INTERMITTENCY IN FREE TURBULENT SHEAR FLOWS

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ABSTRACT

A formalism previously used in the context of deformable porous media and turbulent/nonturbulent intermittent flows is restated here. The behavior of the intermittency function derivatives at the interface gives rise to surface integrals over the latter. The conditioned equations of continuity, momentum, energy, vorticity and conservation of a scalar are derived for the turbulent and irrotational zones. Surface integrals with a precise physical meaning enter the conditioned equations. They can be interpreted as entrainment of mass, momentum, energy and scalar, and as direct interactions between the turbulent and irrotational regions. Use is made of the experimental conditioned measurements for: (i) the plane wake behind a heated flat plate, and (ii) the heated turbulent round jet; the profiles of entrainment of mass, and the combination of direct interactive force and entrainment of momentum are calculated. These derived profiles are compared with models proposed by previous investigators.

NOMENCLATURE

- C - molecular flux of scalar through the interface, Eq. (41).
D - orifice diameter for the jet
dS - interface surface infinitesimal element
E - entrainment of mass per unit mass, Eq. (17).
F - mean force per unit mass of turbulent on irrotational zones
 f_1, f_2, f_3, f_4 - self-preserving functions defined by Eqs. (51)-(64).
 f_Y - interface crossing rate
 g, h^2 - self-preserving functions defined by Eqs. (54) and (63)
I - intermittency function
K - entrainment of total kinetic energy, Eq. (30)
 $l(x)$ - half-width of the wake based on velocity defect
 $\underline{M}$ - average entrainment of momentum, Eq. (21)
 $\underline{n}$ - normal to the interface, pointing towards the turbulent zone
P, Q - any fluid mechanical variables
p - pressure
q - molecular scalar flux vector
r - radial coordinate for the jet
 $r_{1/2}$ - half-radius of the jet based on velocity
 $S(x, t)$ - surface of the turbulent/nonturbulent interface
T - scalar
 T_m - maximum value of T at a section of the jet
t - time
 U_j - jet exit velocity

- U_m - maximum value of the mean velocity at a section of the jet
 U_o - free stream velocity for the wake
 U_s - velocity defect at the centerplane of the wake
 $\underline{u}$ - velocity vector
 $\underline{u}^s$ - interface velocity
 u, v - x and y (or r) velocity components
V - elementary control volume
 v_e - modulus of the velocity of advance of the interface relative to a fluid element at the same point
W - mechanical work done by the turbulent fluid upon the irrotational fluid
w - tangential velocity component for the jet
 $\underline{x}$ - position vector
 x, y - streamwise and normal coordinates
 γ - intermittency factor
 δ_m - momentum thickness of the boundary layer at the trailing edge of the plate
 δ_{ij} - Kronecker delta
 η - similarity variable
 θ - mean entrainment of scalar T
 μ - viscosity of the fluid
 ν - kinematic viscosity of the fluid
 ρ - density of the fluid
 σ_{ij} - viscous stress tensor
 ϕ - $\frac{\sigma_{ij} \partial u_i}{\rho \partial x_j}$ in the kinetic energy equations
 ω_i - vorticity component

Subscripts

- 1, 0 - relative to turbulent and irrotational zone variables
i - vector component
r - axial component of a vector
 x, y - streamwise and normal components of a vector

Special Symbols

- ∇ - gradient operator
 ∇^2 - laplacian operator
 $\sim$ - vector variable
 $\bar{}$ - average
 $\dot{}$ - fluctuating variable or function derivative

INTRODUCTION

Since the discovery of intermittency in free turbulent shear flows by Corrsin (2) and its exploration in subsequent studies by Townsend (10) and Corrsin and Kistler (3), the concept of an intermittency function has been widely used in experimental investigations. The intermittency function allows one to identify separately the turbulent and nonturbulent regions

