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DIMENSIONAL VLASOV EQUATION FOR
A MAGNETIZED PLASMA

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THE INTEGRATION OF THE THREE-DIMENSIONAL
VLASOV EQUATION FOR A MAGNETIZED PLASMA

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A second order splitting scheme is developed to integrate the three dimensional Vlasov equation for a plasma in a magnetic field. In the scheme we divide the integration of the Vlasov equation into a series of intermediate steps and Fourier interpolation and the ASD method with a third order Taylor expansion are used to integrate the fractional equations. Numerical experiments related to cyclotron waves in 2 and 2-1/2D are demonstrated with high accuracy and efficiency. The computer storage requirements are modest; for example, a typical 2D nonlinear electron plasma simulation requires only 4000 "particles".

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1. INTRODUCTION

The numerical integration of the Vlasov equation has been intensely studied in recent years [1,2,3,4,5,6]. However, little progress has been made on fast and accurate integration of the Vlasov equation in two and three dimensions for a magnetized plasma. A knowledge of the nonlinear evolution of the magnetized plasmas in two and three dimensions is indispensable in understanding the plasma physics of controlled thermonuclear fusion.

In this paper we generalize the splitting scheme [6], developed earlier for integrating the Vlasov equation for an unmagnetized plasma, to the integration of the three dimensional Vlasov equation for a magnetized plasma. In the scheme we divide the integration of the Vlasov equation into a series of intermediate steps and the overall scheme is second order in Δt . The resulting fractional equations are computed with Fourier interpolation and the ASD method with a third order Taylor expansion [5]. Since FFT is used to do the Fourier interpolation and Gazdag's ASD calculations, the programming becomes very simple and the accuracy and efficiency are accomplished.

The generalized splitting scheme is described in Section 2. In Section 3 we demonstrate the accuracy and efficiency of the splitting scheme through numerical experiments related to linear and nonlinear cyclotron waves in 2 and 2-1/2 dimensions for a Maxwellian equilibrium plasma. The conclusions are given in Section 4.

2. THE GENERALIZED SPLITTING SCHEME

In this Section we will present the splitting scheme for integrating the Vlasov equation with external electric and magnetic fields. For simplicity, we consider the case with $\vec{B}_{\text{ext}} = B_0 \hat{e}_z$, $\vec{E}_{\text{ext}} = 0$. Then we will show that the numerical procedure can be generalized to include more complicated problems.

Consider a three dimensional, electrostatic collisionless electron plasma immersed in a uniform static magnetic field $\vec{B}_0 = B_0 \hat{e}_z$. The dimensionless forms of the Vlasov-Poisson equations are

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \left(\vec{E}(\vec{x}, t) + \vec{v} \times \frac{\vec{v}}{\omega_c} \right) \cdot \frac{\partial f}{\partial \vec{v}} = 0 \quad (1)$$

$$\nabla \cdot \vec{E} = 1 - \int f(\vec{x}, \vec{v}, t) d\vec{v} \quad (2)$$

where $\omega_c = \frac{eB_0}{mc\omega_{pe}}$ is the ratio of cyclotron frequency to electron plasma frequency ω_{pe} . A periodic boundary condition in $\vec{x}$ is assumed. Following the principles of the splitting scheme used in [6,7], we split Eq. (1) according to the following 4-step procedure:

(i) solve the free streaming equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0 \quad (3)$$

for half a time step with the solution

$$f^*(\vec{x}, \vec{v}) = f^n(\vec{x} - \frac{\vec{v} \Delta t}{2}, \vec{v}) \quad (4)$$

(ii) calculate the electric field from the Poisson equation

$$\nabla \cdot \vec{E}^*(\vec{x}) = 1 - \int f^*(\vec{x}, \vec{v}) d\vec{v} \quad (5)$$

(iii) integrate the acceleration equation

$$\frac{\partial f}{\partial t} - (\vec{E}^*(\vec{x}) + \vec{v} \times \vec{\omega}_c) \cdot \frac{\partial f}{\partial \vec{v}} = 0 \quad (6)$$

for a whole time step with the solution

$$f^{**}(\vec{x}, \vec{v}) = f^*(\vec{x}, \vec{v}^*) \quad (7)$$

where $\vec{v}^*$ is the solution of the characteristics of Eq. (6)

and is given by

$$v_x^*(\vec{x}) = E_y^*(\vec{x})/\omega_c + \left(v_x - \frac{E_y^*(\vec{x})}{\omega_c} \right) \cos \omega_c \Delta t + \left(v_y + \frac{E_x^*(\vec{x})}{\omega_c} \right) \sin \omega_c \Delta t$$

