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DYNAMICAL ZONING WITHIN A LAGRANGIAN MESH
BY USE OF DYN, A STELLAR PULSATION CODE

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ABSTRACT

A method of dynamical zoning within a Lagrangian mesh is used to resolve the motion of the hydrogen ionization front in a time-dependent nonlinear model of a pulsating star. The resulting coupling with the radiative-transfer improves the calculated light curves. The method is described in some detail and then applied to a model of the Cepheid η Aql.

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I. INTRODUCTION

The reliability of nonlinear stellar pulsation calculations has been questioned on the grounds of the zoning's influence on light curve shapes. Keller and Mutschlecner¹ showed that bumps on a certain Cepheid model's light curves are due to traversal of the hydrogen ionization front from one zone to another in the conventional Lagrangian mesh. The conventional Lagrangian technique, as applied to pulsating stars, requires that the ionization front move from zone to zone within the mesh. To resolve the front (maintain 3 or 4 zones within it), one must use a tremendous number of zones in the star's outer regions.

The hydrogen ionization front motion is coupled with the radiative energy transfer, which can be treated with the equilibrium diffusion approximation. The front's motion may be determined from the Rankine-Hugoniot conservation conditions, provided the radiative flux can be found accurately. However, this requires that the ionization front be resolved. Clearly, the Lagrangian technique is inadequate for solving stellar pulsation problems reliably if one must use a small number of zones in such problems owing to limited computational resources. Similarly, we also deduce the inadequacy of the Eulerian techniques.

Here we discuss a hybrid, dynamical zoning technique that is neither Lagrangian, Eulerian, nor mixed Lagrangian-Eulerian, and which allows part of the computational mesh (called the active region) to move with the ionization front in a pulsating star. Furthermore, the front is well enough resolved by this technique so that there can be no question about the accuracy of the radiative flux derived. Quantities near the ionization front change slowly in the coordinate system chosen for the active region. Thus we may use a fine mesh in this region and still use large time steps, provided our differencing technique does not need a Courant-like time-step control.

We describe below a computer code called DYN that uses an implicit differencing scheme in conjunction with the dynamic zoning technique just described. We conclude by demonstrating DYN's application to a model of the Cepheid η Aql.

II. NONLINEAR RADIAL STELLAR PULSATION EQUATIONS

We assume spherical symmetry and adopt as a point of departure the Lagrangian system. The spatial coordinate is the mass within a distance r of the center, M_r . Some of the dependent variables are the radial coordinate r , the flow velocity u , the specific volume V , pressure P , temperature T , internal

energy E , and the radiative luminosity L_r (the total net power crossing a sphere of radius r). The conventional Lagrangian equations are then

$$\left(\frac{\partial r}{\partial t}\right)_{M_r} = u, \quad (1)$$

$$\left(\frac{\partial u}{\partial t}\right)_{M_r} = -\frac{GM_r}{r^2} - 4\pi r^2 \left(\frac{\partial P}{\partial M_r}\right)_t, \quad (2)$$

$$\left(\frac{\partial E}{\partial t}\right)_{M_r} + P \left(\frac{\partial V}{\partial t}\right)_{M_r} = -\left(\frac{\partial L_r}{\partial M_r}\right)_t, \quad (3)$$

$$V = \left(\frac{\partial}{\partial M_r}\right)_t \left(\frac{4\pi}{3} r^3\right), \quad (4)$$

and

$$L_r = -\frac{(4\pi r^2)^2 4\sigma}{3\kappa} \left(\frac{\partial T^4}{\partial M_r}\right)_t, \quad (5)$$

where G is the gravitation constant, σ is the Stefan-Boltzmann constant, and κ is the Rosseland mean opacity.

Because we shall use zonal interfaces that are fixed not in mass, but in some other coordinate that we shall designate by x , a function of M_r and t , we need to convert Eqs. (1)-(5) to the variables x and t , rather than M_r and t . We require the following identity, where f is any physical variable.

$$\left(\frac{\partial f}{\partial t}\right)_{M_r} = \left(\frac{\partial f}{\partial t}\right)_x - \left(\frac{\partial M_r}{\partial t}\right)_x \left(\frac{\partial f}{\partial M_r}\right)_t. \quad (6)$$

(Recall that $\frac{\partial}{\partial t} [f(t, x)]_x = \frac{\partial}{\partial t} [f(t, M_r)]_{M_r} + \frac{\partial}{\partial M_r} [f(t, M_r)]_t \frac{\partial}{\partial t} [M_r(t, x)]_x$ when $M_r = M_r(t, x)$.) Combining Eq. (6) with Eqs. (1)-(3) gives

$$\left(\frac{\partial r}{\partial t}\right)_x = u + \dot{M} \left(\frac{\partial r}{\partial M_r}\right)_t, \quad (7)$$

$$\left(\frac{\partial u}{\partial t}\right)_x = -\frac{GM_r}{r^2} - 4\pi r^2 \left(\frac{\partial p}{\partial M_r}\right)_t + \dot{M} \left(\frac{\partial u}{\partial M_r}\right)_t, \quad (8)$$

and

$$\left(\frac{\partial E}{\partial t}\right)_x + p \left(\frac{\partial V}{\partial t}\right)_x = \dot{M} \left[\left(\frac{\partial E}{\partial M_r}\right)_t + p \left(\frac{\partial V}{\partial M_r}\right)_t \right] - \left(\frac{\partial L_r}{\partial M_r}\right)_t, \quad (9)$$

where

$$\dot{M} \equiv \left(\frac{\partial M_r}{\partial t}\right)_x. \quad (10)$$

Note that Eqs. (7)-(9) are valid regardless of the function of M_r and t that may be chosen for $\dot{M}$. The precise procedure for choosing $\dot{M}$ is described in Sec. III, but, in short, the object is to minimize the right-hand sides of Eqs. (8) and (9). This makes the variables in a given zone, fixed in the x coordinate, slow functions of time.

Equations (7)-(9) are converted to finite difference form to yield a fully implicit scheme. That is, the three basic variables r , u , and T are all specified at instants of time t^n and a time step consists of the simultaneous solution for r , u , and T at time t^{n+1} . The spatial zoning is indicated by a subscript, which is a half-odd integer for zonal quantities such as P , T , V , E , etc., and an integer for interface quantities such as M_r , r , u , L_r , etc. The index increases with radius. An implicit form must be used for Eqs. (7) and (8), as, otherwise, a Courant-type condition for stability is imposed which is more stringent than the condition deriving from the motion of the front.

The finite-difference scheme that has been used most successfully is represented by the following equations:

$$r_i^{n+1} - r_i^n = \frac{(\Delta t)^{n+1/2}}{2} (u_i^{n+1} + u_i^n)$$

$$+ \frac{\dot{M}_i^{n+\frac{1}{2}} (\Delta t)^{n+\frac{1}{2}}}{4 DM_i^{n+1}} \left(r_{i+1}^{n+1} + r_{i+1}^n - r_{i-1}^{n+1} - r_{i-1}^n \right) , \quad (11)$$

$$\begin{aligned} u_i^{n+1} - u_i^n &= - \frac{GM_i^{n+\frac{1}{2}} (\Delta t)^{n+\frac{1}{2}}}{r_i^{n+1} r_i^n} \\ &+ \frac{2\pi (\Delta t)^{n+\frac{1}{2}}}{3 DM_i^{n+1}} \left[(r_i^{n+1})^2 + r_i^{n+1} r_i^n + (r_i^n)^2 \right] \\ &\times \left[(P+q)_{i-\frac{1}{2}}^{n+1} + (P+q)_{i-\frac{1}{2}}^n - (P+q)_{i+\frac{1}{2}}^{n+1} - (P+q)_{i+\frac{1}{2}}^n \right] \\ &+ \frac{\dot{M}_i^{n+\frac{1}{2}} (\Delta t)^{n+\frac{1}{2}}}{4 DM_i^{n+1}} \left(u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - u_{i-1}^n \right) , \quad (12) \end{aligned}$$

