MICROINSTABILITIES IN COMPLEX
MAGNETIC FIELD GEOMETRIES
AND
HIGH- β SHEARED SHEATH STRUCTURE

Progress Report
for Period June 1, 1975 - February 27, 1976

Pradip Bakshi and Gabor Kalman

Boston College
Chestnut Hill, Massachusetts 02167

MASTER

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

February 1976

Prepared For

THE U.S. ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
UNDER CONTRACT NO. E(11-1)-2714

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

Ref

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product or process disclosed or represents that its use would not infringe privately owned rights.

ABSTRACT

A new approach for the solution of the Vlasov equation for complex magnetic field geometries has been developed using operator techniques. The general approach is illustrated by determining the perturbed distribution function and density operator for the problem of shear stabilization of drift waves for transverse and arbitrary directions of propagation. The ensuing corrections to stability criteria of current theories are obtained for certain domains of physical parameters. Preliminary work on the integral equation approach to the dispersion relation has been initiated. As a prelude to the study of particle orbits in complex mirror geometries, the adiabatic and non-adiabatic behaviour of a harmonic oscillator has been studied using operator methods.

High- β , high shear plasma sheath configurations have been studied with the full ion dynamics taken into account and electrons treated in the zero and first order approximation, in the ratio of the electron Larmor radius to the scale length. The resulting sheath structure equation in the lowest order approximation has been solved for certain entering ion distributions, and prepared for computer analysis for others. In this approximation the electron current parallel to magnetic field lines has to be assumed suppressed or predetermined. Equations in the next order approximation include the finite Larmor radius stress tensor. This equation is under study.

INTRODUCTORY REMARKS

This progress report consists of two parts. The studies on "Microinstabilities in Complex Magnetic Field Geometries" are described in Part I by Dr. Pradip Bakshi. The studies on "High- β Sheared Sheath Structure" are described in Part II by Dr. Gabor Kalman. These studies are in compliance with the originally stipulated contract requirements.

Dr. Bakshi has so far devoted approximately 69% of his stipulated time on this contract. The remaining 31% of his stipulated time will be devoted during the remainder of the current term. Dr. Kalman has already devoted 100% of his stipulated time on this contract.

PART I

MICROINSTABILITIES IN COMPLEX MAGNETIC
FIELD GEOMETRIES

Pradip Bakshi

Microinstabilities in Complex Magnetic Field Geometries

The long range objective of this program is to develop a new formalism for the study of microinstabilities in various inhomogeneous magnetic field geometries, involving (possibly strong) variations in density and other plasma parameters. The formalism should be capable of describing not only the slow variation situations such as the tokamaks and mirror machines, but also the sharp variation problems such as the tormac sheath or the 2XIIB radial variation problem.

The first step in this direction is the development of a new approach for the solution of the Vlasov equation for complex magnetic field geometries and inhomogeneities, using operator inversion and other operator techniques based on similarity transforms, Lie algebras, Zassenhaus expansions etc. This approach has been illustrated for the problem of shear stabilization of drift waves¹, by providing an accurate determination of the perturbed distribution function and density operator in terms of a proper time integral representation. The ensuing operator dispersion equation is essentially an integral equation for the perturbed potential function, and appropriate methods have yet to be developed for its accurate treatment. But even when we approximate the full operator dispersion equation as a second order differential equation, it is evident that our more general solution of the Vlasov problem leads to new results; the earlier studies^{2,3} turn out to be special limiting cases of this approach.

The scope of the investigations during the report period encompasses two broad areas. (1) The problem of shear stabilization of drift waves¹, which serves to illustrate the general approach of the formalism, has been further analyzed and: (a) the perturbed distribution function for an arbitrary direction of propagation has been determined and analytical approximations have been considered to obtain the perturbed density operator; (b) the 'first order' corrections to the current theories^{2,3} for transverse propagation have been obtained analytically; (c) an analysis of the corrections to all orders in certain physical parameters has been initiated and (d) some preliminary work on the integral equation approach has been initiated. (2) The problem above directs attention to the determination of a dispersion relation, and is consequently beset with various additional approximations. The efficacy of the solution of the Vlasov problem for involved field geometries can, however, be more directly assessed by trying to determine the orbits in such fields by the use of the operator methods. As an initial study, (a) the problem of adiabatic and non-adiabatic behavior in mirror geometries seems a natural choice, and a simpler related problem is (b) a harmonic oscillator with variable frequency. The latter problem has been analyzed, using operator methods, and a number of interesting results have been obtained.

