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ON THE APPLICATION OF RELAXATION TECHNIQUES
TO THE FACTORIZATION OF ITERATIVE PROCEDURES

Progress Report

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On the application of relaxation techniques
to the factorization iterative procedures^{*}

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Abstract

Relaxation techniques which have been used in connection with the factorization iterative methods are reviewed. Known properties are summarized and new ones are obtained.

A precise meaning is given to the notion of iterative procedures of order q and the two-parameters overrelaxation technique of Woznicki is shown to be a factorization iterative method of the second order; some consequences of this fact are developed.

Other results include the refinement of known properties of the extrapolation principle and estimations of the powers of the Oliphant relaxation technique for consistently ordered matrices and of the Chebyshev technique for Stieltjes matrices.

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1. Introduction

Factorization iterative methods have been recognized as interesting iterative techniques from their introduction by Buleev [1], Oliphant [2], Varga [3] and independently by Woznicki [4]. It was however felt by first users as well as by subsequent workers [5], [6] that the full advantage of these techniques could not be reached without the help of some kind of relaxation procedure. Special techniques were introduced by Buleev [1], Oliphant [2] and later by Stone [5]. The use of the Chebyshev relaxation was proposed by Dupont et al. [6] and a two-parameters overrelaxation technique was introduced by Woznicki [4].

On the basis of the work of Woznicki, recent results concerning the convergence properties of the factorization iterative methods have been developed [7], [8]. Relaxation techniques did not however receive sufficient attention and it is the purpose of the present work to evaluate the possible improvements which can be reached by the various techniques introduced so far.

2. Convergence theorem

2.1 Iterative procedures of the first order

We consider here the solution of the linear system of order n

$$Ax = b \quad (1)$$

by iterative schemes of the form

$$Bx_{m+1} = \sum_{i=1}^q A_i x_{m+1-i} + b \quad (2)$$

where x_m denote the successive iterates and

$$A = B - \sum_{i=1}^q A_i \quad (3)$$

Such schemes will be called iterative procedures of order q .

The evolution of the error vector

$$e^{(m)} = x_m - x \quad (4)$$

where x is the exact solution of the system (1) is given by

$$e^{(m+1)} = \sum_{i=1}^q T_i e^{(m+1-i)} \quad (5)$$

where

$$T_i = B^{-1} A_i \quad i = 1, 2, \dots, q \quad (6)$$

First order procedures will be considered in the present section and general procedures in the next section. For the procedures of the first order, we shall follow the standard notation of Varga [9]:

$$Mx_{m+1} = Nx_m + b \quad (7)$$

$$A = M - N \quad (8)$$

where $q = 1$, $B \equiv M$ and $A_1 \equiv N$.

It is well known [9] that this latter iterative method is convergent if and only if the eigenvalues of the associated iteration matrix

$$T = M^{-1}N \quad (9)$$

are located inside the circle of radius 1, centered at the origin of the complex plane. In this case, we shall say that the splitting $A = M - N$ is convergent.

Another way of presenting this convergence criterium is provided by the following proposition:

Theorem 1

The splitting $A = M - N$ with M non-singular is convergent if and only if the eigenvalues of the pencil of matrices $A - \nu M$ are located inside the circle of radius unity, centered at $z = 1$ in the complex plane.

Proof

If λ is an eigenvalue of $T = M^{-1}N$, let x be an associated eigenvector; then we have successively

$$\begin{aligned} M^{-1}Nx &= \lambda x \\ Nx &= \lambda Mx \\ (M - A)x &= \lambda Mx \\ Ax &= (1 - \lambda)Mx \end{aligned} \tag{10}$$

thus $(1 - \lambda)$ is an eigenvalue of the pencil $A - \nu M$.

Conversely, if ν is an eigenvalue of the pencil $A - \nu M$ and x an associated eigenvector,

$$\begin{aligned} Ax &= \nu Mx \\ (M - N)x &= \nu Mx \\ (I - T)x &= \nu x \\ Tx &= (1 - \nu)x \end{aligned} \tag{11}$$

thus $(1 - \nu)$ is an eigenvalue of the associated iteration matrix.

The proposition follows.

We shall henceforth refer to the pencil $A - \nu M$ as the pencil associated with the splitting $A = M - N$ or associated with the iterative method (7).

If A and M are real symmetric matrices and M is positive definite, the pencil $A - \lambda M$ is called regular [10]. The properties of regular pencils have been much investigated in the classical theory of matrices [10]; if n is the order of the matrices A and M , they have n real eigenvalues which satisfy a number of extremal properties; in particular if λ_1 and λ_n are the lowest and the highest eigenvalues, we have

$$\lambda_1 = \min_x \frac{(x, Ax)}{(x, Mx)} \quad \lambda_n = \max_x \frac{(x, Ax)}{(x, Mx)} \quad (12)$$

where (x, y) denotes the scalar product

$$(x, y) = \sum_{i=1}^n x_i y_i \quad (13)$$

An immediate consequence of these properties and of theorem 1 is

Theorem 2

If A is symmetric and M is positive definite, the splitting $A = M - N$ is convergent if and only if A is positive definite and

$$\max_x \frac{(x, Ax)}{(x, Mx)} < 2 \quad (14)$$

Remark:

Another way of stating this latter property is: if A is symmetric and M positive definite, then the splitting $A = M - N$ is convergent if and only if

$$-(x, Mx) < (x, Nx) < (x, Mx) \quad (15)$$

for all x .

2.2 Iterative procedures of order q

The analysis of the general iterative procedure of order q

$$Bx_{m+1} = \sum_{i=1}^q A_i x_{m+1-i} + b \quad (16)$$

applied to the system (1) of order n , may be reduced to the analysis

of a procedure of the first order applied to a linear system of order nq .

Let M and N be matrices of order nq , partitioned into square blocks

$M_{ij}, N_{ij}, i, j = 1, 2, \dots, q$, of order n with

$$M_{11} = B \quad (17)$$

$$M_{ii} = I \quad i = 2, \dots, q \quad (18)$$

$$M_{ij} = 0 \quad \text{if } i \neq j \quad (19)$$

and

$$N_{1j} = A_j \quad j = 1, 2, \dots, q \quad (20)$$

$$N_{i,i-1} = I \quad i = 2, \dots, q \quad (21)$$

$$N_{i,j} = 0 \quad \text{if } i \neq j \text{ or } j \neq i - 1 \quad (22)$$

The matrices M and N have the structure displayed by the following picture:

$$M = \begin{pmatrix} B & & & \\ & I & & \\ & & \bigcirc & \\ & \bigcirc & & \\ & & & I \end{pmatrix} \quad N = \begin{pmatrix} A_1 & \dots & & A_{q-1} & A_q \\ I & & & & 0 \\ & & \bigcirc & & \\ & \bigcirc & & I & \\ & & & & 0 \end{pmatrix} \quad (23)$$

Let z_m and k be the nq -component vectors defined by

$$z_m = \begin{pmatrix} x_m \\ x_{m-1} \\ \vdots \\ x_{m+1-q} \end{pmatrix} \quad k = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then, the iterative procedure (16) is equivalent to the first order scheme

$$Mz_{m+1} = Nz_m + k \quad (24)$$

Therefore, it is convergent for any initial approximation z_0 if and only if the spectral radius of $T = M^{-1}N$ is less than unity. As a consequence, we have the following property

Theorem 3

The iterative scheme of order q

$$Bx_{m+1} = \sum_{i=1}^q A_i x_{m+1-i} + b \quad (25)$$

where B is non-singular, is convergent for any initial approximations $x_0, x_1, \dots, x_{q-1}$ if and only if the eigenvalues of the generalized eigenvalue problem

$$(\lambda^q - \sum_{i=1}^q \lambda^{q-i} T_i)x = 0 \quad (26)$$

where $T_i = B^{-1}A_i$ are located inside the circle of radius unity centered at the origin of the complex plane.

Proof

Using the notation defined above, let λ be an eigenvalue of $T = M^{-1}N$ and z be an associated eigenvector; let

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix} \quad (27)$$

denote the partitioning of the nq component vector z in q n -vectors

$z_1, z_2, \dots, z_q$. We have

$$\lambda z_1 = \sum_{i=1}^q T_i z_i \quad (28)$$

$$\lambda z_i = z_{i-1} \quad i = 2, 3, \dots, q \quad (29)$$

Therefore

$$\lambda^{i-1} z_i = z_1 \quad (30)$$

and

$$\lambda^q z_1 = \sum_{i=1}^q \lambda^{q-i} T_i z_1 \quad (31)$$

which proves that λ is an eigenvalue of the generalized eigenvalue

problem (26) and z_1 are associated eigenvector.

