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A STATISTICAL APPROACH FOR ESTIMATING THE RELIABILITY OF HIGHLY STRESSED XENON FLASHLAMPS

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MASTER

A STATISTICAL APPROACH FOR ESTIMATING THE RELIABILITY OF HIGHLY STRESSED XENON FLASHLAMPS

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ABSTRACT

Shiva, a neodymium doped glass laser system being built for the Lawrence Livermore Laboratory's Laser Fusion Program, will play an important role in the effort to harness thermonuclear energy. The system will irradiate a microscopic deuterium-tritium pellet with 20 laser beams, all arriving simultaneously along 20 different axes. It will contain 2,200 flashlamps.

This paper describes a statistical modeling technique used to predict failure rates for highly stressed xenon flashlamps used to pump neodymium glass lasers. Because the lamps are mounted in close proximity to the glass laser disks, an exploding lamp can result in costly damage to optical components. The failure data presented is representative of the early period of large bore lamp development. Recent improvements in manufacturing techniques and the implementation of a pulse ionized lamp check (P.I.L.C.) have reduced lamp failure significantly. However, early failure data was used to develop a mathematical method for estimating the reliability of flashlamp units representative of any given design set. Specifically, we estimated the probability that none of these lamps will fail in the course of a given number of shots (say 50, 100, or 200), provided all lamps have been tested for a given number of shots (say 100 or 200). Estimating lamp reliability is complicated by two facts: first, the available data are a mixture of flashlamps which failed (after a certain

number of shots) and which did not fail (after some fixed number of shots); and second, the usual models for failure rate do not hold. Therefore, other techniques were developed. The problem is to determine the failure rate of an individual flashlamp. Once this is done, the overall reliability can be assessed. The likelihood function is used to select the "best" model for the failure rate of a single flashlamp.

The methods described in this paper can be applied to other systems with mixed data and for which the usual failure rate models are not applicable.

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1. INTRODUCTION

Shiva consists of 2,200 flashlamp units. We are interested in evaluating the reliability of such units; more specifically, we want to estimate the probability that none of the units will fail if they are to undergo a certain number of shots (say 50, 100 or 200) provided that all these units have been tested for a given number of shots.

Section 2 describes the data used to develop the method. In Section 3 suitable models for the failure rate of an individual unit are considered and in Section 4 the parameters for these models are computed. In Section 5 the best model is selected and reliability estimates are made using the limited data available. Finally, Section 6 summarizes the steps that were taken to evaluate this estimate.

Two appendices with some technical considerations needed for Section 4 are included. If more data were available then the results of Appendix A could also be used directly to calculate the failure rate of the units.

2. DESCRIPTION OF THE DATA

The original data consisted of 136 units which were assigned to undergo a number of shots, namely: 80 units were assigned to be tested for 1,000 shots,
40 units were assigned to be tested for 10,000 shots and
16 units were assigned to be tested for 13,000 shots.

When a unit — supposed to undergo, say, N shots — failed after, say, S shots, it was replaced and the new unit was tested for N-S shots. None of the units, replacing failed units, happen to fail. Since there were 6 failures, the total number of units that were tested was 142 (i.e. 136 + 6). Let n denote the number of units tested and let m denote the number of failures. Let T_1, . . ., T_m be the *failure times* i.e. the shot numbers that caused failing of a unit and let T_{m+1}, . . ., T_n be the *stopping times* i.e. the number of shots that a non failing unit was supposed to undergo. With this notation the data can be listed as:

$T_1 = 11$, $T_2 = 35$, $T_3 = 55$, $T_4 = 856$, $T_5 = 943$, $T_6 = 3{,}994$

(i.e. these are the failing times) and

$T_7 = 12{,}989$, $T_8 = 965$, $T_9 = 945$, $T_{10} = 9{,}114$, $T_{11} = 9{,}057$, $T_{12} = 9{,}006$,

$T_{13} = \ldots = T_{90} = 1{,}000$, $T_{91} = \ldots T_{128} = 10{,}000$, $T_{129} = \ldots = T_{142} = 13{,}000$.