The main results of these investigations are described below:

1. Shear Stabilization of Drift Waves.

(a) Arbitrary Direction of Propagation.

The perturbed distribution function f_1 and the perturbed density operator n_1 have been derived in our earlier work (Ref. 1, eqs. (11) and (12)) for the case of nearly transverse propagation. Now, the general expression for f_1 , for arbitrary direction of propagation, has been obtained. There are two differences from the case of transverse propagation: (i) the displacement in x direction is now associated with a rotation in k -space rather than a displacement and (ii) the weight factors inside the integral are complicated functions of the velocities. The latter result does not allow an analytic integration over all the velocities and thus various approximations have been considered (according to various domains of the physical parameters) in order to obtain an expression for the perturbed density operator.

(b) Transverse Propagation: First Order Corrections in p .

The parameter $p = (ps)(\rho k)$, where ρ = Larmor radius, s = inverse shear length and k = transverse wave number, plays a significant role in our approach. If we set $p = 0$, our results reduce to those of the current theories^{2,3}. Thus $p \neq 0$ effects provide the departures from current theories. Since the time integration for $p \neq 0$ in Eq. (12), Ref. 1 is not analytically feasible, we have first carried out an expansion to first order in p and obtained analytically the lowest order corrections to the dispersion equations and stability criteria of the current theory³.

(c) Transverse Propagation: p small, q not small.

The results in (b) above are not uniformly valid for all ranges of physical parameters. When the cyclotron frequency $\Omega \gg \omega$, the wave frequency, a new parameter $q = p(\Omega/\omega) = (s/\epsilon)(\omega^*/\omega)$ becomes significant. Here ϵ = inverse length scale of density variation and ω^* = drift velocity. Even if p is quite small, significant contributions arise from the long-time segment of the proper time integral in Eq. (12), Ref. 1 and the departures from the $p = 0$ theory are governed by q rather than p . These effects can be significant even for low shear or long waves. A careful analysis and computation of these effects is being carried out, and their relevance for various fusion devices will be estimated.

(d) The Integral Equation for Φ .

By using an auxilliary integral representation for the term containing the differential (D) operator in Eq. (12), Ref. 1, it is possible to recast the expression for n_1 , the perturbed density into a double integral representation, involving a shift of the arguments of the potential Φ . It is easy to see that for the special case of no shear ($s=0$), the resultant dispersion equation reduces to the usual algebraic form. This is brought about by the appearance of a delta function in performing one of the integrals, which facilitates the second integration, and provides an algebraic dispersion equation involving k_x , the wave number along the x-direction. When $s \neq 0$, k_x is not well defined, and the integral representations allow one to see that

the sharp k_x for the case of no shear is now 'smeared' a little, into a 'wave-packet' and one can estimate the width of this spread in terms of the physical parameters of the system without detailed calculations. Now one can see that the interference of the smeared k_x -modes is responsible for the standing wave normal modes of the problem with shear. If the x -variation of the potential is slow, the usual second order differential equation approximation to the problem arises naturally through a Taylor-expansion. However, the "wave-packet" picture that emerges from the integral equation is generally valid, and provides the proper approach for handling problems with sharp variations. Such an extension is quite necessary for torus and other such geometries. Further investigation of this problem will continue.

In summarizing the work on this problem we note that the most dramatic departures from current theories would occur for short wave lengths, $pk \gtrsim 1$, or $pk \gg 1$. We made an attempt to compare our results for $pk \sim 1$ with those obtained at Princeton by using a particle simulation code for corresponding physical parameters. Unfortunately, for short wave lengths, the non-linear effects set in at too small an amplitude and it was not possible to assess the linearized growth rates from the Princeton-simulation data. A Vlasov code will soon be available at Princeton and a direct comparison of our analytical results with simulation results may now become feasible. We plan to maintain a close contact with the simulation efforts.

2. Orbits and Operator Methods.

(a) Adiabatic and Non-Adiabatic Behavior of a Harmonic Oscillator.