Conversely, let λ be an eigenvalue of (26) and x_1 an associated eigenvector, so that

$$\lambda^q x_1 = \sum_{i=1}^q \lambda^{q-i} T_i x_1 \quad (32)$$

Define the nq -vector

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_q \end{pmatrix} \quad (33)$$

where z_i are the n -component vectors⁽¹⁾

$$z_i = \lambda^{q-i} x_1 \quad (34)$$

We have

$$\lambda z_1 = \sum_{i=1}^q T_i z_i \quad (35)$$

and

$$\lambda z_i = z_{i-1} \quad i = 2, 3, \dots, q \quad (36)$$

Thus, z is an eigenvector of $T = M^{-1}N$ and λ the corresponding eigenvalue.

The proposition follows from the convergence properties of iterative schemes of the first order.

From the equivalence between the iterative procedures (16) and (24), we may also extend the notion of average and asymptotic convergence rates to general iterative procedures.

The average rate of convergence of the general iterative procedure (16) for m iterations will be identified as the average convergence rate for m iterations of the matrix $T = M^{-1}N$, as defined by Varga [9], i.e.,

$$R_m = - \frac{\log \|T^m\|}{m} \quad (37)$$

where $\|A\|$ denotes the spectral norm of A . Its asymptotic value will

be called the asymptotic rate of convergence of the iterative procedure (16);

⁽¹⁾ If $\lambda = 0$, Eq. (34) are understood as $z_i = \delta_q^i x_1$

it is given by

$$R_{\infty} = \lim_{m \rightarrow \infty} R_m = -\log \rho \quad (38)$$

where ρ is the spectral radius of T or, equivalently, by

$$\rho = \max_i |\lambda_i| \quad (39)$$

where λ_i are the nq eigenvalues of the generalized eigenvalue problem (26).

Remarks

Generalizing the definition of factorization iterative methods as introduced in [7] for procedures of the first order, we shall call the scheme (16) a factorization iterative method of order q if Eq. (16) is solved by the (block) Choleski factorization of B ; using the standard form

$$B = (P - E_1)P^{-1}(P - F_1)$$

where P , E_1 and F_1 are respectively (block)diagonal, strictly lower and upper (block) triangular matrices (according to one and the same partitioning), it is seen that a method of this class is completely specified for a given matrix A when P , E_1 , F_1 and A_i for $i = 1, 2, \dots, q-1$ are specified.

3. Relaxation principles

3.1 The extrapolation principle

A general definition of the extrapolation principle may be stated as follows [11] for iteration schemes of the first order. Consider the iterative procedure

$$Mx_{m+1} = Nx_m + b \quad (40)$$

defined by the splitting

$$A = M - N \quad (41)$$

with M nonsingular.

The application of the extrapolation principle consists in multiplying the correction at the m th step by some constant extrapolation parameter ω ; in other words the same iterative procedure with extrapolation is defined by

$$x_{m+1} = x_m + \omega(M^{-1}Nx_m + M^{-1}b - x_m)$$

or

$$Mx_{m+1} = ((1 - \omega)M + \omega N)x_m + \omega b \quad (42)$$

or, equivalently by the splitting

$$A = M_\omega - N_\omega$$

$$M_\omega = \frac{1}{\omega} M \quad (43)$$

$$N_\omega = \frac{1 - \omega}{\omega} M + N$$

The associated iteration matrix is

$$T_\omega = M_\omega^{-1}N_\omega = (1 - \omega)I + \omega T \quad (44)$$

where $T = M^{-1}N$ and the associated pencil is

$$A - \frac{\omega}{\omega} M \quad (45)$$

We shall restrict ourselves here to the consideration of real parameters ω ; making $\omega = 1$ reproduces the original method; $0 \leq \omega \leq 1$ is an interpolation process between the successive iterates of the original

method; $\omega \geq 1$ is an extrapolation. We shall however always refer to this technique as the extrapolation principle because, in many applications, the optimum value of ω is not less than 1. In particular, this property holds for regular splittings and we have [11]:

Theorem 4

If $A = M - N$ is a regular splitting and the associated iterative method is convergent, then $\rho(T_\omega) < 1$ for $0 < \omega \leq 1$. Moreover, if $0 < \omega_1 < \omega_2 \leq 1$, then

$$0 < \rho(T_{\omega_2}) < \rho(T_{\omega_1}) < 1 \quad (46)$$

For non regular splittings such that $|T|$ is convergent we have the following generalization of theorem 4:

Theorem 5

Let the splitting $A = M - N$ be such that M is nonsingular and $\rho(|T|) < 1$. Then $\rho(T_\omega) < 1$ for $0 < \omega \leq 1$. Moreover

$$\rho(T_\omega) \leq \rho(|T|_\omega) < 1 \quad (47)$$

in the same range and, if $0 < \omega_1 < \omega_2 \leq 1$, then

$$0 < \rho(|T|_{\omega_2}) < \rho(|T|_{\omega_1}) < 1 \quad (48)$$

Proof

From Eq. (44) we have, for $0 < \omega \leq 1$,

$$|T_\omega| \leq (1 - \omega)I + \omega|T| = |T|_\omega \quad (49)$$

thus, by applying the Perron-Frobenius theorem

$$\rho(|T_\omega|) \leq \rho(|T|_\omega) = (1 - \omega) + \omega \rho(|T|) \quad (50)$$

By theorem 2.8 of Varga[9], we have

$$\rho(T_\omega) \leq \rho(|T_\omega|) \quad (51)$$

and thus, since $\rho(|T|) < 1$ by hypothesis,

$$\rho(T_\omega) \leq \rho(|T|_\omega) < 1 \quad (52)$$

Moreover we have, for $0 < \omega \leq 1$,

$$\rho(|T|_\omega) = (1 - \omega) + \omega \rho(|T|) = 1 - \omega(1 - \rho(|T|)) \quad (53)$$

and, if $0 < \omega_1 < \omega_2 \leq 1$,

$$1 - \omega_1(1 - \rho(|T|)) > 1 - \omega_2(1 - \rho(|T|)) \quad (54)$$

since $1 - \rho(|T|) > 0$; thus

$$\rho(|T|_{\omega_2}) < \rho(|T|_{\omega_1}) \leq \rho(|T|) < 1 \quad (55)$$

which completes the proof.

Remark

These properties lead to discard interpolation for regular splittings. Negative parameters have not been considered in this case because they lead to divergent processes.

For non regular splittings, interpolation might provide an improved asymptotic rate of convergence, even when $|T|$ is convergent; however, in the latter case, theorem 5 provides an upper bound for $\rho(T_\omega)$ when $0 < \omega \leq 1$ and, if we choose ω so as to minimize this upper bound, we are again led to discard interpolation. It may be observed that this bound becomes higher than 1 for negative ω .

The following theorems give other properties of the extrapolation principle; they are sharpenings of properties reported by Dupont, Kendall and Rachford [6]:

Theorem 6

Let A be symmetric and M positive definite; then the corresponding iterative procedure with extrapolation is convergent if and only if one of the following conditions is satisfied

1. A is positive definite and $0 < \omega < \frac{2}{v_n}$
2. A is negative definite and $-\frac{2}{v_1} < \omega < 0$

where v_1 and v_n are the lowest and the highest eigenvalues of the regular pencil $A - \omega M$ or, equivalently

$$v_1 = \min_x \frac{(x, Ax)}{(x, Mx)} \quad v_n = \max_x \frac{(x, Ax)}{(x, Mx)} \quad (56)$$

Proof

Assume that the iterative procedure with extrapolation is convergent; then from (45) and theorem 1

$$0 < \min \left(\omega \frac{(x, Ax)}{(x, Mx)} \right) \quad \text{and} \quad \max \left(\omega \frac{(x, Ax)}{(x, Mx)} \right) < 2 \quad (57)$$

If $\omega \geq 0$, then

$$\min \left(\omega \frac{(x, Ax)}{(x, Mx)} \right) = \omega v_1 \quad (58)$$

thus

$$\omega v_1 > 0 \quad (59)$$

and this implies simultaneously

$$\omega > 0 \quad \text{and} \quad v_1 > 0 \quad (60)$$

Therefore A is positive definite and $\omega > 0$; moreover we have

$$\max \left(\omega \frac{(x, Ax)}{(x, Mx)} \right) = \omega v_n < 2 \quad (61)$$

thus

$$\omega < \frac{2}{v_n} \quad (62)$$

If $\omega \leq 0$, then

$$\min \left(\omega \frac{(x, Ax)}{(x, Mx)} \right) = \omega v_n \quad (63)$$

thus

$$\omega v_n > 0 \quad (64)$$

and this implies simultaneously

$$\omega < 0 \quad \text{and} \quad \nu_n < 0 \quad (65)$$

Therefore A is negative definite and $\omega < 0$; moreover we have

$$\max (\omega \frac{(x, Ax)}{(x, Mx)}) = \omega \nu_1 < 2 \quad (66)$$

thus

$$\omega > \frac{2}{\nu_1} \quad (67)$$

Conversely, if A is positive definite and $0 < \omega < \frac{2}{\nu_n}$, we have

$$0 < \omega \nu_1 \leq \omega \frac{(x, Ax)}{(x, Mx)} \leq \omega \nu_n < 2 \quad (68)$$

while, if A is negative definite and $\frac{2}{\nu_1} < \omega < 0$, we have

$$0 < \omega \nu_n \leq \omega \frac{(x, Ax)}{(x, Mx)} \leq \omega \nu_1 < 2 \quad (69)$$

In both cases, the conclusion follows from (45) and theorem 1.

