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EXPLICIT INVERSES of SOME SPECIAL MATRICES (WITH A FEW COMPUTER PROGRAMS)

V.R.R. Uppuluri and B.L. Kirk

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FEBRUARY 1976

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EXPLICIT INVERSES OF SOME SPECIAL MATRICES
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ABSTRACT

In this report we present explicit inverses of some classes of special matrices. In Section I, we discuss the tridiagonal matrices. In Section II we discuss an algorithm due to Sherman and Morrison, which is useful in finding the inverse, when an explicit inverse is known for an unperturbed matrix. In Section III we discuss inverses of some patterned matrices. In Appendix A and Appendix B we give computer programs of some of the problems discussed in Sections I, II, and III.

INTRODUCTION AND SUMMARY

Inversion of matrices is an important topic in the theory and applications of matrices. In general, it is too much to expect to find the explicit inverse of an arbitrary matrix. For matrices with special structures, at times it is possible to give the general term of the inverses. Such examples of classes of matrices are scattered in literature [(1), (8), (2), (5)], although there are some attempts to bring them together by Graybill (4), Gregory and Karney (5) among others [(6), (9)].

In this report, we will put together some classes of matrices whose explicit inverses are known and make available computer programs for exact computations of the general term of the inverses. In Section I, we present some results in the case of tridiagonal matrices. Some of these are based on Ukita's theorem (which is not widely known) in the case of symmetric tridiagonal matrices. In Section II, we discuss the

result of Sherman and Morrison (15) and develop a computer program for this purpose. We believe that this is very useful; since in some problems we may know the explicit inverse of a matrix which differs from the original in a few elements, and this program will then enable us to compute the inverse of the desired matrix. The Hilbert matrix, the Vandermonde matrix and a few other special matrices are also discussed in this section. Section III contains a few classes of matrices found in Gregory and Karney (5). Computer programs in Appendix B refer to this section.

SECTION I. TRIDIAGONAL MATRICES

In this section, we focus our attention on tridiagonal matrices.

Let us define a general $n \times n$ tridiagonal matrix by

$$(1.1) \quad A = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \dots & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & a_{n-1, n-1} & \dots & a_{n-1, n} \\ 0 & 0 & 0 & 0 & \dots & \dots & a_{n, n-1} & \dots & a_{nn} \end{bmatrix}$$

It is known (6) that any arbitrary matrix is similar to a tridiagonal matrix. If we have explicit forms for some special tridiagonal matrices, it may be of value to find the exact inverses.

In the special case when all the diagonal elements are equal to x ($a_{ii} = x$, $i = 1, 2, \dots, n$), and all the off-diagonal elements are equal to each other ($a_{ij} = a_{kl} = 1$, $i, j, k, l = 1, 2, \dots, n$, $i \neq j$, $k \neq l$), it is referred to as a continuant. Continuants are found to be useful in problems in physics and chemistry (13). The $n \times n$ continuant

$$(1.2) \quad A = \begin{bmatrix} x & 1 & 0 & \dots & 0 & 0 \\ 1 & x & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & x \end{bmatrix}$$

has the following inverse (5),

$$(1.3) \quad A^{-1} = \frac{1}{\alpha_n} \begin{bmatrix} a^{ij} \end{bmatrix},$$

where

$$(1.4) \quad a^{ij} = \begin{cases} \alpha_{i-1} \alpha_{n-i}, & \text{if } i=j \\ (-1)^{i+j} \alpha_{i-1} \alpha_{n-j}, & \text{if } j>i \\ a^{ji} & \text{if } j<i, \end{cases}$$

and

$$(1.5) \quad \begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= x \\ \alpha_k &= x\alpha_{k-1} - \alpha_{k-2}, \quad k = 2, 3, \dots, n. \end{aligned}$$

These results also appear in a note by Vemuri (17), where the matrix was set up as a model of a one-dimensional resistance network.

In Appendix A3, we give a program for the computation of the general term of A^{-1} .

Upon expansion of the formulas (1.3)-(1.5), one can observe that A^{-1} satisfies the conditions of Ukita's theorem. The following statement is a version of this theorem (4).

Theorem: Let A be an $n \times n$ symmetric nonsingular matrix with $A_{1j} \neq 0$ for $j = 2, \dots, n$. A necessary and sufficient condition for the inverse of the matrix A to be tridiagonal is the following:

$$\begin{aligned}
 \frac{a_{2j}}{a_{1j}} &= \theta_2, \quad 2 \leq j \leq n, \\
 \frac{a_{3j}}{a_{1j}} &= \theta_3, \quad 3 \leq j \leq n, \\
 &\vdots \\
 \frac{a_{tj}}{a_{1j}} &= \theta_t, \quad t \leq j \leq n, \\
 &\vdots \\
 \frac{a_{nn}}{a_{1n}} &= \theta_n, \quad j = n.
 \end{aligned}
 \tag{1.6}$$

The elements of A^{-1} are:

$$a^{11} = -\theta_2(a_{12} - \theta_2 a_{11})^{-1}, \tag{1.7}$$

$$\begin{aligned}
 a^{tt} &= - \frac{a_{t-1,t+1} - \theta_{t+1,t-1}}{(a_{t-1,t} - \theta_{t-1,t-1})(a_{t,t+1} - \theta_{t+1,t})}, \\
 &\text{for } t = 2, 3, \dots, n-1,
 \end{aligned}
 \tag{1.8}$$

$$a^{nn} = \frac{a_{1,n-1}}{a_{1n}(a_{n-1,n} - \theta_{n-1,n-1})}, \tag{1.9}$$

$$\begin{aligned}
 a^{t,t-1} &= a^{t-1,t} = (a_{t-1,t} - \theta_{t-1,t-1})^{-1}, \\
 &\text{for } t = 2, 3, \dots, n,
 \end{aligned}
 \tag{1.10}$$

$$a^{ij} = 0 \quad \text{if } |i-j| > 1. \tag{1.11}$$

These formulas are programmed in Appendix A2.

Matrices which satisfy the conditions of Ukita's theorem occur as variance-covariance matrices of problems in mathematical statistics. One such example is the variance-covariance matrix of order statistics for a sample of size n from an exponential distribution (12):

$$(1.12) \quad A = \begin{bmatrix} \frac{1}{n^2} & \frac{1}{n^2} & \dots & \dots & \dots & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^2} + \frac{1}{(n-1)^2} & \dots & \dots & \dots & \frac{1}{n^2} + \frac{1}{(n-1)^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n^2} & \dots & \dots & \dots & \sum_{i=1}^n \frac{1}{(n-i+1)^2} \end{bmatrix}$$

The inverse is given by:

$$(1.13) \quad A^{-1} = \begin{bmatrix} n^2 + (n-1)^2 & -(n-1)^2 & 0 & \dots & 0 \\ -(n-1)^2 & (n-1)^2 + (n-2)^2 & -(n-2)^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

An interesting class of matrices which satisfy Ukita's theorem is given by

$$(1.14) \quad a_{ij} = \begin{cases} c_j c_i (a_1 + \dots + a_j) & \text{if } i > j, \\ c_j c_i (a_1 + \dots + a_i) & \text{if } j \geq i. \end{cases}$$