and

$$\begin{aligned} E_{i+\frac{1}{2}}^{n+1} - E_{i+\frac{1}{2}}^n &+ 1/2 \left[(P+q)_{i+\frac{1}{2}}^{n+1} + (P+q)_{i+\frac{1}{2}}^n \right] \\ &\times \left(v_{i+\frac{1}{2}}^{n+1} - v_{i+\frac{1}{2}}^n \right) \\ &= \frac{L_i^{n+1} - L_{i+1}^{n+1}}{DM_{i+\frac{1}{2}}^{n+1}} (\Delta t)^{n+\frac{1}{2}} \\ &+ \frac{1}{2} \dot{M}_{i+\frac{1}{2}}^{n+\frac{1}{2}} (\Delta t)^{n+\frac{1}{2}} \left[\frac{E_{i+\frac{3}{2}}^{n+1} - E_{i+\frac{1}{2}}^{n+1}}{DM_{i+1}^{n+1}} + \frac{E_{i+\frac{1}{2}}^{n+1} - E_{i-\frac{1}{2}}^{n+1}}{DM_i^{n+1}} \right. \\ &\left. + (P+q)_{i+\frac{1}{2}}^{n+1} \left(\frac{v_{i+\frac{3}{2}}^{n+1} - v_{i+\frac{1}{2}}^{n+1}}{DM_{i+1}^{n+1}} + \frac{v_{i+\frac{1}{2}}^{n+1} - v_{i-\frac{1}{2}}^{n+1}}{DM_i^{n+1}} \right) \right] . \quad (13) \end{aligned}$$

The boxes emphasize the changes to the basic finite-difference equations which result from the hybrid approach. The auxiliary relations are:

$$v_{i+\frac{1}{2}}^{n+1} = \frac{4\pi}{3} \frac{(r_{i+1}^{n+1})^3 - (r_i^{n+1})^3}{DM_{i+\frac{1}{2}}^{n+1}} , \quad (14)$$

$$L_i^n = \frac{64\pi^2 \sigma}{3} \frac{(r_i^n)^4}{DM_i^n} \left[\frac{(T_{i-\frac{1}{2}}^n)^4}{\kappa_{i-\frac{1}{2}}^n} + \frac{(T_{i+\frac{1}{2}}^n)^4}{\kappa_{i+\frac{1}{2}}^n} \right] \\ \times \frac{(T_{i-\frac{1}{2}}^n)^4 - (T_{i+\frac{1}{2}}^n)^4}{(T_{i-\frac{1}{2}}^n)^4 + (T_{i+\frac{1}{2}}^n)^4} , \quad (15)$$

$$q_{i+\frac{1}{2}}^n = \frac{C_Q F_Q^n}{v_{i+\frac{1}{2}}^n} \left[\text{Max} (0, u_i^n - u_{i+1}^n) \right]^2 , \quad (16a)$$

$$F_Q^n = \exp \left(1 - \text{Max} \left\{ G_Q / \text{Max} \left[(u_i^n - u_{i+1}^n) / \sqrt{p_{i+\frac{1}{2}}^n v_{i+\frac{1}{2}}^n}, G_Q / 100 \right], 1 \right\} \right) , \quad (16b)$$

$$DM_{i+\frac{1}{2}}^n = M_{i+1}^n - M_i^n , \quad (17)$$

$$DM_i^n = 1/2 \left(DM_{i+\frac{1}{2}}^n + DM_{i-\frac{1}{2}}^n \right) , \quad (18)$$

$$\dot{M}_i^{n+\frac{1}{2}} = \frac{M_i^{n+1} - M_i^n}{(\Delta t)^{n+\frac{1}{2}}} , \quad (19)$$

and

$$\dot{M}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = 1/2 \left(\dot{M}_i^{n+\frac{1}{2}} + \dot{M}_{i+1}^{n+\frac{1}{2}} \right) . \quad (20)$$

The algorithm for evolving $\dot{M}_i^n$ is discussed in Sec. III.

The factor F_Q^n in the artificial viscosity relation (16a), given by Eq. (16b), provides an exponential scaling to the artificial viscosity for the range $G_Q/100 \leq \Delta u/a \leq G_Q$, where a is an approximation of the adiabatic sound speed and G_Q is a user-controlled parameter. We have

$$F_Q^n = \begin{cases} 1, & G_Q < \frac{\Delta u}{a} \\ \exp(1 - G_Q a / \Delta u), & G_Q/100 \leq \frac{\Delta u}{a} \leq G_Q \\ \exp(-99), & \frac{\Delta u}{a} < \frac{G_Q}{100} \end{cases}.$$

This heuristic approach is a modification of a similar technique attributed to R. Stellingwerf, which is used to prevent damping in the interior of the star.

Equations (11)-(13), into which the remaining equations have been substituted, may be regarded as a nonlinear system that determines r_i^{n+1} , u_i^{n+1} , and $T_{i+1/2}^{n+1}$. This system is solved, as usual, by use of the Newton-Raphson technique. The resulting linear system has a banded matrix whose band width is 11. It may be solved economically by the standard band matrix procedure.

The boundary conditions for the calculation are unchanged from those usually applied, as it is assumed that $\dot{M}$ vanishes in a region adjacent to each boundary.

A number of comments on the difference equations, (11)-(13), are in order. If the terms containing $\dot{M}$ are omitted, the equations become those introduced by Fraley,² which yield an exact conservation law,

$$\begin{aligned} & \left[\sum_i DM_i \left(\frac{(u_i)^2}{2} - \frac{GM}{r_i} \right) + \sum_i DM_{i+1/2} E_{i+1/2} \right]^{n+1} \\ & - \left[\sum_i DM_i \left(\frac{(u_i)^2}{2} - \frac{GM}{r_i} \right) + \sum_i DM_{i+1/2} E_{i+1/2} \right]^n \\ & = \left(L_1^{n+1} - L_{N+1}^{n+1} \right) (\Delta t)^{n+1/2}, \end{aligned} \quad (21)$$

when the usual boundary conditions are applied. (L_1 and L_{N+1} are the interior and exterior boundary luminosities, respectively.) Unfortunately, it does not seem possible to retain this feature if $\dot{M} \neq 0$. However, energy conservation is more critical in the inner region of the model, where it is possible to set $\dot{M} = 0$, so it seems desirable to use Fraley's equations as a basis.

It is also of interest to consider the extent to which the Rankine-Hugoniot conditions may apply across the ionization front for the finite-difference solution. We suppose that there is a solution of the difference equations which, for several zones near the front, is stationary in the sense that T , V , and u are constant for a given zone or interface, and that $r_i^n = \text{constant}_i + u_F t^n$ where u_F , a constant, is the front velocity. We further suppose that $\dot{M}$ is constant in space for those zones and that $4\pi r^2$ can be considered constant where it multiplies spatial differences. Given these assumptions, Eqs. (11)-(13) yield

$$u_F = u_i + \frac{\dot{M}}{4\pi r^2} \langle V_i \rangle, \quad (22)$$

$$0 = 4\pi r^2 \left[(P+q)_{i-\frac{1}{2}} - (P+q)_{i+\frac{1}{2}} \right] + \dot{M} \frac{u_{i+1} - u_{i-1}}{2}, \quad (23)$$

$$L_i - L_{i+1} + \dot{M} \left[\langle E_{i+1} \rangle - \langle E_i \rangle + (P+q)_{i+\frac{1}{2}} \left(\langle V_{i+1} \rangle - \langle V_i \rangle \right) \right] = 0, \quad (24)$$

where

$$\langle V_i \rangle = \frac{DM_{i-\frac{1}{2}} V_{i-\frac{1}{2}} + DM_{i+\frac{1}{2}} V_{i+\frac{1}{2}}}{DM_{i-\frac{1}{2}} + DM_{i+\frac{1}{2}}}, \quad (25)$$

and

$$\langle E_i \rangle = \frac{DM_{i-\frac{1}{2}} E_{i-\frac{1}{2}} + DM_{i+\frac{1}{2}} E_{i+\frac{1}{2}}}{DM_{i-\frac{1}{2}} + DM_{i+\frac{1}{2}}}. \quad (26)$$

