ISOSPIN RESTRICTIONS ON CHARGE DISTRIBUTIONS IN
CHARMED - PARTICLE DECAYS

Murray Peshkin[†]
Argonne National Laboratory,
Argonne, Illinois 60439

and

Jonathan L. Rosner*
Institute for Advanced Study,
Princeton, New Jersey 08540

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ABSTRACT

The isospin properties of the semileptonic and nonleptonic charm-changing weak interactions lead to restrictions on the branching ratios of the different charge states that can occur in multi-particle decays of charmed particles. Upper and lower bounds imposed by isospin invariance on these ratios, and on the average number of charged decay products, are calculated for various charmed-particle decay modes. Simultaneous bounds on the branching ratios of two different charge distributions in the same decay mode, or in isospin-related decay modes, are presented. Branching ratios are also calculated under the statistical postulate that all the allowed invariant isospin channels contribute equally and incoherently in each decay mode.

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I. Introduction

In the past year, several candidates for charmed particles^{1,2} have been discovered around mass $2 \text{ GeV}/c^2$,³⁻⁸ in striking agreement with theoretical anticipations.^{2,9-11} The charm-changing weak interaction which is responsible for the semileptonic and nonleptonic decays of the charmed particles is believed to have certain simple isospin properties which imply definite values of the total isospins of the decay products. These total isospin restrictions impose useful constraints upon the possible charge distributions in multiparticle decay states, and upon the branching ratios of the allowed states.

In this paper we explore a wide range of constraints upon the multiparticle charge distributions in charmed-particle decays which follow from the assumed isospins of the charmed-particles, the assumed isospin properties of the weak interaction, and the assumed isospin invariance of the strong interactions which govern the evolution of the final state. Section II contains a more precise statement of these assumptions.

Our work is addressed to a problem which has, in part, been responsible for the elusiveness of charmed particles up to now: in any given apparatus, only certain charmed-particle final states (for instance those containing only charged particles) are likely to be visible. The present isospin considerations permit bounds to be placed on branching ratios into the remaining final states, and more generally on any function of the charge distributions. The relations obtained here may be useful in verifying

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the isospins of charmed particles, testing the isospin properties of the charm-changing weak interaction, hunting for missing final states in charmed particle decays, and in planning or interpreting future experiments wherein it may be necessary to allow for unseen neutral decay products.

General consequences of definite total isospin quantum numbers for the charge distributions in multiparticle states have been presented by many authors.¹²⁻²⁷ Nachtmann and Pais have also given some specific bounds on the average ratios of neutral to charged hadrons in charmed-particle decays.²⁷ We concentrate here on the more detailed statements that can be made about charge distributions within each charmed-particle decay mode of definite multiplicity. Specifically, we give the maximum and minimum possible branching ratios of the observable final charge states within such modes, and we give simultaneous bounds for two different final charge states in the same or related decay processes. Other quantities can be formed from charge distributions in charmed particle decays. As an example, we present upper and lower bounds for the number of charged tracks in each decay mode of fixed multiplicity. We also give the branching ratios that follow from a statistical postulate originally due to Fermi,¹² in which all invariant isospin channels allowed by the total isospin restrictions are taken to contribute equally and incoherently in each decay mode. All these results are given in Section III; Section IV contains a brief

discussion of some further applications.

In obtaining bounds, our method is to carry out the required isospin couplings explicitly (Appendix A) and to locate the required bounds upon the operators representing the probabilities of observing specific final states by numerical solution of the implied eigenvalue problems (Appendix B). This approach guarantees that we obtain the tightest bounds that follow from isospin restrictions alone, even in cases where the final state contains several different kinds of particles.

A novel feature of the present work (Appendix C) is a simple method for the introduction of simultaneous bounds upon the probabilities of observing two different charge states. These simultaneous bounds are obtained by bounding linear combinations of the operators representing the two probabilities and making use, in effect, of the convexity of the set of possible density matrices. The results of the simultaneous calculations for several cases are presented through graphs of the allowed regions.

Appendix D contains some details on simultaneous bounds for D^+ and D^0 decays.

In all the discussions and results in this paper, we deal with charge distributions in terms of quantities like the number of charged pions. Stronger results can be obtained by the same methods for charge distributions of kinematically defined charges, such as the charge of the most energetic pion whose velocity lies in a certain spatial cone, but that is not done here.

II. Assumptions

The charm-changing weak current is assumed to induce the quark transition F_1

$$c \leftrightarrow s \cos \theta_c - d \sin \theta_c \quad (1)$$

where $\theta_c \sim 15^\circ$ is the Cabibbo angle.^{1,2,28} The first term, which dominates because θ_c is small, obeys $\Delta S = \Delta C$ and $\Delta I = 0$. Consequently, the corresponding part of the semileptonic weak Hamiltonian $H_W^{s.l.} (\Delta S = \Delta C)$ obeys $\Delta I = 0$.

The nonleptonic weak decays of charmed particles are expected to proceed through the product of the current represented by the quark decay mode (1) and the ordinary Cabibbo current, which is responsible for

$$d \cos \theta_c + s \sin \theta_c \leftrightarrow u \quad (2)$$

Thus the nonleptonic decays of a charmed quark are given by

$$\begin{aligned} c &\rightarrow s u \bar{d} \cos^2 \theta_c \quad (\Delta S = \Delta C) \\ &+ [s u \bar{s} - d u \bar{d}] \sin \theta_c \cos \theta_c \quad (\Delta S = 0) \\ &- d u \bar{s} \sin^2 \theta_c \quad (\Delta S = -\Delta C) \end{aligned} \quad (3)$$

The dominant part of the quark decay mode (3) is the term proportional to $\cos^2 \theta_c$. The corresponding part of the nonleptonic weak Hamiltonian $H_W^{n.l.} (\Delta S = \Delta C)$ obeys $\Delta I = |\Delta I_z| = 1$. It gives rise only to $c \rightarrow s u \bar{d}$, the quark transition that appears to be responsible for the nonleptonic decays of most of the charmed particle candidates

observed up to now.³⁻⁸ In the remainder of this paper, all decays which vanish for $\theta_c = 0$ are neglected and we confine our attention to

$$H_W^{s.l.} (\Delta S = \Delta C) : \Delta I = 0 \quad (4)$$

$$H_W^{n.l.} (\Delta S = \Delta C) : \Delta I = |\Delta I_z| = 1 \quad (5)$$

The charmed particles whose decays we discuss are the following:

- 1) $D^0(1865)$,^{4,7} taken to be a spin-zero $c\bar{u}$ state, i.e., a state with $I = 1/2$.
- 2) $D^+(1876)$,^{5,7} taken to be a spin-zero $c\bar{d}$ state, the isospin partner of D^0 .
- 3) $F^+(2000)$ (not yet discovered),^{F2} taken to be a spin-zero $c\bar{s}$ state, i.e., a state with $I = 0$. (See Refs. 10, 11, 29, 30.)
- 4) $C_0^+(2260)$,^{3,6,8} taken to be a spin-1/2 isoscalar cud state.^{10,31} This particle also is called Λ_c^+ .^{3,6,11} The weak decays of these particles, or of their antiparticles, either have been observed or may be observed in the near future.^{F3} All the final states to which they decay with $\Delta S = \Delta C$ have definite isospin, with the exception of the D^0 nonleptonic decay states, which have a mixture of $I = \frac{1}{2}$ and $I = \frac{3}{2}$.

It is assumed that the final hadron state reached as a result of the effect of $H_W^{s.l.}$ or $H_W^{n.l.}$ [Eqs. (4), (5)] evolves with the conservation of isospin. For instance, certain of our results for multi-pion final states will be invalid if these states contain decay products of the η which are not identified as such. To

avoid such problems in practice it may prove necessary to restrict the kinematic regions under discussion.

