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_c} \int_{\tan^{-1} \frac{\rho_3}{a}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s [(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_c} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s [\rho_2 - \rho_1] \csc \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_{\varphi_m}^{\varphi_c} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s [\rho_2 \csc \theta - a \sec \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta
 \end{aligned}
 \tag{297}$$

Case II, Subcase 4. $\frac{h-R}{a} \geq \frac{h+R}{a+l}$ or $\frac{\rho_1(\varphi)}{a} \geq \frac{\rho_2(\varphi)}{a+l}$ for $\varphi > \varphi_m$; $c \leq \frac{2aR}{a+l}$ is assumed and $\frac{h+R}{a+l} < \frac{h-R-c}{a}$ or $\frac{\rho_2(\varphi)}{a+l} < \frac{\rho_3(\varphi)}{a}$ for $\varphi > \varphi_c > \varphi_m$ where $\frac{\rho_2(\varphi_c)}{a+l} = \frac{\rho_3(\varphi_c)}{a}$. This case and subcase may follow a Case II, subcase 3 configuration.

$$\begin{aligned}
 \Phi_{II.4}(\varphi_c) = & \frac{S_V}{2\pi\mu_s} \int_{\varphi_c}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a+l}}^{\tan^{-1} \frac{\rho_2}{a+l}} \left[1 - e^{-\mu_s[(a+l) \sec \theta - \rho_1 \csc \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_{\varphi_c}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_2}{a+l}}^{\tan^{-1} \frac{\rho_3}{a}} \left[1 - e^{-\mu_s[\rho_2 - \rho_1] \csc \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_c \csc \theta]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_{\varphi_c}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_3}{a}}^{\tan^{-1} \frac{\rho_1}{a}} \left[1 - e^{-\mu_s[\rho_2 - \rho_1] \csc \theta} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta + b_a]} \sin \theta \, d\varphi \, d\theta \\
 & + \frac{S_V}{2\pi\mu_s} \int_{\varphi_c}^{\varphi_0} \int_{\tan^{-1} \frac{\rho_1}{a}}^{\tan^{-1} \frac{\rho_2}{a}} \left[1 - e^{-\mu_s[\rho_2 \csc \theta - a \sec \theta]} \right] e^{-[b_p \sec \theta + b \sec \varphi \csc \theta]} \sin \theta \, d\varphi \, d\theta.
 \end{aligned}$$

The algorithm for finding the flux at the point P in Figure 42, with one finite cylindrical shield and with infinite perpendicular and parallel slab shields which intersect as noted in the last paragraph of Section V.B, is then as follows:

Step 1. If $\frac{h-R}{a} < \frac{h+R}{a+l}$ Case I applies. Otherwise go to Step 4 which treats Case II geometry.

Step 2. If $\frac{h-R}{a+l} \geq \frac{h-R-c}{a}$, find φ_m from (284) and

$$\Phi_{un} = \Phi_{I.1}(\varphi_m) + \Phi_{II.5}(\varphi_m). \quad (299)$$

Step 3. If $\frac{h-R}{a+l} < \frac{h-R-c}{a}$, find φ_m from (284) and φ_c from (294a). Then, if $\varphi_m < \varphi_c < \varphi_0$

$$\Phi_{un} = \Phi_{I.2}(\varphi_m) + \Phi_{II.3}(\varphi_m, \varphi_c) + \Phi_{II.4}(\varphi_c) \quad (300)$$

or, if φ_c does not exist such that $\varphi_m < \varphi_c < \varphi_0$,

$$\Phi_{un} = \Phi_{I.2}(\varphi_m) + \Phi_{II.3}(\varphi_m, \varphi_0). \quad (301)$$

For a Case I geometry, this ends the algorithm.

Step 4. $\frac{h-R}{a} \geq \frac{h+R}{a+l}$ and Case II applies.

If $\frac{h-R-c}{a} < \frac{h-R}{a+l}$,

$$\Phi_{un} = \Phi_{II.5}(0). \quad (302)$$

Step 5. If $\frac{h+R}{a+l} \geq \frac{h-R-c}{a}$ and φ_c can be found from (294a) such that $0 < \varphi_c < \varphi_0$,

$$\Phi_{un} = \Phi_{II.3}(0, \varphi_c) + \Phi_{II.4}(\varphi_c), \quad (303)$$

while if φ_c such $0 < \varphi_c < \varphi_0$ does not exist,

$$\Phi_{un} = \Phi_{II.3}(0, \varphi_0) . \quad (303a)$$

Step 6. If $\frac{h+R}{a+l} < \frac{h-R-c}{a}$,

$$\Phi_{un} = \Phi_{II.4}(0) . \quad (304)$$

This ends the algorithm.

