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MULTICHANNEL WAVE
INTERFEROMETRY

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Multichannel Wave Interferometry

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ABSTRACT

The analysis of multichannel wave interferometry for a variety of density profiles reveals that the phase shift as a function of horn position normalized to that through the center of the plasma, is relatively insensitive to $K = n_0/n_c$ (the ratio of maximum density to cutoff density) when refraction effects are minimized. Thus, this ratio may be easily inverted to obtain a reasonably accurate measurement of the density profile even for values of K approaching 1.

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I. INTRODUCTION

The effects of refraction on multichannel wave measurements of plasma density can be considerable when the peak plasma density is a sizable fraction (≥ 0.1) of the cutoff density for the wave frequency employed. These effects can be avoided by operating at elevated frequencies; this requires the use of infrared lasers for maximum densities in the range of a few times 10^{14} cm^{-3} (e.g., in the TFR and PLT Tokamaks). However, pending the development of multichannel infrared laser interferometers, $\lambda = 2 \text{ mm}$ and 1 mm systems are being used (TFR) or are being contemplated (PLT). For these relatively low-frequency systems, refraction effects must be introduced into the analysis of the phase shift measurements to provide a density profile which is sufficiently accurate.

Following the work of Crenn,¹ an analytical approach for evaluating the refraction effects for a cylindrical plasma column will be presented. In addition to the parabolic case treated by Crenn, the present analysis will be applied to profiles of the form $n(r) = n_0 [1 - (r/r_0)^{2p}]$, p being a positive integer. Following this analytical treatment, a numerical evaluation, which is applicable to arbitrary radial density profiles, will be outlined. With this accurate numerical approach, we will explore a variety of profiles to determine their refraction effects and the most direct way to deduce the density profile from the phase measurements.

Of particular interest in the context of large scale tokamaks are the effects of refraction on the Abel inversion analysis, and

on the wave power intercepted by the receiving horn (or lens). The change in phase along the refracted trajectory relative to the line-of-sight trajectory assumed for Abel inversion^{1,2} tends to give a density profile which is enhanced near the surface of the plasma and depressed at the center of the plasma. In addition, there is generally less power received along the refracted trajectory and this power must be maintained at a sufficient level to permit the phase increment to be measured unambiguously.

Lastly, a novel method for inverting the phase measurements to obtain the density profile will be presented. This method is found to be much less sensitive to refractive effects than the Abel inversion method.

II. CYLINDRICAL MODEL

The ray geometry assumed for the phase measurements is presented in Fig. 1 (one channel). As the plasma density increases from zero to a given level, the straight path d is refracted to the path s which is governed, under the assumption of geometrical optics, by

$$\mu r \sin\phi = r_0 \sin\phi_0 \quad (1)$$

where ϕ is the angle between the ray path and the radius vector, and

$$\mu^2 = 1 - n(r)/n_c \quad (2)$$

with the cutoff density $n_c = m_e \omega^2 / 4\pi e^2$. Choosing $\theta = 0$ along the upward vertical axis

$$i \theta_o = \int_{r_s}^{r_o} \frac{r_o \sin \phi_o dr}{r (\mu^2 r^2 - r_o^2 \sin^2 \phi_o)^{1/2}} \quad (3)$$

which upon changing the variable to $v = r^2/r_o^2$ becomes

$$\theta_o = \frac{1}{2} \int_{v_s}^1 \frac{(\mu_s^2 v_s)^{1/2} dv}{v (\mu^2 v - \mu_s^2 v_s)^{1/2}} \quad (4)$$

with $v_s = r_s^2/r_o^2$, and $\mu_s r_s = r_o \sin \phi_o$ according to Eq. (1). The electrical path lengths along s ($\int \mu ds$) are then

$$s_2 = \frac{r_o}{2} \int_{v_s}^1 \frac{\mu^2 dv}{(\mu^2 v - \mu_s^2 v_s)^{1/2}} \quad (5)$$

and

$$s_1 = r_o (x_h/r_o - \sin \theta_o) / \sin(\theta_o + \phi_o). \quad (6)$$

Similar quantities for path d are

$$r_d = r_o \cos \theta_o + s_1 \cos(\theta_o + \phi_o) \quad (7)$$

$$d_2 = r_o/2 \int_{v_d}^1 \mu dv / (v - v_d)^{1/2} \quad (8)$$

and

$$d_1 = x_h - r_o (1 - v_d)^{1/2} \quad (9)$$

where $v_d = r_d^2/r_o^2$.