If we define

$$v_i \equiv u_F - u_i, \quad (27)$$

we see that

$$v_i = \frac{\dot{M}}{4\pi r^2} (V_i) \quad (28)$$

and

$$(P+q)_{i+\frac{1}{2}} + \frac{\dot{M}}{4\pi r^2} \cdot \frac{v_i + v_{i+1}}{2} \equiv K = \text{constant} . \quad (29)$$

Inserting Eqs. (28) and (29) into Eq. (24) gives

$$\begin{aligned} & (L_i - \dot{M} (E_i)) - (L_{i+1} - \dot{M} (E_{i+1})) \\ & + \dot{M} \left[K - \frac{\dot{M}}{4\pi r^2} \frac{v_i + v_{i+1}}{2} \right] \frac{4\pi r^2}{\dot{M}} (v_{i+1} - v_i) = 0 , \end{aligned} \quad (30)$$

or

$$L_i - \dot{M} \left((E_i) + \frac{4\pi r^2}{\dot{M}} K v_i - 1/2 (v_i)^2 \right) = \text{constant}, \quad (31)$$

or

$$L_i - \dot{M} \left((E_i) + (P_i)(V_i) + 1/2 v_i^2 \right) = \text{constant}, \quad (32)$$

where

$$\begin{aligned} (P_i) & \equiv K - \frac{\dot{M}}{4\pi r^2} v_i = (P+q)_{i+\frac{1}{2}} + \frac{\dot{M}}{8\pi r^2} (v_{i+1} - v_i) \\ & = (P+q)_{i-\frac{1}{2}} + \frac{\dot{M}}{8\pi r^2} (v_{i-1} - v_i) . \end{aligned} \quad (33)$$

Therefore, we see that if the solution for T , V , and u yields smooth spatial variation of these quantities, so that $v_{i+1} - v_i$, $V_{i+1} - V_i$, $E_{i+\frac{1}{2}} - E_{i-\frac{1}{2}}$, and $q_{i+\frac{1}{2}}$ all tend to 0 away from the front, the appropriate conservation laws (Eqs. 28, 29, 32) are obeyed.

A final point about the difference equations is their stability. There is little point in checking stability against thermal disturbances, for the backward-differenced diffusion operator should guarantee stability. (Stability is confirmed by a simple analysis.) The more delicate stability question concerns density and velocity perturbations.

We examine the stability of Eqs. (11) and (12) by considering an analogous linear system with constant coefficients

$$r_i^{n+1} - r_i^n = \frac{\Delta t}{2} \left[u_i^{n+1} + u_i^n + \frac{\dot{M}}{2\Delta M} \left(r_{i+1}^{n+1} + r_{i+1}^n - r_{i-1}^{n+1} - r_{i-1}^n \right) \right] \quad (34)$$

$$u_i^{n+1} - u_i^n = \frac{a^2 \Delta t}{2(\Delta r)^2} \left(r_{i+1}^{n+1} - 2r_i^{n+1} + r_{i-1}^{n+1} + r_{i+1}^n - 2r_i^n + r_{i-1}^n \right) + \frac{\dot{M} \Delta t}{4\Delta M} \left(u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - u_{i-1}^n \right) \quad (35)$$

The eigenvalues p of the amplification matrix (cf. Richtmyer and Morton³) are found by taking

$$r_j^n \propto e^{ikj} p^n,$$

and likewise for u_j^n . We obtain

$$p = \frac{1 + i(\pm L \sin k/2 + \alpha \sin k)}{1 - i(\pm L \sin k/2 + \alpha \sin k)}, \quad (36)$$

where

$$L = \frac{a \Delta t}{\Delta r}, \quad (37)$$

and

$$\alpha = \frac{\dot{M}\Delta t}{2\Delta M} \quad (38)$$

We see that $|p| = 1$, so that the system is stable. If, in Eqs. (34) and (35), the $\dot{M}$ terms had been evaluated at t^{n+1} rather than time centered, we would have found

$$p = \frac{1 \pm i L \sin k/2}{1 - i(2\alpha \sin k \pm L \sin k/2)} \quad (39)$$

which indicates instability if the front is subsonic. If all terms are evaluated at t^{n+1} , the amplification factors are

$$p = \frac{1}{1 - 2i\alpha \sin k \mp 2iL \sin k/2} \quad (40)$$

so that $|p| \leq 1$, giving stability. The very fact that $|p| < 1$ in general for this modification indicates that the equations are dissipative in Richtmyer and Morton's sense, although conservative. Use of such equations gave a large spurious damping which is avoided by use of Eqs. (11) and (12) as written.

III. THE CHOICE OF $\partial M / \partial t$ FOR DYNAMIC ZONING

The routine that drives the motion of the mass interfaces is used for two principal purposes. The first and more basic is to try to keep a certain feature of the calculation stationary with respect to the mesh. The other is to ensure that the mesh remains relatively smooth in the meantime. Both objectives may be met by requiring the mass coordinate to satisfy an inhomogeneous diffusion equation in x and t . That is,

$$\left(\frac{\partial M}{\partial t} \right)_x = B \frac{\partial}{\partial x} \left[\left(\frac{\partial M}{\partial x} \right)_t \right] + A \quad (41)$$

In this equation B , Y , and A are adjustable functions chosen to meet the above objectives best.

So far the precise meaning of x has not been stated. We may suppose that it is a continuous variable that coincides with the zone index i at its integer values; i.e., $x = i$ is the coordinate of interface i . The role of A is that it be the desired value of $\partial M / \partial t$, neglecting the smoothing effect. If A vanishes, the mesh relaxes with time to a state in which $\Delta M \propto 1/Y$. Therefore, Y determines the steady-state zoning. The relaxation rate is fixed by Y and B together. Because it is the term A that distorts the mesh and necessitates the diffusion term action, it is appropriate to take $B \propto \|A\|$ ($\|A\| = \max |A|$) to ensure that the mesh distortion is bounded regardless of the value of A . To make the diffusion term's smoothing effect uniform over the mesh, B is also taken $\propto 1/Y$. Therefore we assume that

$$B = \frac{g \|A\|}{Y} \quad (42)$$

The constant g remains to be determined. The best results have been obtained if the constant g was adjusted so that the time required for the smoothing effect to act over a distance of one zone, $1/g|A|$, was about equal to the time that any part of the active region required to travel in mass an amount equal to the average zone mass in the active region, a time equal to $\Delta M / |A| \Delta i$. In other words, $g \approx \Delta i / \Delta M$. Here ΔM is the mass, and Δi is the number of zones, in the active region. The equation adopted is

$$\frac{\partial M}{\partial t} = \frac{2 \|A\| \Delta i}{\Delta M Y} \frac{\partial}{\partial x} \left(Y \left(\frac{\partial M}{\partial x} \right) + A \right) \quad (43)$$

This is expressed in the finite-difference form

$$M_i^{n+1} = M_i^n + \frac{4 \|A\| \Delta i \Delta t^{n+1/2}}{\Delta M (Y_{i-1/2}^n + Y_{i+1/2}^n)} \left[Y_{i+1/2}^n (M_{i+1}^{n+1} - M_i^{n+1}) - Y_{i-1/2}^n (M_i^{n+1} - M_{i-1}^{n+1}) \right] + A_i \Delta t^{n+1/2} \quad (44)$$

which is linear in the interface masses.