III. Results

In the following, a charge state is specified by giving the numbers of particles of each kind and each charge. A decay mode is the totality of charge states for given numbers of particles of each kind, coming from the decay of a particular charmed particle and having therefore definite total charge. Thus the decay mode $(\bar{K}2\pi)^+$ consists of the charge states $K^-2\pi^+$ and $\bar{K}^0\pi^+\pi^0$. The charge state $\bar{K}^0\pi^+\pi^0$ in turn includes pairs of final states which go into each other under interchange of the π^+ and the π^0 . The fraction f denotes the fractional contribution of a given charge state to its decay mode. In the example, $f(K^-2\pi^+) + f(\bar{K}^0\pi^+\pi^0) = 1$.

A. Bounds on branching ratios of charge states.

Experiments to observe a particular decay mode can usually detect some charge states but not others because unseen neutral particles may prevent kinematic reconstruction. Moreover, the observable charge states may be detected with different efficiencies. For present purposes we assume an experiment in which neutrinos, photons, π^0 's, Σ^0 's, and neutrons are invisible and all other particles are visible. We assume that the experimental count rates have been corrected for the efficiencies in detecting the visible particles. When the final state consists entirely of visible particles, so that a four constraint kinematic fit is possible in principle, we call it a 4C charge state. Similarly, if there is

one invisible particle, we call the final state a 1C charge state.^{F4}

Bounds for the fractions f associated with 4C and 1C charge states have been calculated as explained in Appendix B. The results for decay modes with up to seven hadrons are given in Tables 1 and 2, along with statistical values which are discussed below. Similar tables can easily be produced by the same methods to bound the fractions of charge states that will be visible to future charmed-particle detectors which will be sensitive to some neutral particles. (See, for example, refs. 32-35.) In constructing bounds on sums of fractions of individual charge states, as, for example, in the case of $F^+ \rightarrow (\bar{K}^0 K^0 \pi^+ \text{ or } K^- K^+ \pi^+)$, it would be equally simple to weight these fractions by their detection efficiencies, instead of by correcting the experimental count rates. That procedure could avoid statistical troubles that otherwise might result from low detection efficiencies for some charge states.

For clarity, we now give some specific examples of the results shown in Tables 1 and 2. Semileptonic decays lead to final hadronic states of low isospin, for which the bounds are fairly tight. Thus, for instance,

$$\frac{\Gamma(D^0 \rightarrow \ell^+ \nu_\ell \bar{K}^0 \pi^-)}{\Gamma(D^0 \rightarrow \ell^+ \nu_\ell \bar{K} \pi)} = \frac{2}{3}, \quad (6)$$

and

$$\frac{1}{3} \leq \frac{\Gamma(D^0 \rightarrow \ell^+ \nu_\ell K^- \pi^+ \pi^-)}{\Gamma(D^0 \rightarrow \ell^+ \nu_\ell \bar{K} 2\pi)} \leq \frac{2}{3}. \quad (7)$$

The opposite extreme is provided by the nonleptonic decays of D^0 ,

for which the bounds are very weak as a result of the possibility of both $I = 1/2$ and $I = 3/2$ final states. Here, and in many other cases, a significant improvement is possible when both 4C and 1C final states can be detected.

Some of the bounds are quite tight even for fairly high multiplicity- e.g.,

$$\frac{1}{2} \leq \frac{\Gamma(C_0^+ \rightarrow \Lambda \pi^+ \pi^+ \pi^-)}{\Gamma(C_0^+ \rightarrow \Lambda 3\pi)} \leq \frac{4}{5} \quad (8)$$

$$\frac{3}{5} \leq \frac{\Gamma(C_0^+ \rightarrow \Lambda 2\pi^+ \pi^- \pi^0)}{\Gamma(C_0^+ \rightarrow \Lambda 4\pi)} \leq 1 \quad (9)$$

The latter is another example showing the importance of detecting 1C final states. Of course, semileptonic charge states will be identified only if 1C final states can be detected.

At times it is worthwhile to discuss the sum of two charge states, as in the bounds

$$\frac{1}{2} \leq \frac{\Gamma(C_0^+ \rightarrow \{ \Sigma^+ \pi^- \pi^+ \text{ or } \Sigma^- 2\pi^+ \})}{\Gamma(C_0^+ \rightarrow \Sigma 2\pi)} \leq \frac{4}{5} \quad (10)$$

These involve the same isospin problem as Eq. (8).

B. Bounds on the average numbers of charged tracks.

The same methods can be used to bound any functions of the charge states within a given decay mode. For example, one may

consider the total number n_c of charged tracks, including those that result from the secondary decays of neutral strange particles. Neglecting questions of efficiency and of small deviations from the $\Delta I = \frac{1}{2}$ rule, we take each K^0 or $\bar{K}^0$ to contribute $2/3$, and each Λ or Σ^0 to contribute $4/3$, to n_c .^{F5} Some bounds on the expectation values of n_c have been obtained in this manner and the results are given in Table 3.

The bounds in Table 3 are complementary to ones²⁷ that refer to the sum over decay modes. The present results are expected to be particularly useful in connection with dynamical models which may predict the relative branching ratios into decay modes of various multiplicities.

C. Simultaneous bounds.

In many cases, it is possible to observe more than one charge state in a given decay mode. For example, two distinct 4C charge states occur in $F^+ \rightarrow \bar{K} K \pi \pi$ (see, however, footnote F6) and in $C_0^+ \rightarrow \Sigma \pi \pi$ ($n \geq 2, \text{even}$). In general, the absolute bounds of the two individual charge states cannot simultaneously be achieved. We have calculated the simultaneous bounds on the fractions associated with pairs of charge states, by the method explained in Appendix C, for several cases. The results are shown in Fig. 1, where the simultaneously allowed fractions are represented by points within the convex region for each case.

The results illustrated in Fig. 1 can be used in several ways. For example, in the decay of C_0^+ , the ratio $f(\Sigma^- \pi^+ \pi^+)/f(\Sigma^+ \pi^+ \pi^-)$ may be measured but the values of the individual fractions f may

be unknown. Bounds on the individual fractions f , or equivalently on the fraction F for all unobservable charge states, can then be obtained directly from Fig. 1. Or, the problem of varying detection efficiencies mentioned above in Sec. IIIA could be overcome with the aid of the convex figures presented for simultaneous bounds. For instance, the 4C charge state $D^0 + \bar{K}^0 \pi^+ \pi^-$ may be detected with different efficiency than the 1C charge state in $D^0 + K^- \pi^+ \pi^0$. The absolute bounds on the efficiency-weighted sum of 4C and 1C fractions is determined from Fig. 1d by the projection of the convex curve on a ray whose slope equals the efficiency ratio $\epsilon(1C)/\epsilon(4C)$.

Another possibly useful application of simultaneous bounds would be to the number n_{pc} of prompt charged tracks and the number n_v of visible $\bar{K}^0$ or Λ decays. These quantities are related to the total number of charged tracks by

$$n_c = n_{pc} + 2n_v \quad (11)$$

For example, in the decay mode $D^0 \rightarrow \pi^+ \pi^-$,

$$\begin{aligned} n_{pc} &= 3[f(\pi^+ \pi^-) + f(\pi^+ \pi^0) + f(\pi^0 \pi^0) + f(\pi^+ \pi^-)] \\ &= 1 + 2[f(\pi^+ \pi^-) + f(\pi^+ \pi^0)] \end{aligned} \quad (12)$$

$$n_v = \frac{2}{3} f(\pi^0 \pi^0) \quad (13)$$

The simultaneous bounds for this case are given in Fig. 2.