Theorem 7

Let A be symmetric and definite and M positive definite; then the spectral radius $\rho(T_\omega)$ of the corresponding iterative procedure with the extrapolation parameter

$$\omega = \frac{2}{\nu_1 + \nu_n} \quad (70)$$

has the following upper bound

$$\rho(T_\omega) \leq \frac{\nu_n - \nu_1}{|\nu_1 + \nu_n|} \quad (71)$$

where ν_1 and ν_n are the lowest and the highest eigenvalues of the pencil A - νM .

Proof

If A is positive definite, we have, as in the proof of theorem 1

$$1 - \omega \nu_n \leq \lambda \leq 1 - \omega \nu_1 \quad (72)$$

where λ is any eigenvalue of T_ω ; using (70) we obtain

$$\frac{v_1 - v_n}{v_1 + v_n} \leq \lambda \leq \frac{v_n - v_1}{v_1 + v_n} \quad (73)$$

Similarly, if A is negative definite, we have

$$1 - \omega v_1 \leq \lambda \leq 1 - \omega v_n \quad (74)$$

or

$$\frac{v_n - v_1}{v_1 + v_n} \leq \lambda \leq \frac{v_1 - v_n}{v_1 + v_n} \quad (75)$$

The proposition follows.

Remark

The importance of the last theorems lies in the fact that if A is known to be definite and M positive definite, then it is possible to estimate the range of values of ω where the corresponding method is convergent. Moreover, as observed by Dupont, Kendall and Rachford [6], the same properties and the same estimations allow the application of the Chebyshev relaxation principle (see section 3.2).

When applied to the factorization methods such that $E_1^* = F_1$, these properties reduce the problem of the convergence of the method to the problem of obtaining "factorizability conditions", i.e., here, conditions which insure that P is a positive definite matrix. Such conditions have been investigated in [7] and [8].

3.2 Chebyshev relaxation

We shall briefly summarize here, for reference purposes, some main properties of the Chebyshev relaxation technique. Proofs can be found in the works of Varga [9] and Wachspress [12].

32.1 First form: extrapolation with a variable extrapolation parameter

Applying the extrapolation principle with a variable extrapolation parameter leads to the following iteration scheme

$$Mx_{m+1} = ((1 - \omega_m)M + \omega_m N)x_m + \omega_m b \quad (76)$$

Following Varga [9], we shall call iterative methods of this type semi-iterative methods; they cannot be described simply by a constant splitting of A and they might theoretically give the exact solution in a finite number of steps.

The evolution of the error is given by

$$\epsilon^{(m)} = p_m(T) \epsilon^{(0)} \quad (77)$$

where

$$\epsilon^{(m)} = x_m - x \quad (78)$$

$$p_m(T) = \prod_{i=1}^m ((1 - \omega_i)I + \omega_i T) \quad (79)$$

$$T = M^{-1} N \quad (80)$$

and x is the exact solution of the given linear system (1).

The average convergence rate for m iterations is defined by

$$R_m = - \frac{\log \| p_m(T) \|}{m} \quad (81)$$

where $\|A\|$ denotes the spectral norm of the matrix A .

When T is hermitian, an upper bound of $\|p_m(T)\|$ is provided by

$$\|p_m(T)\| = \max_{1 \leq i \leq m} |p_m(\tau_i)| \leq \max_{a \leq x \leq b} |p_m(x)| \quad (82)$$

where τ_i are the eigenvalues of T and $a \leq \tau_i \leq b$ for $i = 1, 2, \dots, n$.

The Chebyshev relaxation technique consists in choosing the successive extrapolation parameters ω_i such that this upper bound is minimized. Since the polynomials $p_m(x)$ satisfy the normalization condition

$$p_m(1) = 1 \quad (83)$$

this minimization will be realized if the parameters ω_i are such that the error polynomial $p_m(x)$ generated by the semi-iterative procedure is the solution of the following minimization problem

$$\min_{p_m(1)=1} \left\{ \max_{a \leq x \leq b} |p_m(x)| \right\} \quad (84)$$

where $p_m(x)$ is a real polynomial of degree m . The solution of this problem is given by (Varga [9])

$$\tilde{p}_m(x) = \frac{C_m\left(\frac{2x - (b+a)}{b-a}\right)}{C_m\left(\frac{2 - (b+a)}{b-a}\right)} \quad m \geq 0 \quad (85)$$

where $C_m(x)$ denote the Chebyshev polynomials, defined by the recurrence relation

$$C_{m+1}(x) = 2x C_m(x) - C_{m-1}(x) \quad (86)$$

together with

$$C_0(x) = 1 \quad (87)$$

$$C_1(x) = x \quad (88)$$

The corresponding extrapolation parameters are given by (Wachspress [12]):

$$\omega_i = \frac{2}{2 - [(b-a)\cos(\frac{(2i-1)}{2m}\pi) + a + b]} \quad i = 1, 2, \dots, m \quad (89)$$

It should be observed that using this form of the Chebyshev relaxation technique necessitates the a priori knowledge of m ; in practice, this difficulty is circumvented by operating with parameter cycles of small order, thus losing part of the higher convergence rate of the Chebyshev method.

32.2 Second form: the Chebyshev recurrence

The polynomials $\tilde{p}_m(x)$ of (85) satisfy a three-term recurrence relation derived from the Chebyshev recurrence formula (86):

$$C_{m+1}(\alpha) \tilde{p}_{m+1}(x) = 2 \frac{2x - (b+a)}{b-a} C_m(\alpha) \tilde{p}_m(x) - C_{m-1}(\alpha) \tilde{p}_{m-1}(x) \quad (90)$$

where

$$\alpha = \frac{2 - (b + a)}{b - a} \quad (91)$$

Replacing the variable x by the matrix T and using (77), we have

$$C_{m+1}(\alpha) \epsilon^{(m+1)} = 2 \frac{2T - (b + a)}{b - a} C_m(\alpha) \epsilon^{(m)} - C_{m-1}(\alpha) \epsilon^{(m-1)} \quad (92)$$

or

$$\epsilon^{(m+1)} = \frac{2T - (b + a)}{2 - (b + a)} \omega_{m+1} \epsilon^{(m)} - (\omega_{m+1} - 1) \epsilon^{(m-1)} \quad (93)$$

where we have put

$$\omega_{m+1} = 2\alpha \frac{C_m(\alpha)}{C_{m+1}(\alpha)} = 1 + \frac{C_{m-1}(\alpha)}{C_{m+1}(\alpha)} \quad (94)$$

Thus

$$\epsilon^{(m+1)} = \omega_{m+1} \left(\frac{2T - (b + a)}{2 - (b + a)} \epsilon^{(m)} - \epsilon^{(m-1)} \right) + \epsilon^{(m-1)} \quad (95)$$

or, using (78),

$$x_{m+1} = \omega_{m+1} \left(\frac{2T - (b + a)}{2 - (b + a)} (x_m - x) - (x_{m-1} - x) \right) + x_{m-1} \quad (96)$$

$$x_{m+1} = \omega_{m+1} \left(\frac{2T - (b + a)}{2 - (b + a)} x_m + \frac{2}{2 - (b + a)} (I - T)x - x_{m-1} \right) + x_{m-1} \quad (97)$$

$$x_{m+1} = \omega_{m+1} \left(\frac{2T - (b + a)}{2 - (b + a)} x_m + \frac{2}{2 - (b + a)} g - x_{m-1} \right) + x_{m-1} \quad (98)$$

where

$$g = M^{-1}b = x - Tx \quad (99)$$

It should be observed that using this form of the Chebyshev relaxation technique necessitates the storage of two previous iterates, x_m and x_{m-1} ; on the other hand one keeps the full advantage of the higher convergence rate of the Chebyshev method at each step.

