THE PURE TENSION SHAPE OF A
THICK TORUS

BY

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ABSTRACT

This report discusses the forces on an ideal toroidal coil system. It is demonstrated that the total vertical force on the top half of both a thin-shell torus and a thick torus depends only on the inner and outer major radii at the midplane. The total vertical force is divided in half to find the pure tension magnitude from which the shape of the pure tension curve can then be determined. A convenient method of solution is presented with several example curves for different coil thicknesses.

I. INTRODUCTION

The magnetic flux density of a discrete toroidal coil arrangement can be approximated using the analytic formulas for a continuously wound torus. As the number of coils increases the flux density more nearly approaches the analytic value. We have used this approximation to calculate the resulting force distribution for arbitrarily-shaped toroidal cross-sections.

Calculations are made for both negligible winding thickness (thin-shell torus) and finite winding thickness (thick torus). In each case, for a torus with specified inner and outer major radii at the midplane, the total vertical force on the top half of the torus is independent of the shape of the cross-section. Since in pure tension the force is equal at both the inner and outer legs, we can determine the correct pure tension magnitude by halving the total vertical force.

We can then solve for the shape of a pure tension curve by considering the appropriate differential equations. Appendix II presents a method of solution using difference equations. This method is very convenient because the result is a series of circular arcs which can be used as input to magnetic field computer programs.

Appendix I summarizes the equations for both the vertical and centering forces on a circular cross-section torus.

II. THIN-SHELL TORUS

Using rationalized MKSA units in cylindrical coordinates the magnetic flux density at a point inside the bore of an ideal torus is

$$\vec{B}_i = \frac{\mu_0 N_c I_c}{2\pi r} \hat{\phi} \quad (1)$$

where μ_0 is the permeability of free space

r is the radial coordinate of the point

N_c is the number of coils in the torus

I_c is the total current across a coil cross section, i.e., the product of the number of turns in a coil and the current per turn, and

$\hat{\phi}$ is the unit vector in the azimuthal direction.

The net magnetic flux density at the surface of a thin-shell torus can be derived from magnetic pressure considerations as half the value calculated using Eq. (1), because outside the torus the field is zero

$$\vec{B}_{\text{net}} = \frac{\mu_0 N_c I_c}{4\pi r} \hat{\phi} \quad (2)$$

More generally let us initially just restrict $\vec{B}$ to be a function of radius only

$$\vec{B} = B(r) \hat{\phi} \quad (3)$$

Figure 1 shows a filamentary loop carrying current I_c in such a $\vec{B}$ field. The Lorentz force on a differential element of curve length ds is

$$d\vec{F} = I_C (\hat{t} \times \vec{B}) ds \quad (4)$$

where $\hat{r}$, $\hat{\phi}$, and $\hat{z}$ are unit vectors in the x , ϕ , and z directions respectively, and $\hat{t}$ is a unit vector tangent to the curve at ds .

We can calculate the total force on the curve in the z direction by integrating the z component of the force along the length of the curve

$$F_z = I_C \int_C [(\hat{t} \times \vec{B}) \cdot \hat{z}] ds . \quad (5)$$

We can rearrange the integrand as

$$[(\hat{t} \times \vec{B}) \cdot \hat{z}] ds = (\vec{B} \times \hat{z}) \cdot \hat{t} ds = B(r) \hat{r} \cdot \hat{t} ds . \quad (6)$$

Thus the integral is the line integral of the vector field $B(r) \hat{r}$ over the curve

$$F_z = I_C \int_C B(r) \hat{r} \cdot \hat{t} ds . \quad (7)$$

Note that the curl of this field is zero

$$\vec{\nabla} \times [B(r) \hat{r}] = 0 . \quad (8)$$

Therefore F_z is independent of the path of integration.

We can similarly calculate the radial component of the total force F_r by the line integral

$$F_r = -I_C \int_C B(r) \hat{z} \cdot \hat{t} ds . \quad (9)$$

But in this case the curl is generally nonzero

$$\vec{\nabla} \times [B(r) \hat{z}] = - \frac{\partial B(r)}{\partial r} \hat{\phi} . \quad (10)$$

Therefore F_r generally depends on the shape of the curve.

Let us now consider a thin-shell torus composed of N_c identical filamentary coils as illustrated in Fig. 1. The torus is dee-shaped in cross-section and intersects the midplane perpendicularly at inner radius r_1 and outer radius r_2 . We expect the total force on the torus to be zero, so that the z force on the part of torus above the midplane is equal and opposite to the z force on the part below the midplane.