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coexisting in an intermittent flow. A more adequate description of velocity and scalar fields can then be provided. The conditional sampling technique (1,6,12,13) multiplies the random variable to be averaged by a generated signal which is unity in the turbulent and zero in the nonturbulent regions, i.e. an intermittency function. The same device has not received much attention in theoretical studies of turbulent shear flows. Quite recently, Libby (7,8) initiated the first theoretical use of the intermittency concept to condition the conservation equations of fluid mechanics for both turbulent and nonturbulent variables; central to Libby's methodology is the postulate of a conservation equation for the intermittency function with an unknown source term, described as a creation of turbulent fluid. Another crucial point is his definition of conditioned fluctuations in terms of the unconditioned variables. They then have no simply physical meaning. With the assumption of a knowledge of the unconditioned velocity components, a subsequent computation of the unconditioned Reynolds stress and with certain additional modeling assumptions he was able to predict conditioned variables and the intermittency factor for mixing layers, boundary layers and wakes. Tutu (12) made use of Libby's concepts but defined fluctuations in turbulent zone variables as fluctuations about the conditioned mean. He obtained the conditioned equations and proposed modeling approximations for the unknown functions which appear. The meanings ascribed by Libby and Tutu to the source term differ one from the other, and, being highly intuitive, are not entirely clear. Analogous to this conditioning problem is that of obtaining the equations governing the flow through deformable porous media (5). The ideas in (5) were extended to the present problem in (4). Dopazo (4) avoided the postulate of a conservation equation for the intermittency function by examining the spatial and temporal derivatives of the intermittency function at the turbulent/nonturbulent interface; rigorous fundamental formulae were derived and used to obtain the conditioned equations. The main advantage of the latter method over that of Libby is that every unknown entering the zone average equations has a clearly defined physical meaning. One expects this direct connection with physics to be of help in constructing modeling approximations for these unknown terms.

In the present paper the fundamental formulae used by Dopazo (4) are restated. The zone average equations of continuity, momentum, kinetic energy and conservation of a scalar are then presented. The physical meaning of the surface integrals in the conditioned equations is explained; special emphasis is put on these terms since they are the ones which directly couple the motions in the turbulent and nonturbulent regions. The remaining terms in the equations can readily be interpreted as conditioned production, dissipation, convection, diffusion and pressure transport. Such notions as entrainment of mass, momentum, kinetic energy and scalar content clearly appear in the context of this formalism.

The experimental results for the two-dimensional wake behind a heated flat plate (1) and the axisymmetric turbulent heated jet (12) are utilized to evaluate semi-quantitatively the profiles of the unknown terms. Due to the lack of complete experimental data self-preservation of the turbulence (9,11) has been assumed. Our evaluated profiles for entrainment of mass and momentum are compared to those proposed by Tutu (12)

and Libby (7,8). The actual profiles and a better physical understanding of the phenomenon should be helpful in constructing convincing modeling assumptions, namely, the expression of entrainment of mass and momentum in terms of the intermittency, mean conditioned velocity, conditioned Reynolds stress and crossing frequency.

The usefulness of this methodology is discussed and its possible application to reacting and nonreacting transport of scalars in turbulent flows, e.g., the plume of a point source is mentioned.

METHODOLOGY

Fundamental Formulae

Let $I(x,t)$ be the intermittency function defined as

$$I(x,t) = \begin{cases} 1 & \text{if } (x,t) \text{ is in the turbulent zone} \\ 0 & \text{if } (x,t) \text{ is in the nonturbulent zone.} \end{cases}$$

Let $Q(x,t)$ be any fluid mechanical property. The following formulae are presented in Dopazo (4) and their derivation is explained therein.

$$\overline{IVQ} = \overline{VQI} - \lim_{V \rightarrow 0} \frac{1}{V} \int_{S(x,t)} Q \mathbf{n} \, dS \quad (1)$$

$$I \frac{\partial Q}{\partial t} = \frac{\partial}{\partial t} \overline{QI} + \lim_{V \rightarrow 0} \frac{1}{V} \int_{S(x,t)} Q \mathbf{u}^s \cdot \mathbf{n} \, dS \quad (2)$$