The Function A

We wish to fix the value of A so that the ionization front is stationary with respect to the mesh. We also may assume that $\dot{M} = A$ near the front. We showed in Sec. II that if $\dot{M}$ is chosen so that the front is stationary in the mesh, the jump conditions, Eqs. (28), (29), and (32), are obeyed. These relations can be solved for $\dot{M}$ in terms of flow quantities. Conversely, if $\dot{M}$ is determined from one of these relations, the front is stationary in the mesh. Any one of the three relations may be used for this purpose, but as they may not be exactly consistent, the relation in which M is fixed most sensitively is the best choice. This is the energy equation. It gives

$$A_C = \frac{\Delta L}{\Delta(E + PV + \frac{1}{2}v^2)} , \quad (45)$$

where A_C is taken to be the value of A in the central part of the active region. The differences appearing here are taken between a zone close to 8000 K and one close to 15000 K. The values of v come from Eq. (27) where, from Eqs. (27) and (28),

$$U_F = \frac{\Delta(u/V)}{\Delta(T/V)} . \quad (46)$$

The value A_C , given by Eq. (46), is the value we use for A in the center of the active region, where the ionization front is located. At the edges of the region, we use values that maintain a smooth transition to the Lagrangian mesh. Outside the active region, the mesh is uniform in the variable $\ln(M_S - M_i)$, where M_S is the total mass of the star. Letting $Deim$ be the increment in this variable, we set $Dmfac = \exp(Deim)$ and define

$$A_L = A_{i_1+1} = \left[\left(M_S - M_{i_1+1} \right) * Dmfac - \left(M_S - M_{i_1} \right) \right] / \Delta t , \quad (47)$$

$$A_U = A_{i_2-1} = \left[\left(M_S - M_{i_2-1} \right) - \left(M_S - M_{i_2} \right) * Dmfac \right] / \Delta t , \quad (48)$$

where the active region is delimited by the indices i_1 and i_2 .

If $A = A_L$ at the bottom edge of the active region, Eq. (44) tends to restore $(M_S - M_{i_1+1})$ to the value $(M_S - M_{i_1})/Dmfac$. Similarly, Eq. (44) tends to restore $(M_S - M_{i_2-1})$ to $(M_S - M_{i_2}) * Dmfac$ when $A = A_U$ at the top edge; i.e., the increment in $\ln(M_S - M_i)$ just inside the active region is approximately $\Delta \ln m$.

To combine the effects of the values A_C , A_L , and A_U , we define A by

$$A_i = \eta_i A_E + (1 - \eta_i) A_C, \quad (49)$$

where

$$A_E = \begin{cases} A_L & \text{when } i \leq i_0 \\ A_U & \text{when } i > i_0 \end{cases}$$

(i_0 is the nominal coordinate of the ionization front), and

$$\eta_i = \exp \left[(i - i_1 - 1) * (i - i_2 + 1) / (i_2 - i_1 + 2) \right]. \quad (50)$$

We note that $\eta_{i_1+1} = \eta_{i_2-1} = 1$ and that $0 < \eta_{i_0} \ll 1$.

When we use Eq. (45) to calculate A_C and subsequently obtain A from Eq. (49) and $\dot{M}$ from the diffusion equation, Eq. (44), we observe that the ionization front often drifts within the active region. To compensate for this drift, we correct A_C with a feedback term. The ionization front is steepest near 10^4 K; thus the front's location may be assigned the value

$$\langle i \rangle = i_F + (10^4 \text{ K} - T_{i_F}) / (T_{i_F+1} - T_{i_F}), \quad (51)$$

where $i_F \leq \langle i \rangle \leq i_F + 1$.

The deviation of $\langle i \rangle$ from the nominal coordinate i_0 is used to modify A_C .

$$\hat{A}_C = A_C - \epsilon_A |A_C| (\langle i \rangle - i_0), \quad (52)$$

where ϵ_A is a small constant (0.05 - 0.20). Equation (45) usually estimates the actual front motion to within ± 5 or 10% so the front remains within one or two zones of its nominal position in the active region.

The nominal index i_0 of the ionization front is chosen approximately midway between the limiting indices i_1 and i_2 . Equations (45) and (52) keep the ionization front location close to the index i_0 . As the calculation proceeds, the front sweeps through the outer region of the star. This causes mass transfer from one end of the active region to the other, and the front approaches, in mass, one of the edges of the active region, as illustrated in Fig. 1.

In Fig. 1 we see both the effect of the mass motion through the mesh and the effects of Eqs. (47) and (48). The latter are demonstrated by the mesh smoothness, as a function of zone, at the edges of the active region. Figure 1 also shows the mesh fineness at the ionization front.

The ionization front is centered by index in the active region through the action of the feedback term in Eq. (52). To keep the front also centered in mass, we must periodically increment i_1 and i_2 by one unit.

The criterion we have found successful in determining the moment for incrementing i_1 and i_2 is

$$|\ln(M_S - M_{i_0}) - \frac{1}{2} [\ln(M_S - M_{i_1}) + \ln(M_S - M_{i_2})]| > \text{Deviatn} * \text{Delm}, \quad (53)$$

where M_S is the mass of the star, Delm is the (constant) increment in $\ln(M_S - M_i)$ for i outside the active region, and $\text{Deviatn} > 0.5$. We have found that $\text{Deviatn} = 0.6$ provides good results. Note that, when i_1 and i_2 are incremented, the average $\frac{1}{2} [\ln(M_S - M_{i_1}) + \ln(M_S - M_{i_2})]$ changes by a value approximately equal to Delm , but $\ln(M_S - M_{i_0})$ changes by a value that is small compared to Delm .

The value A_C we have chosen for A in the center of the active region serves to minimize the right-hand sides of Eqs. (8) and (9), as we have mentioned before. We imposed this condition to obtain solutions for T , u , and V which vary slowly in time. We showed earlier that this allows us to conclude that the conservation laws are obeyed.

At the edges of the active region, the $\dot{M}$ values obtained from Eqs. (44) and (47)-(49) are not subject to the same restrictions as those obtained at the

center. At the center, we are concerned about steep gradients' effect on the solution as a whole. This problem does not arise at the edges of the active region.

The function η_i , defined by Eq. (50), is small in the center of the active region, approximately equal to 0.001. If $\|A\| = \max |A|$ is large compared to $|A_C|$, the value of A in the center of the active region, calculated from Eq. (49), may differ significantly from the required value A_C . Thus we impose the conditions

Fig. 1.
Ionization front motion.

$$|A_L| \leq 10 \cdot |A_C| \text{ when } |A_L| > |\bar{A}|$$

and

$$|A_U| \leq 10 \cdot |A_C| \text{ when } |A_U| > |\bar{A}|. \quad (54)$$

If $|\bar{A}| = 0$, $\|A\| \leq 10 \cdot |A_C|$.

The final adjustment we apply to A is

$$A'_i = (1 - \eta_i) \left[A'_C - (A_{i_0} + A_{i_0+1})/2 \right] + A_i, \quad (55)$$

where A'_C is obtained from Eq. (52) and A_{i_0} , A_{i_0+1} , and A_i result from Eq. (49).

This adjustment allows us to correct the effect of A_E on the central values of A . We obtain: $A'_i \sim \eta_i^2 A_E + (1 - \eta_i^2) A'_C \sim A'_C$ for $i \sim i_0$.

To summarize, we write our method of obtaining A as an algorithm.

1. Obtain A_C as the central value of A from the Rankine-Hugoniot relation, Eq. (45).
2. Obtain the edge values A_L and A_U by considering the mass changes in the edge cells required to maintain nominal size [Eqs. (47) and (48)].
3. Adjust A_C by a feedback term to keep the ionization front centered in the active region [Eq. (52)].
4. Limit the size of $|A_L|$ and $|A_C|$ by means of Eq. (54).
5. Combine the adjusted values A_L , A_C , and A_U by use of Eq. (49).
6. Correct the effects of the edge values on the central values with Eq. (55).

The Function Y

The hydrogen ionization front is a region of rapid spatial variation of temperature. Therefore, good front resolution is obtained if the relative temperature increment between adjacent zones is controlled.