If we consider only the top part of the torus, the total z force is just the force on the top part of each coil multiplied by N_c . However, F_z for each coil is independent of the shape of the curve and depends only on the beginning and end points of the curve. Thus we can consider the z force on a straight filament from r_1 to r_2 . From Eqs. (2) and (7) we get

$$F_z = I_c \int_{r_1}^{r_2} \frac{\mu_0 N_c I_c}{4\pi r} dr . \quad (11)$$

This integrates to

$$F_z = \frac{\mu_0 N_c I_c^2}{4\pi} \ln \left(\frac{r_2}{r_1} \right) . \quad (12)$$

Since F_z is equal to the sum of the tensions at r_1 and r_2 we have

$$F_z = T_1 + T_2 \quad (13)$$

In pure tension $T_1 = T_2$ so

$$F_z = 2T \quad (14)$$

from which it follows that

$$T = \frac{\mu_o N_c I_c^2}{8\pi} \ln\left(\frac{r_2}{r_1}\right). \quad (15)$$

This is the same result R. G. Mills obtained by a different method in Ref. 1.

We can express the components of the force on an element of curve length in terms of its radius of curvature and the angle θ which the radius of curvature makes with a constant z plane as shown in Fig. 2.

$$f = BI\rho \Delta\theta \quad (16)$$

$$f_r = BI\rho \Delta\theta \cos\theta \quad (17)$$

$$f_z = BI\rho \Delta\theta \sin\theta \quad (18)$$

For pure tension we can write the following equilibrium equation for small $\Delta\theta$

$$f = T\Delta\theta \quad (19)$$

Thus we can express the tension as

$$T = BI\rho \quad (20)$$

If T is constant then the radius of curvature varies as

$$\rho = \frac{T}{BI} \quad . \quad (21)$$

We can find the shape of a pure tension dee-coil by solving the appropriate differential equations numerically. Appendix II describes a convenient method of solution based on difference equations.

III. THICK TORUS

Let us now consider a torus with finite winding thickness. We shall assume the current is uniformly distributed across the winding. Figure 3 shows a circular cross-section torus with major radius r_m , inner minor radius ρ_i , and outer minor radius ρ_o . A filamentary path carrying current ΔI and located at minor radius ρ is indicated as a dashed line. The magnetic flux density $\vec{B}$ along this path is a function of radius only. This is readily seen by considering Ampere's circuital law, i.e., the line integral around a closed path is equal to the current linked by the path multiplied by μ_0

$$\oint_C \vec{B} \cdot \hat{t} \, ds = \mu_0 I_\ell \quad . \quad (22)$$

Any circular path about the z axis which passes through the dashed curve will link the same amount of current

$$I_\ell = \frac{\rho - \rho_i}{\rho_o - \rho_i} N_c I_c \quad . \quad (23)$$

Thus we can express $\vec{B}$ for a given ρ as a function of major radius r only

$$\vec{B}(\rho, r) = \frac{\mu_0 I_\ell}{2\pi r} \quad (24)$$

Note that $\vec{B}$ is not restricted to a circular cross-section torus. Equation (24) holds as long as the ratio of the radii of curvature is maintained as in Eq. (23).

Because $\vec{B}$ is a function of radius only for each filamentary current path, we can apply the observations we made for the thin-shell torus. Let us again restrict our consideration to the top part of the torus. The z component of the force on each filamentary path carrying current ΔI must be independent of the shape of the path. It follows that, since F_z for each coil is the sum of the z forces for each filamentary path, F_z for each coil is a constant.

We can evaluate F_z for a thick torus with a method similar to that used for the thin-shell torus. Since the z component of force on each filamentary element is independent of its shape, we can choose any convenient path to solve for the z force on each element and integrate across the winding thickness to find F_z . Figure 4 demonstrates that an arbitrary cross-section coil #1 which intersects the midplane perpendicularly at its inner and outer leg is equivalent to coil #2 for the purpose of evaluating F_z . Thus we need to find F_z on the upper trapezoidal section of coil #2 shown in Fig. 4.

F_z can be evaluated by dividing the trapezoidal section as illustrated in Fig. 5 into differential straight line elements

each with surface current density K where

$$K = \frac{I_c}{(\rho_o - \rho_i)} \quad (25)$$

and integrating the z force on each element across the winding thickness. As shown in Fig. 5 the length of the elements varies from $\ell = 2\rho_i$ to $\ell = 2\rho_o$ as the winding thickness varies from $t = 0$ to $t = \rho_o - \rho_i$. We can write the required integral using Eqs. (7), (23), (24), and (25)

$$F_z = \int_{t=0}^{t=\rho_o-\rho_i} \frac{I_c dt}{(\rho_o - \rho_i)} \int_{r=r_m-\rho_i-t}^{r=r_m+\rho_i+t} \frac{\mu_o N_c I_c (\rho_o - \rho_i - t)}{2\pi r (\rho_o - \rho_i)} dr \quad (26)$$