This convergence rate can be estimated as follows; from Eq. (82)

and (85), we have

$$\| \tilde{p}_m(T) \| \leq \max_{a \leq x \leq b} | \tilde{p}_m(x) | = \frac{1}{C_m(\alpha)} \quad (100)$$

since

$$\max_{-1 \leq x \leq 1} | C_m(x) | = 1 \quad (101)$$

On the other hand, assuming $b < 1$, we have

$$\alpha - 1 = 2 \frac{1-b}{b-a} > 0 \quad (102)$$

thus

$$C_m(\alpha) = \cosh m\sigma \quad (103)$$

where

$$\alpha = \cosh^{-1} \alpha = \log(\alpha + \sqrt{\alpha^2 - 1}) \quad (104)$$

and therefore

$$C_m(\alpha) = \frac{1}{2} (\alpha + \sqrt{\alpha^2 - 1})^m \left(1 + \frac{1}{(\alpha + \sqrt{\alpha^2 - 1})^{2m}} \right) \quad (105)$$

From the definition (81), together with (100) and (105), we obtain

$$R_m = -\frac{1}{m} \log \| \tilde{p}_m(T) \| \geq \log(\alpha + \sqrt{\alpha^2 - 1}) + \frac{1}{m} \log \frac{1}{2} \left(1 + \frac{1}{(\alpha + \sqrt{\alpha^2 - 1})^{2m}} \right) \quad (106)$$

Thus, when $b < 1$, we have the following estimate of the asymptotic convergence rate:

$$R_{as} = \lim_{m \rightarrow \infty} R_m \geq \log(\alpha + \sqrt{\alpha^2 - 1}) = \cosh^{-1} \alpha = \sigma \quad (107)$$

Theorem 8

Let the iteration matrix T associated with the splitting $A = M - N$ be hermitian with eigenvalues τ_i such that

$$a \leq \tau_i \leq b < 1 \quad (108)$$

Then the asymptotic rate of convergence of the corresponding iterative method with Chebyshev relaxation is not less than

$$\sigma = \cosh^{-1} \alpha \quad (109)$$

where

$$\alpha = \frac{2 - (b + a)}{b - a} \quad (110)$$

Remark

Although described by a three-term recurrence formula, the Chebyshev procedure defined by Eq. (98) is not an iterative procedure of the second order in the sense of section 2, since it contains a variable iteration parameter. It is however defined by a variable "three-term splitting" since Eq. (98) is equivalent to

$$B^{(m)} x_{m+1} = A_1^{(m)} x_m + A_2^{(m)} x_{m-1} + b \quad (111)$$

where

$$B^{(m)} = \frac{1}{\omega_{m+1}} \left(1 - \frac{b+a}{2} \right) M \quad (112)$$

$$A_1^{(m)} = \frac{1}{\omega_{m+1}} \left(1 - \frac{b+a}{2} (1 - \omega_{m+1}) \right) M - N \quad (113)$$

$$A_2^{(m)} = \frac{1 - \omega_{m+1}}{\omega_{m+1}} \left(1 - \frac{b+a}{2} \right) M \quad (114)$$

and it is easily checked that

$$B^{(m)} - A_1^{(m)} - A_2^{(m)} = M - N = A \quad (115)$$

When $b < 1$, we have $\alpha > 1$ and $\sigma > 0$ so that

$$\omega = \lim_{m \rightarrow \infty} \omega_{m+1} = \lim_{m \rightarrow \infty} 2\alpha \frac{C_m(\alpha)}{C_{m+1}(\alpha)} = 2\alpha(\alpha + \sqrt{\alpha^2 - 1}) \quad (116)$$

and the Chebyshev relaxation method tends asymptotically to a second order method in the sense of section 2.

3.3 Special relaxation techniques

In this section, we consider relaxation techniques which are more or less specific to the factorization iterative methods: the overrelaxation technique and the Oliphant-Buleev technique.

33.1 The overrelaxation technique

331.1 The two-parameters technique of Woznicki

First introduced by Frankel [13] and Young [14] in the particular case of the Gauss-Seidel method, the overrelaxation principle was generalized by Woznicki [4] in the case of his EWA and AGA methods, by the introduction of a second relaxation parameter. The same method may evidently be applied to any factorization iterative method and we shall apply it here to procedures of the first order. The idea which leads to the overrelaxation principle consists in applying the extrapolation principle, but using the components of the new estimation as soon as these are computed; since a factorization method involves the solution of two (block) triangular systems, the technique may be applied twice with two relaxation parameters.

Its mathematical definition will now be stated. Factorization iterative procedures of the first order are iterative schemes

$$Mx_{m+1} = Nx_m + b$$

such that M is solved by the block triangular decomposition

$$M = (P - E_1)P^{-1}(P - F_1)$$

where P is a block-diagonal matrix, E_1 and F_1 are strictly lower and upper block-triangular matrices.

Defining

$$y_m = (P - F_1)x_m \tag{117}$$

we have to solve successively at each step the following set of equations

$$\begin{aligned} (P - E_1)P^{-1} y_{m+1} &= Nx_m + b \\ (P - F_1) x_{m+1} &= y_{m+1} \end{aligned} \quad (118)$$

or

$$\begin{aligned} y_{m+1} &= E_1 P^{-1} y_{m+1} + Nx_m + b \\ x_{m+1} &= P^{-1} F_1 x_{m+1} + P^{-1} y_{m+1} \end{aligned} \quad (119)$$

The general two-parameters overrelaxation procedure consists in replacing these equations by

$$\begin{aligned} y_{m+1} &= \omega_1 (E_1 P^{-1} y_{m+1} + Nx_m + b - y_m) + y_m \\ x_{m+1} &= \omega_2 (P^{-1} F_1 x_{m+1} + P^{-1} y_{m+1} - x_m) + x_m \end{aligned} \quad (120)$$

where ω_1 and ω_2 are scalar parameters; otherwise stated, we have at each step

$$\begin{aligned} (I - \omega_1 E_1 P^{-1}) y_{m+1} &= \omega_1 Nx_m + \omega_1 b + (1 - \omega_1) y_m \\ (I - \omega_2 P^{-1} F_1) x_{m+1} &= \omega_2 P^{-1} y_{m+1} + (1 - \omega_2) x_m \end{aligned} \quad (121)$$

It is possible to eliminate y_{m+1} and y_m from the first equation by using the explicit form of y_{m+1} given by the second equation

$$y_{m+1} = \frac{1}{\omega_2} (P - \omega_2 F_1) x_{m+1} - \frac{1 - \omega_2}{\omega_2} Px_m \quad (122)$$

thus giving

$$\begin{aligned} &\frac{1}{\omega_1 \omega_2} (P - \omega_1 E_1) P^{-1} (P - \omega_2 F_1) x_{m+1} \\ &= \frac{1 - \omega_2}{\omega_1 \omega_2} (P - \omega_1 E_1) + \frac{1 - \omega_1}{\omega_1 \omega_2} (P - \omega_2 F_1) + Nx_m - \frac{(1 - \omega_1)(1 - \omega_2)}{\omega_1 \omega_2} Px_{m-1} + b \end{aligned} \quad (123)$$

Defining

$$B = \frac{1}{\omega_1 \omega_2} (P - \omega_1 E_1) P^{-1} (P - \omega_2 F_1) \quad (124)$$

$$A_1 = \frac{1 - \omega_2}{\omega_1 \omega_2} (P - \omega_1 E_1) + \frac{1 - \omega_1}{\omega_1 \omega_2} (P - \omega_2 F_1) + N \quad (125)$$

$$A_2 = - \frac{(1 - \omega_1)(1 - \omega_2)}{\omega_1 \omega_2} P \quad (126)$$

it is seen that

$$\begin{aligned} B - A_1 - A_2 &= \left(\frac{1 - \omega_1}{\omega_1} P - \frac{1}{\omega_1} (P - \omega_1 E_1) \right) P^{-1} \left(\frac{1 - \omega_2}{\omega_2} P - \frac{1}{\omega_2} (P - \omega_2 F_1) \right) - N \\ &= (P - E_1) P^{-1} (P - F_1) - N = M - N = A \end{aligned} \quad (127)$$

In other words, the general overrelaxation procedure is a second order iterative scheme defined by the splitting

$$A = B - A_1 - A_2 \quad (128)$$

where B , A_1 and A_2 are given by (124), (125) and (126). Therefore by applying theorem 3, we obtain after a small amount of algebra, the following convergence criterium.

Theorem 9

The general overrelaxation procedure (120) is convergent if and only if the eigenvalues of the generalized eigenvalue problem

$$\left(\left(\frac{\lambda + \omega_1 - 1}{\omega_1} P - \lambda E_1 \right) P^{-1} \left(\frac{\lambda + \omega_2 - 1}{\omega_2} P - \lambda F_1 \right) - \lambda N \right) x = 0 \quad (129)$$

are less than one in modulus.

Remarks

If we apply the overrelaxation principle in one of the equations (120) only, i.e., if we put one of the parameters ω_1, ω_2 equal to unity, the corresponding second order iteration procedure reduces to a first order process, since, in this case $A_2 = 0$.

The corresponding eigenvalue problem is

$$\lambda \left((P - E_1) P^{-1} \left(\frac{\lambda + \omega_2 - 1}{\omega_2} P - \lambda F_1 \right) - N \right) x = 0 \quad (130)$$

if $\omega_1 = 1$ and

$$\lambda \left(\left(\frac{\lambda + \omega_1 - 1}{\omega_1} P - \lambda E_1 \right) P^{-1} (P - F_1) - N \right) x = 0 \quad (131)$$

if $\omega_2 = 1$.