The problem

$$\ddot{x} = -\omega^2(t)x$$

has received considerable attention⁴ over the past twenty years. We have developed a new approach⁵, using transformations and operator methods which allow a separation of 'adiabatic' and 'non-adiabatic' effects through various orders of successive approximations. (i) For a slowly varying $\omega^2(t)$, with $\omega^2 \rightarrow$ constants as $t \rightarrow \pm \infty$, we have developed a successive transformation scheme for the determination of the phase angle. The leading term in the non-adiabatic change of action can also be determined (ii) With $\omega^2(t) = \omega_0^2 + A \cos \alpha t$, we treat the familiar Mathieu equation in this approach and show that the instability zones are a consequence of the non-adiabatic effects. An accurate determination of the eigen values becomes possible.

(b) Orbits in Mirror Fields.

The first model in (a) provides a simulation of the transition from one constant magnetic field region to another while the second model (the Mathieu or more generally, a Hill type-equation) provides a simulation of the combined effects of Larmor and bounce motions.

A program for the application of the operator methods for the determination of orbits in mirror fields has been initiated. The results of this program would be useful in assessing the role of non-adiabaticity in devices like 2XIIB, with large radial gradients. This effort will be continued, first using simplified mirror-fields and then incorporating various realistic modifications.

The remainder of this term will be primarily devoted to the preparation of a journal article each on topic (1) and topic (2).

References

1. P. Bakshi and W. Bellew, "Shear Stabilization of Drift Waves: A New Approach," AFCRL-TR-74-0619, (1974).
2. W. E. Drummond and K. W. Gentle, Phys. Fluids 10, 2020, (1967).
3. L. D. Pearlstein and H. L. Berk, Phys. Rev. Letters 23, 220, (1969).
4. R. M. Kulsrud, Phys. Rev. 106, 205, (1957)
 A. Lenard, Ann. Phys. 6, 261, (1959)
 C. S. Gardner, Phys. Rev. 115, 791, (1959)
 G. Backus, A. Lenard and R. Kulsrud, Z. Naturforsch 15a, 1007, (1960)
 P. Vandervoort, Ann. Phys. 12, 436, (1961)
 H. R. Lewis, Jr., Phys. Rev. Letters 18, 510, (1967)
 H. R. Lewis, Jr., J. Math. Phys. 9, 1976, (1968)
 J. Howard, Phys. Fluids 13, 2407, (1970).
 and several other articles referred to in these publications.
5. P. Bakshi, Bull. Amer. Phys. Soc. 20, 1385, (1975)
 (abstract reprinted on next page).

Abstract Submitted
for the Seventeenth Annual Meeting of the
Division of Plasma Physics
American Physical Society
St. Petersburg, Florida
November 10-14, 1975

Physical Review
Analytic Subject Index
Number 35

Bulletin Subject Heading
in which Paper should be placed
36. General Theory

Adiabatic and Non-Adiabatic Behavior of a
Harmonic Oscillator.* PRADIP BAKSHI, Princeton Uni-
versity and Boston College -- An operator formalism
is developed which allows an explicit calculation
of the adiabatic and non-adiabatic effects for an
oscillator with a slowly varying time dependent
frequency. As applications, the zones of instabili-
ty in the Mathieu, and more generally the Hill equa-
tion, are shown to be a consequence of the non-adia-
batic effects. The relevance of this approach for
the determination of particle orbits in mirror-
fields and implications for long time confinement
are discussed.

* Work supported by U.S.E.R.D.A.

Printed in:
Bull. Amer. Phys. Soc. 20, 1385 (1975)

PART II

HIGH- β SHEARED SHEATH STRUCTURE

Gabor Kalman

High- β Sheared Sheath Structure

We have studied high- β high-shear sheath structures in approximations schemes of various sophistications. This Report gives a summary of the methods and procedures employed and a preliminary description of the results obtained.

The main feature of the approach employed is the full Vlasov description used for the ions, which is necessary since one is considering scale length L such that $L \simeq r_i$ (ion Larmor-radius), while a fluid or hybrid-fluid description is employed for the electrons for which $r_e \ll L$. The latter condition permits an expansion of the electron Vlasov equation in the small parameter $\epsilon = \frac{r_e}{L}$. To lowest order in the parameter this leads to a model akin to Freidberg's¹ Vlasov-fluid model, while to higher order it is similar to the hybrid-kinetic model recently proposed by D'Ippolito and Davidson².

In our earlier work³ we studied various highly simplified models which brought out the main physical features of the electron and ion dynamics. Continuing along that line it became obvious that the two most important questions are (1) the effect of the ion distribution function; and (2) the coupling of electron currents parallel and perpendicular to the magnetic field.