$$v_y^*(\vec{x}) = - \frac{E_x^*(\vec{x})}{\omega_c} + \left(v_y + \frac{E_x^*(\vec{x})}{\omega_c} \right) \cos \omega_c \Delta t - \left(v_x - \frac{E_y^*(\vec{x})}{\omega_c} \right) \sin \omega_c \Delta t$$

$$v_z^*(\vec{x}) = v_z + E_z^*(\vec{x}) \Delta t \quad (8)$$

(iv) solve the free streaming equation $\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} = 0$

again for half a time step with the solution

$$f^{n+1}(\vec{x}, \vec{v}) = f^{**}(\vec{x} - \frac{\vec{v} \Delta t}{2}, \vec{v}) \quad (9)$$

By substituting Eqs. (7) and (4) into Eq. (9) we obtain

$$f^{n+1}(\vec{x}, \vec{v}) = f^n(\vec{x}', \vec{v}') \quad (10)$$

with

$$\vec{x}' = \vec{x} - \frac{\Delta t}{2} (\vec{v} + \vec{v}')$$

$$\vec{v}' = \vec{v}^* \left(\vec{x} - \frac{\vec{v} \Delta t}{2} \right) \quad (11)$$

where $\vec{v}^*(\vec{x})$ is given by (8).

The characteristics of Eq. (1) are given by

$$\frac{d\vec{v}}{dt} = -[\vec{E}(\vec{x}, t) + \vec{v} \times \vec{\omega}_c] \quad (12)$$

$$\frac{d\vec{x}}{dt} = \vec{v} \quad (13)$$

Their solution is equivalent to the solution of Eq. (1). Now we are going to integrate Eqs. (12) and (13) from t_n to t_{n+1} . Their solutions, correct to second order in Δt , are given by

$$\begin{aligned} \vec{x}^n &= \vec{x}^{n+1} - \frac{\Delta t}{2} (\vec{v}^n + \vec{v}^{n+1}) \\ v_x^n &= \frac{E_y^{n+\frac{1}{2}}}{\omega_c} + \left(v_x^{n+1} - \frac{E_y^{n+\frac{1}{2}}}{\omega_c} \right) \cos \omega_c \Delta t \\ &\quad + \left(v_y^{n+1} + \frac{E_x^{n+\frac{1}{2}}}{\omega_c} \right) \sin \omega_c \Delta t \\ v_y^n &= - \frac{E_x^{n+\frac{1}{2}}}{\omega_c} + \left(v_y^{n+1} + \frac{E_x^{n+\frac{1}{2}}}{\omega_c} \right) \cos \omega_c \Delta t \\ &\quad - \left(v_x^{n+1} - \frac{E_y^{n+\frac{1}{2}}}{\omega_c} \right) \sin \omega_c \Delta t \\ v_z^n &= v_z^{n+1} + E_z^{n+\frac{1}{2}} \Delta t \end{aligned} \quad (14)$$

where the time centered electric field $E^{n+\frac{1}{2}}$ is evaluated at $\vec{x}^{n+\frac{1}{2}} = \vec{x}^{n+1} - \vec{v}^{n+1} \frac{\Delta t}{2} + O(\Delta t^2)$.

A comparison between Eqs. (11) and (14) shows that the splitting scheme (3) - (9) is correct to second order in Δt .

It is clear that after we have shifted f in the $\vec{x}$ space at time $t = 0$ for one half time step, we can solve f in the $\vec{v}$ space for a full time step, then shift f in $\vec{x}$ space for a full time step and so on.

The above procedure can be applied to more complicated situations with magnetic field inhomogeneity and the results remain accurate to second order in Δt . This can be easily proven and will not be presented here. Another advantage of this splitting scheme is that it does not depend on periodic boundary conditions, while the Fourier-Hermite transform method is critically dependent upon spatial periodicity.

From the above analysis we see that our scheme depends on the accuracy of the numerical integration of the linear Eqs. (4) and (7). There are acceptable ways of integrating them, such as Fourier and spline interpolation methods [6], ASD method [5,8], etc. However, the advantage of each method depends upon the geometry and boundary conditions of the system studied. For the free streaming Eq. (4) with spatial periodicity, we will use the Fourier interpolation method to do three subsequent 1D shiftings. The Fourier interpolation can be done more efficiently by employing FFT. Integrating the acceleration Eq. (6) is more difficult. Due to the fact that the electric and magnetic fields accelerate the particles differently, interpolation methods become much more complicated and time consuming in two and three dimensions. Therefore, we will use one more splitting scheme to integrate Eq. (6) (called method 1).