The last terms in equations (1) and (2) arise due to the discontinuities of $I(x,t)$ at the interface $S(x,t) = 0$; VI and $\partial I / \partial t$ are Dirac δ -functions at the interface location and at the time of crossing of the interface respectively. V is an elementary control volume, $\mathbf{n}$ is the normal to the turbulent/nonturbulent interface pointing towards the turbulent region and $\mathbf{u}^s$ is the velocity of the interface relative to a fixed observer. Note that the interface, $S(x,t) = 0$, is not a material surface and its velocity $\mathbf{u}^s$ can be written as

$$\mathbf{u}^s = \mathbf{u} - v \mathbf{n} \quad (3)$$

where $\mathbf{u}$ is the instantaneous velocity of the fluid element at $S(x,t) = 0$ and $-v \mathbf{n}$ is the velocity of advance of an element of the surface relative to the fluid element at the same point (4).

Decomposition of Averages

Let P and Q be two intermittent signals for two fluid mechanical variables. Let their unconditioned averages be $\bar{P}$ and $\bar{Q}$ and their conditioned averages be defined by

$$\bar{P}_1 = \frac{\overline{IP}}{\gamma} \quad (4)$$

$$\bar{Q}_1 = \frac{\overline{IQ}}{\gamma} \quad (5)$$

$$\bar{P}_0 = \frac{\overline{(1-I)P}}{1-\gamma} \quad (6)$$

$$\bar{Q}_0 = \frac{\overline{(1-I)Q}}{1-\gamma} \quad (7)$$

where the subscripts 1 and 0 refer to the turbulent and nonturbulent zone variables respectively and the intermittency, γ , is defined as

$$\gamma = \bar{I} \quad (8)$$

Let P'_1 , Q'_1 , P'_0 and Q'_0 be the fluctuations in the turbulent and irrotational regions relative to the zone

averages of P and Q. Then

$$\overline{IPQ} = \gamma \overline{P_1 Q_1} + \gamma \overline{P_1 Q_1} \quad (9)$$

$$\overline{(1-I)PQ} = (1-\gamma) \overline{P_0 Q_0} + (1-\gamma) \overline{P_0 Q_0} \quad (10)$$

Note that variables with subscript 1 are only defined in the turbulent region while those with 0 are defined in the irrotational zone. The averages of 1 or 0 variables or their products are therefore only over the corresponding domains of definition. The decomposition in (9) and (10) are different from Libby's (7) as was already remarked (4).

Intermittency Variations

Let $Q \equiv 1$ in equations (1) and (2). Then (4)

$$\nabla \gamma = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \mathbf{n} dS \quad (11)$$

$$\frac{\partial \gamma}{\partial t} = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \mathbf{u}^s \cdot \mathbf{n} dS \quad (12)$$

A detailed knowledge of the interface dynamics is needed to evaluate the right side of equations (11) and (12). If the flow is statistically stationary identical positive and negative values of $\mathbf{u}^s \cdot \mathbf{n}$ are equally probable.

FUNDAMENTAL EQUATIONS

The equations governing the motion of a fluid in a turbulent/nonturbulent flow are

$$\frac{\partial \mathbf{u}_i}{\partial x_i} = 0 \quad (13)$$

$$\frac{\partial \mathbf{u}_i}{\partial t} = \frac{\partial \mathbf{u}_i \mathbf{u}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} \quad (14)$$

$$\sigma_{ij} = \mu \left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right) \quad (15)$$

where $\mathbf{u}$ is the velocity, ρ the density, p the pressure, σ_{ij} the viscous stress tensor and μ the viscosity.

To condition the conservation equations one should multiply them by 1 or (1-I), average them and use equations (1) and (2). This method was utilized to obtain the mean conservation equations in reference (4).