In deriving the above equations it has been assumed that $n(\rho)$ is a function only of r . Otherwise, these equations are general for the geometry of Fig. 1.

III. ANALYTICAL APPROACH

The equations of Sec. II are easily solved explicitly in the case of a parabolic density profile ($p = 1$) in terms of well-known functions. Crenn expanded these solutions to second order in $n_o/n_c = K$ to obtain the phase difference between path s and d as a function of v_d ;

$$\frac{\Delta N}{N_{\max}} = K^2 v_d (1 - v_d) \left(\frac{x_h}{2r_o} - \frac{(1 - v_d)^{1/2}}{3} \right), \quad (10)$$

where $\Delta N = \Delta\phi/2\pi$ is the fringe difference compared to $N_{\max} = 2r_o/\lambda$. For $p > 1$, explicit solutions are not straightforward, but solutions to a specified order in K can be obtained by first expanding the integrands of the pertinent equations. This procedure involves considerable algebra and will be considered here for $p = 1$ and 2, for terms up to K^2 .

Pre-expanding the integrands for $p = 1$ gives for the fringe shift N_s along the path s relative to $N_{\max} = 2r_o/\lambda$ the relation:

$$\begin{aligned} \frac{N_s}{N_{\max}} &= \frac{x_h}{r_o} - \frac{s_1 + s_2}{r_o} \\ &= -\frac{x_h}{2r_o} K^2 v_s w_s + w_s^{1/2} \left[\frac{K w_s}{3} + \frac{K^2}{8} \left(\frac{4v_s w_s}{3} + \frac{8w_s^2}{15} \right) \right], \quad (11) \end{aligned}$$

where $w_s = 1 - v_s$. The phase difference between paths s and d is $N_s/N_{\max} - N_d/N_{\max}$ and is found to be that of Eq. (10). However, a limit on the range of validity of Eq. (10) is imposed by the expansion of Eq. (7),

$$\frac{K}{r_o} \frac{2x_h}{w_d^{1/2}} \frac{v_d}{w_d^{1/2}} \ll 1 \quad (12)$$

where $w_d = 1 - v_d$. For example, with $n_o/n_c = 0.2$, $2x_h/r_o \sim 2$, and assuming much less than 1 is ~ 0.2 , v_d must be ≤ 0.64 or $r_d/r_o \leq 0.8$ for Eq. (10) to apply.

With $p = 2$,

$$\begin{aligned} \frac{N_s}{N_{\max}} = & - \frac{x_h}{8r_o} K^2 v_s w_s (4 - 8w_s/3)^2 + w_s^{1/2} \{K(2w_s/3 - 4w_s^2/15) \\ & + K^2/6 [v_s w_s (16 - 32w_s/5 + 149 w_s^2/5 \cdot 7)] \\ & + w_s^2 (8/5 - 48 w_s/5 \cdot 7 + 96 w_s^2/5 \cdot 7 \cdot 9)]\} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \frac{\Delta N}{N_{\max}} = & K^2 v_d w_d \left[2 \frac{x_h}{r_o} \left(1 - \frac{4w_d}{3} + \frac{4w_d^2}{9} \right) \right. \\ & \left. - \frac{4w_d^{1/2}}{3} \left(1 - \frac{6w_d}{5} + \frac{12w_d^2}{35} \right) \right] \end{aligned} \quad (14)$$

The limit on the applicability of Eq. (14) is

$$K \frac{x_h}{r_o} \frac{v_d}{w_d^{1/2}} (4 - 8w_d/3) \ll 1 \quad (15)$$

which gives for the same conditions assumed above in the case of $p = 1$, $v_d \lesssim 0.36$ or $r_d/r_o \lesssim 0.6$.

The leading term $v_d w_d$ in Eqs. (10) and (14) tends to maximize ΔN in the vicinity of $v_d = 1/2$ or $r_d/r_o = 0.7$. The other (bracketed) term pushes the maximum of ΔN to larger v_d and, of course, the added displacement is larger for the larger value of p . The role of x_h/r_o in increasing the difference in phase is clearly displayed and is enhanced near the plasma perimeter ($w_d \rightarrow 0$) as p is increased.