Simultaneous bounds for n_{pc} and the ratio $f(\pi^+ \pi^-)/f(\pi^+ \pi^0)$ can be obtained from Fig. 1d by drawing rays that correspond to constant values of the ratio. Similarly, for $D^0 \rightarrow K^0 \pi^+$, it can be seen that simultaneous bounds on n_{pc} and n_v are equivalent to simultaneous bounds on $f(4C)$ and $f(1C)$, and this may be visualized with the help of Fig. 1a.

D. Relations between D^+ and D^0 decays

The hadrons in final states of semileptonic ($\Delta I = 0$) decays have total isospin 1/2 and are related to each other by an isospin rotation. Thus the partial decay widths obey

$$\Gamma(D^+ \rightarrow \ell^+ \nu_\ell \left\{ \begin{array}{c} \bar{K}^0 \\ K^- \pi^+ \\ \bar{K}^0 \pi^0 \end{array} \right\}) = \Gamma(D^0 \rightarrow \ell^+ \nu_\ell \left\{ \begin{array}{c} \bar{K}^0 \pi^- \\ K^- \pi^0 \end{array} \right\}) \quad (14)$$

and so forth.

Since $H_W^{n,1} (\Delta S = \Delta C = -1)$ transforms like the $(I, I_3) = (1, 1)$ component of an isovector (Eq. (5)), the amplitudes for decays of D^+ and D^0 to $1 + 3/2$ final states are related to one another by vector coupling coefficients. For example,

$$\begin{aligned} \langle \bar{K}^0 \pi^+ | H_W^{n,1} | D^+ \rangle &= A_{3/2} \\ \langle K^- \pi^+ | H_W^{n,1} | D^0 \rangle &= \frac{1}{3} A_{3/2} + \frac{2}{3} A_{1/2} \\ \langle \bar{K}^0 \pi^0 | H_W^{n,1} | D^0 \rangle &= \frac{\sqrt{2}}{3} (A_{3/2} - A_{1/2}) \end{aligned} \quad (15)$$

where the subscripts refer to total isospin.

Eqs. (15) can be combined to give

$$A(D^0 \rightarrow K^- \pi^+) + \sqrt{2} A(D^0 \rightarrow \bar{K}^0 \pi^0) = A(D^+ \rightarrow \bar{K}^0 \pi^+) . \quad (16)$$

By the same reasoning, one obtains

$$\Gamma(D^+ \rightarrow X) \leq 3\Gamma(D^0 \rightarrow X) \quad (17)$$

where the decay width Γ is proportional to $|A|^2$, and X refers to any $S = -1$ decay mode, but not of course to a particular charge state.

The relations (16) can possibly provide dynamical information about nonleptonic decays. The decay Hamiltonian $H_W^{n.l.} (\Delta S = \Delta C)$ transforms as a combination of V -spin triplet and a V -spin singlet. It has been suggested that the V -spin singlet part is enhanced by the mechanism which leads to octet enhancement in nonleptonic decays of strange particles.³⁶ If this suggestion is correct and the V -spin triplet part of $H_W^{n.l.}$ can be neglected, then

$$\Gamma(D^+ \rightarrow \bar{K}^0 \pi^+) = 0 \quad (18)$$

and

$$\Gamma(D^0 \rightarrow K^- \pi^+) = 2 \Gamma(D^0 \rightarrow \bar{K}^0 \pi^0) . \quad (19)$$

It also has been suggested¹⁰ that exotic final states might be suppressed for all D^+ decays to hadrons, in which case

$$\Gamma(D^+ \rightarrow X) \ll \Gamma(D^0 \rightarrow X) \quad (20)$$

for any $S = -1$ mode X . However, this speculation is not supported by compelling theoretical arguments,³⁷ and present data are mute

on the question.

Examples of simultaneous bounds upon related D^+ and D^0 decay processes are given in Figs. 3,4. To make use of these bounds will require independent information about relative production rates and lifetimes of the D^+ and D^0 states, but it appears not to be impossible that such information will soon be experimentally available. That the absolute upper bounds on the ordinates in Figs. 3,4 are never greater than unity follows from Eq. (17).

E. Consequences of a statistical postulate.

Multiparticle states of definite total isospin are characterized by invariant isospin amplitudes labelled by a number of isospins one less than the number of particles. (Appendix A). The number of such amplitudes grows, of course, much more rapidly than the number of particles.^{38,39} Each invariant isospin amplitude depends upon all the kinematic variables.

In Tables 1-4 we give the charge-state fractions f and the numbers of charged particles n_c which follow from the postulate that each allowed isospin channel contributes equally and incoherently to the charge distributions. These results, like the rigorous bounds, are independent of the choice of invariant isospin basis.

This statistical postulate, which is due to Fermi,¹² may be interesting for charmed-particle decays because they probably proceed through many independent invariant isospin and kinematic channels, so that substantial cancellations of interference terms

may occur. Strong deviations from these statistical predictions may represent interesting physics, and the predictions themselves may be useful in planning experiments.

The statistical predictions can be tested experimentally in any decay mode which has at least two experimentally identifiable charge states, since the values of the charge state fractions are predicted unambiguously. Even in cases where only one charge state of a decay mode is experimentally accessible, it may be possible to test the statistical postulate by using the charge state fraction to deduce the total rate expected for the decay mode, and by summing over decay modes to obtain a prediction for the average number of charged tracks and for other observable quantities.²³ In this manner, for example, one could predict the relative rates for

$$D^0 \rightarrow \left\{ \begin{array}{l} K^- + (1,3,5,\dots\pi^+) \\ \bar{K}^0 + (2,4,6,\dots\pi^+) \end{array} \right\} + (\text{any number of } \pi^0), \quad (21)$$

given only the decays of D^0 into 4C states of $\bar{K} + \text{pions}$. There does exist at least one instance where the statistical postulate is already known to fail on the basis of such a test. In $\bar{p}n$ pionic annihilations at rest, the postulate underestimates the average ratio of neutral to charged particles.^{24,40-42}

For D^0 decays, Table 4 shows that the statistical postulate implies a definite relation between the charge-state fractions

and the ratio $\rho^{(n)}$, defined for the decay mode in question as the intensity ratio

$$\rho^{(n)} \equiv \frac{\Gamma(D^0 \rightarrow \bar{K}n\pi | I=3/2)}{\Gamma(D^0 \rightarrow \bar{K}n\pi)} \quad (22)$$

For instance,

$$\frac{1}{2} \leq \frac{f(D^0 \rightarrow \bar{K}^0\pi^0)}{f(D^0 \rightarrow K^-\pi^+)} = \frac{1 + \rho^{(1)}}{2 + \rho^{(1)}} \leq 2 \quad (23)$$

and

$$\frac{2}{3} \leq \frac{f(D^0 \rightarrow K^-\pi^+\pi^0)}{f(D^0 \rightarrow \bar{K}^0\pi^+\pi^-)} = \frac{2}{3} \frac{5 + 2\rho^{(2)}}{5 - \rho^{(2)}} \leq \frac{7}{6} \quad (24)$$

For $n \geq 3$, as Table 4 shows, the predictions of the statistical postulate for $D^0 \rightarrow \bar{K}n\pi$ are almost totally insensitive to $\rho^{(n)}$.

IV. Limitations and Extensions

The application of isospin theorems to experimental data depends upon the possibility of correcting those data to compensate for any charge-dependent detection probabilities. For instance, a final state involving two pions may be observed with the aid of charged-particle detectors set at certain positions. Isospin invariance implies certain statements about the number of neutral pions entering the same detectors. However, neutral pions may be detected by observing the missing mass recoiling against visible charged particles, so that the experiment is sensitive to charged and neutral pions with different values of their kinematic variables. Isospin invariance has nothing to say about such an experiment unless additional assumptions are made, for instance that angular distributions are isotropic. A second example is an experiment involving two pions, where the positive pion is detected in one direction and the negative pion in another direction. Isospin invariance alone does not relate those two counting rates, but permutation symmetry can under some circumstances be joined with isospin invariance to produce such a relation.²¹ That possibility is not included in our present calculations.