Equations (295) - (304) and the conditions under which they apply constitute a fairly complicated algorithm requiring much computation to numerically integrate the required functions. Any further refinements of these flux formulas result in still more complicated algorithms and expressions. For these cases a more general program, such as SPAN4 (Ref. 8) should be used, since the computational advantages of using two-dimensional flux formulas with implicit ray-tracing are not great for formulas such as (300).

VI. APPENDIX - SOME USEFUL INTEGRAL FUNCTIONS

Tabulations and/or plots of these functions may be found in the cited references and in many other publications. Compact tables of most of the functions given here and some discussion of their uses are given in Ref. 25.

1. Exponential Functions (References 1, 2, 3, 22, 23, 24, 25)

Definition:

$$E_n(b) = b^{n-1} \int_b^{\infty} \frac{e^{-t}}{t^n} dt, \quad n = 1, 2, 3, \dots \quad (A1)$$

If we set $t = by$, we get an equivalent definition

$$E_n(b) = \int_1^{\infty} \frac{e^{-by}}{y^n} dy, \quad n = 0, 1, 2, \dots \quad (A2)$$

$E_n(b)$ is related to the incomplete gamma function

$$\Gamma(a, b) = \int_b^{\infty} e^{-t} t^{a-1} dt$$

by the expression

$$E_n(b) = b^{n-1} \int_b^{\infty} \frac{e^{-t}}{t^n} dt = b^{n-1} \Gamma(1-n, b) .$$

Integrating (A1) by parts gives the recursion relation

$$E_{n+1}(b) = \frac{1}{n} [e^{-b} - b E_n(b)], \quad n > 1, \quad (A3)$$

while differentiating (A2) with respect to b gives

$$\frac{dE_{n+1}}{db} = -E_n(b) . \quad (A4)$$

$$\text{Thus } E_n(b+\delta) = E_n(b) - \delta E_{n-1}(b) + \frac{\delta^2}{2!} E_{n-2}(b) - \frac{\delta^3}{3!} E_{n-3}(b) + \dots .$$

Also, we have specifically, from equations (A1) through (A4)

$$E_0(b) = \frac{e^{-b}}{b} \quad (A5)$$

$$E_1(b) = \int_b^{\infty} \frac{e^{-t} dt}{t} \quad (A6)$$

$$E_2(b) = b \int_b^{\infty} \frac{e^{-t} dt}{t^2} \quad (A7)$$

$$E_3(b) = b^2 \int_b^{\infty} \frac{e^{-t} dt}{t^3} \quad (A8)$$

$$\int_{a_1}^{a_2} E_1(y) dy = \int_1^{\infty} \int_{a_1}^{a_2} \frac{e^{-yt} dy dt}{t} = E_2(a_1) - E_2(a_2) \quad (A9)$$

and

$$\int_{a_1}^{a_2} E_2(y) dy = E_3(a_1) - E_3(a_2)$$

$$E_1(0) \rightarrow \infty \quad (A10)$$

$$E_1(\infty) = 0 \quad (A11)$$

$$E_2(0) = 1 \quad (A12)$$

$$E_2(\infty) = 0 \quad (A13)$$

Use of the substitution $\rho = b \sec \theta$ enables us to show that

$$\int_0^{\theta} \tan \theta e^{-b \sec \theta} d\theta = E_1(b) - E_1(b \sec \theta_0), \quad (A14)$$

$$\int_0^{\theta} \sin \theta e^{-b \sec \theta} d\theta = E_2(b) - \cos \theta_0 E_2(b \sec \theta_0), \quad (A15)$$

and

$$\int_0^{\theta_0} \sin \theta \cos \theta e^{-b \sec \theta} d\theta = E_3(b) - \cos^2 \theta_0 E_3(b \sec \theta_0) \quad (A16)$$

(A14) through (A16) may be generalized to give

$$E_n(b) = \int_0^{\frac{\pi}{2}} \sin \theta (\cos \theta)^{n-2} e^{-b \sec \theta} d\theta \text{ for } n=0,1,2,\dots \quad (A17)$$

Some useful approximations for $E_n(b)$ are:

For $b > 10$

$$E_n(b) = \frac{e^{-b}}{b} \left[1 - \frac{n}{b} + \frac{n(n+1)}{b^2} - \frac{n(n+1)(n+2)}{b^3} + \dots \right], \quad (A18)$$

and for small values of b

$$E_1(b) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{b^n}{n \cdot n!} - r - \ln |b| \quad (A19)$$

where $r = 0.577216\dots$ is the Euler-Mascheroni Constant.