The inverse of this matrix is given by $A^{-1} = (a^{ij})$, where

$$(1.15) \quad a^{ij} = \begin{cases} \frac{1}{c_i^2} \left(\frac{1}{a_i} + \frac{1}{a_{i+1}} \right) & \text{if } i=j, \\ -\frac{1}{c_{i-1} c_i a_i} & \text{if } j = i+1 \text{ or } i = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

If the matrix A is defined by

$$(1.16) \quad A = \begin{bmatrix} 1 & q & q^2 & \dots & q^n \\ q & 1 & q & \dots & q^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q^n & q^{n-1} & q^{n-2} & \dots & 1 \end{bmatrix},$$

then it is obvious that A satisfies Ukita's theorem again. Explicitly, its inverse is

$$(1.17) \quad A^{-1} = \frac{1}{1-q^2} \begin{bmatrix} 1 & -q & 0 & \dots & 0 \\ -q & 1+q^2 & -q & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1+q^2 & -q \\ 0 & \dots & \dots & -q & 1 \end{bmatrix}$$

The applicability of Ukita's theorem can also be noticed in the matrix $A = (a_{ij})$, $i, j = 1, \dots, n$, where

$$(1.18) \quad a_{ij} = \begin{cases} i/j & \text{if } i \leq j, \\ j/i & \text{if } i > j. \end{cases}$$

The elements of A^{-1} are supplied by the following formulas:

$$(1.19) \quad a^{ij} = \begin{cases} 4i^3/(4i^2-1) & \text{if } i=j, i < n, \\ n^2/(2n-1) & \text{if } i=j=n, \\ -i(i+1)/(2i+1) & \text{if } j=i+1, \\ -j(j+1)/(2j+1) & \text{if } i=j+1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

Recently, the inverse for any symmetric tridiagonal matrix was treated by B. Bukhberger and G. A. Emel'yanenko (2). If A is written as

$$(1.20) \quad A = \begin{bmatrix} b_1 & a_2 & 0 & \dots & 0 \\ a_2 & b_2 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_n & b_n \end{bmatrix},$$

then the inversion of A is accomplished by means of a decomposition process.

The matrix A is factored in the following manner (16):

$$(1.21) \quad \begin{aligned} A &= DQC, \\ D &= (d_{ij}), \\ Q &= (q_{ij}), \\ C &= (c_{ij}), \end{aligned}$$

where

$$(1.22) \quad d_{ij} = \begin{cases} b_i + c_i a_i & \text{if } i=j, \\ 0 & \text{if } i \neq j, \end{cases}$$

$$(1.23) \quad q_{ij} = \begin{cases} \frac{a_i}{b_i + c_i a_i} & \text{if } i=j+1, j=1, \dots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting by a^{ij} the inverse of the $(i-j)$ th element in A, the recursion formulas for obtaining A^{-1} are:

$$(1.24) \quad a^{ij} = c_{j+1} a^{ij+1} - a_{j+1}^{-1} \prod_{k=i+1}^{j+1} c_k,$$

for $i \leq j$, $j=n-1, n-2, \dots, 1$;

$$(1.25) \quad a^{in} = \alpha \prod_{k=i+1}^n c_k, \quad i=n-1, n-2, \dots, 1;$$

$$(1.26) \quad a^{nn} = \frac{1}{b_n + C_n a_n} = \alpha.$$

The C_i 's in the preceding are defined using the following recursion formulas:

$$(1.27) \quad \begin{aligned} C_1 &= 0 \\ C_i &= -\frac{a_i}{b_{i-1} + C_{i-1} a_{i-1}}, \quad 2 \leq i \leq n. \end{aligned}$$

Explicit formulas for the inverses are available in the case of even order nonsymmetric tridiagonal matrices with zero diagonal elements. The result due to P. Clement (3) may be stated as follows.

Let the matrix A be given the form

$$(1.28) \quad A = \begin{bmatrix} 0 & a_1 & 0 & \dots & 0 \\ b_1 & 0 & a_2 & \dots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & a_n \\ 0 & \dots & \dots & b_n & 0 \end{bmatrix}$$

where n is odd. Then the elements of A^{-1} are:

$$(1.29) \quad a^{ij} = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \frac{1}{b_i} & \text{if } j \text{ is even and } \frac{j-(i+1)}{2} = 0, \\ \frac{j-(i+1)}{2} \frac{\frac{j-(i+1)}{2}-1}{\pi} a_{2(\alpha+\frac{i+1}{2})} & \text{if } \frac{j-(i+1)}{2} \neq 0, \\ (-1)^{\frac{j-(i+1)}{2}} \frac{b_{2\alpha+i}}{\pi} & \text{if } \frac{j-(i+1)}{2} \neq 0, \end{cases}$$

$$(1.32) \quad A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 - \frac{n-1}{n} \end{bmatrix}.$$

The inverse of the next matrix closely resembles (1.32). Given

$$(1.33) \quad A = \begin{bmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & n-1 & n-2 & \dots & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix},$$

its inverse assumes the form:

$$(1.34) \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

SECTION II. THE SHERMAN-MORRISON FORMULA. THE HILBERT MATRIX. THE
VANDERMONDE MATRIX. OTHER SPECIAL MATRICES.

From (1.34), we note that the diagonal elements are all equal to each other with the exception of the first term in the leading diagonal. There is an algorithm due to Sherman and Morrison (15) which is useful in finding the inverse in this case.

Let $A = (a_{ij})$, $i, j = 1, \dots, n$; $A^{-1} = (a^{ij})$; $B = (b_{ij})$, where b_{ij} 's denote the elements of B which differ from A in one element, say b_{RS} ; $B^{-1} = (b^{ij})$. Further, let

$$(1.35) \quad b_{RS} = a_{RS} + \Delta a_{RS}.$$

Then the set of equations by means of which B^{-1} may be computed from Δa_{RS} and A^{-1} is

$$(1.36) \quad b^{rj} = a^{rj} - \frac{a^{rR} a^{Sj} \Delta a_{RS}}{1 + a^{SR} \Delta a_{RS}}$$

$r = 1, 2, \dots, n$; $j = 1, 2, \dots, n$; and $1 + a^{SR} \Delta a_{RS} \neq 0$.

The preceding algorithm is applicable where the inverse of the unperturbed matrix A is originally known. This method is also used by Murthy Prabhakar (7), to obtain the explicit inverse of the covariance matrix of a first-order moving average model. A program involving this algorithmic procedure appears in Appendix A6.

An interesting class of matrices with explicit inverses was considered by Chandan Mustafi (8). Let I_n be the $n \times n$ identity matrix and $W_n^{(k)}$ be an $n \times n$ matrix whose upper left $k \times k$ submatrix consists of elements equal to 1 and remaining elements equal to 0. Let

$$(1.37) \quad A = I_n + \sum_{k=1}^n C_k W_n^{(k)},$$

with $0 < C_k < \infty$ ($k = 1, 2, \dots, n$).