The classical block S.O.R. method is obtained as a particular case of the general procedure, by setting

$$\omega_1 = \omega \quad \omega_2 = 1 \quad (132)$$

and by choosing

$$E_1 = E \quad F_1 = 0 \quad (133)$$

where $-E$ is the strictly lower block triangular part of A . Letting D and $-F$ be the block diagonal and strictly upper block triangular parts of A we have then

$$P = D \quad (134)$$

and the generalized eigenvalue problem (129) reduces to

$$\left(\lambda \left(\frac{\lambda + \omega - 1}{\omega} D - \lambda E \right) - \lambda F \right) x = 0 \quad (135)$$

or, eliminating the trivial solutions $\lambda = 0$,

$$((\lambda + \omega - 1)D - \lambda \omega E - \omega F) x = 0 \quad (136)$$

i.e.,

$$(\lambda I - \mathcal{L}_\omega) x = 0 \quad (137)$$

where

$$\mathcal{L}_\omega = (D - \omega E)^{-1} ((1 - \omega)D + \omega F) \quad (138)$$

is the well-known block S.O.R. iteration matrix.

Returning to the general case, let

$$\Phi(\lambda) = \det(\lambda^2 I - \lambda T_1 - T_2) \quad (139)$$

where

$$T_i = B^{-1} A_i \quad i = 1, 2 \quad (140)$$

Thus, the eigenvalues of the generalized eigenvalue problem (129) are the roots of the algebraic equation of order $2n$.

$$\Phi(\lambda) = 0 \quad (141)$$

Since the leading coefficient of $\Phi(\lambda)$ is unity, the product of these roots is given by

$$\prod_{i=1}^{2n} \lambda_i = \Phi(0) = \det(T_2) \quad (142)$$

But

$$\begin{aligned} \det(T_2) &= \det(B^{-1}A_2) = \det((P - \omega_2 F_1)^{-1} P (P - \omega_1 E_1)^{-1} (1 - \omega_1)(1 - \omega_2)P) \\ &= (1 - \omega_1)^n (1 - \omega_2)^n \end{aligned} \quad (143)$$

Thus

$$\prod_{i=1}^{2n} \lambda_i = (1 - \omega_1)^n (1 - \omega_2)^n \quad (144)$$

As a consequence we have:

Theorem 10

Let ρ be the modulus of the largest eigenvalue of the generalized eigenvalue problem corresponding to the general overrelaxation procedure with parameters ω_1 and ω_2 ; then

$$\rho \geq (|(\omega_1 - 1)(\omega_2 - 1)|)^{1/2} \quad (145)$$

with equality if and only if all eigenvalues are of modules $(|(\omega_1 - 1)(\omega_2 - 1)|)^{1/2}$

This property is a generalization of the theorem of Kahan (cf. Varga [9]) for the S.O.R. method. A practical consequence is that we have to choose ω_i such that $|(\omega_1 - 1)(\omega_2 - 1)| < 1$ in order to have a convergent process.

As a particular case, we have

Corollary

If the general overrelaxation procedure with equal relaxation

parameters $\omega_1 = \omega_2 = \omega$ is convergent, then

$$|\omega - 1| < 1 \quad (146)$$

In other words, when equal parameters are used, their common value must lie inside the circle of radius unity centered at $z = 1$ in the complex plan; thus, for real parameters we must have

$$0 < \omega < 2 \quad (147)$$

as in the classical S.O.R. method.

331.2 Further generalizations

Other generalizations of the overrelaxation procedure have been proposed in the litterature, aiming to replace the scalar parameter ω by a non singular matrix Ω with

$$\mathcal{L}'_{\Omega} = (D - \Omega E)^{-1} ((I - \Omega)D + \Omega F) \quad (148)$$

as iteration matrix when this technique is applied to the block Gauss-Seidel method. Examples of this generalized technique have been analyzed by Taylor [15].

The eigenvalues of $\mathcal{L}'_{\Omega}$ are the roots of its characteristic polynomial

$$\Phi(\lambda) = \det(\lambda I - \mathcal{L}'_{\Omega}) \quad (149)$$

and this product is given by

$$\prod_{i=1}^n \lambda_i = \det(-\mathcal{L}'_{\Omega}) \quad (150)$$

If $\Omega = (\omega_i)$ is a diagonal matrix which commutes with D , we have

$$\prod_{i=1}^n \lambda_i = \prod_{i=1}^n (\omega_i - 1) \quad (151)$$

and, therefore, the following generalization of the theorem of Kahan:

Theorem 11

Let $\Omega = (\omega_i)$, $i = 1, 2, \dots, n$, be a diagonal matrix which commutes with D and $\mathcal{L}_\Omega$ be the iteration matrix defined by (148); then

$$\rho(\mathcal{L}_\Omega) \geq \left(\prod_{i=1}^n |\omega_i - 1| \right)^{1/n} \quad (152)$$

with equality if and only if all eigenvalues have the same modulus.

Other properties of the classical S.O.R. method can be generalized under the same assumptions; in particular, we have the following generalization of the sufficient part of the Ostrowski-Reich theorem (see Varga [9]):

Theorem 12

Let A be positive definite, D be its block-diagonal part and $\Omega = (\omega_i)$, $i = 1, 2, \dots, n$ a diagonal matrix which commutes with D ; if $|\omega_i - 1| < 1$ for $i = 1, 2, \dots, n$, then $\rho(\mathcal{L}_\Omega) < 1$.

Proof

The proof is a straightforward application of theorem 1.11 of Wachspress [12].⁽¹⁾

Remark

Attempt to generalize both parts of the Ostrowski-Reich theorem gives somewhat less complete results which, however, hold under still weaker assumptions; let

$$A = D - E - E^*$$

be any decomposition of the $n \times n$ matrix A , let Ω be a non-singular $n \times n$ matrix such that $\Omega^{-1}D - E$ is non-singular and define the generalized S.O.R. method by the splitting

⁽¹⁾ This criterium must probably be attributed to John [16] and Householder [17].

$$A = M - N$$

with

$$M = \Omega^{-1}D - E$$

$$N = (\Omega^{-1} - I)D + E^*$$

Let $\mathcal{L}_\Omega$ be the corresponding iteration matrix:

$$\mathcal{L}_\Omega = M^{-1}N = (D - \Omega E)(I - \Omega)D + \Omega E^*$$

Then, on the basis of theorem 4.17 of Householder [17], we have

Theorem 13

Let $A = D - E - E^*$ be hermitian, Ω and $\Omega^{-1}D - E$ be non-singular and $(D^*\Omega^{*-1} + \Omega^{-1}D - D)$ be positive definite. Then $\rho(\mathcal{L}_\Omega) < 1$ if and only if A is positive definite.

Clearly, theorem 12 and therefore the sufficient part of the Ostrowski-Reich theorem are particular cases of theorem 13; the necessary part of the Ostrowski-Reich theorem is however stronger.

Under similar assumptions, another result is worthwhile to cite, although suffering the same limitations. For any positive definite $n \times n$ matrix S , let the S -norm of a n -vector x be

$$||x||_S = ||Sx||$$

and define the S -norm of a $n \times n$ matrix B as

$$||B||_S = \sup_x \frac{||Bx||_S}{||x||_S}$$

Then, on the basis of the theorem of Feingold and Spohn [18], we have

Theorem 14

Let $A = D - E - E^*$ be positive definite, let Ω and $\Omega^{-1}D - E$ be non-singular; then $||\mathcal{L}_\Omega||_A < 1$ if and only if $\Omega^*D^{*-1} + \Omega^{-1}D - D$ is positive definite.

Similar techniques can be applied to the more general two-parameters overrelaxation procedure, by replacing both parameters ω_1 and ω_2 by non singular matrices Ω_1 and Ω_2 with

$$B = \Omega_1^{-1}(P - \Omega_1 E_1) \Omega_2^{-1}(I - \Omega_2 P^{-1} F_1) \quad (153)$$

$$A_1 = \Omega_1^{-1}(P - \Omega_1 E_1)(\Omega_2^{-1} - I) + (\Omega_1^{-1} - I)P\Omega_2^{-1}(I - \Omega_2 P^{-1} F_1) + N \quad (154)$$

$$A_2 = -(\Omega_1^{-1} - I)P(\Omega_2^{-1} - I) \quad (155)$$

replacing Eqs. (124)-(126). It is easily checked that Eqs. (153)-(155) entail

$$B - A_1 - A_2 = A \quad (156)$$

and that we have the following generalization of theorem 10 when

$\Omega_1 = (\omega_{1i})$ and $\Omega_2 = (\omega_{2i})$ are diagonal matrices which commute with P :

Theorem 15

Let ρ be the modulus of the largest eigenvalue of the generalized eigenvalue problem

$$(\lambda^2 B - \lambda A_1 - A_2)x = 0 \quad (157)$$

where B , A_1 and A_2 are defined by Eq. (153)-(155) and $\Omega_1 = (\omega_{1i})$,

$\Omega_2 = (\omega_{2i})$, $i = 1, 2, \dots, n$ be diagonal matrices which commute with

P ; then

$$\rho \geq \left(\prod_{i=1}^n |(\omega_{1i} - 1)(\omega_{2i} - 1)| \right)^{1/2n} \quad (158)$$

with equality if and only if all eigenvalues have the same modulus.