For any value of b , the following inequality brackets the value of $E_n(b)$:

$$\frac{e^{-b}}{b+n-1} > E_n(b) > \frac{e^{-b}}{b+n}, \quad n \geq 1. \quad (A20)$$

Finally, for $b < 0$

$$E_1(b) = -E_1(-b) \quad (A21)$$

where
$$E_1(b) = \int_{-\infty}^b \frac{e^t}{t} dt \quad (-\infty < b < \infty) \quad (A22)$$

and
$$E_1(b) = \frac{e^b}{b} \left(1 + \frac{1}{b} + \frac{2!}{b^2} + \frac{3!}{b^3} + \dots \right) \text{ for } b \geq 10. \quad (A23)$$

2. Sievert's Integral of the First Kind (Refs 1, 2, 3, 12, 13 and 25)

$$F(\theta_0, b) = \int_0^{\theta_0} e^{-b \sec \theta} d\theta, \quad b \geq 0 \quad (A24)$$

$$0 \leq \theta_0 \leq \frac{\pi}{2}.$$

For $\theta_0 < 5^\circ$,

$$F(\theta_0, b) \doteq \theta_0 e^{-b} \quad (A25)$$

and for a small angle δ ,

$$F(\theta_0 + \delta, b) \doteq F(\theta_0, b) + \delta e^{-b \sec \theta_0}. \quad (A26)$$

The substitution $\rho = b \sec \theta$ gives

$$F(\theta_0, b) = b \int_b^{b \sec \theta_0} \frac{e^{-\rho} d\rho}{\rho \sqrt{\rho^2 - b^2}}. \quad (A27)$$

Note that

$$F(\theta_0, 0) = \theta_0.$$

3. Sievert's Integral of the Second Kind (Ref. 13)

$$F'(\theta_0, b) = b \int_0^{\theta_0} \sec \theta e^{-b \sec \theta} d\theta \quad (A28)$$

The substitution $\rho = b \sec \theta$ gives

$$F'(\theta_0, b) = b \int_b^{b \sec \theta_0} \frac{e^{-\rho} d\rho}{\sqrt{\rho^2 - b^2}}. \quad (A29)$$

Note that

$$F'(\theta_0, 0) = b \log (\sec \theta_0 + \tan \theta_0).$$

4. The G-Function Defined by Ono and Tsuro (References 4, 5, 6, and 25)

$$G(\theta_0, b) = \int_0^{\theta_0} \cos \theta e^{-b \sec \theta} d\theta \quad (A30)$$

which becomes, using $\rho = b \sec \theta$,

$$G(\theta_0, b) = b^2 \int_b^{b \sec \theta_0} \frac{e^{-\rho} d\rho}{\rho^2 \sqrt{\rho^2 - b^2}} \quad (A31)$$

$$G(\theta_0, 0) = \int_0^{\theta_0} \cos \theta d\theta = \sin \theta_0 \quad (A32)$$

The corresponding function for an infinitely high cylinder is defined Ono and Tsuro as

$$H(b) = G\left(\frac{\pi}{2}, b\right) = \int_0^{\frac{\pi}{2}} \cos \theta e^{-b \sec \theta} d\theta. \quad (A33)$$

$G(\theta_0, \mu a)/2\mu_s$ represents unit uncollided flux at the center of the base of an infinitely thick annular source of inner radius a , where the macroscopic self-attenuation coefficient of the source material is μ_s and the macroscopic attenuation coefficient of the shielding material filling the interior of the annulus is μ . The top surface of the annular source is conical, being defined by a line passing thru the center at the angle θ_0 and the height of the inner surface of the annular source is $l = a \tan \theta_0$. (See Option 15 of Ref. 7).