Then

$$A^{-1} = I_n - Q$$

where $Q = (q_{ij})$ is an $n \times n$ matrix whose elements have the following form:

$$(1.39) \quad q_{ij} = \begin{cases} \lambda_i q_j, & i \leq j \\ \lambda_j q_i, & i > j \end{cases}$$

$i, j = 1, 2, \dots, n$.

The quantities $\lambda_3, \dots, \lambda_n$, and $q_1, \dots, q_n$ are obtained from the following equations:

$$(1.40) \quad C_{r-1} \lambda_{r+1} - \lambda_r (C_{r-1} + C_r + C_{r-1} C_r) + \lambda_{r-1} C_r = 0$$

($r = 2, 3, \dots, n$), $\lambda_1 = 1$, $\lambda_2 = 1 + C_1$;

$$(1.41) \quad \lambda_{r-1} q_r - \lambda_r q_{r-1} + C_{r-1} = 0$$

($r = 2, 3, \dots, n$);

$$(1.42) \quad q_n = \frac{C_{n-1} C_n}{C_{n-1} (1 + C_n) \lambda_n + C_n (\lambda_n - \lambda_{n-1})}.$$

The formulas for A^{-1} are documented in a program contained in Appendix A3.

It is well known that finite segments of the Hilbert matrix are ill-conditioned and are generally used for test purposes in matrix inversion algorithms. In 1954, Savage and Lukacs (14) presented the Hilbert matrix and its inverse in the following forms:

$$(1.43) \quad A = \left(\frac{1}{i+j-1} \right), \quad i, j = 1, \dots, n,$$

$$(1.44) \quad A^{-1} = (a^{ij}), \text{ where}$$

$$a^{ij} = \frac{(-1)^{i+j}}{i+j-1} \cdot \frac{(n+i-1)!(n+j-1)!}{[(i-1)!(j-1)!]^2(n-i)!(n-j)!}$$

$i, j = 1, \dots, n.$

Again a program for these formulas appears in Appendix A5.

Inverses of Vandermonde matrices appear in curve fitting problems for numerical differentiation and difference equations. The $n \times n$ Vandermonde matrix is defined as (11)

$$(1.45) \quad A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix}$$

The inverse of A exists iff $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct.

Let $S_{n-j}(i)$ denote the sum of all possible products of the numbers $\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n$ taken $(n-j)$ at a time.

The elements of the inverse of A are given by

$$(1.46) \quad a^{ij} = \frac{(-1)^{n-j} S_{n-j}(i)}{\prod_{\substack{k=1 \\ k \neq i}}^n (\lambda_i - \lambda_k)}$$

with $S_0(i) = 1.$

When the matrix A is written as a sum of matrices, B. Noble's tearing formula comes into play (9). That is, if $A = B + CDE$, where B is $n \times n$,

C is $n \times p$, D is $p \times p$ and E is $p \times n$, then

$$(1.47) \quad A^{-1} = B^{-1} - B^{-1}C(D^{-1} + EB^{-1}C)^{-1}EB^{-1}.$$

An application of the preceding formula is seen in the following. Let

$$(1.48) \quad A = B + \lambda e_i e_j^T,$$

where e_i 's are unit vectors. Then from the inverse formula (1.47),

$$(1.49) \quad A^{-1} = B^{-1} - \frac{\lambda p_i q_j^T}{1 + \alpha_{ji}}$$

α_{ji} is the (j, i) th element of B^{-1} , $p_i = B^{-1}e_i$,
the i th column of B^{-1} ,

$q_j^T = e_j^T B^{-1}$, the j th row of A^{-1}

SECTION III. EXPLICIT INVERSES OF OTHER FORMS OF MATRICES

Another useful source of examples of matrices with explicit inverses is Chapter III in the book by R. T. Gregory and David L. Karney (5). We selected the following examples and present computer programs in Appendix B.

EXAMPLE I. (Appendix B1)

Let $A = (a_{ij})$, $a_{ij} = n - |i-j|$.

Then $A^{-1} = (a^{ij})$ is given by

$$a^{ij} = \begin{cases} \frac{n+2}{2n+2}, & \text{if } i = j = 1 \text{ or } i = j = n \\ 1, & \text{if } i = j \text{ and } 1 < i < n \\ -\frac{1}{2}, & \text{if } |i-j| = 1 \text{ and } n \neq 2 \\ -\frac{1}{3}, & \text{if } |i-j| = 1 \text{ and } n = 2 \\ \frac{1}{2n+2}, & \text{if } |i-j| = n-1 \neq 1 \\ 0, & \text{if } 1 < |i-j| < n-1. \end{cases}$$

EXAMPLE 2. (Appendix B2)

Let $A = (a_{ij})$, where

$$a_{1j} = 1, \quad j = 1, 2, \dots, n,$$

$$a_{ij} = \frac{1}{i+j-1}, \quad i = 2, 3, \dots, n, \quad j = 1, 2, \dots, n.$$

Then $A^{-1} = (a^{ij})$, where

$$a^{i1} = (-1)^{n-i} \binom{n+i-1}{i-1} \binom{n}{i}, \quad i = 1, 2, \dots, n,$$

$$a^{i, j+1} = (-1)^{i-j} \binom{i+j}{j} \binom{i+j-1}{j-1} \binom{n+i-1}{i+j} \binom{n+j}{i+j} i, \quad$$

for $i = 1, 2, \dots, n, \quad j = 1, 2, \dots, n-1$.

EXAMPLE 3. (Appendix B3)

$$A = \begin{bmatrix} I_{n-1} & \vdots & 1 \\ & \vdots & 2 \\ & \vdots & \vdots \\ & \vdots & n-1 \\ \dots\dots\dots & & \\ 1 & 2 & \dots & n-1 & \vdots & n \end{bmatrix}$$

$$A^{-1} = (a^{ij}), \text{ where}$$

$$a^{ii} = 1 - ki^2, \quad 1 \leq i \leq n-1$$

$$a^{nn} = -k$$

$$a^{ij} = -kij, \text{ if } i \neq j, \quad 1 \leq i \leq n-1, \text{ and } 1 \leq j \leq n-1$$

$$a^{in} = a^{ni} = ki, \quad 1 \leq i \leq n-1$$

$$\text{and } k = \frac{6}{n(n+1)(2n-5)}$$

EXAMPLE 4. (Appendix B4)

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ \dots\dots\dots & & & & & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix} = (a_{ij}).$$

$$A^{-1} = (a^{ij}), \text{ where}$$

$$a^{ij} = \begin{cases} \frac{i(n-i+1)}{n+1}, & \text{if } i = j \\ \frac{a^{i, j-1} - i}{n+1}, & \text{if } j > i \\ \frac{a^{ji}}{n+1}, & \text{if } j < i. \end{cases}$$