Remark

Practical consequences can be derived from theorem 11 and theorem 15, similar to the corollary of theorem 10, giving relations which the

parameters ω_{1i} and ω_{2i} must satisfy when the method is convergent. However, we have not succeeded to extend the sufficient condition contained in the Ostrowski-Reich theorem to the two-parameters over-relaxation technique of Woznicki.

33.2 The Oliphant and Buleev relaxation techniques

By the Oliphant, Buleev and Oliphant-Buleev relaxation techniques, we denote the techniques used in [7] and in [8] in order to generate the composite methods of Oliphant-Woznicki, Buleev-Woznicki and Oliphant-Buleev-Woznicki respectively from the family of Woznicki methods.⁽¹⁾ Definitions and properties have been investigated in [7] and [8]. Some complements will be given here and in the next section.

Indications on the possible range to consider for the iteration parameters may be obtained by applying these techniques to the block Gauss-Seidel method. Considering the Buleev technique in this particular case, let $\beta = (\beta_{ii})$ be the block diagonal matrix of iteration parameters; if we assume that all non-zero entries of each block β_{ii} are equal to a scalar parameter β_i , then we have

$$\beta \cdot D = \Omega^{-1} D \quad (159)$$

where the element by element product is used on the left hand side and the ordinary matrix multiplication on the right hand side and where Ω is the block diagonal matrix $\Omega = (\Omega_i)$ with $\Omega_i = \beta_i^{-1} I$. Under these assumptions, the Buleev technique reduces to an extension of the one-parameter overrelaxation technique considered above and we refer to section 33.2 for the corresponding results.

We shall consider here the application of the Oliphant technique to

⁽¹⁾ We keep here the terminology introduced in [7]; however, as pointed out in [8], the Woznicki methods have first been introduced by Varga [3].

the block Gauss-Seidel method; the resulting method might be called the Oliphant-Gauss-Seidel method.

Its associated iteration matrix is

$$T = (D - E_{\beta})^{-1} (E - E_{\beta} + F) \quad (160)$$

where the notations of ref. [7] are used. We have the following sufficient condition for positive definite matrices:

Theorem 16

If A is positive definite and the non zero entries of the blocks of parameters β_{ij} belong to the interval $[1,2]$, then the Oliphant-Gauss-Seidel method is convergent.

Proof

The proof follows immediately from theorem 1.11 of Wachspress [12] and from the fact that, if A is positive definite, then

$$D - (E_{\beta} - E) - (E_{\beta}^* - E^*) \quad (161)$$

is also positive definite when the non zero entries of the blocks β_{ij} belong to the interval $[1,2]$.

4. Consistently ordered matrices

4.1 Summary

The theory of consistently ordered matrices has received much attention during the recent years [15], [19], [20], [21] and we briefly summarize here the results which shall need later.

Let the $n \times n$ matrix A be partitioned into submatrices A_{ij} , $i, j = 1, 2, \dots, N$ where the A_{ii} are square and nonsingular; let

$$A = D - E - F \quad (162)$$

be the decomposition of A into its block-diagonal, strictly lower and upper block-triangular parts according to this partitioning.

Definition (Verner and Bernal [20])

The matrix A is consistently ordered relative to the given partitioning if the eigenvalues of the matrix

$$B(\alpha) = \alpha^r D^{-1} E + \alpha^{-q} D^{-1} F \quad (163)$$

are independent of α for $\alpha \neq 0$, where r and q are relatively prime integers.

Remark

With $p = r + q$, the definition given by Varga [9] is comparable with a special case of this definition when $r = 1$ and $q = p - 1$.

The following theorem shows that when A is consistently ordered, the eigenvalues of the block Jacobi matrix have a cyclic structure.

Theorem 17 (Verner and Bernal [20])

If A is consistently ordered with $p = r + q$ and if $\theta^p = 1$ and μ is an eigenvalue of the block Jacobi matrix

$$B = D^{-1}(E + F) \quad (164)$$

then $\theta^j \mu$, $j = 0, 1, \dots, p - 1$ are also eigenvalues of B .

Remark

This theorem provides a simplification of the theory of consistent

orders as it was exposed in Varga [9] since, with its help, it becomes unnecessary to refer a priori to the theory of cyclic matrices.

The next theorem establishes a functional relation between the overrelaxation factor ω , the eigenvalues of the block Jacobi matrix

$$B = D^{-1}(E + F) \quad (165)$$

and the eigenvalues of the block S.O.R. iteration matrix

$$\mathcal{L}_\omega = (D - \omega E)^{-1} ((1 - \omega)D + \omega F) \quad (166)$$

Theorem 18 (Verner and Bernal [20])

Let the matrix A be consistently ordered with $p = r + q$. If $\omega \neq 0$, and λ is a nonzero eigenvalue of $\mathcal{L}_\omega$ and if μ satisfies

$$(\lambda + \omega - 1)^p = \lambda^q \omega^p \mu^p \quad (167)$$

then μ is an eigenvalue of the block Jacobi matrix B ; conversely, if μ is an eigenvalue of B and λ satisfies (167), then λ is an eigenvalue of $\mathcal{L}_\omega$.

By making $\omega = 1$ in theorem 18, we obtain the following consequence; here

$$R_\omega(T) = -\log \rho(T) \quad (168)$$

is the asymptotic rate of convergence of an iterative method whose associated iteration matrix is T (Varga [9]):

Corollary

Let A be consistently ordered with $p = r + q$. If μ is an eigenvalue of the block Jacobi matrix B , then $\mu^{p/r}$ is an eigenvalue of $\mathcal{L}_1$; if λ is a nonzero eigenvalue of $\mathcal{L}_1$ and $\lambda^r = \mu^p$, then μ is an eigenvalue of B . Thus, the block Jacobi method converges if and only if the block Gauss-Seidel method converges, and if both converge, then

$$\rho(\mathcal{A}_1) = (\rho(B))^{p/r} < 1 \quad (169)$$

and consequently

$$R_\infty(\mathcal{A}_1) = \frac{p}{r} R_\infty(B) \quad (170)$$

The following theorem gives the theoretical value of the optimum relaxation factor ω_b of the S.O.R. method for consistently ordered matrices, when the eigenvalues of B^p are real and nonnegative.

Theorem 18 (Varga [9], exercise 2 p. 109, Nichols and Fox [21])

If the matrix A is consistently ordered with parameters q and $p = r + q$ and if $\mu = \rho(B) < 1$ and the eigenvalues of B^p are real and nonnegative, then

a/ if $q \neq p - 1$, then $\omega_b = 1$ and

$$\min_{\omega} \rho(\mathcal{A}_\omega) = \rho(\mathcal{A}_1) = |\mu|^{p/r} < 1 \quad (171)$$

b/ if $q = p - 1$, then ω_b is the unique positive real root between 1 and $p/(p - 1)$ of the equation

$$(\mu\omega_b)^p = p^p (p - 1)^{-(p-1)} (\omega_b - 1) \quad (172)$$

and

$$\min_{\omega} \rho(\mathcal{A}_\omega) = \rho(\mathcal{A}_{\omega_b}) = (\omega_b - 1)(p - 1) \quad (173)$$

Moreover, the block successive overrelaxation matrix $\mathcal{A}_\omega$ is convergent for all ω with $0 < \omega < p/(p - 1)$.

Corollary (Varga [9])

Let the matrix A be consistently ordered with parameters $p = 1 + q$, and let the eigenvalues of the p th power of the associated block Jacobi matrix B be real nonnegative with $0 \leq \rho(B) < 1$. As $\rho(B) \rightarrow 1^-$,

$$R_\infty(\mathcal{A}_{\omega_b}) \rightarrow \left(\frac{2p}{p-1}\right)^{1/2} (R_\infty(\mathcal{A}_1))^{1/2} \quad (174)$$

$$R_{\infty}(\omega_b) \rightarrow \left(\frac{2p^2}{p-1}\right)^{1/2} (R_{\infty}(B))^{1/2} \quad (175)$$

4.2 The Oliphant and Buleev relaxation techniques

General properties of the Oliphant and Buleev relaxation techniques have been investigated in [7] and [8] with the help of the properties of regular splittings. Some complements have been obtained in section 33.2. We shall consider here the problem of evaluating the improvement which can be reached by the use of optimal parameters.

Although it does not seem feasible to consider the optimization of these techniques in their full complexity, the power of the optimal procedures may be estimated by analyzing some particular cases. As observed in section 33.2, the one parameter overrelaxation procedure is a particular case of the Buleev relaxation method and the results summarized in section 4.1 give an estimation of the improvement reached when the optimal Buleev technique with one parameter is applied to the Gauss-Seidel method for consistently ordered matrices.

We shall consider here, as another example, the application of the Oliphant technique with one parameter to the Gauss-Seidel iterative scheme and compare the resulting method against the optimal S.O.R. technique for the same class of matrices.