The cluster of the three early failures clearly indicates that they are of a different nature than the failures, say, after a 100 shots. Therefore, it is not desirable to use the data before a 100 shots since they indicate some malfunction of the units and a different process is therefore occurring. Besides that, all units will be tested for at least a 100 shots before being put into operation. Thus, the relevant data in computing the reliability of Shiva will be all of the previous data minus the first three failures. Let

$t = T - 100$, then the data in terms of t is

$t_1 = 756$, $t_2 = 843$, $t_3 = 3{,}894$

(i.e. m = 3 failures) and

$t_4 = 12{,}889$, $t_5 = 865$, $t_6 = 845$, $t_7 = 9{,}014$, t_8, $8{,}957$, $t_9 = 8{,}906$,

$t_{10} = \ldots = t_{87} = 900$, $t_{88} = \ldots = t_{125} = 9{,}900$, $t_{126} = \ldots = t_{139} = 12{,}900$.

3. A MODEL FOR THE FAILURE RATE OF A UNIT

As pointed out in the introduction, we are interested in evaluating P(S,T), the probability of no failure on the first T shots given that the units have been tested for S shots. Let G(T) be the probability that a unit will have a life time of T or longer, then G(S+T)/G(S) is the probability that a unit has no failures on the first T shots given that it has been tested for S shots and hence

$$P(S,T) = \left(\frac{G(S+T)}{G(S)}\right)^{2,200} . \qquad (1)$$

Let $\lambda(T)$ be the failure rate of a unit, i.e.

$$\lambda(T) = \lim_{\Delta T \to 0} \left\{ \frac{1}{\Delta T} \cdot \text{Probability that the unit fails between } T \text{ and } T+\Delta T \text{ given that it survives up to time } T \right\}$$

$$= \lim_{\Delta T \to 0} \frac{G(T) - G(T+\Delta T)}{G(T)} = - \frac{G'(T)}{G(T)} .$$

From this relation one can easily derive that

$$G(T) = e^{-\int_0^T \lambda(u)du} . \qquad (2)$$

In order to construct a model for the failure rate it is necessary to have some more knowledge about this rate. From the physical properties of the unit it has been established that the failure rate curve looks like a "bath-tub" (see the Figure below) i.e. the failure rate is very high for

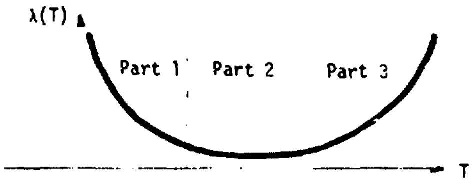

Figure 1.

Failure rate for one unit

small values of T, it then decreases, having a flat part (i.e. a part where the unit operates well) and finally it increases again as the lifetime of the unit is reached. The data belongs to part 1, therefore it is reasonable to use as a model a function such as $\lambda(T) = \lambda/(1+\alpha T)$ or $\lambda(T) = \lambda/(1+\alpha T)^{(1+\beta)}$ where λ, α and β are parameters to be determined from the data. If more data were available, one could also compute $\tilde{G}(T)$, the empirical probability that a unit will have a lifetime of T or longer. From $\tilde{G}(T)$ an estimate of $\lambda(T)$, say, $\tilde{\lambda}(T)$, can be obtained, or if necessary, a smoother version of it, say, $\tilde{\tilde{\lambda}}(T)$. As mentioned in Section 2, we shall work with the translated data (i.e. the t-values). In this new system let $\gamma(t)$ be the failure rate, i.e. $\lambda(T) = \gamma(T-100)$, and let $H(t)$ be the probability that a unit will have a lifetime of t or longer. Note that

$$\frac{G(S+T)}{G(S)} = e^{-\int_S^{S+T} \lambda(u)du} = e^{-\int_{S-100}^{S+T-100} \gamma(u)du} = \frac{H(S+T-100)}{H(S-100)} .$$

In Appendix A we illustrate how $\tilde{H}(t)$, $\tilde{\gamma}(t)$ and $\tilde{\tilde{\gamma}}(t)$ can be obtained (i.e. an empirical estimate of $H(t)$, an estimate of $\gamma(t)$ and a smoothed estimate of $\gamma(t)$ respectively). Though there are not enough data available to explicitly use these estimates, they will be helpful in Appendix B for establishing initial guesses for the parameters λ, α and β.