EXAMPLE 5. (Appendix B5)

$$A = \begin{bmatrix} x+b & 1 & & & & \\ & 1 & x & 1 & & \\ & & 1 & x & 1 & \\ & \dots & \dots & \dots & \dots & \\ & & & & 1 & x & 1 \\ & & & & & 1 & x+a \end{bmatrix}$$

$$A^{-1} = (a^{ij}), \text{ where}$$

$$a^{ij} = a^{ji} = \frac{(-1)^{i+j} r_{j-1} s_{n-i}}{(x+a) r_{n-1} - r_{n-2}}, \quad j \geq i,$$

where

$$r_0 = 1$$

$$r_1 = x + b$$

$$r_k = x r_{k-1} - r_{k-2}, \quad k = 2, 3, \dots, n-1$$

and

$$s_0 = 1$$

$$s_1 = x + a$$

$$s_k = x s_{k-1} - s_{k-2}, \quad k = 2, 3, \dots, n-1$$

EXAMPLE 6. (Appendix B6)

$$A = (a_{ij}), \quad a_{ij} = |i-j|.$$

$$A^{-1} = \begin{pmatrix} -1/2 \end{pmatrix} \begin{bmatrix} (1 - \frac{1}{n-1}) & -1 & & & -\frac{1}{n-1} \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ \dots & \dots & \dots & \dots & \dots \\ & & & -1 & 2 & -1 \\ -\frac{1}{n-1} & & & & -1 & (1 - \frac{1}{n-1}) \end{bmatrix}$$

EXAMPLE 7. (Appendix B7)

$A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } j \geq i \\ a_j & \text{if } j < i. \end{cases}$$

Then

$A^{-1} = (a^{ij})$, where

$$a^{ij} = \begin{cases} \frac{1}{1-a_i} & \text{if } i = j, i \neq n, \\ \frac{-1}{1-a_i} & \text{if } j = i + 1, i \neq n, \\ (a_{j-1} - a_j)/[(1-a_j)(1-a_{j-1})] & \text{if } i = n, j \neq 1, n, \\ -a_1/(1-a_1) & \text{if } i = n, j = 1, \\ 1/(1-a_{n-1}) & \text{if } i = j = n, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 8. (Appendix B8)

$$A = \begin{bmatrix} (x+y) & y & y & . & . & . & y \\ y & (x+y) & y & . & . & . & y \\ \\ y & y & y & . & . & . & (x+y) \end{bmatrix}$$

If $A^{-1} = (a^{ij})$, then

$$a^{ij} = \frac{\delta_{ij}(x+ny) - y}{x(x+ny)}$$

where δ_{ij} is the Kronecker delta.

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ACKNOWLEDGMENT

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APPENDIX A

28

```

ISN 0036      Q1 = P1
ISN 0037      P2 = 1.
ISN 0038      Q2 = 1.
ISN 0039      P3 = 1.
ISN 0040      Q3 = 1.
ISN 0041      DO 420 KK=1,L
ISN 0042      KP = KK*(I+1)/2-1
ISN 0043      P2 = X(2*KP)*P2
ISN 0044      Q2 = Y(2*KP)*Q2
ISN 0045      CONTINUE
ISN 0046      L2 = L+1
ISN 0047      DO 430 KK=1,L2
ISN 0048      KP = 2*(KK-1)+1
ISN 0049      P3 = P3*Y(KP)
ISN 0050      Q3 = Q3*X(KP)
ISN 0051      CONTINUE
ISN 0052      B(I,K) = (P1*P2)/P3
ISN 0053      B(K,I) = (Q1*Q2)/Q3
ISN 0054      GO TO 500
ISN 0055      B(I,K) = 1./Y(I)
ISN 0056      B(K,I) = 1./X(I)
ISN 0057      CONTINUE
C
C
C
C
      PRINT OUT A AND ITS INVERSE

ISN 0058      PRINT 600
ISN 0059      FORMAT (1H1,'A MATRIX')
ISN 0060      DO 660 I=1,N
ISN 0061      PRINT 650,(A(I,J),J=1,N)
ISN 0062      FORMAT ('0',10(F11.6,2X))
ISN 0063      CONTINUE
ISN 0064      PRINT 700
ISN 0065      FORMAT ('//','0THE INVERSE OF A')
ISN 0066      DO 760 I=1,N
ISN 0067      PRINT 650,(B(I,J),J=1,N)
ISN 0068      CONTINUE
ISN 0069      GO TO 1
ISN 0070      STOP
ISN 0071      END

```

30

ISN 0037		PRINT 220
ISN 0038	220	FORMAT (//, '0THE INVERSE CF A')
ISN 0039		DO 240 I=1,K
ISN 0040		PRINT 180.(B(I,J),J=1,K)
ISN 0041	240	CONTINUE
ISN 0042		GO TO 1
ISN 0043	900	STOP
ISN 0044		END

CCCCC

ISN 0002
ISN 0003

CCC

```

ISN 0004
ISN 0005
ISN 0006
ISN 0007
ISN 0008
ISN 0009
ISN 0010
ISN 0011
ISN 0012
ISN 0013
ISN 0014
ISN 0015

```

C
1
2

CCCCC

ISN 0016
 ISN 0017
 ISN 0018
 ISN 0019
 ISN 0020
 ISN 0021
 ISN 0022
 ISN 0023
 ISN 0024
 ISN 0026
 ISN 0028
 ISN 0030
 ISN 0031
 ISN 0032
 ISN 0033
 ISN 0034

1
1
1
1

CCCC

ISN 0035
ISN 0036
ISN 0037

```
PRINT 140
FORMAT (1H1,'THE MATRIX A')
DO 180 I=1,N
```

ISN 0038		PRINT 160, (A(I,J), J=1,N)
ISN 0039	160	FORMAT ('0'.10F12.3)
ISN 0040	180	CONTINUE
ISN 0041		PRINT 200
ISN 0042	200	FORMAT ('//.0THE INVERSE OF A')
ISN 0043		DO 240 I=1,N
ISN 0044		PRINT 220, (C(I,J), J=1,N)
ISN 0045	220	FORMAT ('0'.8F15.6)
ISN 0046	240	CONTINUE
ISN 0047		GO TO 1
ISN 0048	500	STOP
ISN 0049		END

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```

ISN 0034      LAMBDA(I+1) = (LAMBDA(I)*S-LAMBDA(I-1)*C(I))/C(I-1)
ISN 0035      CONTINUE
ISN 0036      T = C(M-1)*(1+C(M))*LAMBDA(M) + C(M)*(LAMBDA(M)-LAMBDA(M-1))
ISN 0037      J(M) = (C(M-1)*C(M))/T
ISN 0038      DO 180 I=2,M
ISN 0039      J = M+1-I
ISN 0040      J(I) = (LAMBDA(J)*Q(J+1)+C(J))/LAMBDA(J+1)
ISN 0041      CONTINUE
ISN 0042      DO 200 I=1,M
ISN 0043      DO 200 J=1,M
ISN 0044      IF (I.LE.J) Q(I,J) = LAMBDA(I)*Q(J)
ISN 0045      IF (I.GT.J) Q(I,J) = LAMBDA(J)*Q(I)
ISN 0046      AI(I,J) = -Q(I,J)
ISN 0047      IF (I.EQ.J) AI(I,J) = 1. + AI(I,J)
ISN 0048      CONTINUE
ISN 0049      C
ISN 0050      C
ISN 0051      C
ISN 0052      PRINT A AND A INVERSE
ISN 0053      PRINT 220
ISN 0054      FORMAT(1H1,'THE MATRIX A')
ISN 0055      DO 240 I=1,M
ISN 0056      PRINT 230,(A(I,J),J=1,M)
ISN 0057      FORMAT('0'.1P7E17.7)
ISN 0058      CONTINUE
ISN 0059      C
ISN 0060      PRINT 260
ISN 0061      FORMAT(1H1,'THE INVERSE OF A')
ISN 0062      DO 280 I=1,M
ISN 0063      PRINT 230,(AI(I,J),J=1,M)
ISN 0064      CONTINUE
ISN 0065      STOP
ISN 0066      END

```