It is necessary to minimize d/r_o to minimize the refraction effect on the measured phase shift. As a practical example, this can be done for a microwave system by placing the foci of the horns on a circle of radius r_h where $r_h = r_o + \Delta r$ with $\Delta r \sim 2$ cm or so to avoid excessive plasma bombardment. The absolute minimum effect would be achieved if the horn foci could be placed at the surface of the plasma. In this hypothetical case, the phase shift is given by replacing x_h/r_o by $w_d^{1/2}$ in the formulas above. Then for this case,

$$p = 1: \Delta N/N_{\max} = K^2 v_d w_d^{3/2}/6, \quad (16)$$

and

$$p = 2: \Delta N/N_{\max} = K^2 v_d w_d^{3/2} \left(\frac{2}{3} - \frac{16}{15} w_d + \frac{136}{315} w_d^2 \right). \quad (17)$$

Equations (16) and (17) show that in absolute value $\Delta N/N_{\max}$ for $p = 2$ is 4 times that for $p = 1$ near the perimeter of the plasma ($w_d \rightarrow 0$). Relative to the locally measured phase shift, $\Delta N/N_s$ for $p = 2$ is twice that for $p = 1$.

Equations (16) and (17) also reveal that ΔN cannot be sufficiently minimized by reducing r_h/r_o to 1 if $K = n_o/n_c$ is too large. In fact, since $\Delta N/N_s \propto K = n_o 4\pi e^2/m_e \omega^2$, the most effective way to eliminate refractive effects is to increase ω . In particular, an HCN laser system with $\lambda = 0.336$ mm reduces the refraction phase increment $\Delta N/N_s$ by a factor of ~ 40 relative to that for a 2 mm tube.

IV. NUMERICAL SOLUTION

To solve the equations of Sec. II for an arbitrary radial profile of density, the Runge-Kutta numerical method of integration³ is employed to find θ_o , s_2/r_o , and d_2/r_o , starting with the differential equations

$$dR/d\theta = R(\mu^2 v - \mu_s^2 v_s)^{1/2} / (\mu_s^2 v_s)^{1/2} \quad , \quad (18)$$

$$dR/d(s_2/r_o) = (\mu^2 v - \mu_s^2 v_s)^{1/2} / \mu^2 R \quad , \quad (19)$$

and

$$dR/d(d_2/r_o) = (v - v_d)^{1/2} / \mu R \quad , \quad (20)$$

with $R = r/r_o$. Each integral is performed by increasing the independent variable in each case (θ , s_2/r_o , or d_2/r_o) in sufficiently small steps so as to permit the dependent variable in each case (R) to increase from r_s/r_o [Eqs. (18) and (19)] or from r_d/r_o [Eq. (20)] to 1. The values of s_1 and d_1 then follow directly from Eqs. (6), (7), and (9) and the solution is completed.

Numerical results obtained with this approach for the analytical density profiles with $p = 1$ and 2 are precisely those found using the formulas of Sec. III when $K = n_o/n_c$ is sufficiently small. Even at $K \sim 0.3$ the analytical formulas are valid to within a few percent except near the edge of the plasma column.

V. REFRACTION EFFECTS FOR THE TFR MICROWAVE MULTICHANNEL INTERFEROMETER

In order to illustrate the refraction effects which may be important in any large tokamak, we present here calculations appropriate to the microwave system presently employed on TFR.⁴ This system operates at a frequency of 134 GHz ($\lambda = 2.24$ mm) and the horns are located at a constant vertical distance (x_h) of 24 cm from the plasma midplane. Since Thomson scattering measurements indicate that the density persists at radii beyond that of the limiter,⁵ we will assume that it goes to zero at the vessel shield, or at $r_o = 22$ cm. This permits us to display the refraction effects as universal curves with $x_h/r_o = 1.1$.

Ray trajectories for the 2.24 mm system are presented in Fig. 2 for a parabolic profile with a central density of $n_o = 10^{14} \text{ cm}^{-3}$ ($K = n_o/n_c = 0.449$). This example clearly illustrates the enormous refraction of the rays which can be encountered at the higher operating densities in TFR. The displacement of the rays from the line-of-sight trajectory in the plasma midplane is plotted versus horn position in Fig. 3. (This displacement scales approximately linearly with n_o .) This displacement peaks to a value of $\sim 0.2 r_o$ or 4.4 cm. A sizable fraction of this

displacement can be eliminated simply by placing the horns on a constant radius around the plasma as is shown by the dashed curve in Fig. 3 for which $r_h/r_0 = 1.1$.