It is obvious from Eq. (8) that Eq. (6) is equivalent to the following sequence of equations:

$$\frac{\partial f}{\partial t} - \frac{E_y}{\omega_c \Delta t} \frac{\partial f}{\partial v_x} + \frac{E_x}{\omega_c \Delta t} \frac{\partial f}{\partial v_y} = 0 \quad (15a)$$

$$\frac{\partial f}{\partial t} + \omega_c \frac{\partial f}{\partial \phi} = 0 \quad (15b)$$

$$\frac{\partial f}{\partial t} + \frac{E_y}{\omega_c \Delta t} \frac{\partial f}{\partial v_x} - \frac{E_x}{\omega_c \Delta t} \frac{\partial f}{\partial v_y} = 0 \quad (15c)$$

$$\frac{\partial f}{\partial t} - E_z \frac{\partial f}{\partial v_z} = 0 \quad (15d)$$

where Eqs. (15a), (15c) must be integrated for the time step Δt . We will use the ASD method together with the third order truncation of the Taylor series to solve (15a) and (15c) in cylindrical coordinates. Doing this, (15a) and (15c) become

$$\frac{\partial f}{\partial t} + \frac{E_x \sin \phi - E_y \cos \phi}{\omega_c \Delta t} \frac{\partial f}{\partial v} + \frac{E_x \cos \phi + E_y \sin \phi}{\omega_c \Delta t v} \frac{\partial f}{\partial \phi} = 0 \quad (16a)$$

and

$$\frac{\partial f}{\partial t} - \frac{E_x \sin \phi - E_y \cos \phi}{\omega_c \Delta t} \frac{\partial f}{\partial v} - \frac{E_x \cos \phi + E_y \sin \phi}{\omega_c \Delta t v} \frac{\partial f}{\partial \phi} = 0 \quad (16b)$$

respectively. Equations (15b) and (15d) will be solved by

employing a one dimensional Fourier interpolation. The stability and error analysis of using the third order Taylor series with ASD method on integrating the convective type equations has been investigated by Gazdag [5,8]. It has been applied with success in solving several nonlinear partial differential equations including the Vlasov-Poisson system of equations in three phase space variables [5].

We have also applied Gazdag's ASD method to integrate Eq. (6) directly (called method 2). In cylindrical coordinates, Eq. (6) can be written as

$$\frac{\partial f}{\partial t} - (E_x \cos\phi + E_y \sin\phi) \frac{\partial f}{\partial v} + \left(\frac{E_x \sin\phi - E_y \cos\phi}{v} + \omega_c \right) \frac{\partial f}{\partial \phi} = 0$$

$$\frac{\partial f}{\partial t} - E_z \frac{\partial f}{\partial v_z} = 0 \tag{17}$$

However, the stability criteria forces us to use a small time step of integration with this method because of the strong magnetic field. Employing method 1 allows us to use a larger time step because the fast gyration (15b) of the particle has been separated from the slow guiding center drift motion (15a) and (15c). Also by choosing $\omega_c \Delta t = \Delta\phi$, we don't need to perform interpolation to solve (15b). Thus, for the overall computation, method 1 is superior to method 2.

In the following, we will demonstrate the accuracy and efficiency of the splitting scheme by integrating the 2 and 2-1/2D Vlasov equations.

3. NUMERICAL RESULTS

In this Section we will present the results of integrating the 2 and 2-1/2D Vlasov equation. The external magnetic field $\vec{B}$ is chosen to lie in the y-z plane with $\vec{B} \cdot \hat{e}_z = B \cos\theta$. We use a rectangular mesh in the x-y plane. In the velocity space we use cylindrical coordinates such that $v_{||}$ is taken along $\vec{B}$ and $\vec{v}_{\perp}$ along $\hat{e}_x$. No mesh point is placed at the origin and smallest velocities associated with the mesh points in the $v_{||}$ direction and $v_{\perp}$ plane are $\Delta v_{||}/2$ and $\Delta v_{\perp}/2$ respectively as shown in Fig. 1. The computational domain is

$$R = \{(\vec{x}, \vec{v}) \mid 0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq \phi \leq 2\pi, \\ |v_{||}| \leq v_{||\max}, |v_{\perp}| \leq v_{\perp\max}\}.$$

$N_x, N_y, 2N_p, N_{\phi}$, and N_r designate the number of mesh points used in the x, y, $v_{||}$, ϕ , and $v_{\perp}$ directions respectively.