More precisely, we consider the following iterative scheme

$$Mx_{m+1} = Nx_m + b \quad (176)$$

with

$$M = D - \beta E \quad (177)$$

$$N = M - A = (1 - \beta)E + F \quad (178)$$

where β is a scalar parameter and other notations are those of section 4.1.

The associated iteration matrix is

$$\mathcal{C}_\beta = M^{-1}N = (D - \beta E)^{-1} ((1 - \beta)E + F) \quad (179)$$

For the block Jacobi and block S.O.R. methods, the notations of the preceding section are used.

This method has been analyzed by Sisler [22]; we give here an alternate proof of his result⁽¹⁾:

Theorem 20 (Sisler [22])

Let the matrix A be consistently ordered with $p = r + q$. If λ is a nonzero eigenvalue of $\mathcal{C}_\beta$ and if μ satisfies

$$\lambda^p = (1 - \beta + \beta\lambda)^q \mu^p \quad (180)$$

then μ is an eigenvalue of the block Jacobi matrix B; conversely, if μ is an eigenvalue of B and λ satisfies (180), then λ is an eigenvalue of $\mathcal{C}_\beta$.

Proof

Let

$$\Phi(\lambda) = \det(\lambda I - \mathcal{C}_\beta) \quad (181)$$

be the characteristic polynomial of $\mathcal{C}_\beta$; we have

$$\Phi(\lambda) = \det(\lambda I - (I - \beta L)^{-1} ((1 - \beta)L + U)) \quad (182)$$

where

$$L = D^{-1}E \quad U = D^{-1}F \quad (183)$$

Thus

$$\begin{aligned} \Phi(\lambda) &= \det(\lambda(I - \beta L) - (1 - \beta)L - U) \\ &= \det(\lambda I - (1 - \beta + \beta\lambda)L - U) \end{aligned} \quad (184)$$

and, letting

⁽¹⁾We were informed of the work of Sisler only after completion of the present study.

$$\delta^P = 1 - \beta + \beta\lambda \quad (185)$$

we obtain

$$\begin{aligned} \Phi(\lambda) &= \det(\lambda I - \delta^q(\delta^r L + \delta^{-q} U)) \\ &= \det(\lambda I - \delta^q B) \end{aligned} \quad (186)$$

since A is consistently ordered with parameters r,q. Therefore, applying theorem 11,

$$\begin{aligned} \Phi(\lambda) &= \delta^{qn} \det(\lambda \delta^{-q} I - B) \\ &= \delta^{qn} (\lambda \delta^{-q})^m \prod_{i=1}^s ((\lambda \delta^{-q})^p - \mu_i^p) \\ &= \lambda^m \prod_{i=1}^s (\lambda^p - (\delta^P)^q \mu_i^p) \end{aligned} \quad (187)$$

where n is the order of A, m is the multiplicity of the possible zero eigenvalue of B, and $\theta^j \mu_i$, $i = 1, \dots, s$, $j = 0, 1, \dots, p-1$ are the eigenvalues of B where $\theta^P = 1$; we note that use has been made of the relation

$$n = m + ps \quad (188)$$

Thus

$$\Phi(\lambda) = \lambda^m \prod_{i=1}^s (\lambda^p - (1 - \beta + \beta\lambda)^q \mu_i^p) \quad (189)$$

Then, if λ is a non zero-eigenvalue of $\mathcal{C}_\beta$, there exists one indice i for which

$$\lambda^p = (1 - \beta + \beta\lambda)^q \mu_i^p \quad (190)$$

and, if μ satisfies (180), we conclude that

$$\mu^p = \mu_i^p \quad (191)$$

By theorem 11, μ is then an eigenvalue of B.

Conversely, if μ is an eigenvalue of B, say

$$\mu = \theta^j \mu_i \quad (192)$$

then

$$\mu^P = \theta^{jp} \mu_i^P = \mu_i^P \quad (193)$$

so that, if λ satisfies (180), one of the factors of the right hand side of Eq. (189) is zero, thus $\phi(\lambda) = 0$ and λ is an eigenvalue of $\mathcal{Z}_\beta$.

Remarks

If one considers the scheme (176) with

$$M = D - \beta F \quad (194)$$

in the place of (177), the functional relations corresponding to (180) becomes

$$\lambda^P = (1 - \beta + \beta\lambda)^r \mu^P \quad (195)$$

If one applies the Oliphant-Buleev technique with two parameters, α, β to the Gauss-Seidel method, i.e., if

$$M = \alpha D - \beta E \quad (196)$$

then theorem 14 still holds but the functional relation (180) becomes

$$(1 - \alpha + \alpha\lambda)^P = (1 - \beta + \beta\lambda)^q \mu^P \quad (197)$$

It may be observed that this latter method does not differ from the simultaneous use of the one parameter Oliphant technique together with the one parameter overrelaxation procedure, ω and α being related by

$$\alpha = \frac{1}{\omega} \quad (198)$$

and the relation (197) reduces to (167) when $\beta = 1$.

As in the classical theory of the S.O.R. process, the functional relation (180) may be taken as the basis of the theoretical determination of the optimal Oliphant relaxation parameter $\beta^{(1)}$. For simplicity, consider the particular case where $r = q = 1$ and where the eigenvalue of β^2 are

(1) See also Sisler [22].

real and non-negative. In this case, the eigenvalues of B occur in pairs $\pm \mu_i$ (by theorem 11) and the relation (180) writes

$$\lambda^2 = (1 - \beta + \beta\lambda)\mu^2 \quad (199)$$

Defining the functions

$$y_1(\lambda) = \left(\frac{\lambda}{\mu}\right)^2 \quad (200)$$

$$y_2(\lambda) = 1 - \beta + \beta\lambda \quad (201)$$

the equation (199) can be geometrically interpreted as the intersection of the parabola $y_1(\lambda)$ and the straight line $y_2(\lambda)$ which passes through the point (1,1) and whose slope is β , as illustrated in Figure 1.

Figure 1

It is thus seen that the largest abscissa of the two points of intersection decreases when β increased until $y_2(\lambda)$ becomes tangent to $y_1(\lambda)$, which occurs when

$$\beta = \tilde{\beta} \equiv \frac{2}{\mu^2}(1 - \sqrt{1 - \mu^2}) \quad (202)$$

For $\beta > \tilde{\beta}$, the equation (199) has two conjugate roots of modulus $\mu\sqrt{\beta - 1}$ which now increases when β increases until $y_2(\lambda)$ becomes anew tangent to $y_1(\lambda)$, which occurs for $\beta = \frac{2}{\mu^2}(1 + \sqrt{1 - \mu^2})$ but this corresponds to divergent processes as well as higher values of β .

Since the curve

$$y(\lambda) = \left(\frac{\lambda}{\rho}\right)^2 \quad (203)$$

is the envelope of all curves $\left(\frac{\lambda}{\mu}\right)^2$ for $0 \leq \mu \leq \rho$, the optimal

Oliphant relaxation parameter β_b is given by

$$\beta_b = \frac{2}{\rho^2} (1 - \sqrt{1 - \rho^2}) \quad (204)$$

where ρ is the spectral radius of the block Jacobi matrix B . The spectral radius of the corresponding iteration matrix $\mathcal{G}_\beta$ is given by

$$\min_{\beta} \rho(\mathcal{G}_\beta) = \rho(\mathcal{G}_{\beta_b}) = \frac{\beta_b \rho^2}{2} = (1 - \sqrt{1 - \rho^2}) \quad (205)$$

Thus we have

Theorem 21

Let the matrix A be consistently ordered with $r = q = 1$. If all the eigenvalues of the square of the block Jacobi matrix B are real and non-negative and $0 < \rho < 1$, then

$$1. \quad \rho(\mathcal{G}_{\beta_b}) = (1 - \sqrt{1 - \rho^2}) \quad (206)$$

$$2. \quad \rho(\mathcal{G}_\beta) > \rho(\mathcal{G}_{\beta_b}) \quad \text{for all } \beta \neq \beta_b \quad (207)$$

where ρ is the spectral radius of B and β_b is defined by (204).

In the same case, the spectral radius of the optimal S.O.R. method is given by (Varga [9]):

$$\rho(\omega_b) = \frac{\rho^2}{(1 + \sqrt{1 - \rho^2})^2} = \frac{1 - \sqrt{1 - \rho^2}}{1 + \sqrt{1 - \rho^2}} \quad (208)$$

Figure 2 displays the behavior of $\rho(\omega_b)$ and $\rho(\mathcal{G}_{\beta_b})$

versus $x = \sqrt{1 - \rho^2}$, together with $\rho(\omega_1)$ and $\rho(B)$.