The ray trajectories for an HCN system do not deviate perceptibly from the straight line trajectories over the dimensions considered in Fig. 2. The displacement in the plasma midplane for the HCN system is also illustrated in Fig. 3.

More important to the analysis of the multichannel microwave interferometer measurements than the actual ray path are the difference in phase shift and the change in incident (and exit) angle of the ray produced by the refraction. This difference in phase shift must be subtracted from the measured phase shift to permit an Abel inversion and the change in incident angle affects the power emitted (and received) along the refracted ray path.

In Fig. 4, we present $N_s/N_s(0)$ (to be measured), N_d/N_s , and the incident angle γ (Fig. 1) for a parabolic density profile with $n_0 = 10^{14} \text{ cm}^{-3}$ and for both the microwave and HCN laser interferometer systems. The measured phase shift N_s is clearly enhanced towards the plasma boundary relative to the line-of-sight phase shift N_d [Figs. 4(a) and (b)] in the case of the microwave system. This complicates considerably the Abel inversion of the measured N_s curve to obtain $n(r)$ since it is N_d which must be deduced to be used in the inversion formula.

The incident angle γ is found to have a maximum of $\sim 13^\circ$ for the microwave system as compared to $\sim 0.3^\circ$ for the HCN laser system [Fig. 4(c)]. (These angles scale approximately linearly with n_0 .) As in the case of the phase shift, the change in

incident angle is completely tolerable for the HCN laser system. However, $\gamma_{\max}$ falls outside the first null angle of the normally used "long" optimized H-plane horns.⁶ Thus, for long optimized horns aligned along the trajectory d (Fig. 1), the incident (received) power would fall by several orders of magnitude relative to that without a plasma and additional phase shift would be introduced by the change from the main lobe to the side lobes of the radiation pattern.

In TFR, oversized waveguide structures are used in place of horns and these have a main lobe with nulls at $\sim 5^\circ$ and first and second lobes out to $\sim 17^\circ$ with intensities down by ~ 12 dB from the maximum of the main lobe. Much larger input power is required in the outer channels to provide detectable signals as n_0 approaches 10^{14} since the rays traverse the side lobes.⁷

VI. MINIMIZATION OF REFRACTION EFFECTS

Two steps are again indicated in the example of Sec. V for minimizing the effects of refraction on the multichannel microwave interferometer. First, the horns should be placed as close as possible to the surface of the plasma, or more precisely, they should be placed at a radius r_h and r_h/r_0 should be made as close to 1 as possible. Secondly, the horn design and orientation should be chosen to assure that the detected rays are located well inside the main lobe of the horn radiation pattern for the densities of interest.

The limiting case of $r_h/r_0 = 1$ is of special interest because it is more easily approached as machines increase in size (larger r_0) since presumably a "safe" distance between the horn

and plasma surface is relatively independent of machine size. In Figs. 5-7, we present $N_s/N_s(o)$, N_d/N_s , and γ as functions of the horn position for three density profiles: $n(r) = n_o [1 - (r/r_o)^{2p}]$ with $p = 1/2$ (linear), 1 (parabolic), and 2 (quartic). The several curves in each case correspond to the different values of the parameter $K = n_o/n_c$ indicated.

With the horn at the plasma surface the value of $N_s/N_s(o)$ now approaches zero near the surface of the plasma and thus a large part of the refraction effect on this quantity as experienced for the constant horn positioning (x_h constant) in the previous section has been eliminated. The refraction effect on $N_s/N_s(o)$ which remains is a weak function of n_o/n_c . For example, in the case of the parabolic density profile, expansion of N_s and $N_s(o)$ to powers up to $(n_o/n_c)^2$ gives

$$N_s/N_s(o) \approx [1 - v_d]^{3/2} [1 + 0.3 K v_d] \quad (21)$$

The maximum effect of n_o/n_c in this equation occurs at $v_d = (r_d/r_o)^2 = 0.4$ and is equal to $\Delta(N_s/N_s(o)) = 0.056 K$. This equation predicts rather well the maximum displacements found in Fig. 6(a).