Our specific interest in this paper is restricted to charmed particle decays, but the method is of course more general. It can be applied, for example, to hadronic decays of (isoscalar) $c\bar{c}$ states (cf. Ref.25), and to weak decays of hadrons containing even heavier quarks, if such hadrons are found to exist. Even within the limits of charmed particle final states, the methods presented here are not confined to answering the specific

questions we have asked. The definitions of observable final states will change as detectors improve. Many types of simultaneous bounds will become possible in addition to the few we have noted. Functions of charge distributions other than those we have introduced (n_c, n_{pc}, n_v , in Sec. III. B) may be of interest.

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Appendix A. Invariant Isospin Basis

Consider an N-particle state with individual isospin quantum numbers I_k, m_k for $k=0, 1, \dots, N$. The particle labels may be chosen in any way that does not depend upon any charge state. For instance, the particles may be numbered according to their kind and energy. The isospin wave function $\phi(I, M)$ for a state of definite total isospin ($I = \sum I_k$) can be written in the form

$$\phi(I, M) = \sum_{\underline{t}} x_{\underline{t}}(I) \phi_{\underline{t}}(I, M) \quad (A.1)$$

where $\underline{t}$ stands for the invariant isospin quantum numbers $(t_2, t_3, \dots, t_{N-1})$ of the operators $\underline{T}_k^2$ defined by

$$\begin{aligned} \underline{T}_2 &= \underline{T}_1 + \underline{T}_2 \\ \underline{T}_k &= \underline{T}_{k-1} + \underline{T}_k \quad \text{for } 3 \leq k \leq N-1 \\ \underline{T} &= \underline{T}_{N-1} + \underline{T}_N \end{aligned} \quad (A.2)$$

The coefficients $x_{\underline{t}}(I)$ may depend upon kinematic variables but not upon any isospin z components, and the basis functions $\phi_{\underline{t}}(I, M)$ are the simultaneous eigenfunctions of $\underline{T}^2, I_z$, and all the $\underline{T}_k^2$.

Eqs. (A.2) define one particular choice for the invariant isospin basis. In practice, it is sometimes computationally convenient to use other basis sets (e.g. $\underline{T}_{12} = \underline{T}_1 + \underline{T}_2$, $\underline{T}_{34} = \underline{T}_3 + \underline{T}_4$, etc.). All our results are independent of the invariant basis chosen. We present our methods in terms of the $\underline{t}$ basis defined by Eqs. (A.2).

The amplitude for finding precisely the charge values $\underline{m} = \{m_1, m_2, \dots, m_N\}$ is given by

$$\langle \underline{m} | \phi(I, M) \rangle = \sum_{\underline{t}} x_{\underline{t}}(I) U_{\underline{m}\underline{t}}(I, M), \quad (A.3)$$

where

$$\begin{aligned} U_{\underline{m}\underline{t}}(I, M) &= \langle \underline{m} | \phi_{\underline{t}}(I, M) \rangle \\ &= (I_1, m_1; I_2, m_2 | t_2, m_1 + m_2) \\ &\times (t_2, m_1 + m_2; I_3, m_3 | t_3, m_1 + m_2 + m_3) \\ &\dots (t_{N-1}, m_1 + \dots + m_{N-1}; I_N, m_N | I, M) \end{aligned} \quad (A.4)$$

and the quantities on the right-hand side are the Clebsch-Gordan coefficients for the vector couplings of Eqs. (A.2). The matrix elements $U_{\underline{m}\underline{t}}(I, M)$ vanish unless $M = \sum m_k$ and unless $\underline{t}$ obeys all the triangle inequalities implied by Eqs. (A.2). We eliminate the vanishing rows and columns to leave a rectangular matrix $U(I, M)$ which usually has more rows ($\underline{m}$) than columns ($\underline{t}$). This matrix satisfies

$$[\tilde{U}(I, M) U(I, M)]_{\underline{t}\underline{t}'} = \delta_{\underline{t}\underline{t}'} \quad (A.5)$$

Appendix B. Bounds Upon One Function of the Charge Distribution

Any function of the N-particle charge distribution is represented by some matrix R which is diagonal in the π representation. For instance, the total number of charged particles is represented by

$$R_{\pi\pi'}^{(a)} = \delta_{\pi\pi'} \sum_k h(\pi_k) \quad (B.1)$$

where $h(\pi_k) = 1$ if π_k has a value for which particle k is charged and $h(\pi_k) = 0$ otherwise. Similarly, the probability of finding a certain charge state $\bar{m}$ is represented by the projection

$$R_{\pi\pi'}^{(b)} = \delta_{\pi\pi'} \delta_{\pi\bar{m}} \quad (B.2)$$

The expectation $\langle R \rangle$ of any such matrix R is obtained by combining Eqs. (A.1) and (A.3).

$$\langle R \rangle = \langle \psi(I, M) | R | \psi(I, M) \rangle$$

$$= \sum_{\pi\pi'} x_{\pi}^* x_{\pi'} R_{\pi\pi'}(I, M), \quad (B.3)$$

where the square matrix $R(I, M)$ is given by

$$R(I, M) = \bar{U}(I, M) R U(I, M). \quad (B.4)$$

The upper and lower bounds upon $\langle R \rangle$ are respectively the maximum and minimum eigenvalues of $R(I, M)$. These bounds are

achievable by choosing x_{π} to be the appropriate eigenvectors of $R(I, M)$, provided that no constraints other than those imposed by isospin invariance supervene.

The matrix $R(I, M)$ is also useful for statistical models. For instance, if it is assumed that

$$\text{Av}(x_{\pi}(I)^* x_{\pi'}(I)) = D^{-1} \delta_{\pi\pi'}, \quad (B.5)$$

where D is the number of basis states $\phi_{\pi}(I, M)$, then it follows that

$$\text{Av}(\langle R \rangle) = D^{-1} \text{Trace} \{R(I, M)\} \quad (B.6)$$

Appendix C. Simultaneous Bounds Upon Two Functions of the Charge Distribution

Stronger results can sometimes be obtained by considering simultaneously two functions of the charge distribution, represented by matrices R and S . For instance, R could represent the probability that all pions above a certain energy are charged, and S could represent the sum of the charges of all particles moving in a certain direction. The matrices R and S necessarily commute; they are both diagonal in the π representation. There is no implication of any other relation between the measurements they represent. However the smaller matrices $R(I, M)$ and $S(I, M)$ which result from projection onto the space of definite I and M according to Eq. (B.4) do not necessarily commute, even if R and S represent the probabilities of realizing mutually exclusive conditions.

The result of a simultaneous measurement of R and S is

represented by a point

$$\begin{aligned} r &= A_V \langle R \rangle \\ s &= A_V \langle S \rangle \end{aligned} \quad (C.1)$$

in the two-dimensional (r,s) plane. The expectation values $\langle \rangle$ in Eqs.(C.1) refer to a single wave function $\psi(I,M)$ with fixed values of all variables other than the isospin quantum numbers. The indicated averages $A_V \langle \rangle$ in Eqs.(C.1) may then be incoherent averages with respect to some kinematic variables, and possibly also with respect to some ensemble of initial states. The isospin-allowed region of the (r,s) plane must be convex: any point (r,s) on the straight-line segment connecting allowed points (r_1, s_1) and (r_2, s_2) must be allowed because it represents some weighted average of the conditions leading to (r_1, s_1) and those leading to (r_2, s_2) . Examples of such convex regions appear in Figs.1-4.