5. The L_0 Function Defined by Ono and Tsuro (Ref. 4, 5, and 25)

$$L_0(\varphi_0, b) = \frac{\int_0^{\varphi_0} G\left(\frac{\pi}{2}, b \sec \varphi\right) d\varphi}{\varphi_0 G\left(\frac{\pi}{2}, b\right)} \quad (A34)$$

where $G\left(\frac{\pi}{2}, b\right)$ and $G\left(\frac{\pi}{2}, b \sec \varphi\right)$ are as defined by (A33). $L_0(\varphi_0, b)$ gives an approximation to the ratio between the attenuation of a slab shield of optical thickness b and the attenuation of an annular shield (also of optical thickness b) which is curved around the dose point as a center. Both the slab and annular shields are assumed to be of infinite height, while subtending the angle φ_0 . (See option 15 of Ref. 7). As noted in Ref. (25), the value of $L_0(\varphi_0, b)$ is only weakly dependent on the assumption that the slab and annular shields are infinitely high and so this function is valid for shields of any height.

6. The G-Function Defined by Rockwell (Ref. 1, 2, 3, and 25)

$$G(\mu_s h, \mu_s R) = \int_0^{\mu_s R} \int_0^{\tan^{-1} \frac{\mu_s h}{\mu_s X}} e^{-\mu_s X \sec \theta} d\theta d(\mu_s X) \quad (A35)$$

$$G(\mu_s h, \mu_s R) = \int_0^{\mu_s R} F\left(\tan^{-1} \frac{\mu_s h}{\mu_s X}, \mu_s X\right) d(\mu_s X) \quad (A36)$$

where the function $F\left(\tan^{-1} \frac{\mu_s h}{\mu_s X}, \mu_s X\right)$ is defined by (A24).

$G(\mu_s h, \mu_s R)/2\mu_s$ represents unit uncollided flux at the center of the base of a cylindrical source of height h and radius R . μ_s is the macroscopic self-attenuation coefficient of the source. (See option 12 of Ref. 7).

7. Relations Between Functions

In terms of a cylinder of radius R , height h and macroscopic self-attenuation coefficient μ_s , finding the flux at the center of the base in different ways, as is done in Section II starting at equation (47), permits us to write the following relation between several of these functions:

$$G(\mu_s h, \mu_s R) = 1 - G\left(\tan^{-1} \frac{h}{R}, \mu_s R\right) - E_2(\mu_s h) + \frac{h}{\sqrt{R^2 + h^2}} E_2\left(\mu_s \sqrt{R^2 + h^2}\right) \quad (A37)$$

where $G(\mu_s h, \mu_s R)$ is given by (A36) and $G(\tan^{-1} \frac{h}{R}, \mu_s R)$ is given by (A30).

To derive (A37) the relation

$$\int_{\tan^{-1} \frac{h}{R}}^{\frac{\pi}{2}} \cos \theta e^{-\mu_s h \csc \theta} d\theta = \int_0^{\tan^{-1} \frac{R}{h}} \sin \psi e^{-\mu_s h \sec \psi} d\psi$$

$$= E_2(\mu_s h) - \frac{h}{\sqrt{R^2 + h^2}} E_2(\mu_s \sqrt{R^2 + h^2}) \quad (A38)$$

is needed. Here $\psi = \frac{\pi}{2} - \theta$.

The left-hand side of (A37) may also be expressed as

$$G(\mu_s h, \mu_s R) = 1 - E_2(\mu_s h) - \int_0^{\mu_s h} E_1(\sqrt{y^2 + (\mu_s R)^2}) dy, \quad (A39)$$

where $y = \mu_s r$.

Equating the right-hand sides of (A37) and (A39) gives another useful relationship:

$$\int_0^{\mu_s h} E_1(\sqrt{y^2 + (\mu_s R)^2}) dy = G(\tan^{-1} \frac{h}{R}, \mu_s R) - \frac{h}{\sqrt{R^2 + h^2}} E_2(\mu_s \sqrt{R^2 + h^2}). \quad (A40)$$

Finally, if we consider a cylinder of infinite height, (A37) reduces to

$$G(\infty, \mu_s R) + G(\frac{\pi}{2}, \mu_s R) = 1.0. \quad (A41)$$

This simple relationship furnishes us with the justification for assigning the letter G to represent both of these functions.