```

LEVEL 21.6 (DEC 72)                                OS/360    FORTRAN H

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=50,SIZE=000K,
SOURCE,EBCDIC,MOLIST,NODECK,LOAD,NAP,NOED IT,NOID,XREF

C
C
C APPENDIX A5.
C THIS PROGRAM COMPUTES THE INVERSE OF THE GENERALIZED HILBERT
C MATRIX USING FORMULAS BY SAVAGE AND LUKACS.
C THE USER SHOULD PROVIDE SUBROUTINE FACT WHICH
C COMPUTES THE FACTORIAL OF A NUMBER.
C
C
      IMPLICIT REAL*8(A-M,O-Z)
      READ*8 A(10,10),I(10,10)
      READ 100,N
      P=0.00
111 DO 01 J=1,N
      DO 01 I=1,N
      EL2 = DFLDAT(I+J-1)
      A(I,J) = 1.D0/(P+EL2)
      EL1 = (-1.D0)*((I+J)
      PROD1 = (P+1) * (P+J)
      N1 = N - 1
      DO 02 K=1,N1
      PROD1 = PROD1 * (P+I+K) * (P+J+K)
02 CONTINUE
      EL3 = PROD1
      I1 = I - 1
      CALL FACT(I1,P1)
      I2 = N - I
      CALL FACT(I2,P2)
      I3 = J - 1
      CALL FACT(I3,P3)
      I4 = N - J
      CALL FACT(I4,P4)
      EL4 = P1*P2*P3*P4
      I(I,J)=(EL1*EL3)/(EL2*EL4)
01 CONTINUE
03 FORMAT ('1A MATRI X')
DO 04 I = 1,N
PRINT 05, (A(I,J),J=1,N)
04 CONTINUE
05 FORMAT (7(E17.8,2X))
PRINT 06
06 FORMAT ('//','0A INVERSE')
DO 07 I=1,N
PRINT 05, (T(I,J),J=1,N)
07 CONTINUE
N=N+N
IF (N.LE.10) GO TO 111
100 FORMAT (15)
STOP
END
ISN 0002
ISN 0003
ISN 0004
ISN 0005
ISN 0006
ISN 0007
ISN 0008
ISN 0009
ISN 0010
ISN 0011
ISN 0012
ISN 0013
ISN 0014
ISN 0015
ISN 0016
ISN 0017
ISN 0018
ISN 0019
ISN 0020
ISN 0021
ISN 0022
ISN 0023
ISN 0024
ISN 0025
ISN 0026
ISN 0027
ISN 0028
ISN 0029
ISN 0030
ISN 0031
ISN 0032
ISN 0033
ISN 0034
ISN 0035
ISN 0036
ISN 0037
ISN 0038
ISN 0039
ISN 0040
ISN 0042
ISN 0043
ISN 0044
```

EVEL 21.6 (DEC 72)

OS/360 FORTRAN H

```
COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EB,CDIC,NCLIST,NODECK,LCAD,PAP,KDED IT,NOID,XREF
ISN 0002      SUBROUTINE FACT(IF,P)
ISN 0003      IMPLICIT REAL*8(A-F,O-Z)
ISN 0004      PROD = 1.
ISN 0005      IF (IF.EQ.0) GO TC 20
ISN 0007      I      DO 10 I=1,IF
ISN 0008      10     PROD = PROD * I
ISN 0009      CONTINUE
ISN 0010      P = PROD
ISN 0011      GO TO 100
ISN 0012      20     P = 1.
ISN 0013      100    RETURN
ISN 0014      END
```

```

COMPILER OPTIONS - NAME= MAIN,OPT=07,LINENR=50,SIZE=0000K,
SOURCE=EB,CDIC,NCLIST,NODECK,LCAD,NAP,NODEIT,NOID,XREF
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(50,50),APER(50,50),B(50,50),BPER(50,50)

      C
      C
      C      APPENDIX A6.
      C      THIS PROGRAM IS BASED ON SHERMAN AND MORRISON'S *ADJUSTMENT
      C      OF AN INVERSE MATRIX CORRESPONDING TO A CHANGE IN ONE ELEMENT
      C      OF A GIVEN MATRIX *.
      C
      C      READ M (THE DIMENSION OF A); READ THE MATRIX A AND ITS INVERSE B.
      C
      C
ISN 0004      READ 20,M
ISN 0005      FORMAT (I5)
ISN 0006      DO 60 I=1,M
ISN 0007      READ 40,(A(I,J),J=1,M)
ISN 0008      FORMAT (8F10.0)
ISN 0009      DO 80 I=1,M
ISN 0010      READ 40,(B(I,J),J=1,M)
ISN 0011      CONTINUE
ISN 0012      CONTINUE

      C
      C
      C      PERTURB ONE ELEMENT OF A AND CALL THE NEW MATRIX APER;
      C      A(NR,NC) IS THE NR-TH ROW, NC-TH COLUMN ELEMENT OF A THAT
      C      WILL BE PERTURBED. DELTA IS THE AMOUNT OF PERTURBATION.
      C
      C
ISN 0013      READ (5,120,END=500) NR,NC,DELTA
ISN 0014      FORMAT (2I5,F10.0)
ISN 0015      DO 140 I=1,M
ISN 0016      DO 140 J=1,M
ISN 0017      APER(I,J) = A(I,J)
ISN 0018      CONTINUE
ISN 0019      APER(NR,NC) = A(NR,NC) + DELTA

      C
      C
      C      COMPUTE THE INVERSE OF THE PERTURBED A MATRIX AND CALL THE INVERSE BPER.
      C
      C
ISN 0020      DO 160 I=1,M
ISN 0021      DO 160 J=1,M
ISN 0022      BNUM = B(I,NR) * E(NC,J) * DELTA
ISN 0023      BDEN = 1. + B(NC,NR) * DELTA
ISN 0024      BPER(I,J) = B(I,J) - (BNUM/BDEN)
ISN 0025      CONTINUE