(A) 2D and 2-1/2D Linear Cases

We first applied the splitting scheme to integrate the linearized Vlasov-Poisson system in 2D and 2 1/2D for one-and two-species plasmas. The equilibrium distribution is a Maxwellian, f_{oj} , with two or three velocity components and the initial condition is

$$f_j(\vec{x}, \vec{v}, 0) = f_{oj}(\vec{v}) (1 + A_j \cos \vec{k} \cdot \vec{x})$$

The subscript $j = e, i$ refers to electrons and ions. In this linear case the acceleration equation can be solved analytically by integrating along the characteristics. For the one-species plasma case, the ions are considered to be an immobile background. Fig. 2 shows the power spectrum of the 2D electron

Bernstein modes with $k_{\perp}\lambda_D = 1.02$, $\omega_{ce}/\omega_{pe} = 1$. The computational parameters are $N_x = N_y = 4$, $N_{\phi} = N_r = 8$, $N_p = 0$, $\omega_{pe}\Delta t = \pi/4$, $v_{\parallel\text{max}} = 0$, and $v_{\perp} = 4.0$. The sequence of the Fourier Mode of potential $\phi(k_{\perp}, t)$ over 512 time steps was Fourier transformed with respect to time in order to obtain the power spectrum $p(k_{\perp}, \omega) \propto |\phi(k_{\perp}, \omega)|^2$. We see from Fig. 2 that the result agrees very well with the linear theory [9]. We should point out the existence of the strong DC mode ($\omega=0$) in 2D collisionless magnetized plasmas which was predicted by Hsu and Montgomery [10]. We also note that the recurrence effect, which occurs in the numerical integration of 1D Vlasov equation for an unmagnetized plasma [4,6], disappears for a 2D magnetized plasma. This is due to the fact that the magnetic field plays a dominant role in the particle motion. For particle motion along the $\vec{B}$ field we would still expect recurrence effects. However, for most of the physically interesting cases, the wavelength along the field lines are very large compared to the perpendicular wavelength ($k_{\parallel} \ll k_{\perp}$). Therefore, the recurrence time would be very long for a modest number of grid points in the $v_{\parallel}$ direction.

Fig. 3 shows the power spectrum of 2D 2-species plasma with $k_{\parallel}\lambda_D = 0$, $k_{\perp}\lambda_D = 1.414$, $\omega_{ce}/\omega_{pe} = 1$, $T_e/T_i = 1$, $m_i/m_e = 25$. The computation parameters are the same as in Fig. 2. Besides the electron Bernstein modes and DC mode, we have also obtained the ion Bernstein modes. Fig. 4 shows the power spectrum of a 2-1/2D electron plasma with $k_{\parallel}\lambda_D = 0.1$, $k_{\perp}\lambda_D = 1.414$, $\omega_{ce}/\omega_{pe} = 1$.

The computation parameters are $N_x = N_y = 4$, $N_\phi = N_r = N_p = 8$, $v_{\perp \max} = 4.1$, $v_{\parallel \max} = 4.2$, $\omega_{pe} \Delta t = \pi/4$. The result agrees very well with Tataronis' linear solutions [11]. Fig. 5 shows the power spectrum of a 2-1/2D, 2-species plasma with $k_{\parallel} \lambda_D = 0.01$, $k_{\perp} \lambda_D = 1.414$, $\omega_{ce}/\omega_{pe} = 1$, $T_e/T_i = 1$, $m_i/m_e = 25$. The computation parameters are the same as in Fig. 4. For the above computations, the total density, total momentum and total kinetic energy are conserved except due to roundoff errors.

(B) 2D Nonlinear Case

A 2D electron plasma with $\omega_{ce}/\omega_{pe} = 1$ has been studied. The initial condition is taken to be

$$f(\vec{x}, \vec{v}, 0) = 1/2\pi \exp(-v^2/2) (1 + A \cos kx)$$

where $A = 0.001$ and $k\lambda_D = 1.02$. The computation parameters are $N_x = N_y = 8$, $N_\phi = N_r = 8$, $v_{\perp \max} = 4.5$, and $\omega_{pe} \Delta t = \pi/4$. We used method 1 described in Section 2 to integrate the acceleration Eq. (6). Method 1 is superior to method 2 because it separates the fast gyro motion from the slow guiding center drift. If we use method 2, we have to take $\omega_{pe} \Delta t$ as small as $\omega_{pe} \Delta t = 0.05$ to obtain the same accuracy.

Figs. 6 and 7 show the power spectrum and autocorrelation of the potential $\phi(k_{\perp}, t)$. The computation is carried out for 128 time steps with $\omega_{pe} T_{\max} = 32\pi$. Comparing Figs. 6 and 2, we see that the results of the nonlinear calculation agree very well with the linear case. The autocorrelation of the potential shows the phase mixing effects of the Bernstein modes, which generate the damping [12] and recurrence phenomena.