4. ESTIMATION OF THE PARAMETERS APPEARING IN THE FAILURE RATE

The maximum likelihood method is used to estimate the parameters. Let $L(t_1, \ldots, t_m, t_{m+1}, \ldots, t_n)$ be the likelihood function, then we have

$$L(t_1, \ldots, t_m, t_{m+1}, \ldots, t_n)$$

$$\propto \prod_{i=1}^{m} H(t_i)\, \gamma(t_i) \prod_{i=m+1}^{n} H(t_i) = \prod_{i=1}^{n} H(t_i) \prod_{i=1}^{m} \gamma(t_i)$$

Hence

$$\ell = \ell n\, L \propto - \sum_{i=1}^{n} \Gamma(t_i) + \sum_{i=1}^{m} \ell n\, \gamma(t_i)$$

where

$$\Gamma(t) = \int_0^t du\, \gamma(u).$$

We first consider the two-parameter model i.e. $\lambda(t) = \lambda/(1+\alpha t)$. For this model $\Gamma(t) = (\lambda/\alpha)\, \ell n(1+\alpha t)$ and thus

$$\ell \propto - \frac{\lambda}{\alpha} \sum_{i=1}^{n} \ell n(1+\alpha t_i) + m\, \ell n\, \lambda - \sum_{i=1}^{m} \ell n(1+\alpha t_i).$$

The maximum likelihood solution, say, $\hat{\lambda}$ and $\hat{\alpha}$ of the parameters λ and α respectively is obtained by solving the system of equations $\frac{\partial \ell}{\partial \lambda} = 0$ and $\frac{\partial \ell}{\partial \alpha} = 0$. The equation $\frac{\partial \ell}{\partial \lambda} = 0$ is equivalent to

$$\frac{m}{\lambda} = \frac{1}{\alpha} \sum_{i=1}^{m} \ell n(1+\alpha t_i). \tag{3}$$

Using $\frac{\partial \ell}{\partial \alpha} = 0$ and (3) we get

$$\frac{m}{\alpha} - \frac{m}{\sum_{i=1}^{n} \ell n(1+\alpha t_i)} \sum_{i=1}^{n} \frac{t_i}{1+\alpha t_i} - \sum_{i=1}^{m} \frac{t_i}{1+\alpha t_i} = 0\,. \tag{4}$$

Once $\hat{\alpha}$ is obtained by solving (4), we can immediately derive $\hat{\lambda}$ from (3). Because of the complexity of the equation (4), it is necessary to obtain a good initial guess for $\hat{\alpha}$, say $\hat{\alpha}_0$. The initial guess for $\hat{\alpha}_0$ is given by (B3) in Appendix B. Using the 139 data points we obtain

$$\hat{\lambda} = 1.354 \times 10^{-5} \text{ and } \hat{\alpha} = 7.504 \times 10^{-4}.$$

If we use the three parameter model i.e. $\lambda(t) = \lambda/(1+\alpha t)^{(1+\beta)}$ we get $\Gamma(t) = (\lambda/\alpha\beta)\ (1 - 1/(1+\alpha t)^{\beta})$ and

$$\ell \sim -\frac{\lambda}{\alpha\beta} \sum_{i=1}^{n} \left(1 - \frac{1}{(1+\alpha t_i)^{\beta}}\right) + m \ln \lambda - (1+\beta) \sum_{i=1}^{m} (1+\alpha t_i).$$

Let $\hat{\hat{\lambda}}$, $\hat{\hat{\alpha}}$, $\hat{\hat{\beta}}$ be the solution of the likelihood equations $\frac{\partial \ell}{\partial \lambda} = 0$, $\frac{\partial \ell}{\partial \alpha} = 0$ and $\frac{\partial \ell}{\partial \beta} = 0$. The equation $\frac{\partial \ell}{\partial \lambda} = 0$ can be written as

$$\frac{m}{\lambda} = \frac{1}{\alpha\beta} \sum_{i=1}^{n} \left(1 - \frac{1}{(1+\alpha t_i)^{\beta}}\right). \tag{5}$$

Using $\frac{\partial \ell}{\partial \alpha} = 0$ and (5) we get

$$\frac{m}{\alpha} - m\beta \frac{\sum_{i=1}^{n} t_i (1+\alpha t_i)^{-(1+\beta)}}{\sum_{i=1}^{n} t_i [1-(1+\alpha t_i)^{-\beta}]} - (1+\beta) \sum_{i=1}^{m} \frac{t_i}{1+\alpha t_i} = 0. \tag{6}$$