The coupling of the ion dynamics with the field configuration has been followed by introducing a "pseudopotential"^{3,4} $\Psi(u,v)$ where u and v are the ion velocity components in the sheath-plane, and operating in the $(u,v; \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y})$ phase plane (y being

the direction perpendicular to the sheath). This method reduces the problem to that of a representative particle moving under the influence of the force derived from the pseudopotential. The distribution of the parallel electron-current on the different field lines in the Vlasov-fluid model is arbitrary for an infinite sheath. Such a model is appropriate for closed-line high- β configurations (θ -pinch or the recently proposed Surmac⁵). In open-line configurations (Tormac) it is more appropriate to assume that the sheath is finite and no parallel current flow is possible. In the actual physical situation the current flow is not arbitrary because of the finite Larmor radius viscosity effect. This is properly taken into account in the hybrid-kinetic model.

We considered $j_{||} = 0$ cases with various ion distributions. The simplest case which contains most of the physical effects associated with more general ion-distributions, arises when the ions enter the sheath almost perpendicularly. The resulting mathematical problem is the solution of two coupled second order differential equations of the type

$$u'' = F(u, v; u'/v')$$

$$v'' = G(u, v; u'/v')$$

For certain cases this equation has been solved analytically in terms of elliptic functions; other situations have been studied without the explicit solution of the differential equation in terms of phase space analysis. In this case explicit integrals for global variables (sheath, width, etc.) have been

derived; the formulation of the results for computer evaluation is in progress.

For a moderate distribution of the entrance angle of the ions around the perpendicular direction a perturbation method has been worked out, where one starts with the sheath structure determined by perpendicular entry and calculates the modifications due to the angular distribution. This effect turns out to be not substantial, (in contrast to the distribution along the perpendicular direction, which fundamentally affects the sheath structure).

For the case of semi-Maxwellian entrance angle distribution (i.e. for Maxwellian distribution for $w > 0$, where w is the velocity direction perpendicular to the sheath) the method of using particle dynamical equations for the ions did not turn out to be very convenient. Instead, one can use the Vlasov equation directly. The analytic solution of the resulting equation does not seem to be feasible although qualitative conclusions can be arrived at. The appropriate formulation of the equation for computer solution is in progress.

We have also studied the previous cases without the quoted restriction on the parallel electron current and the simple sheath structure solutions have been studied as a function of the electron current. As expected, a prescribed electron current can easily mask the ion effects, and the details of the ion distribution function become insignificant. On the other hand there are physical limitations on the parallel electron current coming from sources other than referred to

(stability being one of them) which should be taken into account in determining the physical significance of such solutions.

As we have already pointed out, the unphysical arbitrariness of the parallel electron current is the result of the lowest order expansion in r_e/L . Similar arbitrariness results in many other equilibrium calculations based on similar models. We have made further progress by formulating the sheath-structure equations in the hybrid-kinetic formalism of D'Ippolito and Davidson.² These equations now contain the Roberts-Taylor stress tensor, appropriately coupling parallel and perpendicular electron currents. Other effects of the same order in ϵ appear also in the equations: these are first order fluid drifts and heat flow contributions to the electron pressure tensor. A perturbation solution of the equations in the same smallness parameter ϵ is now possible around a given $\epsilon^0(j_{||})$ configuration. We expect that only certain $j_{||}$ domains lead to a convergent iteration indicating the physically possible $j_{||}$ values.

The study of the high- β high shear sheath structure has originally been motivated by the problems of the Tormac geometry. The results, however, are of interest to all sharp boundary, high- β situations, and the method of pseudopotentials applies to other high or moderate β equilibria (including toroidal ones) where the ordering $r_i/L \simeq 1$, $r_e/L \ll 1$ is appropriate.

The results reported here are being compiled for journal publication.

References

1. J. P. Freidberg, Phys. Fluids 15, 1102, (1972).
2. D. A. D'Ippolito and R. C. Davidson, Phys. Fluids 18, 1494, (1975).
3. G. Kalman, "Studies on the Sheath Structure in the TORMAC Geometry." AFCRL-TR-74-0620, (1974).
4. E. N. Parker, J. Geophys. Res. 72, 2315, (1967).
5. A. Y. Wong, Y. Nakamura, B. H. Quon, J. R. Dawson, Phys. Rev. Letters 35, 1156, (1975).