The conservation of the total energy is very good with a relative error 10^{-5} at $\omega_{pe}t = 32\pi$. This simulation corresponds to the use of 4096 "particles" as far as computer storage is concerned. As a reference the computation time is 5 min. 30 sec. for an unoptimized Fortran code on an IBM 360/91. Thus a great saving of computer time and storage is achieved.

4. CONCLUSION

In this work we have generalized the second order splitting scheme given by Cheng and Knorr [6] to integrate the three dimensional, collisionless, magnetized Vlasov system. The resulting free streaming equations are integrated by employing Fourier interpolation as before, but the acceleration equation is integrated by a new splitting scheme, and Gazdag's ASD method and Fourier interpolation are employed to solve the final fractional equations. As a result, the time step Δt can be chosen to be large because the fast gyromotion of the plasma has separated from the slow guiding center drift and is integrated exactly.

The splitting scheme was tested on the linear and nonlinear Vlasov equation in 2D and 2-1/2D with a Maxwellian equilibrium distribution. The results of these experiments are in very good agreement with the analytic theory. The important feature is not only that the computation time is quite low, but also that good results can be obtained with a few number of "particles". Thus, the 3D integration of the Vlasov equation can be carried out by using presently available computers.

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REFERENCES

1. T.P. Armstrong, R.C. Harding, G. Knorr, and D. Montgomery,
in "Methods in Computational Physics"
Vol. 9, B. Alder, S. Fernbach, M. Rotenberg, eds.,
Academic Press, New York, 1970.
2. J. Nuhrenberg, J. Appl. Math. and Phys.
(ZAMP) 22, 1057 (1971).
3. J. Denavit and W.L. Kruer, Phys. Fluids 14, 1782 (1971).
4. G. Knorr, J. Comp. Phys. 13, 165 (1973).
5. J. Gazdag, J. Comp. Phys. 19, 77 (1975).
6. C.Z. Cheng, and G. Knorr, U. Iowa Report 75-24,
(submitted to J. Comp. Phys. 1975).
7. N.N. Yanenko, "The Method of Fractional Steps",
Springer Verlag, New York, 1970.
8. J. Gazdag, J. Comp. Phys. 20, 196 (1976).
9. I.B. Bernstein, Phys. Rev. 108, 10 (1958).
10. J.Y. Hsu and D. Montgomery, Phys. Lett. 48A, 39 (1974).
11. J.A. Tataronis and F.W. Crawford, J. Plasma Phys. 4, 231 (1970).
12. D.E. Baldwin and G. Rowlands, Phys. Fluids 9, 2444 (1966).

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Fig. 1. The 2-1/2D configuration. (a) shows the spatial domain and (b) shows the computational mesh in the velocity space.

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Fig. 2. Power spectrum of the 2D linearized electron electrostatic waves in a magnetized plasma with $k_{\perp} \lambda_{De} = 1.02$, $\omega_{ce}/\omega_{pe} = 1$, $k_{\parallel} \lambda_{De} = 0$.

752290

Fig. 3. Power spectrum of the 2D linearized electrostatic waves in a 2-species magnetized plasma with $k_{\parallel} \lambda_{De} = 0$, $k_{\perp} \lambda_{De} = 1.414$, $\omega_{ce}/\omega_{pe} = 1$, $T_e/T_i = 1$, $m_i/m_e = 25$.

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Fig. 4. Power spectrum of the 2-1/2D linearized electron electrostatic waves in a magnetized plasma with $k_{\parallel} \lambda_{De} = 0.1$, $k_{\perp} \lambda_{De} = 1.414$, $\omega_{ce}/\omega_{pe} = 1$.

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Fig. 5. Power spectrum of the 2-1/2D linearized electrostatic waves in a 2-species magnetized plasma with $k_{\parallel} \lambda_{De} = 0.01$, $k_{\perp} \lambda_{De} = 1.414$, $\omega_{ce}/\omega_{pe} = 1$, $T_e/T_i = 1$, and $m_i/m_e = 25$.

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Fig. 6. Power spectrum of the 2D nonlinear electron electrostatic waves in a magnetized plasma with $k_{\parallel} \lambda_{De} = 0$, $k_{\perp} \lambda_{De} = 1.02$, $\omega_{ce}/\omega_{pe} = 1$.

Fig. 7. The autocorrelation of the potential $\phi(k_{\perp}, t)$ for the nonlinear magnetized electron plasma as in Fig. 6. 762086