We have seen that Eq. (41) tends to smooth out the mass zoning in the active region. It also controls the zone sizes in such a way that $\Delta M \sim 1/Y$. This suggests that Y should be a function that is large in the center of the active region (near the ionization front) and is small and approximately constant towards the edges of the active region. This latter requirement arises from a need to allow the values A_L , A_U to control the zoning at the edges by Eqs. (44) and (49). Furthermore, Y should be a smooth function of x. Considering these requirements, we define

$$Y_i^n = (Y_{\max}^n - 1) \exp \left[-|C_S (i - \langle i \rangle)|^\alpha \right] + 1, \quad (56)$$

where

$$Y_{\max}^n = F_S^n \left(\exp \left\{ \left[C_S (i_2 - i_1)/3 \right]^\alpha \right\} + 1 \right), \quad (57)$$

$$F_S^n = \text{Average} \left\{ F_S^{n-1}, \exp \left[W_S - (x_B^n - x_A^n)/F_S^{n-1} \right] \right\}, \quad (58)$$

$$C_S = 2/W_S. \quad (59)$$

W_S is the nominal width of the ionization front in the x coordinate, and x_B^n, x_A^n are the interpolated x -values where $T = 8000$ and $15\,000$ K, respectively. $\langle i \rangle$ was defined previously as the interpolated x -value corresponding to $T = 10\,000$ K. $\langle i \rangle$ is also evaluated at time n , but we omit the superscript for the sake of notational clarity.

Some remarks are in order. First, when $F_S = 1$ and $|i - \langle i \rangle| > (i_2 - i_1)/3$, $1 < Y_i^n < 2$. Furthermore, as $|i - \langle i \rangle|$ increases (as i moves farther from $\langle i \rangle$), Y_i^n decreases smoothly towards 1. We see, then, that Y_i^n is small and approximately constant towards the edges of the active region.

Next, we have $Y_{\max}^n = Y_{\langle i \rangle}^n = F_S^n \left(\exp \left\{ \left[C_S (i_2 - i_1)/3 \right]^\alpha \right\} + 1 \right)$. As we choose to set $W_S = (i_2 - i_1)/4$ and $C_S = 2/W_S$, we obtain

$$Y_{\max}^n = F_S^n \left\{ \exp \left[(1/3)^\alpha \right] + 1 \right\}. \quad (60)$$

If $F_S^n \sim 1$, we see that $Y_{\max}^n$ is a rapidly increasing function of α . Thus, for $\alpha \sim 2$ and $F_S^n \sim 1$, Y_i^n meets the requirements stated above (large in the vicinity of the ionization front and small and constant towards the edges of the active region).

F_S^n , defined by Eq. (58), is a feedback term. It adjusts the value of $Y_{\max}^n$ to maintain the relation $W_S \sim (x_B - x_A)$. In actual practice, we do not obtain $W_S \sim (x_B - x_A)$ for all α . $(x_B - x_A)$ increases as α increases; i.e., the number of zones in the ionization front is controlled by α . We have found that $W_S = (i_2 - i_1)/4 = (x_B - x_A)$ for some α between 1 and 2. We also have found that smaller values of α allow the code to run faster (with less resolution of the ionization front).

In developing the feedback term F_S^n , we considered using $F_S^n = \exp \left[W_S - (x_B^n - x_A^n) \right]$. We rejected this choice, as it resulted in alternate-period oscillations in the value of $(x_B^n - x_A^n)$. Equation (58) is a variant of the Newton iteration on the relation $(F_S)^2 - \exp \left[W_S - (x_B - x_A) \right] = 0$.

IV. GENERAL DYN CODE FEATURES

The DYN code incorporates the features described in Secs. II and III. It also includes a static-model-generating subroutine (Static), problem dump save and read subroutines (Save and Restor), a pressure-opacity table-reading subroutine (Initial), a table traverser (EOS), and plotting routines (Plotter, Profils, etc). Figure 2 is a flowchart of the code.

Time Step Control

In subroutine Solve, the quantities $\delta_{3i+1} = \Delta T/T$, $\delta_{3i+2} = \Delta R/R$, and $\delta_{3i+3} = \Delta u/a$ are calculated, where $a = \sqrt{PV}$ is approximately equal to the sound speed. From these, we get $|\delta| = \max|\delta_i|$, which may decrease the time step when $|\delta|$ takes on certain values. (See loop B-D in Fig. 2. On loop B-D, $\epsilon < |\delta| < \epsilon'$ and $I_S \geq I_{S(\max)}$ or $|\delta| > \epsilon'$, where $\epsilon \sim 10^{-6}$, $\epsilon' \sim 0.3$, and $I_{S(\max)} \sim 6$. (ϵ, ϵ' , and $I_{S(\max)}$ are input parameters.)

In subroutine Accept, $|\delta_T| = \max |\Delta T/T|$ and $|\delta_V| = \max |\Delta V/V|$ are calculated. The time step is cut whenever $|\delta_T|/L_T > 1$ or $|\delta_V|/L_V > 1$ where L_T and L_V are input tolerances ($L_T \sim L_V \sim 0.1$). It is increased when $|\delta_T|/L_T < 0.5$, $|\delta_V|/L_V < 0.5$, and $\Delta t/Dtlim < 0.5$. $Dtlim$ is an input parameter ($Dtlim \sim 1.-5$).

Static Model

The characteristics of the static model that the DYN code produces are controlled by the star mass, luminosity, and effective temperature parameters. The zoning is controlled by the Dmfac parameter, an upper limit to the ratio $(M_S - M_i)/(M_S - M_{i+1})$, where M_S is the star mass. The outer zone temperature is obtained from $T^4 = T_e^4(3/4)(q_0 + \tau_0)$, where q_0 is the Hopf function, evaluated at τ_0 , the optical depth of the surface zone. (See Mihalas,⁴ Ch. 2). τ_0 and q_0 are input parameters.

Initial Velocity Distribution

The initial velocity distribution for the dynamic calculation is obtained from $u_i = u_0 (R_i/R_S)^{6.3}$, where R_S is the surface radius and u_0 is an input parameter.

Mass Zoning Control

The mass zoning is controlled by four input parameters. The coefficient ϵ_A of the feedback term in Eq. (52) is the first of these. A nominal value for ϵ_A is 0.1. We have found that the code sometimes runs faster when small ϵ_A values are used (0.05 as opposed to 0.2). This is understandable because the uncorrected value of A_C is the value that minimizes the changes in the problem

Fig. 2.

DYN program flowchart. Loop A is an iteration on R,U,T to obtain a small value of $|\delta|$. Loop B - D is an iteration on Δt . Loop C - D is an iteration on the problem step counter.

variables, as we have seen. Some feedback is necessary, however, to line the center of the active region to the ionization front position.

The second zoning control parameter is the Deviatn factor in Eq. (53). This parameter controls the timing of the active region shift in the Lagrangian mesh which is necessary to keep the ionization front approximately centered, in mass, in the active region. A satisfactory Deviatn value is 0.6.

The third parameter is the quantity $|\bar{A}|$ in Eq. (54). $|\bar{A}|$ is the floor value for $|A_E| = \max(|A_L|, |A_U|)$ below which the dominance of $|A_E|$, compared to $|A_C|$, is not controlled. When $|A_E| > |\bar{A}|$, A_L and A_U are adjusted so that $|A_E| \leq 10 \cdot |A_C|$.

The last zoning control parameter is the exponent α in Eq. (56). α controls the distribution of mass within the active region. Larger α values bring more zones into the ionization front, increasing its resolution. A suggested α value at low amplitudes is 1. At higher amplitudes, it may be necessary to increase α to 1.5 or 2 to keep the velocity curves smooth.

Plotting Facilities

DYN has facilities for plotting several parameters as functions of time, mass, and zone number. The plotting subroutines Ploter and Profils may be modified easily to provide other plots.

For convenience, we have associated a small integer with each plot.