      C
      C
      C      PRINT THE MATRICES A, APER, B, BPER.
      C
      C
ISN 0026      PRINT 180
ISN 0027      FORMAT (1H1,'C A MATRIX')
ISN 0028      DO 220 I=1,M
ISN 0029      PRINT 200,(A(I,J),J=1,M)

```

ISN 0030	200	FORMAT ('0',10(F10.5,2X))
ISN 0031	220	CONTINUE
ISN 0032		PRINT 230,NR,NC
ISN 0033	230	FORMAT (//,' THE PERTURBED ELEMENT IS IN THE',15,'-TH ROW', 15,'-TH COLUMN')
ISN 0034		PRINT 240
ISN 0035	240	FORMAT (//,' THE PERTURBED MATRIX')
ISN 0036		DO 260 I=1,M
ISN 0037		PRINT 200,(APER(I,J),J=1,M)
ISN 0038	260	CONTINUE
ISN 0039		PRINT 280
ISN 0040	280	FORMAT (//,'0 THE INVERSE OF A')
ISN 0041		DO 300 I=1,M
ISN 0042		PRINT 200,(B(I,J),J=1,M)
ISN 0043	300	CONTINUE
ISN 0044		PRINT 320
ISN 0045	320	FORMAT (//,'0 THE INVERSE OF THE PERTURBED MATRIX A')
ISN 0046		DO 340 I=1,M
ISN 0047		PRINT 200,(BPER(I,J),J=1,M)
ISN 0048	340	CONTINUE
ISN 0049		DO 360 I=1,M
ISN 0050		DO 360 J=1,M
ISN 0051		A(I,J) = APER(I,J)
ISN 0052		B(I,J) = BPER(I,J)
ISN 0053	360	CONTINUE
ISN 0054		GO TO 110
ISN 0055	500	STOP
ISN 0056		END

1

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APPENDIX B

COMPILER OPTIONS - NAME=MAIN,OPT=02,LINENCT=60,SIZE=0000K,
SOURCE,EB,CDIC,NCLIST,NODECK,LCAD,WAP,NODEIT,NOID,XREF

```

C
C
C   APPENDIX B1.
C   THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE
C   3.6, PAGE 31 OF R. T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A
C   COLLECTION OF MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20)
C
C   READ N, THE DIMENSION OF A.
C
ISN 0004      1 READ (5,20,END=50) N
ISN 0005      20 FORMAT (I5)
ISN 0006      DO 40 I=1,N
ISN 0007      DO 40 J=1,N
ISN 0008      A(I,J) = N - IABS(I-J)
ISN 0009      40 CONTINUE
C
C   COMPUTE B, THE INVERSE OF A.
C
ISN 0010      R = 1./3.
ISN 0011      DO 60 I=1,N
ISN 0012      DO 60 J=1,N
ISN 0013      IJ = IABS(I-J)
ISN 0014      IF ((I.EQ.J).AND.((I.GT.1).AND.(I.LT.N))) B(I,J) = 1.
ISN 0016      IF ((IJ.EQ.1).AND.(N.NE.2)) B(I,J) = -.5
ISN 0018      IF ((IJ.EQ.1).AND.(N.EQ.2)) B(I,J) = R
ISN 0020      IF ((IJ.EQ.N-1).AND.(IJ.NE.1)) B(I,J) = 1./(2*N+2)
ISN 0022      IF ((IJ.GT.1).AND.(IJ.LT.N-1)) B(I,J) = 0.
ISN 0024      60 CONTINUE
ISN 0025      B(1,1) = DFLOAT(N+2)/DFLOAT(2*N+2)
ISN 0026      B(N,N) = B(1,1)
ISN 0027      PRINT 80
ISN 0028      80 FORMAT (1H1,'A MATRIX')
ISN 0029      DO 120 I=1,N
ISN 0030      PRINT 100,(A(I,J),J=1,N)
ISN 0031      100 FORMAT ('0',1P10E12.4)
ISN 0032      120 CONTINUE
ISN 0033      PRINT 140
ISN 0034      140 FORMAT (//,'THE INVERSE OF A')
ISN 0035      DO 160 I=1,N
ISN 0036      PRINT 100,(B(I,J),J=1,N)
ISN 0037      160 CONTINUE
ISN 0038      GO TO 1
ISN 0039      500 STOP
ISN 0040      END

```

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EB,CDIC,NCLIST,NODECK,LLAD,MAP,NODEIT,NOID,XREF

```

C
C
C      APPENDIX B2.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE
C      3.9 OF R. T. GREGORY AND DAVID L. KARNEYHS BOOK, 'A
C      COLLECTION OF MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20)
C
C      READ N, THE DIMENSION OF A.
C
ISN 0004      1 READ (5,20,END=50()) N
ISN 0005      20 FORMAT (I5)
ISN 0006      DO 40 J=1,N
ISN 0007      A(1,J) = 1.
ISN 0008      40 CONTINUE
ISN 0009      DO 60 I=2,N
ISN 0010      DO 60 J=1,N
ISN 0011      A(I,J) = 1./(I+J-1)
ISN 0012      60 CONTINUE
C
C
C      COMPUTE B, THE INVERSE OF A.
C
ISN 0013      DO 80 I=1,N
ISN 0014      P = (-1.)*I*(N-I)
ISN 0015      IQ = N+I-1
ISN 0016      IR = I
ISN 0017      IS = I-1
ISN 0018      IT = N-I
ISN 0019      CALL FACT(IQ,Q)
ISN 0020      CALL FACT(IR,R)
ISN 0021      CALL FACT(IS,S)
ISN 0022      CALL FACT(IT,T)
ISN 0023      B(I,1) = (P*Q)/(R*S*T)
ISN 0024      80 CONTINUE
ISN 0025      N1 = N-1
ISN 0026      DO 100 I=1,N
ISN 0027      DO 100 J=1,N1
ISN 0028      O = I+J
ISN 0029      P = (-1.)*I*(N-J)
ISN 0030      IQ = N+I-1
ISN 0031      IR = N+J
ISN 0032      IS = J
ISN 0033      IT = I
ISN 0034      IU = I-1
ISN 0035      IV = J-1
ISN 0036      IW = N-J-1
ISN 0037      IX = N-I
ISN 0038      CALL FACT(IQ,Q)
ISN 0039      CALL FACT(IR,R)
ISN 0040      CALL FACT(IS,S)
ISN 0041      CALL FACT(IT,T)

```

ISN 0042		CALL FACT(IU,U)
ISN 0043		CALL FACT(IV,V)
ISN 0044		CALL FACT(IW,W)
ISN 0045		CALL FACT(IX,X)
ISN 0046		DNUM = P*Q*R
ISN 0047		DDEN = O*S*T*L*V*U*X
ISN 0048		B(I,J+1) = DNUM/DDEN
ISN 0049	100	CONTINUE
	C	
	C	
	C	PRINT A AND ITS INVERSE
	C	
	C	
ISN 0050		PRINT 120
ISN 0051	120	FORMAT (1H1, 'A MATRIX')
ISN 0052		DO 160 I=1,N
ISN 0053		PRINT 140,(A(I,J),J=1,N)
ISN 0054	140	FORMAT ('0',1P10E12.4)
ISN 0055	160	CONTINUE
ISN 0056		PRINT 180
ISN 0057	180	FORMAT (//, 'THE INVERSE OF A')
ISN 0058		DO 220 I=1,N
ISN 0059		PRINT 200,(B(I,J),J=1,N)
ISN 0060	200	FORMAT ('0',1P8E15.6)
ISN 0061	220	CONTINUE
ISN 0062		GO TO 1
ISN 0063	500	STOP
ISN 0064		END

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EB,CDIC,NCLIST,NODECK,LCAD,PAP,NODEDIT,NOID,XREF

```

C
C
C      APPENDIX B3.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE 3.16
C      OF R.T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A COLLECTION OF TEST
C      MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20)

C
C      READ N, THE DIMENSION OF A.
C
ISN 0004      READ (5,20,END=50) N
ISN 0005      FORMAT (I5)
ISN 0006      N1 = N-1
ISN 0007      DO 40 I=1,N1
ISN 0008      DO 40 J=1,N1
ISN 0009      A(I,J) = 0.
ISN 0010      A(I,I) = 1.
ISN 0011      CONTINUE
ISN 0012      DO 60 J=1,N
ISN 0013      A(N,J) = DFLOAT(J)
ISN 0014      A(J,N) = A(N,J)
ISN 0015      CONTINUE
ISN 0016      PN = 6./((N*(N+1))*(2.*N-5.))