This integrates to

$$F_z = \frac{\mu_o}{4\pi} \frac{N_c I_c^2}{(\rho_o - \rho_i)^2} \left[(r_m^2 + \rho_o^2) \ln \left(\frac{r_m + \rho_o}{r_m - \rho_o} \right) - (r_m^2 + 2\rho_i \rho_o - \rho_i^2) \ln \left(\frac{r_m + \rho_i}{r_m - \rho_i} \right) \right. \\ \left. + 2r_m \rho_o \ln \left(\frac{r_m^2 - \rho_o^2}{r_m^2 - \rho_i^2} \right) - 2r_m (\rho_o - \rho_i) \right] \quad (27)$$

Although Eq. (27) uses parameters of an equivalent circular cross-section torus, it is not restricted to this shape.

Recall that in pure tension

$$F_z = 2T \quad (14)$$

from which it follows that

$$T = \frac{1}{2} F_z \quad (28)$$

As one would expect when ρ_i approaches ρ_o , Eq. (28) reduces to Eq. (15).

We can express the components of force on the cross section of a thick coil in terms of an element of curve length on the inside edge of the coil as shown in Fig. 6. The inner radius of curvature is ρ_i , the radial coordinate of the element is r_i , the thickness of the coil is a , where $a = \rho_o - \rho_i$, and θ is the angle which the inner radius of curvature makes with a constant z plane. The force is the integral of the differential forces across the cross section

$$f = \frac{\mu_0 N_c^2 I_c^2 \Delta \theta}{2\pi(\rho_o - \rho_i)^2} \int_{\rho=\rho_i}^{\rho=\rho_o} \frac{\rho(\rho_o - \rho) d\rho}{r_i + (\rho - \rho_i) \cos \theta} \quad (29)$$

Let

$$b = \cos \theta \quad (30)$$

$$c = \ln \left(\frac{r_i + ab}{r_i} \right) \quad (31)$$

$$d = cr_i - (a - ac)b \quad (32)$$

$$e = \frac{a^2 b}{2} + (a - ac)r_i - \frac{c}{b} r_i^2 \quad (33)$$

then we can integrate Eq. (29) to obtain

$$f = \frac{\mu_0 N_c^2 I_c^2}{2\pi a^2 b^2} (d\rho_i + e) \Delta \theta \quad [b \neq 0] \quad (34)$$

$$f = \frac{\mu_0 N_c I_c^2}{4\pi r_i} \left(\rho_i + \frac{a}{3} \right) \Delta\theta \quad [b = 0] \quad (35)$$

$$f_r = f \cos\theta \quad (36)$$

$$f_z = f \sin\theta. \quad (37)$$

If we desire to have the cross section as close to pure tension as possible, we can neglect any bending moments across the cross section and use the same equilibrium equation as for the thin-shell case

$$f = T\Delta\theta \quad (19)$$

If T is constant then the inner radius of curvature varies as

$$\rho_i = \frac{1}{d} \left(\frac{2\pi a^2 b^2 T}{\mu_0 N_c I_c^2} - e \right) \quad [b \neq 0] \quad (38)$$

$$\rho_i = \frac{4\pi r_i T}{\mu_0 N_c I_c^2} - \frac{a}{3} \quad [b = 0] \quad (39)$$

We can use these equations to find the shape of a pure tension dee-coil. As in the thin-shell case we can use the method in Appendix II.

Figures 7 and 8 show how the centerline of the coil moves down as the thickness of the coil increases for two different inverse aspect ratios ϵ where

$$\epsilon = \frac{\rho_o - \rho_i}{\rho_o + \rho_i} \quad (40)$$

The curves in Figs. 7 and 8 were generated by first solving for the shape of the inside edge and then adding $a/2$ to the calculated radii of curvature to find the centerline.

IV. CONCLUSION

We initially considered the total vertical force on the top half of a thin-shell torus and found that it depends only on the inner and outer major radii at the midplane. Similarly by considering the vertical force on a thick torus as the sum of the forces on a series of nested thin-shell tori, we found that for this case too the total vertical force was independent of the shape of the minor cross-section. We determined the correct pure tension magnitude in each case by dividing the total vertical force in half. Knowing the tension enabled us to solve for the shape of the pure tension curves for different coil thicknesses.

Figures 7 and 8 demonstrate, for inverse aspect ratios $\epsilon = 0.6$ and $\epsilon = 0.8$ respectively, the variation of the centerlines as the thickness of a coil increases. No significant departure is observed from the thin-shell case for practical winding thicknesses. Since both derivations were based on non-discrete coil systems, in many practical cases the thin-shell formulization will provide quantitative results with essentially the same accuracy as the thick torus.