Because of the convexity of the allowed regions, every boundary point is a point of maximum or minimum projection on a ray drawn at some angle θ to the r axis, with $0 \leq \theta < \pi$. To locate the boundary points, we define rotated matrices

$$\begin{aligned} V(\theta) &= R(I,M) \cos\theta + S(I,M) \sin\theta \\ W(\theta) &= -R(I,M) \sin\theta + S(I,M) \cos\theta \end{aligned} \quad (C.2)$$

The expectation value $\langle V(\theta) \rangle$ represents the projection of the point (r,s) on the ray at angle θ to the r axis. If the extreme

eigenvalue of $V(\theta)$ is non-degenerate, its eigenvector $\psi(\theta)$ determines the corresponding boundary point through

$$\begin{aligned} r &= \langle \psi(\theta) | R(I,M) | \psi(\theta) \rangle \\ s &= \langle \psi(\theta) | S(I,M) | \psi(\theta) \rangle \end{aligned} \quad (C.3)$$

If the extreme eigenvalue of $V(\theta)$ is degenerate, the convex region may be bounded by a straight-line segment perpendicular to the ray at angle θ . In that case, the endpoints of the straight-line segment can be determined by finding the extreme eigenvalues of $W(\theta)$ under the constraint that $V(\theta)$ has its maximum or minimum value, i.e. by defining $\psi(\theta)$ as the extreme eigenvector of the projection of $W(\theta)$ upon the subspace generated by the degenerate eigenvectors belonging to the maximum or minimum eigenvalue of $V(\theta)$. The allowed regions in Figs. 1-4 were determined by calculating the extreme eigenvectors for θ in $0 \leq \theta < \pi$ and using Eqs.(C.3).

Appendix D. Simultaneous Bounds Upon Two Functions of the Charge Distributions in the Decays of D^+ and D^0 Mesons

The method described in Appendix C applies also to the case where R and S represent measurements on isospin-related states instead of two functions of the same charge distribution. Suppose $\psi(I,M)$ and $\phi(I,M')$ are related by an isospin rotation, so that

$$\begin{aligned} \psi(I,M) &= \sum_{M'} x_{M'} \langle I' | \phi_{M'} \rangle \langle I, M \rangle \\ \phi(I,M') &= \sum_{M} x_{M'} \langle I' | \psi \rangle \langle I, M' \rangle. \end{aligned} \quad (D.1)$$

Let R represent a measurement on the state $\psi(I, M)$ and S on the state $\psi(I, M')$:

$$\begin{aligned} r &= \text{Av}\{\langle R(I, M) \rangle\} = \text{Av}\{\langle \tilde{U}(I, M) R U(I, M) \rangle\} \\ s &= \text{Av}\{\langle S(I, M') \rangle\} = \text{Av}\{\langle \tilde{U}(I, M') S U(I, M') \rangle\} \end{aligned} \quad (D.2)$$

Then $V(\theta)$ and $W(\theta)$ are defined as in Eqs. (C.2), except that $S(I, M')$ replaces $S(I, M)$ and all the rest of Appendix C is unchanged.

The relations between charge distributions in D^+ and D^0 nonleptonic decays are more complicated because D^+ decays by the weak interaction into states of $I = \frac{3}{2}$, $M = \frac{3}{2}$ only, while D^0 decays weakly into a mixed state with $I = \frac{3}{2}$ or $\frac{1}{2}$ and $M = \frac{1}{2}$.

The isospin wave functions are given, for D^+ or $D^0 \rightarrow \bar{K} n \pi$, by

$$\begin{aligned} \psi(D^+) &= [\rho^{(n)}]^{-\frac{1}{2}} \sum_{I, I'} x_{I, I'} \left(\frac{3}{2}\right) \psi_{I, I'} \left(\frac{3}{2}, \frac{3}{2}\right) \\ \psi(D^0) &= \sum_{I, I'} x_{I, I'} \left(\frac{3}{2}\right) \psi_{I, I'} \left(\frac{3}{2}, \frac{1}{2}\right) + \sum_{I, I'} x_{I, I'} \left(\frac{1}{2}\right) \psi_{I, I'} \left(\frac{1}{2}, \frac{1}{2}\right), \end{aligned} \quad (D.3)$$

We use the normalization condition

$$\sum_{I, I'} |x_{I, I'}(I)|^2 = 1 \quad (D.4)$$

and we assume that the branching ratio

$$\rho^{(n)} = \sum_{I, I'} |x_{I, I'} \left(\frac{3}{2}\right)|^2 = \frac{\Gamma(D^0 \rightarrow \bar{K} n \pi) I=3/2}{\Gamma(D^0 \rightarrow \bar{K} n \pi)} \quad (D.5)$$

is unknown; we do not constrain the value of $\rho^{(n)}$ in finding the

boundaries of the isospin-allowed region of the (r, s) plane.

Charge measurements R on decay states of the D^0 are treated just as in Appendix B, except that the applicable invariant isospin basis states $\psi_{I, I'}(I, M)$ now involve two values of I . The transformation matrix $U_{m, I'}(D^0)$ has the columns of $U_{m, I'}(\frac{3}{2}, \frac{1}{2})$ defined in Eq. (A.4), followed by the columns of $U_{m, I'}(\frac{1}{2}, \frac{1}{2})$. Then

$$R(D^0) = \tilde{U}(D^0) R U(D^0) \quad (D.6)$$

The absolute bounds of $r = \text{Av}\{\langle R \rangle\}$ are the extreme eigenvalues of the $R(D^0)$, as in Appendix B.

To obtain the joint bounds for R and a measurement S on the decay products of D^+ , we define a matrix $U_{m, I'}(D^+)$ which consists of the columns of $U_{m, I'}(\frac{3}{2}, \frac{3}{2})$ followed by columns of zeroes for $I = \frac{1}{2}$, and we define

$$B = \rho^{(n)} \text{Av}\{\langle \psi(D^+) | S | \psi(D^+) \rangle\} \quad (D.7)$$

$$= \sum_{I, I'} x_{I, I'}^*(I) x_{I', I'}(I') S_{I, I', I', I'}(D^+) \quad (D.8)$$

where

$$S(D^+) = \tilde{U}(D^+) S U(D^+) \quad (D.9)$$

contains $S(\frac{3}{2}, \frac{3}{2})$ in its upper-left corner and is otherwise filled out with zeroes to give it the same dimension as $R(D^0)$.

Now the process of determining the boundary of the isospin-allowed region in the (r, s) plane is again that of Appendix C,

except that $R(D^0)$ and $S(D^+)$ replace $R(I,M)$ and $S(I,M)$. However the meaning of the results is different in this case because, according to Eqs. (D.3) and (D.7), s represents the result of a measurement on a D^+ decay system relative to one on a D^0 decay system. For instance, consider the decay mode $D \rightarrow \bar{K} 2\pi$, and assume equal numbers of D^+ and D^0 decays into the $\bar{K} 2\pi$ mode. Let $R_{\pi\pi} = 1$ if both pions are charged in the D^0 decay, and let $R_{\pi\pi} = 0$ otherwise. Similarly, let $S_{\pi\pi} = 1$ if both pions are charged in the D^+ decay and let $S_{\pi\pi} = 0$ otherwise. In terms of the fractions f of the possible results of a charge measurement,

$$r = f(D^0 \rightarrow \bar{K}^0 \pi^+ \pi^-) \quad (D.10)$$

$$s = f(D^+ \rightarrow K^- \pi^+ \pi^+) \frac{\Gamma(D^+ \rightarrow \bar{K} 2\pi)}{\Gamma(D^0 \rightarrow \bar{K} 2\pi)} \quad (D.11)$$

$$= \frac{\Gamma(D^+ \rightarrow K^- \pi^+ \pi^+)}{\Gamma(D^0 \rightarrow \bar{K} 2\pi)}$$

The isospin-allowed regions in Figs. 3-4 were calculated as described here. Tighter bounds can in principle be obtained by extending the calculation to constrain the branching ratios $\rho^{(n)}$, either within some model or for experimental reasons. The calculations would be significantly more elaborate because they would in effect involve joint bounds upon three functions of the charge distributions.