Finally using $\frac{\partial \ell}{\partial \beta} = 0$ and (5) we get

$$\frac{m}{\beta} - m \frac{\sum_{i=1}^{n} (1+\alpha t_i)^{-\beta} \ln(1+\alpha t_i)}{\sum_{i=1}^{n} [1-(1+\alpha t_i)^{-\beta}]} - \sum_{i=1}^{m} \ln(1+\alpha t_i) = 0. \tag{7}$$

In order to solve the system of equations (6) and (7) in α and β, initial guesses for α and β are needed, say $\hat{\hat{\alpha}}_0$ and $\hat{\hat{\beta}}_0$. We use for $\hat{\hat{\alpha}}_0 = \hat{\alpha}$ and for $\hat{\hat{\beta}}_0$ the expression (B4) of Appendix B. Once $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$ are obtained, one can directly compute $\hat{\hat{\lambda}}$ from (5). We obtained

$$\hat{\hat{\lambda}} = 1.354 \times 10^{-5},\ \hat{\hat{\alpha}} = 7.501 \times 10^{-4} \text{ and } \hat{\hat{\beta}} = -2.783 \times 10^{-7}.$$

5. CHOOSING THE BEST MODEL AND ESTIMATING THE RELIABILITY

Let $L(\lambda,\alpha)$ and $L(\lambda,\alpha,\beta)$ denote the likelihood functions when the two-parameter model (i.e. $\gamma(t) = \lambda/(1+\alpha t)$) and the three-parameter model (i.e. $\gamma(t) = \lambda/(1+\alpha t)^{(1+\beta)}$) are used respectively. Then the three-parameter model will be superior to the two-parameter model if

$$\frac{L(\hat{\hat{\lambda}},\hat{\hat{\alpha}},\hat{\hat{\beta}})}{L(\hat{\lambda},\hat{\alpha})} >> 1.$$

From the data we obtain that $L(\hat{\hat{\lambda}},\hat{\hat{\alpha}},\hat{\hat{\beta}}) \simeq L(\hat{\lambda},\hat{\alpha})$ and therefore the three-parameter model is no improvement over the two-parameter model. Therefore, the latter model is adequate.

For the model $\gamma(t) = \lambda/(1+\alpha t)$ we have that $H(t) = 1/(1+\alpha t)^{\frac{\lambda}{\alpha}}$ and therefore

$$\frac{G(S+T)}{G(S)} = \frac{H(S+T-100)}{H(S-100)} = \left(\frac{1+\alpha(S-100)}{1+\alpha(S+T-100)}\right)^{\frac{\lambda}{\alpha}} .$$

Thus

$$P(S,T) = \left(\frac{1+\alpha(S-100)}{1+\alpha(S+T-100)}\right)^{2,200\,\frac{\lambda}{\alpha}}$$

and an estimate of this probability is given by

$$\tilde{P}(S,T) = \left(\frac{1+\hat{\alpha}(S-100)}{1+\hat{\alpha}(S+T-100)}\right)^{2,200\,\frac{\hat{\lambda}}{\hat{\alpha}}}$$

The numerical results of $\tilde{P}(S,T)$ for various values of S and T are given in Table 1.

T =	50.00	
	S	$\tilde{P}(S,T)$
	100.00	2.317E-01
	500.00	3.232E-01
	1000.00	4.150E-01
	2000.00	5.436E-01
	3000.00	6.273E-01
T =	100.00	
	S	$\tilde{P}(S,T)$
	100.00	5.654E-02
	500.00	1.077E-01
	1000.00	1.755E-01
	2000.00	2.983E-01
	3000.00	3.956E-01
T =	200.00	
	S	$\tilde{P}(S,T)$
	100.00	3.881E-03
	500.00	1.307E-02
	1000.00	3.315E-02
	2000.00	9.219E-02
	3000.00	1.599E-01

Table 1.

$\tilde{P}(S,T)$ for S = 100, 500, 1,000, 2,000, 3,000
and T = 50, 100, 200.

Note that P(S,T) is for fixed S a decreasing function of T and for fixed T it is an increasing function of S. The figure below is self-explanatory.