C
C      COMPUTE B, THE INVERSE OF A.
C
ISN 0017      B(N,N) = -PN
ISN 0018      DO 80 I=1,N1
ISN 0019      B(I,I) = 1.-PN*(I**2)
ISN 0020      B(I,N) = PN*I
ISN 0021      B(N,I) = B(I,N)
ISN 0022      CONTINUE
ISN 0023      DO 120 I=1,N1
ISN 0024      DO 120 J=1,N1
ISN 0025      IF (I.EQ.J) GO TO 120
ISN 0027      B(I,J) = -PN * I * J
ISN 0028      CONTINUE

C
C      PRINT A AND ITS INVERSE.
C
ISN 0029      PRINT 140
ISN 0030      FORMAT (I1,'THE MATRIX A')
ISN 0031      DO 180 I=1,N
ISN 0032      PRINT 160, (A(I,J),J=1,N)
ISN 0033      FORMAT ('0',10F12.3)
ISN 0034      CONTINUE
ISN 0035      PRINT 200

```

ISN 0036	200	FORMAT (//,'0THE INVERSE OF A')
ISN 0037		DO 240 I=1,N
ISN 0038		PRINT 220, (B(I,J),J=1,N)
ISN 0039	220	FORMAT ('0',8F15.6)
ISN 0040	240	CONTINUE
ISN 0041		GO TO I
ISN 0042	500	STOP
ISN 0043		END

COMPILER OPTIONS - NAME=MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EB,CDIC,NOLIST,NODECK,LCAD,MAP,NOEDIT,NOID,XREF

```

C
C
C      APPENDIX B4.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE 3.18
C      OF R.T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A COLLECTION OF TEST
C      MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20)
C
C      READ N, THE DIMENSION OF A.
C
ISN 0004      READ (5,20,END=50) N
ISN 0005      FORMAT (I5)
ISN 0006      N1 = N-1
ISN 0007      DO 40 I=1,N
ISN 0008      DO 40 J=1,N
ISN 0009      A(I,J) = 0.
ISN 0010      A(I,I) = 2.
ISN 0011      40 CONTINUE
ISN 0012      DO 60 I=1,N1
ISN 0013      A(I,I+1) = -1.
ISN 0014      A(I+1,I) = -1.
ISN 0015      60 CONTINUE
C
C      COMPUTE B THE INVERSE OF A.
C
ISN 0016      DO 100 I=1,N
ISN 0017      DO 100 J=1,N
ISN 0018      IF (I.EQ.J) B(I,J) = 1 * (N-I+1.)
ISN 0020      IF (I.LT.J) B(I,J) = B(I,J-1) - 1
ISN 0022      IF (J.LT.I) B(I,J) = B(J,I)
ISN 0024      100 CONTINUE
ISN 0025      DO 120 I=1,N
ISN 0026      DO 120 J=1,N
ISN 0027      B(I,J) = B(I,J)/(N+1.)
ISN 0028      120 CONTINUE
C
C      PRINT A AND ITS INVERSE.
C
ISN 0029      PRINT 140
ISN 0030      140 FORMAT (1H1,'THE MATRIX A')
ISN 0031      DO 180 I=1,N
ISN 0032      PRINT 160, (A(I,J),J=1,N)
ISN 0033      160 FORMAT ('0',10F12.3)
ISN 0034      180 CONTINUE
ISN 0035      PRINT 200
ISN 0036      200 FORMAT (//,'THE INVERSE OF A')
ISN 0037      DO 240 I=1,N

```

```
ISN 0038      220  PRINT 220.(B(I,J),J=1..N)
ISN 0039      220  FORMAT (10.8F15.6)
ISN 0040      240  CONTINUE
ISN 0041      240  GO TO 1
ISN 0042      500  STOP
ISN 0043      500  END
```

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=00000,
SOURCE,EB CDIC,NCLIST,NODECK,LCAD,PAP,NCE DIT,NOID,XREF

```

C
C
C      APPENDIX B5.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE 3.20
C      OF R.T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A COLLECTION OF TEST
C      MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20),R(20),S(20)

C
C      READ N, THE DIMENSION OF A.
C
ISN 0004      READ (5,20,END=500) N,AV,BV,X
ISN 0005      20  FORMAT (I5,3F10.0)
ISN 0006      N1 = N-1
ISN 0007      N2 = N+1
ISN 0008      DO 40 I=1,N
ISN 0009      DO 40 J=1,N
ISN 0010      A(I,J) = 0.
ISN 0011      A(I,I) = X
ISN 0012      40  CONTINUE
ISN 0013      A(1,1) = X + BV
ISN 0014      A(N,N) = X + AV
ISN 0015      DO 60 I=1,N1
ISN 0016      A(I,I+1) = 1.
ISN 0017      A(I+1,I) = 1.
ISN 0018      60  CONTINUE

C
C      COMPUTE B THE INVERSE OF A.
C
ISN 0019      R(1) = 1.
ISN 0020      R(2) = X + BV
ISN 0021      S(1) = 1.
ISN 0022      S(2) = X + AV
ISN 0023      DO 100 I=3,N
ISN 0024      R(I) = X*R(I-1) - R(I-2)
ISN 0025      S(I) = X*S(I-1) - S(I-2)
ISN 0026      100 CONTINUE
ISN 0027      DO 120 I=1,N
ISN 0028      DO 120 J=1,I
ISN 0029      P = ((-1.)**(I+J))*R(J)+S(N-I+1)
ISN 0030      Q = S(2) * R(N) - R(N-1)
ISN 0031      B(I,J) = P/Q
ISN 0032      B(J,I) = B(I,J)
ISN 0033      120 CONTINUE

C
C      PRINT A AND ITS INVERSE.
C
ISN 0034      PRINT 140

```


ISN 0035	140	FORMAT (1H1,'THE MATRIX A')
ISN 0036		DO 180 I=1,N
ISN 0037		PRINT 160, (A(I,J),J=1,N)
ISN 0038	160	FORMAT ('0',10F12.3)
ISN 0039	180	CONTINUE
ISN 0040		PRINT 200
ISN 0041	200	FORMAT (//,'THE INVERSE OF A')
ISN 0042		DO 240 I=1,N
ISN 0043		PRINT 220, (B(I,J),J=1,N)
ISN 0044	220	FORMAT ('0',8F15.6)
ISN 0045	240	CONTINUE
ISN 0046		GO TO 1
ISN 0047	500	STOP
ISN 0048		END

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EB CDIC,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,XREF