Many of the results in these appendices can be obtained in a wide variety of ways. In particular, the literature on bounds in the presence of constraints is quite extensive (see, e.g., Ref. 43, 44).

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FOOTNOTES

- F1. The charmed quark c is taken to have charm quantum number $C = 1$. All the present discussion is, of course, valid for $C = -1$ states as well, and they will not be treated separately.
- F2. The notation is that of Ref. 10. The mass estimates of Refs. 11, 29, and 30 are used.
- F3. This evaluation is based in part on the mass estimates of Refs. 29-31, indicating which particles should be stable against weak decays, and in part on the estimates of production rates of Ref. 31. An important omission, made for the sake of brevity, is the $A^{0,+}$ (~ 2470) (not yet discovered), a spin-1/2 csu or csd state belonging to a $\bar{3}$ of $SU(3)$. Its dominant decays will be to states of $S = -2$. The isospin properties of A decays are exactly those of D decays, so certain of our results can be carried over directly. The A should not be hard to produce in e^+e^- or photon-induced reactions. Less likely to be seen, but also expected to be weakly decaying, are a spin-1/2 T^0 (~ 2740) = css , and doubly - or

triply-charmed baryons.

F4. If all the final particles are observed and if their source is a particle of known four-momentum (this may not always be so) then the conservation of energy and momentum implies four constraints. If one final particle is not observed but its identity is hypothesized (so that $p^2 = m^2$), we lose three of these constraints.

F5. Half of the neutral kaons materialize as K_L^0 , and for the purpose of Table I we assume that their charged decay products are not seen in the experiment. Of the half which decay as K_S^0 , two-thirds produce two charged tracks each.

F6. In decays involving two neutral kaons, or in decays of the form $D^0 \rightarrow \bar{K}^0 + \text{pions}$ where the D^0 is known to be in a state of definite CP, it may not always be true that a $\bar{K}^0$ can be detected with known probability a priori. For example, if $F^+ \rightarrow \bar{K}K\pi$ were to be dominated by $F^+ \rightarrow \pi\pi$, the ϕ never would decay to $K_S^0 K_S^0$ so the charge state $\bar{K}^0 K^0 \pi^+$ could not be classified as $4C$. We shall indicate in the tables where effects of this sort can arise, but ignore them in quoting results.

F7. Another test of the V-spin singlet nature of $K_v^{n,1}$ ($\Delta S = \Delta C$) is provided by the relation $\sqrt{2}A(C_0^+ + p \bar{K}^0) + \sqrt{3}A(C_0^+ + \Lambda\pi^+) = -A(C_0^+ + \bar{K}^0\pi^+)$, separately for S-waves and P-waves. (See Ref. 36.)

Fig. We have performed such a test using the data on $\bar{p}n$ pionic annihilations at rest quoted in Ref. 42. For more complete discussions, see Refs. 40-42.

FIGURES

Fig. 1. Simultaneous bounds on fractions of a given mode in specific charge states. Allowed regions lie within closed curves.

a) $D^0 \rightarrow \bar{K} n \pi$, $n=2$: $4C = \bar{K}^0 \pi^+ \pi^-$; $1C = K^- \pi^+ \pi^0$, $n=3$: $4C = K^- 2\pi^+ \pi^-$; $1C = \bar{K}^0 \pi^+ \pi^- \pi^0$, $n=4$: $4C = \bar{K}^0 2\pi^+ 2\pi^-$; $1C = K^- 2\pi^+ \pi^- \pi^0$.
b) $D^+ \rightarrow \bar{K} n \pi$, $n=3$: $4C = \bar{K}^0 2\pi^+ \pi^-$; $1C = K^- 2\pi^+ \pi^0$, $n=4$: $4C = K^- 3\pi^+ \pi^-$; $1C = \bar{K}^0 2\pi^+ \pi^- \pi^0$. c) $F^+ \rightarrow \bar{K} K n \pi$, $n=1$: $4C = K^- K^+ \pi^+$; $4C' = \bar{K}^0 K^0 \pi^+$, $n=2$: $4C = \bar{K}^0 K^+ \pi^+ \pi^-$; $4C' = K^- K^0 2\pi^+$. d) $C_0^+ \rightarrow l 2\pi$.

Fig. 2. Simultaneous bounds on n_{pc} = number of prompt charged tracks and n_v = number of visible Λ decays for $C_0^+ \rightarrow l 2\pi$. Allowed region lies within closed curve. [Cf. Ref. 18.]

Fig. 3. Simultaneous bounds on D^0 and D^+ decays to $\bar{K} n$. Allowed region lies within closed curve.

Fig. 4. Simultaneous bounds on D^0 and D^+ decays to $\bar{K} n \pi$. Allowed regions lie within closed curves. $4C$ and $1C$ final states defined as in Figs. 1a and 1b; for $D^+ \rightarrow \bar{K} 2\pi$, $4C = K^- 2\pi^+$, $1C = \bar{K}^0 \pi^+ \pi^0$.
a) $4C$ final states. b) $4C$ or $1C$ final states.

Table 1. Isospin bounds on fractions $f(1C)$ of charmed particle semileptonic decay modes.

Decay	n	$f(1C)$		
		min.	max.	stat. value
$D^{+,0} + l^+ \nu_l \bar{K} n \pi$: a)		$(I, I_z; S) = (\frac{1}{2}, \pm \frac{1}{2}; -1)$		
$1C: no \pi^0$	0	1	1	1
	1	2/3	2/3	2/3
	2	1/3	2/3	1/2
	3	0	8/15	3/10
	4	0	8/15	2/9
	5	0	16/35	20/147
	6	0	16/35	5/51

$F^+ + l^+ \nu_l \bar{K} K n \pi$: $(I, I_z; S) = (0, 0; 0)$
 $1C: no \pi^0$. b) Entries same as $D^{+,0} + l^+ \nu_l \bar{K} n \pi$, above.

$F^+ + l^+ \nu_l n \pi$: $(I, I_z; S) = (0, 0; 0)$

$1C: no \pi^0$. Entries same as $C_0^+ + l^+ \nu_l \Lambda n \pi$, below.

$C_0^+ + l^+ \nu_l \{\Lambda n \pi, l(n-1)\pi\}$ c)		$(I, I_z; S) = (0, 0; -1)$		
$1C: no \pi^0$	0	1	1	1
or π^0	2	2/3	2/3	2/3
	4	1/3	8/15	2/5
	6	0	16/35	4/21

$C_0^+ + l^+ \nu_l \bar{K} n \pi$: $(I, I_z; S) = (0, 0; -1)$

1C: no π^0
or neutron

Divide entries in table for
 $D^{0,+} + l^+ \nu_l \bar{K} n \pi$, above, by 2.

- a) Also applies to $F^+ + l^+ \nu_l \bar{K} n \pi$ (but see note (b).)
When values of $f(1C)$ are divided by 2, also applies
to $C_0^+ + l^+ \nu_l \bar{K} n \pi$, with 1C = (no π^0 or neutron).
- b) In identifying 1C final states for $F^+ + l^+ \nu_l \bar{K} n \pi$
we assume that the $\bar{K}^0 K^0$ system is detectable; see,
however, footnote F6.
- c) $n=1$ forbidden by isospin; odd values of $n \geq 3$ always
have a π^0 and hence do not contribute to $f(1C)$. Also
applies to $F^+ + l^+ \nu_l n \pi$.