The relative insensitivity of $N_s/N_s(o)$ to K is most directly attributable to the increase in the physical path length s of the ray with increasing density. However, since the path length does not change for the calculation of N_d in this case, N_d/N_s still varies considerably with n_o/n_c as shown. As p increases the deviation of N_d/N_s from 1 is enhanced closer to the plasma

perimeter [as expected from Eqs. (16) and (17)] and its maximum value increases.

Finally, we find that γ has a maximum value for a given n_o/n_c which increases with p and whose location also moves toward the surface of the plasma as p increases. This complicates the horn design and orientation problem somewhat since it suggests the use of a rather wide horn radiation pattern with its associated low directivity and gain to make a system applicable to arbitrary density profiles at large values of n_o/n_c .

It can be noted here, that for $r_h/r_o = 1$, our coordinate frame (Fig. 1) may be transformed to that used by Shmoys⁸ in his treatment detailing the use of the diffraction pattern (angular distribution) to determine the density profile:

$$b/r_o = \sin\phi_o ,$$

and

$$\theta(b) = \pi - 2(\theta_o + \phi_o) ,$$

where b is the impact parameter in his case. It follows that our results for γ with $r_h/r_o = 1$ are related to his results according to

$$r_d/r_o = \sin(\phi_o + \theta(b)/2) ,$$

and

$$\gamma = \theta(b)/2 .$$

Due to our different coordinate frame γ peaks for all n_o/n_c at a roughly constant value of r_d/r_o whereas in the frame used by Shmoys the $\theta(b)$ peak approaches $b/r_o = 0$ as n_o/n_c increases (for the smooth profiles considered here).

As r_h/r_o increases above 1, the added refraction effect of the vacuum region enhances N_s somewhat, whereas the refraction effect of the plasma region is rather insensitive to small changes in r_h/r_o since the increase in path length partially compensates the reduction in the electrical path length in the denser plasma region. Results calculated for $r_h/r_o = 1.1$ are presented in Figs. 8-10. Note the enhancement in phase N_s at the plasma boundary. This enhancement is especially marked in the ratio N_d/N_s .

VII. INVERSION OF THE MEASURED PHASE SHIFT

Several approaches can be used to invert $N_s(v_d)$ to obtain $n(r)$. Those considered here include Abel inversion, comparison to curves of $N_s/N_s(o)$ for trial analytic curves of density, and a novel approach of treating the normalized density profile separately from the magnitude of the density.

Abel inversion is not straightforward for N_s in contrast to $\theta(b)$ due to the fact that in our coordinate frame N_s continues to be a direct function of b which is a priori an undetermined function of v_d . However, Abel inversion can be applied directly to N_d if this phase shift can be deduced from $N_s^{1,2}$:

$$1 - (1 - n(r_d)/n_o)^{1/2} = -(\lambda/\pi) \int_{r_d}^{r_{oo}} (dN_d/dr) dr / (r^2 - r_d^2)^{1/2} \quad (22)$$

This equation may be solved through iteration by starting the iteration process with the choice of an N_d curve which passes

through the measured N_s values. The resulting $n(r_d)$ is then used to calculate N_d/N_s which is multiplied by the measured N_s values to obtain a second N_d curve. Equation (22) is solved for a second $n(r_d)$ curve, and so forth, until the density profile ceases to change, or rather, until the calculated N_s curve passes through the measured N_s values.

The choice of an initial N_d curve which ignores refraction as above may cause the iteration process to be slow to converge. A considerable improvement to this approach is obtained by starting the use of Eq. (22) with an initial N_d curve which is much closer to the final curve. Families of curves of $N_s/N_s(o)$ with n_o/n_c as a parameter as in Figs. 5-10, calculated for different analytic density profiles, may be compared directly to the measured $N_s/N_s(o)$ points to provide a reasonable first choice of the relative profile $n(r)/n_o$. Then n_o is selected to produce the measured $N_s(o)$ value. Finally, this starting $n(r)$ profile is used to calculate N_d/N_s and the iteration is begun and continued as detailed above.