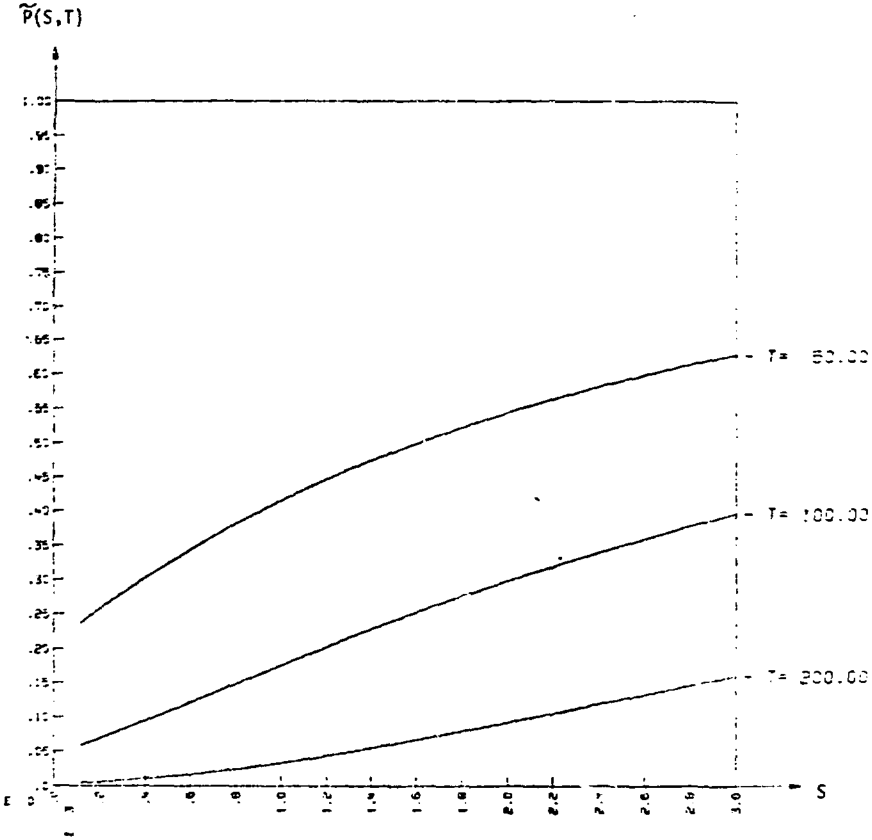

Figure 2.

$\widetilde{P}(S,T)$: The estimated probability that all the units in Shiva will not fail for T shots given that they have been tested for S shots.

6. SUMMARY

This technical memorandum evaluated $P(S,T)$, the probability of no failure on the first T shots of flashlamp units given that the units were tested for S shots. Table 1 gives values of $\widetilde{P}(S,T)$, the estimate of $P(S,T)$, for various values of S and T. Figure 2 illustrates the behavior of $\widetilde{P}(S,T)$ as a function of S. Since

(i) Shiva consists of 2,200 flashlamp units,

(ii) $P(S,T) = (G(S+T)/G(S))^{2,200}$ where $G(T)$ is the probability that a unit will have a life of T or longer,

(iii) $G(T) = \exp\left\{- \int_0^T \lambda(u)\, du\right\}$ where $\lambda(u)$ denotes the failure rate of a unit as a function of u,

it suffices to estimate $\lambda(u)$.

Because of the nature of the data it was necessary to work with the failure rate γ where $\lambda(T) = \gamma(T-100)$. It was shown that $\gamma(t) = \lambda/(1+\alpha t)$ was a suitable model for the failure rate. Given this model, $P(S,T)$ can be expressed as

$P(S,T) = [(1+\alpha(S-100))/(1+\alpha(S+T-100))]^{2,200 \frac{\lambda}{\alpha}}$ and $\widetilde{P}(S,T)$ is the value of $P(S,T)$ obtained by replacing λ and α by their estimates.

The technique used in this memorandum is fairly general, it can be employed to evaluate the reliability of other devices.

No inferences should be drawn regarding the reliability of finalized flashlamp designs, full flashlamp circuits, or of the Shiva laser system itself. The method described in this paper was developed using preliminary test data which have since been superseded by more extensive data on much more reliable flashlamps of more recent design and in improved circuits.