Turbulent Zone Continuity

$$\frac{\partial \gamma}{\partial t} + \frac{\partial \gamma \mathbf{u}_i}{\partial x_i} = E \quad (16)$$

where $\mathbf{u}_i$ is the turbulent zone velocity signal and

$$E = \lim_{V \rightarrow 0} \frac{1}{V} \int_S v^e dS \quad (17)$$

is the volumetric entrainment per unit volume, or the massic entrainment per unit mass.

Irrotational Zone Continuity

$$\frac{\partial (1-\gamma)}{\partial t} + \frac{\partial (1-\gamma) \mathbf{u}_0}{\partial x_i} = -E \quad (18)$$

where $\mathbf{u}_0$ is the irrotational zone velocity signal.

Unconditioned Continuity

Adding equations (16) and (18) and taking into

account that $\mathbf{u}_i = \gamma \mathbf{u}_1 + (1-\gamma) \mathbf{u}_0$ one gets

$$\frac{\partial \mathbf{u}_i}{\partial x_i} = 0 \quad (19)$$

which is the conventional mean continuity equation. While $\mathbf{u}_i$ is solenoidal, $\mathbf{u}_1$ and $\mathbf{u}_0$ are not.

Turbulent Zone Momentum

$$\begin{aligned} \frac{\partial \gamma \mathbf{u}_1}{\partial t} + \frac{\partial \gamma \mathbf{u}_1 \mathbf{u}_1}{\partial x_j} + \frac{\partial}{\partial x_j} \gamma \mathbf{u}_1 \mathbf{u}_1 = -\frac{1}{\rho} \frac{\partial \gamma p_1}{\partial x_i} \\ + \nu \nabla^2 [\gamma \mathbf{u}_1] + \mathbf{M}_i - \mathbf{F}_i \end{aligned} \quad (20)$$

where $\mathbf{u}_1$ is the turbulent zone velocity fluctuation, p_1 is the turbulent zone mean pressure and

$$\mathbf{M}_i = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \rho \mathbf{u}_i v^e dS \quad (21)$$

$$\begin{aligned} \mathbf{F}_i = \frac{1}{\rho} \lim_{V \rightarrow 0} \frac{1}{V} \int_S (-p \delta_{ij} + \sigma_{ij}) \mathbf{n}_j dS \\ + \nu \frac{\partial}{\partial x_j} \lim_{V \rightarrow 0} \frac{1}{V} \int_S (\mathbf{u}_i \mathbf{n}_j + \mathbf{u}_j \mathbf{n}_i) dS \end{aligned} \quad (22)$$

$\mathbf{M}$ is the average entrainment of momentum, i.e. the momentum flux through the interface. $\mathbf{F}$ is the mean force per unit mass that the turbulent fluid exerts upon the irrotational fluid. The meaning of the last term is indicated in reference (4). $\mathbf{u}_1 \mathbf{u}_1$ is the conditional Reynolds stresses.

Irrotational Zone Momentum

$$\begin{aligned} \frac{\partial (1-\gamma) \mathbf{u}_0}{\partial t} + \frac{\partial (1-\gamma) \mathbf{u}_0 \mathbf{u}_0}{\partial x_j} + \frac{\partial}{\partial x_j} (1-\gamma) \mathbf{u}_0 \mathbf{u}_0 = \\ = -\frac{1}{\rho} \frac{\partial (1-\gamma) p_0}{\partial x_i} + \nu \nabla^2 [(1-\gamma) \mathbf{u}_0] - \mathbf{M}_i + \mathbf{F}_i \end{aligned} \quad (23)$$

where $\mathbf{u}_0$ is the irrotational zone velocity fluctuations, and p_0 is the irrotational zone mean pressure. Note that the entrainment of momentum $\mathbf{M}$ and the mean force $\mathbf{F}$ have opposite signs in equations (20) and (23), i.e. fluid is entrained into the turbulent region at the expense of the irrotational zone, and the action of the turbulent on the irrotational zones has opposite sign to the reaction $\mathbf{F}$ of the irrotational upon the turbulent regions.