The relative insensitivity of $N_s/N_s(o)$ to n_o/n_c revealed in Figs. 5-10 even for r_h/r_o somewhat greater than 1 suggests a novel method for obtaining the density profile. This method consists of deducing the normalized density profile $n(v_d)/n_o$ from the measured points $N_s/N_s(o)$ assuming the low density limit. Once this relative profile is known and the appropriate value of n_o is obtained to provide the measured $N_s(o)$, the resulting density profile can be used either in the Abel inversion technique or more directly to calculate $N_s/N_s(o) \Big|_{n_o} - N_s/N_s(o) \Big|_{n_o \rightarrow 0}$ to subtract

from the measured $N_s/N_s(o)$ values to yield values more appropriate to the low density limit. This new set of $N_s/N_s(o)$ values is inverted to yield $n(r)/n_o$ in the low density limit, and so forth. As we shall see, two steps of this process are usually sufficient even at very large values of n_o/n_c .

As a model for exploring the use of this final approach, we assume that we have 4 measurements of N_s ; one at or near the center of the discharge and 3 at horn positions spaced away from the central plasma plane. Further, we assume that the density falls to zero at the vessel wall (or shield) radius r_o . Then the 4 points are sufficient to determine n_o and

$$n(r)/n_o = 1 - A (r/r_o) - B (r/r_o)^2 - C (r/r_o)^3 - D (r/r_o)^4, \quad (23)$$

with $A + B + C + D = 1$. In the low density limit

$$\left. \frac{N_s}{N_s(o)} \right|_{n_o \rightarrow 0} - \left. \frac{N_d}{N_d(o)} \right|_{n_o \rightarrow 0} = \frac{1}{C'} \int_{R_d}^1 \frac{[n(R)/n_o] R dR}{(R^2 - R_d^2)^{1/2}} \quad (24)$$

with $C' = \int_0^1 [n(R)/n_o] dR$ and $R_d = r_d/r_o$ is the minimum normalized radius at the horn position (Fig. 1). Clearly, $C' = 1 - A/2 - B/3 - C/4 - D/5$. Then solution of Eq. (24) gives

$$\begin{aligned} \left. N_s/N_s(o) \right|_{n_o \rightarrow 0} &+ 12(T_3 - T_2) \\ &= 1/C' [s - T_3 + 9(T_3 - T_2)] \\ &+ D/C' [T_3 - T_4 + 3/5 (T_3 - T_2)] \\ &+ A/C' [T_3 - T_1 - 3(T_3 - T_2)] \end{aligned} \quad (25)$$

with

$$\begin{aligned}
 B &= 9 - 12 C' - 3 A + 3/5 D , \\
 C &= 1 - A - B - D , \\
 s &= (1 - R_d^2)^{1/2} , \\
 T_1 &= s/2 + R_d^2/2 \ln [(1 + s)/R_d] , \\
 T_2 &= s^3/3 + R_d^2 s , \\
 T_3 &= s/4 + 3/4 R_d^2 T_1 ,
 \end{aligned} \tag{26}$$

and

$$T_4 = s^5/5 + 2/3 R_d^2 s^3 + R_d^4 s .$$

Note that s , T_1 , T_2 , T_3 , and T_4 are functions only of the position of the horns, so that the 3 measurements of $N_s/N_s(o)$ assumed at the start equal to $N_s/N_s(o) |_{n_o \rightarrow 0}$ suffice to give a simple 3-by-3 matrix soluable for $1/C'$, D/C' , and A/C' , and subsequently A , D , B , and C . Finally, n_o for the profile is determined from the requirement that $N_s(o)$ be produced at $R_h = 0$.

Profiles of density thus deduced from the $N_s/N_s(o)$ curves of the cases treated in the previous section are shown in Figs. 11-16 where we have taken the "measured" values at $R_h = 0.376$, 0.59 , and 0.79 , corresponding to nominal horn positions for PLT. Obviously this technique is very powerful since a quite accurate profile is obtained to $n_o/n_c = 0.4$ even with $r_h/r_o = 1.1$ and with only one step. (N. B. all the curves of Figs. 11-16 are plotted as absolute values since they are normalized to the true n_o/n_c). Improving the profile by subtraction of the refractive effect on $N_s/N_s(o)$ using the first profile, results in curves which are relatively accurate even at densities up to the value

$n_o/n_c = 0.8$ (except for $r_h/r_o = 1.1$ and $p = 2$ where the calculated density exceeds n_c in the vicinity of $r = 0$ preventing a simple scaling of the curve).

Comparison of Figs. 11-13 with 14-16 reveals that the inversion is less accurate for the same value of K as r_h/r_o increases above 1, as expected. In fact, $r_h/r_o = 1.2$ gives inversions at $K = 0.4$ with errors comparable to the $K = 0.8$ case for $r_h/r_o = 1$. Therefore it is advantageous to keep $r_h/r_o < 1.1$ if possible.