APPENDIX A

Computations of the Empirical Probability that a Unit will have a Lifetime of t or Longer and Computation of the Estimated Failure Rate

In computing $\tilde{H}(t)$, it will be helpful to consider first the case where there are no stopping times (i.e. the data consists exclusively of the times $t_1, \ldots, t_m$. Since these times are ordered we have

$$\tilde{H}(t_i) = \frac{m-i+1}{m+1}\ , \ i=1, \ldots, m,$$

and

$$\tilde{H}(0) = H(0) = 1.$$

For convenience, let $t_0 = 0$. The following table illustrates in detail how $\tilde{H}(t)$ is estimated.

k	$\tilde{H}(t_k)$	Number of Units Being Tested Just Before Shot # t_{k+1}	Mass to be Removed
k=0	1	m	$1 \times \frac{1}{m+1} = \frac{1}{m+1}$
k=1	$1 - \frac{1}{m+1} = \frac{m}{m+1}$	m-1	$\frac{m}{m+1} \times \frac{1}{(m-1)+1} = \frac{1}{m+1}$
k=2	$\frac{m}{m+1} - \frac{1}{m+1} = \frac{m-1}{m+1}$	m-2	$\frac{m-1}{m+1} \times \frac{1}{(m-2)+1} = \frac{1}{m+1}$
. . .	. . .	. . .	
k=i	$\frac{m-(i-2)}{m+1} - \frac{1}{m+1} = \frac{m-i+1}{m+1}$	m-i	$\frac{m-i+1}{m+1} \times \frac{1}{(m-i)+1} = \frac{1}{m+1}$
. . .	. . .	. . .	
k=m-1	$\frac{3}{m+1} - \frac{1}{m+1} = \frac{2}{m+1}$	1	$\frac{2}{m+1} \times \frac{1}{1+1} = \frac{1}{m+1}$
k=m	$\frac{2}{m+1} - \frac{1}{m+1} = \frac{1}{m+1}$		

Table A1.
Detailed evaluation of $\tilde{H}(t)$ when all units are tested until failure.

We now consider the case of interest i.e. m ordered failure times and n-m stopping times. Let n_i be the number of units being tested just before shot # t_k (i.e. n_i=n-# of failures before t_i-# of units stopped before t_i). Then a Table similar to Table A1 can be constructed.

k	$\tilde{H}(t_k)$	Number of Units Being Tested Just Before Shot # t_{k+1}	Mass to be Removed
k=0	1	n_1	$1 \times \frac{1}{n_1+1} = \frac{1}{n_1+1}$
k=1	$1 - \frac{1}{n_1+1} = \frac{n_1}{n_1+1}$	n_2	$\frac{n_1}{n_1+1} \times \frac{1}{n_2+1} = \tilde{H}(t_1) \times \frac{1}{n_2+1}$
k=2	$\tilde{H}(t_1) - \tilde{H}(t_1) \times \frac{1}{n_2+1}$	n_3	$\tilde{H}(t_2) \times \frac{1}{n_3+1}$
	$= \tilde{H}(t_1)\ \frac{n_2}{n_2+1} = \tilde{H}(t_2)$		
. . .			
k=i	$\tilde{H}(t_{i-1})\ \frac{n_i}{n_i+1}$	n_{i+1}	$\tilde{H}(t_i) \times \frac{1}{n_{i+1}+1}$
. . .	. . .	. . .	. . .
k=m-1	$\tilde{H}(t_{m-2})\ \frac{n_{m-1}}{n_{m-1}+1}$	n_m	$\tilde{H}(t_{m-1}) \times \frac{1}{n_m+1}$
k=m	$\tilde{H}(t_{m-1})\ \frac{n_m}{n_{m+1}}$		

Table A2.

Detailed evaluation of $\tilde{H}(t)$ when all units are tested until failure time or stopping time is reached.

Thus the emperical $\widetilde{H}(t)$ is given by

$$\left.\begin{aligned} \widetilde{H}(t_i) &= \widetilde{H}(t_{i-1})\,\frac{n_i}{n_i+1}\,, \quad i=1,\ .\ .\ .\ .,\ m \\ \text{and} \qquad & \\ \widetilde{H}(0) &= 1. \end{aligned}\right\} \tag{A1}$$

There is a very natural interpretation for $\widetilde{H}(t)$, namely of the probability $\widetilde{H}(t_{i-1})$ remaining after t_{i-1} a fraction $\frac{1}{n_i+1}$ lies between t_{i-1} and t_i (based on the principle that the n_i failure times, if they were all observed, would divide the probability beyond t_{i-1} equally). Now we shall use $\widetilde{H}(t)$ to obtain an estimate, $\widetilde{\gamma}(t)$, of the failure rate. Let $\Delta \geq 0$, then, since

$$\frac{H(t)}{H(t+\Delta)} = e^{\int_t^{t+\Delta} \gamma(u)du},$$

we have for Δ small

$$\ell n \left(\frac{H(t)}{H(t+\Delta)}\right) \approx \gamma(t)\Delta. \tag{A2}$$