I	Ordinate	Abscissa
1	Radius at $\tau = 2/3$ and at $T = 10^4$ K	Time
2	Velocity at $\tau = 2/3$ and at $T = 10^4$ K	Time
3	Luminosity at $\tau = 2/3$ and at the surface	Time
4	Mass depth of ionization front (at $T = 10^4$ K)	Time
5	Kinetic energy (total)	Time
11	Temperature	Mass
12	Velocity	Mass
13	Density	Mass
14	Ln (pressure)	Mass
15	Mass	Zone number
16	Opacity	Zone number
17	$\left(\frac{\partial \ln T}{\partial M}\right)_t = \frac{3L_s}{16\pi\sigma G M_s} \left(\frac{\kappa P}{T^4}\right)$	Zone number
21	Old temperature	Old mass
22	Old velocity	Old mass

<u>I</u>	<u>Ordinate</u>	<u>Abscissa</u>
23	Old density	Old mass
24	Ln (old pressure)	Old mass
25	Old mass	Zone number

The plots indexed as 11-17 are intended primarily for diagnostic use and are accessed by a call to the Profils subroutine. Those indexed as 21 to 25 are no different from those indexed as 11 to 17 when Profils is called from subroutine Accept, as usually is the case

Plot selection is controlled by a single input integer. The integer's bit pattern, considered as a sequence of indexed bins, is examined for bits in the positions tabulated above. Thus, for example, the integer 14, with bit pattern 1110 conjures up the plots indexed as 1, 2, and 3 (the right-most bit has index 0). By this arrangement, we are able to choose among 30 plots with a 10-digit integer.

Plot timing is controlled by input parameters, those indexed as 1 to 5 being treated separately from those indexed as 11 to 25. Examples of the use of these parameters are given in Sec. V.

V. SAMPLE PROBLEM

We illustrate DYN code use by running an actual problem, an attempt to construct a model of η Aql, a nearby galactic Cepheid for which there are ample observational data. From Pel,⁵ we get $\theta_e = 0.898$, which corresponds to an effective temperature $T_e = 5613$ K. We calculate a mean effective temperature of 5633 K from data in Whitney's Table 4,⁶ and of 5659 K from data in Oke's Table 3.⁷ Schmidt⁸ gives 5943 K. Although the trend seems to be upward, except for Pel's result, we adopt $T_e = 5610$ K as a starting value. Christy⁹ describes a nonlinear model with $T_e = 5600$ K, so we may be on familiar ground.

We derive the luminosity from the relation

$$L_T = 4\pi\sigma R_s^2 T_e^4, \quad (61)$$

where the radius R_s is $4.75 (10^{12})$ cm, as reported by Whitney.⁶ This differs from Christy's value of R_s , $4.44 (10^{12})$ cm, but his model oscillates in the

first harmonic mode (we anticipate oscillation in the fundamental mode). We get $L_T = 1.59 (10^{37})$ ergs/s from Eq. (61).

Keeping L_T and T_e fixed, we guess the star mass M_S using Eq. (5) by King, Cox, Eilers, and Davey.¹⁰

$$\log \frac{L_T}{L_0} = 3.48 \log \frac{M_S}{M_0} + 0.68, \quad (62)$$

where L_0 and M_0 are the respective solar mass and luminosity. We get

$$M_S = 1.38 (10^{34}) \text{ g.}$$

From the period mean density relation, Cox's Eq. (2.3),¹¹ we derive

$$\Pi \sqrt{M_S} \sim K, \text{ a constant,} \quad (63)$$

for fixed R_S .

As we do not expect Eq. (62) to give us a model with the correct period on the first try, we use Eq. (63) in an iterative scheme to obtain a model with the correct period. To determine the pulsation period that corresponds to a given set of parameters (T_e , L_T , M_S), we use a linear, nonadiabatic model code and iterate on the mass. Our algorithm is as follows.

1. Accept T_e and R_S values given above.
2. Calculate L_T using Eq. (61).
3. Calculate M_S using Eq. (63).
4. Determine the period Π that corresponds to (T_e , L_T , M_S) using the linear code.
5. If Π is close enough to the period Π_η of η Aql (7.18 days), stop; otherwise, proceed to 6.
6. Calculate a new mass using Eq. (63).
 - a. $M'_S = M_S$
 - b. $M_S = (\Pi/\Pi_\eta)^2 M'_S$
7. Return to step 4 with the new mass value.

Using this algorithm, we get $M_S = 1.84(10^{34})$ g, which yields $\Pi = 7.20$ days for the (pulsationally unstable) fundamental mode.

For clarity, we repeat that the values we accept for DYN code starting parameters are $(T_e, L_T, \text{ and } M_s) = (5610 \text{ K}, 1.59(10^{37}) \text{ ergs/s}, \text{ and } 1.84(10^{34}) \text{ g}.$

We are now ready to start the nonlinear calculation with the DYN code. The parameters T_e, L_T, M_s are entered on card 1 (see Fig. 3 and Table I) of the data, in the proper units. Parameter 2a is appropriate to the King 4a mixture. Card 4 indicates that we will run the problem for 2500 steps. Because parameter 5d is TRUE, 4b is 250, and 4f is $2^{20}-1$ (first 19 bits of the PLCTRL parameter turned "on"), we will get at least $7(2500/250) = 70$ plots (the diagnostic plots indexed as 11 to 17, at step intervals of 250). Thus, some provision must be made for microfilm or microfiche output.

Using the input cards shown in Fig. 3, we get a successful run that produces the partial listings shown in Figs. 4-7.

We continue the run by pumping for 4000 cycles (set parameter 5c to TRUE, parameter 2h to 1.20). Finally, we reset the pump parameter 5c to FALSE and run the model for a few periods to get what we accept as a limiting amplitude result, shown in Fig. 8.

VI. DYN CODE LISTING

Owing to its length (31 pages), the DYN code listing is not included in this report. However, anyone wanting a copy, may get it from the Los Alamos Scientific Laboratory, Group J-15 Group office.

7	9E AOL 4A2	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
6	ETA AOL -- KING 4A (FROM LASL TABLE) -- NUMBER 2	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
5	READFL F SAVEFL T PUMFPL F DIAGFL T	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
4	2500 250 1 500 0 1048575	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
3	1.0E-6 0.300 6 4 1.000 .010 1.414214 0.100 0.100 -10.000 4.000	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
2	0.90715640 1.53000 0.001 0.577350263 .500 .100 0.1.00	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
1	1.8400E10 1.5900E6 5.6100E-1	10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80

Fig. 3. DYN program input cards. The above parameters are described in Table 1.

TABLE I

THE PARAMETERS THAT APPEAR ON THE DYN INPUT CARDS
(The names given are those within the program. The routine in which the parameter is used is given in the last column.)

Card	Param.	Name	Description	Routine
1	a	MT	Star mass M_s	Static
	b	LT	Luminosity L_T	Static
	c	TE	Effective temperature T_e	Static
2	a	XH	Tabular helium abundance. See initial routine	Initial
	b	DMFAC	Nominal value of $(M_s - M_{i+1})/M_s - M_i$ in the Lagrangian region of the mesh. See Eqs. (47) and (48).	Static, Emfix
	c	TAUO	Optical depth of outermost zone.	Static
	d	QO	Value of the Hopf function that corresponds to TAUO.	Static
	e	DEVIATN	Criterion for moving the active region within the Lagrangian mesh. See Eq. (53).	Zonefix
	f	AEPS	Centering feedback coefficient ϵ_A . See Eq. (52).	Emfix
	g	AMX	$ A_E $ dominance criterion $ A $. See Eq. (54).	Emfix
	h	Yex	Scaling exponent α . See Eq. (56).	Emfix
3	a	EPS	Convergence criterion ϵ . See Fig. 2.	DYN
	b	TOL	Convergence criterion ϵ' . See Fig. 2.	DYN
	c	ICMAX	Solve iteration counter limit $I_{s(max)}$. See Fig. 2.	DYN
	d	ICTSM	Attempt counter limit $I_{A(max)}$. See Fig. 2.	DYN
	e	CQ	Artificial viscosity coefficient. See Eq. (16).	Solve
	f	CQCUT	Artificial viscosity cutoff G_0 . See Eq. (16).	Solve
	g	DTFAC	Time step factor.	Reset, Accept
	h	DTTOL	Temperature change tolerance.	Accept
	i	DVTOL	Specific volume change tolerance.	Accept
	j	UZERO	Initial velocity parameter.	Start
	k	DTLIM	Limit on Δt .	Accept

TABLE I (Continued)