VII. REFERENCES

1. D. Van Nostrand, "Reactor Shielding Design Manual," T. Rockwell, Ed., Princeton, N. J. (1956).
2. A. Foderaro and F. Obenshain, "Fluxes from Regular Geometric Sources," WAPD-TM-508, June 1955.
3. Engineering Compendium on Radiation Shielding, Vol. I, "Shielding Fundamentals and Methods," R. Jaeger, Ed., Springer-Verlag, New York (1968).
4. H. Ono and A. Tsuro, "An Approximate Calculation Method of Flux for Spherical and Cylindrical Sources with a Slab Shield," J. Nucl. Sci. Tech., 2(7), p. 229-235 (June 1965).
5. A. Tsuro, "Unscattered Flux from Spherical and Cylindrical Sources with Shell-Shaped Shields," J. Nucl. Sci. Tech., 2(7), p. 261-69 (July 1965).
6. D. F. Rhodes, R. A. Stallwood, and W. E. Mott, "Intensity of Unscattered Gamma-Rays Inside Cylindrical Self-Absorbing Sources," Nucl. Sci. Eng., 9, p. 41-46 (1961).
7. O. J. Wallace, "SPAR1, A Semi-Analytic Point Kernel Computer Program for Shielding," WAPD-TM-1196, June 1976.
8. O. J. Wallace, "SPAN4 - A Point-Kernel Computer Program for Shielding," WAPD-TM-809, December 1969.
9. D. P. Osanov and E. E. Kovalev, "Radiation Field Due to a Cylindrical Source Placed Behind a Plane Screen," Soviet Journal of Atomic Energy, Volume 8, No. 4., pp. 374-376, April 1960.
10. E. E. Kovelev, "Radiation from a Cylindrical Source," Soviet Journal of Atomic Energy, Vol. 2, p. 659.
11. D. P. Osanov, "Shielding Against γ -Radiation Leaving Through the Base of a Cylindrical Source," Soviet Journal of Atomic Energy, Vol. 6, p. 217.
12. H. E. Hungerford, "Tables of Secant Integrals of the First and Second Kinds," Nucl. Sci. Eng., 14, 312 (1962).
13. C. Farmer, D. S. Gooden, and J. Hogarth, "On a Gauss-Based Integration Formula for the Secant Integral Function," Nuclear Engineering and Design, 15, 265-272, (1971).

14. F. W. Kruger and E. Atze, "Abschirmung der Strahlung einer Zylindrischen selbstabsorbierenden Quelle in Achsenrichtung," *Kernenergie*, 5, No. 10, p. 787-791, 1962.
15. F. W. Kruger and P. Seeligman, "Abschirmung der Strahlung einer Zylindrischen selbstabsorbierenden Quelle in Achsenrichtung," *Kernenergie*, 12, No. 12, p. 390-394, 1969.
16. V. S. Grammatikati, U. Ya. Margulis, and V. G. Kruschev, "Dose Field of a Linear Source," *Soviet Journal of Atomic Energy*, Vol. 8, Number 2, pp. 154-155, February 1960.
17. U. Ya. Margulis and A. V. Khrustalev, "Calculations and Measurement of γ -Field from a Plane Source," *Soviet Journal of Atomic Energy*, Vol. 3, p. 1169.
18. E. E. Konalev, V. I. Popov, and L. N. Smirenniy, "Radiation Field of a Rectangular Source," *Soviet Journal of Atomic Energy*, Vol. 2, p. 208.
19. J. H. Smith and M. L. Storm, "Generalized Off-Axis Distributions from Disk Sources of Radiation," *J. App. Phys.*, 25, No. 4, p. 519-529, April 1954.
20. J. R. Hubbell, R. L. Bach, and C. J. Lamkin, "Radiation Field from a Rectangular Source," *Journal of Research of the National Bureau of Standards*, Vol. 64C, No. 2, pp. 121-138, April-June 1960.
21. L. N. Korzhov, "Radiation of a Spherical Self-Absorbing Source," *Soviet Atomic Energy (English Translation)* 37, No. 2, 870-871 (August 1974).
22. L. N. Posik, "Radiation Field of a Rectangular Parallelepiped with Self-Absorption Taken into Account," *Soviet Journal of Atomic Energy*, Vol. 4, No. 5, p. 609.
23. Carl Kaplan, "On Some Functions Related to the Exponential Integrals," ARL-70-0097, June 1970. AD-710217
24. I. K. Abu-Shumays, "Transcendental Functions Generalizing the Exponential Integrals," *Tech. Inst. Northwestern Univ. Report No. 73-8*, November 1973. C00-2280-6
25. K. Shure and O. J. Wallace, "Compact Tables of Functions for Use in Shielding Calculations," *Nucl. Sci. Eng.*, 56, p. 84-94 (1975).