Using (A1) we have

$$\ell n \left(\frac{H(t_{i-1})}{H(t_i)}\right) = \ell n(1 + \frac{1}{n_i}) \approx \frac{1}{n_i}\,, \quad i=1,\ .\ .\ .\ m. \tag{A3}$$

Thus by combining (A2) and (A3), an estimate of $\gamma(t)$ is given by

$$\widetilde{\gamma}\left(\frac{t_{i-1}+t_i}{2}\right) \approx \frac{\left(\ell n\, \dfrac{\widetilde{H}(t_{i-1})}{\widetilde{H}(t_i)}\right)}{t_i - t_{i-1}} \approx \frac{1}{n_i(t_i - t_{i-1})}\,, \quad i=1,\ \ldots\ .,\ m. \tag{A4}$$

From the data we obtain $n_1 = 139$, $n_2 = 138$, $n_3 = 57$; the $\tilde{\gamma}\left(\frac{t_{i-1}+t_i}{2}\right)$, $i=1, \ldots, m$, are given in the table below.

$\underline{t}$	$\underline{\tilde{\gamma}(t)}$
378	9.52×10^{-6}
799.5	8.33×10^{-5}
2,368.5	5.75×10^{-6}

Table A3.

Estimated failure rate.

Table A3 indicates that more smoothing is required. Using once more (A1) we get

$$\frac{\tilde{H}(t_{i-2})}{\tilde{H}(t_i)} = \frac{(n_{i-1}+1)(n_i+1)}{n_{i-1}n_i}, \quad i=2, \ldots, m$$

and, if we call $\tilde{\tilde{\gamma}}(t)$ the smoothed failure rate: we get

$$\tilde{\tilde{\gamma}}(t_{i-1}) = \frac{\left(\ln \dfrac{\tilde{H}(t_{i-2})}{\tilde{H}(t_i)}\right)}{t_i - t_{i-2}}, \quad i=2, \ldots, m,$$

and thus

$$\tilde{\tilde{\gamma}}(t_{i-1}) \approx \frac{\left(\ln \dfrac{(n_{i-1}+1)(n_i+1)}{n_{i-1}n_i}\right)}{t_i - t_{i-2}}, \quad i=2, \ldots, m.$$

From the data the numerical values of $\tilde{\tilde{\gamma}}(t_{i-1})$, $i=2, \ldots, m$ are computed and given below.

t	$\breve{\breve{\gamma}}(t)$
756	1.65×10^{-4}
843	1.07×10^{-5}

Table A4.

Smoothed estimated failure rate.

APPENDIX B

Initial Guess $\hat{\alpha}_0$ for $\hat{\alpha}$ and Initial Guess $\hat{\hat{\beta}}_0$ for $\hat{\hat{\beta}}$

From (A4) we get

$$n_i(t_i - t_{i-1}) \approx \frac{1}{\tilde{\gamma}\left(\frac{t_{i-1}+t_i}{2}\right)}, \quad i=1, \ldots, m,$$

and hence, since $\hat{\gamma}(t) = \hat{\lambda}/(1+\hat{\alpha}t)$,

$$n_i(t_i - t_{i-1}) \approx \frac{1+\hat{\alpha}\left(\frac{t_{i-1}+t_i}{2}\right)}{\hat{\lambda}}, \quad i=1, \ldots, m. \tag{B1}$$

This can be written as

$$\langle y_i \rangle = \hat{\alpha}\, x_i, \quad i=1, \ldots, m, \tag{B2}$$

where $y_i = \hat{\lambda}\, n_i(t_i - t_{i-1}) - 1$, $x_i = (t_{i-1}+t_i)/2$ and < > is used to denote expectation. Since (B2) is a typical regression model, a reasonable guess for $\hat{\alpha}$ is

$$\hat{\alpha}_0 = \frac{\sum_{i=1}^{m} x_i y_i}{\sum_{i=1}^{m} x_i^2}$$

or, equivalently,

$$\hat{\alpha}_0 = \frac{2\left[\hat{\lambda} \sum_{i=1}^{m} n_i(t_i^2 - t_{i-1}^2) - 2\sum_{i=1}^{m} t_i + t_m\right]}{\sum_{i=1}^{m} (t_{i-1}+t_i)^2}.$$