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APPENDIX I

NET FORCES ON HALF-COIL IN CIRCULAR TORUS

THIN-SHELL TORUS

Equations (17) and (18) can be integrated in terms of the parameters in Fig. 3. to obtain

$$F_r = \frac{\mu_0}{4} N_c I_c^2 \left(1 - \frac{r_M}{(r_M^2 - \rho^2)^{1/2}} \right) \quad (I-1)$$

$$F_z = \frac{\mu_0}{4\pi} N_c I_c^2 \ln \left(\frac{r_M + \rho}{r_M - \rho} \right) \quad (I-2)$$

THICK TORUS

Equations (36) and (37) can be integrated in terms of the parameters in Fig. 3 to obtain

$$F_r = \frac{\mu_0}{4} N_c I_c^2 \left\{ 1 - \frac{2}{(A_o - A_i)^2} [A_o (\sin^{-1} A_o - \sin^{-1} A_i) - (1 - A_i^2)^{1/2} + (1 - A_o^2)^{1/2}] \right\} \quad (I-3)$$

where $A_i = \rho_i / r_M$, $A_o = \rho_o / r_M$

$$F_z = \frac{\mu_0}{4\pi} \frac{N_c I_c^2}{(\rho_o - \rho_i)^2} \left[(r_M^2 + \rho_o^2) \ln \left(\frac{r_M + \rho_o}{r_M - \rho_o} \right) - (r_M^2 + 2\rho_i \rho_o - \rho_i^2) \ln \left(\frac{r_M + \rho_i}{r_M - \rho_i} \right) + 2r_M \rho_o \ln \left(\frac{r_M^2 - \rho_o^2}{r_M^2 - \rho_i^2} \right) - 2r_M (\rho_o^2 - \rho_i^2) \right] \quad (I-4)$$

APPENDIX II

ALGORITHM FOR GENERATING PURE TENSION CURVE

We can approximate the pure tension curve as a series of circular arcs with coincident tangents. We need to know the radius of curvature as a function of the major radius and the direction of the negative normal at a point on the curve

$$\rho = \rho(x, \theta) \quad (II-1)$$

Thus for the thin-shell torus we can use Eqs. (21) and for the thick torus we can use Eqs. (38) and (39).

Figure 9 illustrates the two-step method for generating each arc at each point (x_i, y_i) on the curve, $i = 1$ to N . Each arc subtends an equal angle $\Delta\theta = \pi/N$. An initial prediction ρ_i' is made for the radius of curvature from the parameters at (x_i, y_i) and this determines the predicted location of (x'_{i+1}, y'_{i+1}) . A second radius of curvature ρ_i'' is calculated at (x'_{i+1}, y'_{i+1}) and the average of ρ_i' and ρ_i'' is used as the value for ρ_i . The formulas are summarized as follows:

$$\Delta\theta = \pi/N$$

$$\theta_i = (i-1)\Delta\theta$$

$$x_1 = r_2$$

$$y_1 = 0$$

$$\rho_i' = \rho(x_i, \theta_i)$$

$$x'_C(i) = x_i - \rho_i' \cos \theta_i$$

$$x'_i = x'_C(i) + \rho_i' \cos \theta_{i+1}$$

$$\rho_i'' = \rho[x'_i(i), \theta_{i+1}]$$

$$\rho_i = \frac{\rho_i' + \rho_i''}{2}$$

$$x_c(i) = x_i - \rho_i \cos \theta_i$$

$$y_c(i) = y_i - \rho_i \sin \theta_i$$

$$x_{i+1} = x_c(i) + \rho_i \cos \theta_{i+1}$$

$$y_{i+1} = y_c(i) + \rho_i \sin \theta_{i+1}$$

where $i = 1, N$.

REFERENCES

¹J. File, R. G. Mills, and G. V. Sheffield, Princeton Plasma Physics Laboratory Report MATT-848 (1971).

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Fig. 1. Cross-section of thin-shell torus. Filamentary coil.

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Fig. 2. Components of force on an element of a filamentary coil.

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Fig. 3. Circular cross-section of thick torus. Filamentary path indicated by dashed line.

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Fig. 4. Demonstration of equivalency of arbitrarily shaped coil #1 with coil #2 for vertical force evaluation.

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Fig. 5. Parameters used to evaluate vertical force on trapezoidal section.

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Fig. 6. Components of force on an element of a thick coil.

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Fig. 7. Pure tension centerline for various coil thicknesses a with constant $\epsilon = 0.6$.

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Fig. 8. Pure tension centerline for various coil thicknesses a with constant $\epsilon = 0.8$.

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Fig. 9. Two-step method for generating a series of arcs approximating a pure tension curve.