Unconditioned Momentum

Adding (20) and (23) and taking into account that

$$\begin{aligned} \mathbf{u}_1 \mathbf{u}_j + \mathbf{u}_i \mathbf{u}_j = \gamma \mathbf{u}_1 \mathbf{u}_j + (1-\gamma) \mathbf{u}_0 \mathbf{u}_j \\ + (1-\gamma) \mathbf{u}_1 \mathbf{u}_j + (1-\gamma) \mathbf{u}_0 \mathbf{u}_j \end{aligned} \quad (24)$$

$$\text{and } \bar{p} = \gamma \bar{p}_1 + (1-\gamma) \bar{p}_0 \quad (25)$$

one obtains

$$\frac{\partial \mathbf{u}_i}{\partial t} + \frac{\partial \mathbf{u}_i \mathbf{u}_j}{\partial x_j} + \frac{\partial \mathbf{u}_i \mathbf{u}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \nabla^2 \mathbf{u}_i \quad (26)$$

which is the conventional mean momentum equation.

Unconditioned Kinetic Energy

If equation (14) is multiplied by $\mathbf{u}_i$ one readily obtains

$$\frac{\partial \overline{u_i u_i}}{\partial t} + \frac{\partial \overline{u_j u_i u_i}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \overline{p u_i}}{\partial x_i} + \frac{\partial}{\partial x_j} \overline{u_i \sigma_{ij}} - \frac{1}{\rho} \sigma_{ij} \frac{\partial \overline{u_i}}{\partial x_j} \quad (27)$$

Ensemble averaging of equation (27) yields

$$\frac{\partial}{\partial t} \frac{1}{2} \overline{u_i u_i} + \frac{\partial}{\partial x_j} \overline{u_j \frac{1}{2} u_i u_i} + \frac{\partial}{\partial x_j} \overline{u_j u_i u_i} = -\frac{1}{\rho} \frac{\partial \overline{p u_i}}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \overline{u_i \sigma_{ij}} - \frac{1}{\rho} \sigma_{ij} \frac{\partial \overline{u_i}}{\partial x_j} \quad (28)$$

which is the unconditioned total kinetic energy equation. Multiplication of equation (27) by I and $(1-I)$, averaging and use of formulae (1) and (2) yield the:

Turbulent Zone Total Kinetic Energy

$$\frac{\partial}{\partial t} \overline{I u_i u_i} + \frac{\partial}{\partial x_j} \overline{I u_j u_i u_i} = -\frac{1}{\rho} \frac{\partial \overline{I p u_i}}{\partial x_i} + \overline{v^2 I u_i u_i} - \overline{I \phi} + K - W \quad (29)$$

where

$$\overline{I \phi} = \frac{1}{\rho} \overline{I \sigma_{ij} \frac{\partial u_i}{\partial x_j}}$$

is the conditioned energy dissipation in the turbulent region and

$$K = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{u_i u_i v^e} dS \quad (30)$$

$$W = \frac{1}{\rho} \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{(-p \delta_{ij} + \sigma_{ij}) u_i n_j} dS - v \frac{\partial}{\partial x_j} \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{(u_i n_j + u_j n_i) u_i} dS \quad (31)$$

K is the entrainment of total kinetic energy into the turbulent zone and W is the mechanical work done by the turbulent fluid upon the nonturbulent.

Irrotational Zone Total Kinetic Energy

$$\frac{\partial}{\partial t} \overline{(1-I) u_i u_i} + \frac{\partial}{\partial x_j} \overline{(1-I) u_j u_i u_i} = -\frac{1}{\rho} \frac{\partial \overline{(1-I) p u_i}}{\partial x_i} + \overline{v^2 (1-I) u_i u_i} - \overline{(1-I) \phi} - K + W \quad (32)$$

Equation (28) can be recovered by adding equations (29) and (32).