Table 2. Isospin bounds on fractions $f(4C)$ and $f(4C \text{ or } 1C)$
of charmed particle nonleptonic decay modes.^{a1}

Decay	n	f(4C)			f(4C or 1C)		
		min.	max.	stat. value	min.	max.	stat. value
$D^0 + \bar{K} n \pi : (I, I_z; S) = (\frac{1}{2}, \frac{1}{2}; -1)^b)$							
4C: no π^0	1	0	1	1/2	1	1	1
	2	0	1	9/20	2/5	1	17/20
4C or 1C:	3	0	4/5	4/15	1/5	1	11/15
at most	4	0	4/5	10/49	0	1	138/245
one π^0	5	0	24/35	15/119	0	1	55/119
	6	0	24/35	35/381	0	4/5	905/2667
$D^+ + \bar{K} n \pi : (I, I_z; S) = (\frac{3}{2}, \frac{3}{2}; -1)$							
4C: no π^0	1	1	1	1	1	1	1
	2	0	4/5	2/5	1	1	1
4C or 1C:	3	0	4/5	2/5	2/5	1	10/25
at most	4	0	24/35	4/21	1/5	1	2/3
one π^0	5	0	24/35	1/6	0	4/5	19/42
	6	0	64/105	5/57	0	4/5	15/38
$F^+ + n \pi$ or $n + n \pi$: see $C_0^+ + n \pi$, below.							
$F^+ + \bar{K} n \pi^0 : (I, I_z; S) = (1, 1; 0)$							
	0	1	1	1	1	1	1
	1	1/2	1	3/4	1	1	1
4C: no π^0	2	0	4/5	9/20	3/5	1	9/10
4C or 1C:	3	0	4/5	1/3	1/2	1	11/15
at most	4	0	24/35	10/49	1/5	1	30/49
one π^0	5	0	24/35	5/34	0	4/5	55/119

$C_O^+ \rightarrow \Lambda n \pi$ ^{d)}: $(I, I_z; S) = (1, 1; -1)$

	1	1	1	1	1	1	1
	2	0	0	0	1	1	1
4C: no π^0	3	1/2	4/5	3/5	1/2	4/5	3/5
4C or 1C:	4	0	0	0	3/5	1	4/5
at most	5	0	24/35	2/7	0	24/35	2/7
one π^0	6	0	0	0	1/5	4/5	15/28
	7	0	64/105	5/39	0	64/105	5/39

$C_O^+ \rightarrow \bar{K} N n \pi$: $(I, I_z; S) = (1, 1; -1)$

	0	1	1	1	1	1	1
4C: no π^0	1	0	3/4	3/8	1	1	1
or neutron	2	0	7/10	3/10	1/4	1	27/40
4C or 1C:	3	0	3/5	1/6	1/2 - 1/10	1	3/5
at most one	4	0	4/7	6/49	0	3/4	20/49
π^0 or neutron	5	0	18/35	5/58	0	7/10	40/119

- a) For 4C kinematic fits, all hadrons must be detected; for (4C or 1C) fits, all but at most one of the hadrons must be detected.
- b) Statistical values for $D^0 \rightarrow \bar{K} n \pi$ are quoted for equal weights for all $I = 1/2$ and $I = 3/2$ channels. More general results, in which $I = 1/2$ and $I = 3/2$ channels may be weighted differently, may be obtained from Table 4.

- c) In identifying 4C or 1C final states for $F^+ \rightarrow \bar{K} K n \pi$ we assume that the $\bar{K}^0 K^0$ system is detectable; see, however, footnote F6.
- d) These entries also are valid for the decays $F^+ \rightarrow n \pi$, $F^+ \rightarrow \eta + n \pi$ (the 4C column in Table 2 corresponds to 1C for $F^+ \rightarrow \eta + n \pi$, and for $C_O^+ \rightarrow \pi + (n-1)\pi$ (4C: no π^0 or π^0 ; 1C: at most one π^0 or π^0 .) The decay $F^+ \rightarrow \pi^+ \pi^0$ is forbidden by Bose statistics. [See Ref. 10.]

Table 3. Isospin bounds on average number of charged tracks in decays of charmed particles (see text for definition of n_c , which includes charged products of decays of K_S^0 and Λ .)

<u>Decay</u>	<u>n</u>	<u>n_c</u>		<u>stat. value</u>
		<u>min.</u>	<u>max.</u>	
<u>D⁰ + l⁺ν_l K̄nπ: (I, I_z; S) = (1/2, -1/2; -1)</u>				
	0	2.00	2.00	2.00
	1	2.44	2.44	2.44
	2	3.11	3.33	3.22
	3	3.78	4.00	3.83
	4	4.44	4.67	4.52
	5	5.11	5.33	5.17
	6	5.78	6.00	5.84
<u>D⁺ + l⁺ν_l K̄nπ: (I, I_z; S) = (1/2, 1/2; -1)</u>				
	0	1.67	1.67	1.67
	1	2.56	2.56	2.56
	2	3.00	3.22	3.11
	3	3.67	3.89	3.83
	4	4.33	4.56	4.48
	5	5.00	5.22	5.16
	6	5.67	5.89	5.82

$$\underline{F^+ + \ell^+ \nu_\ell \bar{K} K n \pi}^a: (I, I_z; S) = (0, 0; 0)$$

$$n_c = (8 + 2n)/3$$

$$\underline{C_0^+ + \ell^+ \nu_\ell \{A n \pi \text{ or } \bar{K} n n \pi\}}: (I, I_z; S) = (0, 0; -1)$$

$$n_c = (7 + 2n)/3$$

$$\underline{C_0^+ + \ell^+ \nu_\ell \Lambda n \pi}: (I, I_z; S) = (0, 0; -1)$$

$$n_c = (19 + 6n)/9 \quad (n \geq 1)$$

$$\underline{D^0 + \bar{K} n \pi}^b: (I, I_z; S) = (\frac{1}{2} \text{ or } \frac{3}{2}, \frac{1}{2}; -1)$$

1	0.67	2.00	1.33
2	1.26	2.67	2.10
3	1.47	3.71	2.76
4	2.03	4.35	3.45
5	2.27	5.32	4.12
6	2.82	5.96	4.79

$$\underline{D^+ + \bar{K} n \pi}: (I, I_z; S) = (\frac{3}{2}, \frac{3}{2}; -1)$$

1	1.67	1.67	1.67
2	1.67	2.73	2.20
3	2.33	3.27	2.89
4	2.78	4.33	3.53
5	3.40	4.87	4.20
6	3.87	5.93	4.86

$P^+ + n\pi$: c) $(I, I_z; S) = (1, 1; 0)$

1	1	1	1
2	1	1	1
3	2.00	2.60	2.20
4	2.20	3.00	2.60
5	3.00	4.20	3.40
6	3.40	4.60	4.00
7	4.20	5.80	4.60

$P^+ + \bar{K}K\pi$: a) $(I, I_z; S) = (1, 1; 0)$

0	1.67	1.67	1.67
1	2.07	2.76	2.42
2	2.57	3.44	3.02
3	3.21	4.32	3.70
4	3.78	5.05	4.35
5	4.41	5.91	5.02

$C_O^+ + A\pi$: $(I, I_z; S) = (1, 1; -1)$

See entries above for $P^+ + n\pi$ and add

$4/3 = 1.33$ average track.

$C_O^+ + \bar{K}\pi$: $(I, I_z; S) = (1, 1; -1)$

1	1.67	1.67	1.67
2	2.16	2.84	2.56
3	2.74	3.52	3.07
4	3.29	4.45	3.83
5	3.94	5.13	4.44
6	4.49	6.06	5.13

$C_O^+ + \bar{K}N\pi$: $(I, I_z; S) = (1, 1; -1)$

0	1.67	1.67	1.67
1	1.67	2.67	2.17
2	2.09	3.24	2.77
3	2.75	4.16	3.42
4	3.31	4.77	4.07
5	3.95	5.70	4.73

a) Independent of possible correlations between $\bar{K}$ and K ; true for $C(K^0\bar{K}^0) = \pm$.

b) Statistical values for $D^0 \rightarrow \bar{K}\pi$ quoted for equal weights for all $I = 1/2, 3/2$ channels. More general results may be obtained from Table 4.

c) For $n = \pi$ add 0.58 average track; for $n' = \pi$ add 1.90 average track. ($n = 1, 2$ relevant only for n or $n' = \pi$.) For $C_O^+ - \bar{K}\pi$ add $4/3 = 1.33$ average track.