```

C
C
C      APPENDIX B6.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE 3.23
C      OF R.T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A COLLECTION OF TEST
C      MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20)
C
C      READ N, THE DIMENSION OF A.
C
C
ISN 0004      READ (5,20,END=500) N
ISN 0005      FORMAT (I5)
ISN 0006      N1 = N-1
ISN 0007      N2 = N+1
ISN 0008      DO 40 I=1,N
ISN 0009      DO 40 J=I,N
ISN 0010      A(I,J) = DABS(DFLGAT(I-J))
ISN 0011      CONTINUE
C
C      COMPUTE B, THE INVERSE OF A.
C
C
ISN 0012      DO 100 I=1,N
ISN 0013      DO 100 J=1,N
ISN 0014      B(I,J) = 0.
ISN 0015      B(I,I) = -1.
ISN 0016      CONTINUE
ISN 0017      DO 120 I=1,N1
ISN 0018      B(I,I+1) = .5
ISN 0019      B(I+1,I) = .5
ISN 0020      CONTINUE
ISN 0021      B(1,N) = .5/(N-1)
ISN 0022      B(N,1) = B(1,N)
ISN 0023      B(1,1) = (-.5*(N-2))/(N-1)
ISN 0024      B(N,N) = B(1,1)
C
C      PRINT A AND ITS INVERSE.
C
C
ISN 0025      PRINT 140
ISN 0026      FORMAT (1H1,'THE MATRIX A')
ISN 0027      DO 180 I=1,N
ISN 0028      PRINT 160, (A(I,J),J=1,N)
ISN 0029      FORMAT ('0',10F12.3)
ISN 0030      CONTINUE
ISN 0031      PRINT 200
ISN 0032      FORMAT (//,'THE INVERSE OF A')
ISN 0033      DO 240 I=1,N
ISN 0034      PRINT 220, (B(I,J),J=1,N)

```

ISN 0035	220	FORMAT ('0'.8F15.6)
ISN 0036	240	CONTINUE
ISN 0037		GU TO 1
ISN 0038	500	STOP
ISN 0039		END

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EBCDIC,NCLIST,NODECK,LC70,PAP,KCEDIT,NOID,XREF

```

C
C
C      APPENDIX B7.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE 3.24
C      OF R.T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A COLLECTION OF TEST
C      MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20),C(20)

C
C      READ N, THE DIMENSION OF A.
C
ISN 0004      READ (5,20,END=50) N
ISN 0005      20  FORMAT (I5)
ISN 0006      N1 = N-1
ISN 0007      N2 = N+1
ISN 0008      READ 25, (C(I),I=1,N1)
ISN 0009      25  FORMAT (20F4.0)
ISN 0010      DO 40 I=1,N
ISN 0011      DO 40 J=I,N
ISN 0012      A(I,J)= 1.0
ISN 0013      40  CONTINUE
ISN 0014      DO 60 J=1,N1
ISN 0015      JJ = J+1
ISN 0016      DO 60 I=JJ,N
ISN 0017      A(I,J) = C(J)
ISN 0018      60  CONTINUE

C
C      COMPUTE B, THE INVERSE OF A.
C
ISN 0019      DO 80 I=1,N
ISN 0020      DO 80 J=1,N
ISN 0021      B(I,J) = 0.
ISN 0022      80  CONTINUE
ISN 0023      B(N,N) = 1./(1.-C(N-1))
ISN 0024      B(N,1) = -C(1)/(1.-C(1))
ISN 0025      DO 100 J=2,N1
ISN 0026      B(N,J) = (C(J-1)-C(J))/(1.-C(J))*(1.-C(J-1))
ISN 0027      100 CONTINUE
ISN 0028      DO 110 I=1,N1
ISN 0029      B(I,I) = 1./(1.-C(I))
ISN 0030      B(I,I+1) = -B(I,I)
ISN 0031      110 CONTINUE

C
C      PRINT A AND ITS INVERSE.
C
ISN 0032      PRINT 140
ISN 0033      140  FORMAT (1H1,'THE MATRIX A')
ISN 0034      DO 180 I=1,N

```

ISN 0038		PRINT 160, (A(I,J),J=1,N)
ISN 0039	160	FORMAT ('0',10F12.3)
ISN 0040	180	CONTINUE
ISN 0041		PRINT 200
ISN 0042	200	FORMAT (//,'*0THE INVERSE OF A*')
ISN 0043		DO 240 I=1,N
ISN 0044		PRINT 220, (C(I,J),J=1,N)
ISN 0045	220	FORMAT ('0',8F15.6)
ISN 0046	240	CONTINUE
ISN 0047		GO TO 1
ISN 0048	500	STOP
ISN 0049		END

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EB,CDIC,NOLIST,NODECK,LCAD,PAP,KCREDIT,NOID,XREF

```

C
C
C      APPENDIX B8.
C      THIS PROGRAM COMPUTES THE INVERSE OF THE MATRIX IN EXAMPLE 3.25
C      OF R.T. GREGORY AND DAVID L. KARNEY'S BOOK, 'A COLLECTION OF TEST
C      MATRICES FOR TESTING COMPUTATIONAL ALGORITHMS'.
C
ISN 0002      IMPLICIT REAL*8(A-H,O-Z)
ISN 0003      DIMENSION A(20,20),B(20,20)
C
C      READ N, THE DIMENSION OF A.
C
ISN 0004      READ (5,20,END=50) N,X,Y
ISN 0005      20  FORMAT (I5,2F5.0)
ISN 0006      DO 40 I=1,N
ISN 0007      DO 40 J=1,N
ISN 0008      A(I,J) = Y
ISN 0009      40  CONTINUE
ISN 0010      DO 60 I=1,N
ISN 0011      A(I,I) = Y+X
ISN 0012      60  CONTINUE
C
C      COMPUTE B, THE INVERSE OF A.
C
ISN 0013      DO 100 I=1,N
ISN 0014      DO 100 J=1,N
ISN 0015      B(I,J) = -Y/(X*(X+N*Y))
ISN 0016      100 CONTINUE
ISN 0017      DO 120 I=1,N
ISN 0018      B(I,I) = (1./X) + B(I,I)
ISN 0019      120 CONTINUE
C
C      PRINT A AND ITS INVERSE.
C
ISN 0020      PRINT 140
ISN 0021      140  FORMAT (IHI,'THE MATRIX A')
ISN 0022      DO 180 I=1,N
ISN 0023      PRINT 160, (A(I,J),J=1,N)
ISN 0024      160  FORMAT ('0',10F12.3)
ISN 0025      180  CONTINUE
ISN 0026      PRINT 200
ISN 0027      200  FORMAT (//,'THE INVERSE OF A')
ISN 0028      DO 240 I=1,N
ISN 0029      PRINT 220, (B(I,J),J=1,N)
ISN 0030      220  FORMAT ('0',8F15.6)
ISN 0031      240  CONTINUE
ISN 0032      GO TO 1
ISN 0033      500 STOP
ISN 0034      END

```

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