(a) Jacobi	$\rho(B) = (1 - x^2)^{1/2}$
(b) Gauss-Seidel	$\rho(\mathcal{A}_1^p) = 1 - x^2$
(c) Optimal $\mathcal{Z}_{\beta}$	$\rho(\mathcal{Z}_{\beta_b}) = 1 - x$
(d) Optimal S.O.R.	$\rho(\mathcal{A}_{\omega_b}^p) = \frac{1 - x}{1 + x}$

Figure 2

We observe that both optimal relaxed methods give improvements of the same order of magnitude over the Gauss-Seidel and Jacobi procedures, in the region of practical interest (x small); indeed, letting $R(T)$ be the asymptotic convergence rate of the iteration matrix T (cf. Eq. (38)), we have

$$\frac{R(\mathcal{Z}_{\beta_b})}{\rho(\mathcal{A}_1^p)} \rightarrow \infty \text{ as well as } \frac{R(\mathcal{A}_{\omega_b}^p)}{R(\mathcal{A}_1^p)} \rightarrow \infty \text{ for } x \rightarrow 0. \text{ However}^{(1)}, \rho(\mathcal{Z}_{\beta_b})$$

is always higher than $\rho(\mathcal{A}_{\omega_b}^p)$ and

the classical overrelaxation technique is thus more powerful than the one-parameter Oliphant technique applied to the Gauss-Seidel scheme, at least

⁽¹⁾ as pointed out by Varga [23].

for consistently ordered matrices with $r = q = 1$.

Remembering that we analyzed these particular cases only as simplified examples of the general Oliphant and Buleev relaxation techniques, we can take the corresponding results as indications for the more general situation: both techniques must probably yield improvements of the same order of magnitude; however, the optimal Buleev technique should be expected more powerful than the optimal Oliphant technique, at least for consistently ordered matrices with $r = q = 1$.

5 The block EWA method with Chebyshev relaxation

We shall consider, in this section, the application of the Chebyshev relaxation technique to factorization iterative methods, taking the block EWA method as a typical example. We refer to [7] for its detailed definition, which may be briefly summarized as

$$Mx_{m+1} = Nx_m + b \quad (209)$$

$$A = D - E - F = M - N \quad (210)$$

$$M = (P - E)P^{-1}(P - F) \quad (211)$$

$$P = D - \text{blockdiag} (EP^{-1}F) \quad (212)$$

We denote by $\mathcal{E}$ the corresponding iteration matrix

$$\mathcal{E} = M^{-1}N = (P - F)^{-1}P(P - E)^{-1}(\text{Offblockdiag}(EP^{-1}F)) \quad (213)$$

and by ρ_e its spectral radius. Other notations are defined in section 2 of ref. [7].

In order to apply the Chebyshev relaxation technique, the iteration matrix must be symmetric and we shall henceforth assume that A is symmetric. Therefore,

$$E = F^T \quad (214)$$

and P is symmetric.

A symmetric and essentially non-positive matrix is an M -matrix if and only if it is positive definite. This well known result is an immediate consequence of theorem 6 of [8]. Matrices of this class are known as Stieltjes matrices (Varga [9]).

Assuming that A is a Stieltjes matrix, it follows from theorem 7 of [7] that the corresponding block EWA method is convergent. Thus, the eigenvalues e_i of the iteration matrix $\mathcal{E}$ satisfy the relation

$$-\rho_e \leq e_i \leq \rho_e < 1 \quad (215)$$

and we may apply the Chebyshev relaxation technique with $a = -\rho_e$ and $b = \rho_e$. A lower bound of the corresponding asymptotic convergence rate is obtained from theorem 8; thus, we have

Theorem 22

If A is a Stieltjes matrix, the asymptotic rate of convergence of the block EWA method with Chebyshev relaxation where $b = -a = \rho_e$ is not less than

$$\sigma = \cosh^{-1} \left(\frac{1}{\rho_e} \right) = -\log \left(\frac{\rho_e}{1 + \sqrt{1 - \rho_e^2}} \right) \quad (216)$$

For comparison purposes, it is useful to introduce an equivalent spectral radius

$$\rho_{eq} = \exp(-R_{as}) \quad (217)$$

where R_{as} is the asymptotic convergence rate of any semi-iterative method. Since $\exp(-x)$ is a monotonically decreasing function of x for $-\infty < x < \infty$, lower bounds for R_{as} will give, using the relation (217), upper bounds for ρ_{eq} . Since $\cosh^{-1} \left(\frac{1}{x} \right)$ is a monotonically decreasing function of x for $0 \leq x \leq 1$, upper bounds for ρ_e will give upon using the relations (216) and (217) upper bounds for ρ_{eq} .

For Stieltjes matrices, it follows from theorem 7 of [7] that an upper bound for ρ_e is given by

$$\rho_e \leq \rho(\mathcal{L}_1) \quad (218)$$

where $\mathcal{L}_1$ is the block Gauss-Seidel iteration matrix. Thus we have

Theorem 23

Let A be a Stieltjes matrix, and ρ_{eq} be the equivalent spectral radius of the block EWA method with Chebyshev relaxation; then

$$\rho_{eq} \leq \frac{\rho(\mathcal{A}_1)}{1 + \sqrt{1 - (\rho(\mathcal{A}_1))^2}} \quad (219)$$

where $\rho(\mathcal{A}_1)$ is the spectral radius of the block Gauss-Seidel iterative method.

Bounds for $\rho(\mathcal{A}_1)$ may be deduced from theorem 12-2.2 of Young [24] under the same assumptions: if A is a Stieltjes matrix, then

$$\rho^2(B) \leq \rho(\mathcal{A}_1) \leq \frac{\rho(B)}{2 - \rho(B)} \quad (220)$$

with left equality if B is a consistently ordered matrix with $r = q = 1$.

Therefore we have

$$\rho_{eq} \leq \frac{\rho(B)}{2 - \rho(B) + 2\sqrt{1 - \rho(B)}} \quad (221)$$

if A is a Stieltjes matrix and

$$\rho_{eq} \leq \frac{(\rho(B))^2}{1 + \sqrt{1 - (\rho(B))^4}} \quad (222)$$

if A is moreover consistently ordered with parameters $r = q = 1$.

The comparison with the block S.O.R. method using the parameter

$$\omega_b = 1 + \left(\frac{\rho(B)}{1 + \sqrt{1 - (\rho(B))^2}} \right)^2 \quad (223)$$

can be made on the basis of theorem 12-2.1 of Young [24] from which we have: if A is a Stieltjes matrix, then

$$\left(\frac{\rho(B)}{1 + \sqrt{1 - (\rho(B))^2}} \right)^2 \leq \rho(\mathcal{A}_{\omega_b}) \leq \left(\frac{\rho(B)}{1 + \sqrt{1 - (\rho(B))^2}} \right) \quad (224)$$

with left equality if A is consistently ordered with parameters $r = q = 1$.

These results are illustrated in Figure 3 which displays the behavior of the upper bounds

$$\frac{\rho_{eq}}{\rho(B)} \leq \frac{1}{2 - \rho(B) + 2\sqrt{1 - \rho(B)}} \quad \text{and} \quad \frac{\rho(\mathcal{L}_{\omega_b})}{\rho(B)} \leq \frac{1}{1 + \sqrt{1 - (\rho(B))^2}} \quad (225)$$

versus $\rho(B)$ when A is a Stieltjes matrix and in Figure 4 which displays the upper bound

$$\frac{\rho_{eq}}{(\rho(B))^2} \leq \frac{1}{1 + \sqrt{1 - (\rho(B))^4}} \quad (226)$$

together with the exact value

$$\frac{\rho(\mathcal{L}_{\omega_b})}{(\rho(B))^2} = \left(\frac{1}{1 + \sqrt{1 - (\rho(B))^2}} \right)^2 \quad (227)$$

versus $\rho(B)$ where A is, in addition, consistently ordered with parameters $r = q = 1$.

Figure 3

a: exact value of $\rho(\omega_b)/(\rho(B))^2$

b: upper bound for $\rho_{eq}/(\rho(B))^2$

for consistently ordered ($r = q = 1$) Stieltjes matrices.

Figure 4

It is thus seen that, in the general case, we may expect that the Chebyshev procedure applied to block EWA method will achieve at least comparable improvements as the block S.O.R. method with $\omega = \omega_b$ (using the same partitioning). However, although these results give an idea of the convergence rate which can be reached by the Chebyshev-EWA method, with respect to S.O.R., no positive conclusion can be drawn; we can only state that Chebyshev-EWA might be superior if A is a Stieltjes matrix while S.O.R. might be superior if A is consistently ordered with $r = p = 1$.

A weakness of the preceeding comparisons should be stressed: we have assumed that the same partitioning is used in both methods. In practice however, computational difficulties will make hard to use the block EWA method with a large blocksize.

6. Concluding remarks

In the present report, we attempted to review the various relaxation techniques which have been used or proposed for improving the convergence rate of the factorization iterative methods.

Refinements of known properties have been obtained for the extrapolation procedure and some estimation of the efficiency of the Chebyshev technique has been provided. Most useful complements would be given by numerical experimentation in this area.

Some applications of the theory of consistently ordered matrices have been reviewed and some new applications have been indicated.

Finally, in order to assess the properties of the general two-parameters overrelaxation procedure, the necessity of studying iterative procedures of the second order, in the sense of section 2, has been shown and first results obtained; we think however that this method of analysis should deserve more consideration in later studies.

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