Table 4. Fractions f for individual charge states inferred from statistical postulate of Ref. 12.^{a)}

$$D^{+,0} + \ell^+ \nu_{\ell} \bar{K} n \pi; (I, I_z; S) = (\frac{1}{2}, \pm \frac{1}{2}; -1)^b)$$

$n(\pi^+) + n(\pi^-) =$		0	1	2	3	4	5	6
$N(1/2) (c)$								
$n=0$	1	1	-	-	-	-	-	-
1	1	1/3	2/3	-	-	-	-	-
2	2	1/6	1/3	1/2	-	-	-	-
3	4	1/20	1/5	9/20	3/10	-	-	-
4	9	1/45	4/45	14/45	16/45	2/9	-	-
5	21	1/147	2/49	26/147	44/147	50/147	20/147	-
6	51	1/357	2/119	11/119	24/119	40/119	30/119	5/51

$$F^+ + \ell^+ \nu_{\ell} \bar{K} n \pi;^d) (I, I_z; S) = (0, 0; 0)$$

and $C_O^+ + \ell^+ \nu_{\ell} \bar{K} n \pi; (I, I_z; S) = (0, 0; -1)$

Divide each entry of table for $D^{+,0} + \ell^+ \nu_{\ell} \bar{K} n \pi$, above, by 2.

This gives f for the charge state with indicated value of n and $n(\pi^+) + n(\pi^-)$, and for either a K^- or a $\bar{K}^0$ present.

$$C_O^+ + \ell^+ \nu_{\ell} \bar{K} n \pi;^e) (I, I_z; S) = (0, 0; -1)$$

$n(\pi^+) + n(\pi^-) = 0$		2	4	6
$n=0$	1	-	-	-
1	forbidden by isospin			
2	1/3	2/3	-	-
3	0	1	-	-
4	1/15	8/15	2/5	-
5	0	1/3	2/3	-
6	1/105	6/35	22/35	4/21

$$C_O^+ + \ell^+ \nu_{\ell} \bar{K} n \pi; (I, I_z; S) = (0, 0; -1)$$

$n(\pi^+) + n(\pi^-) = 0$		1	2	3	4	5	6
$Q (Z) = 0$		- or +	0	- or +	0	- or +	0

$n=0$		forbidden by isospin					
1	1/3	1/3	-	-	-	-	-
2	0	1/3	1/3	-	-	-	-
3	1/15	2/15	4/15	1/5	-	-	-
4	0	1/15	1/5	4/15	2/15	-	-
5	1/105	1/35	4/35	22/105	22/105	2/21	-
6	0	1/84	5/84	1/7	3/14	5/28	5/84

$$D^0 \rightarrow \bar{K}n\pi \text{ (isospin-1/2 channel)}: (I, I_2; S) = (\frac{1}{2}, \frac{1}{2}; -1)$$

See table for $D^{*+}, D^0 \rightarrow \pi^+ \pi^- \bar{K}n\pi, n \geq 1$.

$$D^0 \rightarrow \bar{K}n\pi \text{ (isospin-3/2 channel)}: (I, I_2; S) = (\frac{3}{2}, \frac{1}{2}; -1)$$

$n(\pi^+) + n(\pi^-) = 0$	1	2	3	4	5	6
$N^{(3/2)} (c)$						
n=1	1	2/3	1/3	-	-	-
2	2	2/15	7/15	2/5	-	-
3	5	2/25	1/5	12/25	6/25	-
4	12	2/105	11/105	34/105	38/105	4/21
5	30	1/105	3/70	4/21	32/105	1/3
6	76	1/399	5/266	13/133	4/19	45/133

$$D^{*+} \rightarrow \bar{K}n\pi: (I, I_2; S) = (\frac{3}{2}, \frac{3}{2}; -1)$$

$n(\pi^+) + n(\pi^-) = 1$	2	3	4	5	6
n=1	1	-	-	-	-
2	3/5	2/5	-	-	-
3	7/25	6/25	2/5	-	-
4	13/105	22/105	10/21	4/21	-
5	11/210	4/35	8/21	2/7	1/6
6	17/798	23/399	100/399	110/399	35/114

$$F^+ \rightarrow n\pi \text{ or } \bar{K}n\pi: (I, I_2; S) = (1, 1; 0)$$

$$\text{and } C_O^+ \rightarrow \bar{K}n\pi: (I, I_2; S) = (1, 1; -1)$$

$n(\pi^+) + n(\pi^-) =$	1	3	5	7
n=1	1	-	-	-
2	1	-	-	-
3	2/5	3/5	-	-
4	1/5	4/5	-	-
5	3/35	22/35	2/7	-
6	1/28	3/7	15/28	-
7	4/273	23/91	55/91	5/39

$$F^+ \rightarrow \bar{K}n\pi: (I, I_2; S) = (1, 1; 0)$$

$$\text{and } C_O^+ \rightarrow \bar{K}n\pi: (I, I_2; S) = (1, 1; -1)$$

$n(\pi^+) + n(\pi^-) =$	0	1	2	3	4	5
$Q(\bar{K}) =$	0	- or 0	- or 0	- or 0	- or 0	- or 0
n=0	1	-	-	-	-	-
1	1/4	3/8	-	-	-	-
2	1/10	9/40	3/20	3/10	-	-
3	1/30	7/60	2/15	4/15	1/6	-
4	3/245	13/245	22/245	44/245	10/49	4/49
5	1/238	11/476	6/119	12/119	20/119	15/119

$$C_0^+ + \text{nn}: (I, I_z; S) = (1, 1; -1)$$

$n(\pi^+) + n(\pi^-) = 0$	1	2	3	4	5	6
$Q(\Sigma) = +$	0	-	+	0	-	+
$n=1$	1/2	1/2	-	-	-	-
2	2/15	4/15	1/5	2/5	-	-
3	1/20	3/20	1/5	2/5	1/5	-
4	3/175	12/175	22/175	44/175	44/175	4/35
5	1/168	3/168	1/14	1/7	3/14	5/28
6	4/1911	8/637	23/637	46/637	92/637	110/637

- a) See Refs. 15 and 16 for extensive tables and methods of computation.
- b) Also applies to $D^0 \rightarrow \bar{K} n \pi$, $(I, I_z; S) = (\frac{1}{2}, \frac{1}{2}; -1)$ and with modifications noted in table to $F^+ \rightarrow \ell^+ \nu_\ell \bar{K} K n$ and $C_0^+ \rightarrow \ell^+ \nu_\ell \bar{K} n \pi$.
- c) $N^{(I)}$ denotes the number of independent isospin channels with total isospin $I = 1/2$ or $3/2$. To obtain results for D^0 non-leptonic decays in Tables 2 and 3, results of table 4 for $I = 1/2$ or $3/2$ should be weighted by $N^{(I)} [N^{(1/2)} + N^{(3/2)}]$ before combining.
- d) Neglects possible correlations between $\bar{K}$ and K ; see footnote F6.
- e) This table is implied by the corresponding one for $\Sigma(n-1)\pi$; for the convenience of the reader it is presented separately.

Figure 1a.

Figure 1b.

Figure 1c.

Figure 1d.

Figure 2.

Figure 3.

Figure 4a.

Figure 4b.