Equations for the unconditioned, turbulent zone and irrotational zone fluctuating kinetic energies can easily be obtained following the traditional method (9,11). The actual equations will not be presented here. One important remark in connection with the energy balance is that previous investigators [e.g., Wygnanski and Fiedler (13)] do forget K and W in the energy balance. Therefore, the turbulent zone energy balance must necessarily be in error, and the pressure transport computed from the measured convection, diffusion, production and dissipation may not be reliable.

Turbulent Zone Vorticity

Since $(1-I)\omega_i \equiv 0$, $\omega_i \equiv I\omega_i$ and the equation for the conditioned vorticity is the same as that for the unconditioned vorticity. This equation is

$$\frac{\partial \overline{I \omega_i}}{\partial t} + \frac{\partial \overline{I \omega_i u_j}}{\partial x_j} = \overline{\omega_j \frac{\partial I u_i}{\partial x_j}} + \overline{v^2 I \omega_i} \quad (33)$$

this implies a balance between "entrainment of vorticity", "vortex line stretching" and "vorticity viscous propagation terms" at the interface, namely,

$$\lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{\omega_i v^e} dS - \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{u_i \omega_j n_j} dS - v \left(\frac{\partial}{\partial x_j} \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{\omega_i n_j} dS + \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{\frac{\partial \omega_i}{\partial x_j} n_j} dS \right) = 0 \quad (34)$$

Generalized Corrsin-Kistler equation

The following identity is always true (7)

$$(1-I)\omega_i \equiv 0 \quad (35)$$

Use of equation (35) multiplied by u_j and averaged yields (4)

$$\frac{\partial}{\partial x_j} (1-\gamma) \overline{(u_0)_i (u_0)_j} - \frac{1}{2} \frac{\partial}{\partial x_i} (1-\gamma) \overline{(u_0)_j (u_0)_j} + \frac{\partial}{\partial x_j} (1-\gamma) \overline{(u_0)_i (u_0)_j} - \frac{1}{2} \frac{\partial}{\partial x_i} (1-\gamma) \overline{(u_0)_j (u_0)_j} + \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{(u_i u_j n_j - u_j u_i n_i) dS} = 0 \quad (36)$$

The last term in equation (36) is associated with momentum fluxes through the interface.

Corrsin-Kistler equation (3) relating Reynolds stresses and turbulent kinetic energy in the irrotational zone can be recovered from equation (36) in the limit $\gamma \rightarrow 0$ and $\overline{u_0} \rightarrow$ constant vector; in that limit since there are no interface crossings the last term in (36) tends to zero and one recovers (4)

$$\frac{\partial}{\partial x_j} \overline{(u_0)_i (u_0)_j} = \frac{1}{2} \frac{\partial}{\partial x_i} \overline{(u_0)_j (u_0)_j} \quad (37)$$

Turbulent Zone Scalar

Let T be a scalar (e.g. the temperature field in a turbulent heated jet) satisfying the equation

$$\frac{\partial T}{\partial t} + \frac{\partial u_j T}{\partial x_j} = \frac{\partial q_j}{\partial x_j} \quad (38)$$

where q is the "scalar flux vector" due to molecular agitation. Multiplication by I , averaging and use of equations (1) and (2) yields

$$\frac{\partial \overline{I T}}{\partial t} + \frac{\partial \overline{I u_j T}}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{\gamma (u_0)_j T} = -\frac{\partial \overline{\gamma (q_1)_j}}{\partial x_j} + \Theta + C \quad (39)$$

where T_1 is the turbulent zone scalar signal, T_1' are the turbulent zone scalar fluctuations about $\overline{T_1}$ and q_1 is the turbulent zone mean "scalar flux vector" due to molecular effects and

$$\Theta = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{T v^e} dS \quad (40)$$

$$C = \lim_{V \rightarrow 0} \frac{1}{V} \int_S \overline{q_j n_j} dS \quad (41)$$

Θ is the mean entrainment of scalar T into the turbulent region and C is the molecular flux of scalar through the interface.