<u>Card</u>	<u>Param.</u>	<u>Name</u>	<u>Description</u>	<u>Routine</u>
4	a	NSTEP	Limit on number of steps to calculate (number of times through C-D loop in Fig. 2).	DYN
	b	NPRINT	Modular print interval. Also controls diagnostic plot interval. Zone-by-zone prints and diagnostic plots arise when Mod (ISTEP,NPRINT) = 0. ISTEP is the step counter. See also 5d.	Accept
	c	PLOTMOD	Data for time plots (those indexed as 1-5 in the plot list) are stored in 500-element arrays that are entered when Mod (ISTEP,PLOTMOD) = 0. Plots are made when the arrays are full.	Accept
	d	SAVEMOD	Modular dump interval. DYN dumps onto the file MZSAVE whenever Mod (ISTEP, SAVEMOD) = 0. See also 5b.	DYN
	e	DMPSTEP	Dump file step at which to begin problem (currently a dummy variable). See also 5a.	Restor
	f	PLTCTRL	Integer whose bit pattern is analyzed to determine which plots to process.	Profils, Ploter
5	a	READFL	Flag that controls decision to construct a static model or to read the model from the file MZSAVE. See Fig. 2.	Start
	b	SAVEFL	Flag that controls decision to make a dump on the save file MZSAVE. See Fig. 2.	DYN
	c	PUMPFL	When this flag is set to TRUE, velocities are increased 25% at every kinetic energy maximum. Thus one may "pump" a model rapidly to limiting amplitude.	Accept
	d	DIAGFL	Flag that controls whether diagnostic plots are made. It overrides the selection implied by the PLTCTRL variable, parameter 4f, but does not interfere with the time plots. Time plots are avoided by turning the lower 11 bits of PLTCTRL "off".	Accept
6	a - h	X	Listing title card.	Initial
7	a	NCHMONA	Number of characters (9 or fewer) in the model name. See 7b.	Start
	b	MODEL	Model name that appears on the plots.	Start

090676

TARGET DATE 083176

ETA AUL -- KING 4A (FROM LASL TABLE) -- NUMBER 2

```

STAR MASS, LUMINOSITY, EFF. TEMP.= 1.8400E+10 1.5900E+06 5.6100E-01
TABULAR HELIUM ABUNDANCE= .90715640
DM FACTOR, TAU0, Q0= 1.97000 .001 .577350269
DEVIATION, AEDS, AMX, YEX= .600 .100 0. 1.00
EPSILON, EPSILON PRIME= 1.0E-08 .300
SOLVE ITERATION AND ATTEMPT COUNTER LIMITS= 6 4
CW, CCUT= 1.000 .010 UT MULTIPLIER, LIMIT= 1.414 4.000
TEMP. AND SP. VEL. CHANGE TOLERANCES= .100 .100
INITIAL SURFACE VELOCITY= -10.000
NUMBER STEPS AND PRINT, PLOT, AND SAVE INTERVALS= 2500 250 1 500
DUMP STEP FOR PROBLEM START= 0
PLCT CTRL = 1748574 (OCTAL CODE= 00000000000003777777)
FLAGS READ SAVE PUMP DIAGNOSTIC
      F T F T

```

Fig. 4.

Listing of program output heading. The PLCTCTRL parameter (4f of Fig. 3) is listed both in integer and octal code. The latter is included for simple dissection of the integer PLCTCTRL into its bit pattern.

Fig. 5.
Static model produced

STEP	TIME	DT	DE	IS	IA	M(2/3)	U(2/3)	L(2/3)	TE	KE	M(T=1)	ECHK	I1	JJ0	JJ1	Z2/3	ZT=1	PHASE	
////	KINETIC ENERGY MAXIMUM, PERIOD = 617.73																		
979	2165.72	2.2627	-2.48E+08	5	0	47576.07	11.247	1811720	.57721	2.289E+08	10.0996	-7.86E+04	28	39	3	44.8	37.8	.004	
980	2167.99	2.2627	-2.48E+08	4	0	47600.73	11.084	1807780	.57672	2.283E+08	10.1083	-8.11E+04	28	39	3	44.8	37.8	.007	
981	2170.25	2.2627	-2.48E+08	4	0	47625.06	10.948	1802935	.57617	2.275E+08	10.1171	-8.36E+04	28	37	2	44.7	37.8	.011	
982	2172.51	2.2627	-2.47E+08	4	0	47649.08	10.815	1797576	.57560	2.265E+08	10.1257	-8.58E+04	28	39	3	44.7	37.7	.015	
983	2174.78	2.2627	-2.47E+08	4	0	47672.79	10.676	1791984	.57501	2.253E+08	10.1343	-8.81E+04	28	38	3	44.7	37.7	.018	
984	2177.04	2.2627	-2.45E+08	4	0	47696.20	10.534	1786135	.57441	2.238E+08	10.1428	-9.04E+04	28	39	3	44.7	37.7	.022	
985	2179.30	2.2627	-2.44E+08	4	0	47719.29	10.386	1779996	.57378	2.221E+08	10.1512	-9.27E+04	28	38	3	44.7	37.7	.026	
986	2181.56	2.2627	-2.43E+08	5	0	47742.04	10.219	1773844	.57315	2.202E+08	10.1594	-9.49E+04	28	39	3	44.6	37.6	.029	
987	2183.83	2.2627	-2.41E+08	4	0	47764.37	10.029	1767793	.57252	2.180E+08	10.1674	-9.75E+04	28	22	2	44.6	37.6	.033	
988	2186.09	2.2627	-2.39E+08	5	0	47786.25	9.845	1761819	.57189	2.154E+08	10.1754	-9.98E+04	28	37	2	44.6	37.6	.037	
989	2188.35	2.2627	-2.37E+08	4	0	47807.78	9.669	1756070	.57130	2.131E+08	10.1831	-1.02E+05	28	39	3	44.6	37.6	.040	
990	2190.61	2.2627	-2.34E+08	4	0	47828.49	9.521	1750934	.57075	2.107E+08	10.1908	-1.05E+05	28	24	2	44.6	37.6	.044	
991	2192.88	2.2627	-2.32E+08	5	0	47849.77	9.312	1746393	.57025	2.073E+08	10.1984	-1.08E+05	28	37	2	44.6	37.5	.048	
992	2195.14	2.2627	-2.29E+08	5	0	47870.06	9.107	1741334	.56970	2.042E+08	10.2060	-1.10E+05	28	37	2	44.5	37.5	.051	
993	2197.40	2.2627	-2.26E+08	4	0	47889.95	8.943	1736823	.56907	2.008E+08	10.2136	-1.13E+05	28	37	2	44.5	37.5	.055	
994	2199.67	2.2627	-2.23E+08	4	0	47909.51	8.787	1732420	.56839	1.971E+08	10.2210	-1.14E+05	28	37	2	44.5	37.5	.059	
995	2201.93	2.2627	-2.19E+08	4	0	47929.73	8.598	1719899	.56769	1.934E+08	10.2281	-1.18E+05	28	38	3	44.6	37.4	.062	
996	2204.19	2.2627	-2.16E+08	4	0	47947.51	8.386	1712679	.56699	1.898E+08	10.2351	-1.21E+05	28	38	3	44.5	37.4	.066	
997	2206.45	2.2627	-2.12E+08	4	0	47965.81	8.177	1705657	.56630	1.864E+08	10.2419	-1.23E+05	28	38	3	44.4	37.4	.070	
998	2208.72	2.2627	-2.08E+08	5	0	47983.66	7.975	1698734	.56563	1.814E+08	10.2485	-1.26E+05	28	37	2	44.5	37.4	.073	
999	2210.98	2.2627	-2.04E+08	4	0	48001.08	7.769	1692079	.56498	1.773E+08	10.2550	-1.28E+05	28	39	3	44.5	37.4	.077	
1000	2213.24	2.2627	-2.00E+08	4	0	48018.03	7.553	1685742	.56436	1.729E+08	10.2613	-1.31E+05	28	37	2	44.5	37.3	.081	

Fig. 6.