But this guess depends on $\hat{\lambda}$. Since $\gamma(t) = \lambda/(1+\alpha t)$ and $\gamma(0)=\lambda$, a reasonable guess for $\hat{\lambda}$ is, $\hat{\gamma}_0 = \tilde{\gamma}\left(\frac{t_0+t_1}{2}\right) = 1/(n_1 t_1)$ because this is the estimated failure rate at the smallest t-value available (See A4). Therefore, we recommend as initial guess

$$\hat{\alpha}_0 = \frac{2\left[\frac{1}{n_1 t_1} \sum_{i=1}^{m} n_i(t_i^2 - t_{i-1}^2) - 2\sum t_i + t_m\right]}{\sum_{i=1}^{m} (t_{i-1}+t_i)^2} . \tag{B3}$$

Next we discuss the initial guess $\hat{\hat{\beta}}_0$ for $\hat{\hat{\beta}}$. In this case the model is $\gamma(t) = \lambda/(1+\alpha t)^{(1+\beta)}$ and we have a relation analogous to (B1), namely

$$n_i(t_i - t_{i-1}) \approx \frac{\left(1+\hat{\hat{\alpha}}\left(\frac{t_{i-1}+t_i}{2}\right)\right)^{1+\hat{\hat{\beta}}}}{\hat{\hat{\lambda}}}, \; i=1, \ldots, m.$$

Again, this can be written as

$$\langle y_i \rangle = x_i^{1+\hat{\hat{\beta}}}, \; i=1, \ldots, m$$

where $y_i = \hat{\hat{\lambda}}\, n_i(t_i - t_{i-1})$ and $x_i = 1+\hat{\hat{\alpha}}\left(\frac{t_{i-1}+t_i}{2}\right)$ which corresponds to a regression model of the form

$$y_i \approx x_i^{1+\hat{\hat{\beta}}} + \varepsilon_i$$

or

$$y_i \doteq x_i^{1+\hat{\hat{\beta}}}\left(1 + \frac{\varepsilon_i}{x_i^{1+\hat{\hat{\beta}}}}\right) .$$

Taking logarithm we have

$$\ln y_i \approx (1+\hat{\hat{\beta}}) \ln x_i + \ln \left(1 + \frac{\varepsilon_i}{x_i^{1+\beta}}\right)$$

or

$$\ln y_i \doteq (1+\hat{\hat{\beta}}) \ln x_i + \frac{\varepsilon_i}{x_i^{1+\hat{\hat{\beta}}}} .$$

Thus we have a weighted least squares problem, namely

$$\ln y_i - \ln x_i \approx \hat{\hat{\beta}} \ln x_i + \delta_i$$

where $\delta_i = \varepsilon_i / x_i^{1+\hat{\hat{\beta}}}$ and $\mathrm{var}(\delta_i) \approx \delta^2/y_i^2$. Hence

$$\hat{\hat{\beta}}_0 = \frac{\sum_{i=1}^{m} y_i^2 (\ln y_i - \ln x_i) \ln x_i}{\sum_{i=1}^{m} (y_i \ln x_i)^2} .$$

Since $\hat{\hat{\beta}}_0$ depends on $\hat{\hat{\alpha}}$ and $\hat{\hat{\lambda}}$, an obvious guess for these parameters are the maximum likelihood solutions $\hat{\alpha}$ and $\hat{\lambda}$ obtained from using the model $\gamma(t) = \lambda/(1+\alpha t)$. Thus we shall use for *initial guess* of $\hat{\hat{\beta}}$

$$\hat{\hat{\beta}}_0 = \frac{\sum_{i=1}^{m} \left(\hat{\lambda}\, n_i\, (t_i - t_{i-1})\right)^2 \left[\ln \left(\hat{\lambda}\, n_i (t_i - t_{i-1})\right) - \ln \left(1+\hat{\alpha}\, \frac{(t_{i-1}+t_i)}{2}\right)\right] \ln\left(1+\hat{\alpha}\, \frac{(t_{i-1}+t_i)}{2}\right)}{\sum_{i=1}^{m} \left[\hat{\lambda}\, n_i (t_i - t_{i-1}) \ln \left(1+\hat{\alpha}\, \frac{(t_{i-1}+t_i)}{2}\right)\right]^2} . \tag{B4}$$

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