Irrotational Zone Scalar

$$\frac{\partial(1-\gamma)\bar{T}_0}{\partial t} + \frac{\partial}{\partial x_j}(1-\gamma)(\bar{u}_0)_j \bar{T}_0 + \frac{\partial}{\partial x_j}(1-\gamma)(\bar{u}_0)_j \bar{T}_0' = -\frac{\partial(1-\gamma)(\bar{q}_0)_j}{\partial x_j} - \theta - c \quad (41)$$

$\bar{T}_0$, $\bar{T}_0'$ and $\bar{q}_0$ are the counterpart of $\bar{T}_1$, $\bar{T}_1'$ and $\bar{q}_1$ in the irrotational region.

Unconditioned Scalar

Addition of (39) and (41) yields

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j} \bar{u}_j \bar{T} + \frac{\partial}{\partial x_j} \bar{u}_j \bar{T}' = -\frac{\partial \bar{q}_j}{\partial x_j}$$

which can also be obtained by directly averaging (38). Recall that $\bar{T} = \gamma \bar{T}_1 + (1-\gamma)\bar{T}_0$ and $\bar{q} = \gamma \bar{q}_1 + (1-\gamma)\bar{q}_0$.

ENTRAINMENT PROFILES

The above equations have been particularized for the following two cases:

1. The plane wake behind a heated flat plate (1). A thin aluminum plate was mounted in a low speed wind tunnel, with the plate horizontal and parallel to the mean flow direction. The upper and lower side boundary layers formed were tripped near the leading edge of the plate. The plane turbulent wake behind the trailing edge was studied. A thermal wake was also generated by heating the plate, but only the velocity measurements are utilized here.
2. The heated turbulent round jet (12). An axisymmetric horizontal heated jet emerges into a room. The exit diameter is $D = 9$ in. A set of screens and a honeycomb were introduced to reduce the exit turbulence intensity level to a minimum. The diffuser was insulated with fiber glass and a collar heater was provided at the end of the contraction.

The statistically stationary equations to be considered are the following:

Plane Wake Turbulent Zone

$$\frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{v}_1}{\partial y} = E \quad (42)$$

$$\gamma \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + \gamma \bar{v}_1 \frac{\partial \bar{u}_1}{\partial y} + \frac{\partial}{\partial x} \gamma \bar{u}_1'^2 + \frac{\partial}{\partial y} \gamma \bar{u}_1' \bar{v}_1' = -\frac{1}{\rho} \frac{\partial \gamma \bar{p}_1}{\partial x} + \nu \nabla^2 \gamma \bar{u}_1 - F_x + M_x - \bar{u}_1 E \quad (43)$$

$$\gamma \bar{u}_1 \frac{\partial \bar{v}_1}{\partial x} + \gamma \bar{v}_1 \frac{\partial \bar{v}_1}{\partial y} + \frac{\partial}{\partial x} \gamma \bar{u}_1' \bar{v}_1' + \frac{\partial}{\partial y} \gamma \bar{v}_1'^2 = -\frac{1}{\rho} \frac{\partial \gamma \bar{p}_1}{\partial y} + \nu \nabla^2 \gamma \bar{v}_1 - F_y + M_y - \bar{v}_1 E \quad (44)$$

where x is the streamwise coordinate and y is perpendicular to x and the flat plate, $\bar{u}_1$ and $\bar{v}_1$ are the x and y mean turbulent zone velocities, respectively, u_1' and v_1' are the zone fluctuating velocities along x and y , respectively.