Partial listing immediately preceding the variable listing at step 1000. Kinetic energy maxima are flagged on the output listing. Here we see that the program calculates a period of 7.15 days from kinetic energy maximum data. The data under the heading PE are a normalized combination of gravitational and internal energies. I_S and I_A are the solve and attempt counters indicated in Fig. 2. The "2/3" after the R, u, and T headings indicates that the listed radii, velocities, and temperatures are measured at optical depth $\tau = 2/3$. $M(T=1)$ is the mass depth of the point at which $T = 10\,000$ K (the location of the hydrogen ionization front). ECHK is a normalized energy check. I1 is the lower edge of the active region. JJ0 and JJ1 are the indexes of the zone controlling the timing and of the corresponding parameter ($1 \leftrightarrow u$, $2 \leftrightarrow T$, $3 \leftrightarrow R$). Z2/3 and Zt = 2 are the locations of optical depth $\tau = 2/3$ and of the hydrogen ionization front (where $T = 10\,000$ K).

I	EM	R	U	L	EMDOT	T	P	V	E	XAP	0
1	22.66908	10181.6191	0.000	1590000	0.000	101.88323	5.458490E+00	2.73305E+03	24313.52	1.1446E+08	0.
2	21.91847	15120.7022	.021	1589487	0.000	72.80128	1.24482E+00	8.26205E+03	17960.59	1.442E+08	0.
3	21.29419	16595.6411	.051	1588464	0.000	55.96286	4.11498E+01	1.91198E+04	14288.33	1.491E+08	0.
4	20.74118	21491.4647	.096	1587420	0.000	45.47040	1.00713E+01	3.994756E+04	11965.09	1.860E+08	0.
5	20.23417	25052.9744	.163	1584781	0.000	37.91481	7.34130E+02	7.422647E+04	10324.49	1.866E+08	0.
6	19.76017	29130.5533	.256	1582982	0.000	32.16333	3.55416E+02	1.31204E+05	9065.54	1.902E+08	0.
7	19.30780	29108.8236	.379	1579609	0.000	27.87216	1.81151E+02	2.24400E+05	8162.12	2.068E+08	0.
8	18.87147	29927.0483	.540	1571961	0.000	24.36967	9.06555E+03	3.70941E+05	7422.99	2.401E+08	0.
9	18.44463	31621.1030	.735	1568402	0.000	21.30291	5.31134E+03	5.93943E+05	6741.05	2.642E+08	0.
10	18.02144	33218.6175	.998	1566804	0.000	18.04085	2.97156E+03	9.33340E+05	6133.07	2.789E+08	0.
11	17.59646	34713.7773	1.283	1558369	0.000	16.35369	1.69471E+03	1.44480E+06	5900.19	2.944E+08	0.
12	17.17108	36108.6171	1.672	1551194	0.000	14.35465	9.75981E+04	2.40589E+06	5119.10	3.146E+08	0.
13	16.74740	37395.8500	1.762	1543096	0.000	12.59161	5.10125E+04	3.31337E+06	4681.44	3.281E+08	0.
14	16.32270	38580.4814	2.647	1540367	0.000	11.04842	3.37486E+04	4.91496E+06	4292.74	3.274E+08	0.
15	15.90281	39665.8550	2.401	1530601	0.000	9.73552	2.03130E+04	7.27149E+06	3965.87	3.236E+08	0.
16	15.47823	40675.5883	2.779	1517961	0.000	8.62879	1.41073E+04	1.07453E+07	3697.06	3.270E+08	0.
17	15.05811	41571.6402	3.217	1507706	0.000	7.65020	7.40243E+05	1.58918E+07	3474.27	3.30E+08	0.
18	14.64011	42421.4911	3.699	1437034	0.000	6.88719	4.48718E+05	2.32426E+07	3287.34	3.503E+08	0.
19	14.22642	43207.9712	4.232	1369734	0.000	6.18614	2.70045E+05	3.43940E+07	2967.89	3.849E+08	0.
20	13.81741	43934.6683	4.827	1341311	0.000	5.56200	1.72745E+05	5.49833E+07	2697.04	4.295E+08	0.
21	13.41769	44603.9300	5.507	1300379	0.000	4.97054	1.04496E+05	7.07458E+07	2407.04	4.699E+08	0.
22	13.02977	45203.8976	6.140	1302040	0.000	4.47054	7.0047E+06	9.70314E+07	2301.54	4.772E+08	0.
23	12.65912	45735.9275	6.715	1415262	0.000	3.90619	4.7111E+06	1.26953E+08	2592.14	4.444E+08	0.
24	12.31255	46173.3061	7.159	1415262	0.000	3.50143	3.16104E+06	1.63729E+08	2047.02	3.870E+08	0.
25	11.94747	46580.5674	7.559	1594555	0.000	3.16623	2.13710E+06	2.14283E+08	1907.05	3.341E+08	0.
26	11.60519	46974.7324	8.018	1553232	0.000	2.87155	1.49473E+06	2.75908E+08	1824.11	3.22E+08	0.
27	11.30335	47222.1352	8.449	1543227	0.000	2.60974	1.00168E+06	3.51170E+08	1753.00	3.23E+08	0.
28	10.97819	47503.5105	8.749	1399304	0.000	2.37287	7.82299E+07	4.29459E+08	1666.90	3.91E+08	0.
29	10.73500	47693.4391	8.933	1340054	0.000	2.14693	5.56374E+07	5.12490E+08	1567.48	3.78E+08	0.
30	10.58248	47840.3171	8.984	1340054	0.000	2.05064	4.36149E+07	4.63305E+08	1632.12	4.370E+08	0.
31	10.47646	47872.6373	8.935	1544434	0.000	1.94631	3.50162E+07	5.12490E+08	1567.48	5.115E+08	0.
32	10.40549	47917.8614	8.845	1531170	0.000	1.84251	2.70473E+07	5.12490E+08	1567.48	6.107E+08	0.
33	10.35442	47947.6050	8.747	1529433	0.000	1.74304	2.13710E+07	5.12490E+08	1495.31	7.566E+08	0.
34	10.31742	47947.1007	8.674	1566500	0.000	1.64128	1.60745E+07	4.76420E+08	1356.00	8.975E+08	0.
35	10.29136	47974.7300	8.586	1574351	0.000	1.52709	1.10071E+07	4.41913E+08	1277.09	1.398E+09	0.
36	10.27392	47994.7804	8.471	1582408	0.000	1.37410	8.42872E+07	3.92872E+08	1198.98	2.432E+09	0.
37	10.26157	47994.7231	8.222	1510942	0.000	1.26228	6.24409E+07	2.63790E+08	727.89	4.592E+09	0.
38	10.25331	47994.8842	7.745	1653270	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
39	10.24573	47994.1851	7.615	1674713	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
40	10.22745	47997.9073	7.579	1683171	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
41	10.20332	48003.2645	7.572	1684337	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
42	10.16944	48003.5243	7.552	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
43	10.12007	48007.9815	7.549	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
44	10.06956	48014.0140	7.545	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
45	9.94955	48022.0173	7.562	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
46	9.81419	48032.6564	7.564	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
47	9.63242	48045.3033	7.495	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
48	9.43195	48061.6335	7.559	1684695	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
49	9.20733	48084.0844	7.474	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
50	8.96191	48104.8064	7.478	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
51	8.69645	48134.9740	7.500	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
52	7.91118	48155.9004	7.544	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
53	7.68611	48194.3007	7.345	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
54	7.06094	48233.5722	7.239	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
55	6.63548	48255.1170	7.339	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
56	6.21011	48289.0204	7.524	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.
57	5.78544	48309.5012	7.247	1686637	0.000	1.03700	4.22041E+07	1.33341E+08	190.95	4.190E+08	0.

Fig. 7. Variable listing at step 1000. EM is $\ln(M_S - M_i)$. EMDOT is $\left(\frac{\delta M}{\delta t}\right)_x$. The temperature data show that the ionization front contains four zones.

Fig. 8.
Plots of radius, velocity, luminosity, and mass depth at limiting amplitude.
NOTE: The scales have been redone for clarity.

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