VIII. DISCUSSION

Numerical integration of a variety of density profiles has revealed that the phase shift along the refracted path relative to that through the center of the plasma $N_s/N_s(0)$ is only weakly dependent on n_o/n_c when the horns are positioned near the perimeter of the plasma. Thus, the inversion of the phase shift measurements is greatly simplified for large values of n_o/n_c (≥ 0.1) by treating the $N_s/N_s(0)$ profile and $N_s(0)$ separately as detailed above. This novel inversion technique avoids the need to employ the derivative of a curve specified by only a few measured points as is required in the Abel inversion technique, and is useful even for values of n_o/n_c approaching 1.

However, the use of microwave interferometer systems at large values of n_o/n_c requires that considerable attention be given to the horn design and orientation to assure that the phase shift is measured accurately. It is especially important to use "short"

optimized horns with wide main lobes so that the detected rays stay within the main lobe of the radiation pattern. This necessitates that the microwave power available be much larger ($\sim 10 \times$) than required for the use of conventional optimized "long" horns.

In this paper, the idealized case of geometrical optics has been assumed throughout. The extent to which this case is valid for a given microwave interferometer system also depends on the horn design and location.⁹ It should be possible, generally, to have the near field of the horn essentially unperturbed by the plasma while maintaining $r_h/r_o \leq 1.1$, thus yielding a far field pattern appropriate to the geometrical optics analysis. This criterion is, of course, easier to satisfy in the larger machines.

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Fig. 2. Ray trajectories for a parabolic density profile with $n_0 = 10^{14} \text{ cm}^{-3}$, $\lambda = 2.24 \text{ mm}$ ($K = 0.449$), and $x_F/r_0 = 1.1$.

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Fig. 3. Refraction displacement of the rays in the plasma midplane, $r_d/r_0 - r_s/r_0$ versus horn position for a parabolic density profile with $n_0 = 10^{14} \text{ cm}^{-3}$. $\lambda = 2.24 \text{ mm}$ ($K = 0.449$) and 0.337 mm ($K = 0.010$) for the microwave and HCN systems, respectively.

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Fig. 4. Plots of (a) $N_s/N_s(0)$, (b) N_d/N_s , and (c) γ versus horn position for the case of Fig. 3.

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Fig. 5. Plots of (a) $N_s/N_s(0)$, (b) N_d/N_s , and (c) γ versus horn position for $n(r) = n_0 [1 - (r/r_0)^{2p}]$ with $p = 1/2$ (linear profile) and for the $K = n_0/n_c$ values indicated. $r_h/r_0 = 1$.

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Fig. 6. Plots as in Fig. 5 for $n(r) = n_0 [1 - (r/r_0)^2]$ ($p = 1$) and $r_h/r_0 = 1$.

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Fig. 7. Plots as in Fig. 5 for $n(r) = n_0 [1 - (r/r_0)^4]$
 ($p = 2$) and $r_h/r_0 = 1$.

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Fig. 8. Plots for case of Fig. 5 with $r_h/r_0 = 1.1$.

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Fig. 9. Plots for case of Fig. 6 with $r_h/r_0 = 1.1$.

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Fig. 10. Plots for case of Fig. 7 with $r_h/r_0 = 1.1$.

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Fig. 11. Inversions of Fig. 5(a) ($p = 1/2$, $r_h/r_0 = 1$) using Eq. (25). (a) Direct use of $N_s/N_s(o)$ for $N_s/N_s(o) \big|_{n_o \rightarrow 0}$. (b) After one subtraction of refraction effect. (N.B. n_o is the true value of density on axis.) Dashed curve is the true profile.

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Fig. 12. Inversions as in Fig. 11 for Fig. 6(a) ($p = 1$, $r_h/r_0 = 1$).

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Fig. 13. Inversions for Fig. 7(a) ($p = 2$, $r_h/r_0 = 1$).

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Fig. 14. Inversions for Fig. 8(a) ($p = 1/2$, $r_h/r_o = 1.1$).

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Fig. 15. Inversions for Fig. 9(a) ($p = 1$, $r_h/r_o = 1.1$).

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Fig. 16. Inversions for Fig. 10(a) ($p = 2$, $r_h/r_o = 1.1$).