Heated Turbulent Round Jet

$$\frac{\partial \bar{u}_1}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \gamma \bar{v}_1) = E \quad (45)$$

$$\gamma \bar{u}_1 \frac{\partial \bar{u}_1}{\partial x} + \gamma \bar{v}_1 \frac{\partial \bar{u}_1}{\partial r} + \frac{\partial}{\partial x} \gamma \bar{u}_1'^2 + \frac{1}{r} \frac{\partial}{\partial r} r \gamma \bar{u}_1' \bar{v}_1' = -\frac{1}{\rho} \frac{\partial \gamma \bar{p}_1}{\partial x} + \nu \nabla^2 \gamma \bar{u}_1 - F_x + M_x - \bar{u}_1 E \quad (46)$$

$$\gamma \bar{u}_1 \frac{\partial \bar{v}_1}{\partial x} + \gamma \bar{v}_1 \frac{\partial \bar{v}_1}{\partial r} + \frac{\partial}{\partial x} \gamma \bar{u}_1' \bar{v}_1' + \frac{1}{r} \frac{\partial}{\partial r} r \gamma \bar{v}_1'^2 - \frac{\gamma}{r} \bar{u}_1'^2 = -\frac{1}{\rho} \frac{\partial \gamma \bar{p}_1}{\partial r} + \nu \left(\nabla^2 \gamma \bar{v}_1 - \frac{\gamma \bar{v}_1}{r^2} \right) - F_r + M_r - \bar{v}_1 E \quad (47)$$

$$\gamma \bar{u}_1 \frac{\partial \bar{T}_1}{\partial x} + \gamma \bar{v}_1 \frac{\partial \bar{T}_1}{\partial r} + \frac{\partial}{\partial x} \gamma \bar{u}_1' \bar{T}_1' + \frac{1}{r} \frac{\partial}{\partial r} r \gamma \bar{v}_1' \bar{T}_1' = -\frac{\partial \gamma (\bar{q}_1)_x}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} r \gamma (\bar{q}_1)_r + C + \theta - \bar{T}_1 E \quad (48)$$

The equations are in cylindrical coordinates x being the streamwise and r the radial directions, w_1' is the fluctuating velocity in the angular direction and $(\bar{q}_1)_x$ and $(\bar{q}_1)_r$ are the x and r components of the "mean scalar molecular zone flux".

Assumptions

Extensive data are available for conditioned and unconditioned variables for the heated turbulent jet at a single downstream location $x/D = 15$. The same is true for the plane wake at several downstream locations $x/\delta_m = 200$ and 345 . D and δ_m are, respectively, the jet exit diameter and the momentum thickness of the boundary layers at the trailing edge of the plate. Although self-preservation has almost surely not been achieved at the measuring stations, in the absence of experimental data to compute the x and y (or x and r) derivatives self-preservation for the jet and wake has been assumed. Direct measurements of the x -derivatives in the heated jet facility will hopefully soon be available and one will then be able to make a decision on the effect of the assumption of self-preservation. The traditional boundary layer approximation (9,11) were used to simplify equations (42) through (48). Moreover, we assumed that $\bar{p}_1$ and $\bar{v}_1'^2$ satisfy the unconditioned cross-stream momentum equation, i.e.

$$\frac{\partial \bar{v}_1'^2}{\partial y} = -\frac{1}{\rho} \frac{\partial \bar{p}_1}{\partial y} \quad (49)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{v}_1'^2) - \frac{w_1'^2}{r} = -\frac{1}{\rho} \frac{\partial \bar{p}_1}{\partial r} \quad (50)$$

and that $(-F_y + M_y - \bar{v}_1 E)$ and $(-F_r + M_r - \bar{v}_1 E)$ depend only weakly on x . E and the right sides of (43) and (44) and (46)-(48) were then computed by calculating the remaining terms in the equations from the experimental data.

Plane Wake

The following definitions were used

$$\gamma = f_1(\eta) \quad (51)$$

$$\frac{U_0 - \bar{u}_1}{U_s(x)} = f_2(\eta) \quad (52)$$

$$-\bar{u}_1' \bar{v}_1' = U_s^2(x) g(\eta) \quad (53)$$

$$\bar{v}_1'^2 = U_s^2(x) h^2(\eta) \quad (54)$$

$$\eta = \frac{y}{\delta(x)} \quad (55)$$

$$\frac{U_s(x)}{U_0} = 1.58 \frac{\delta_m^{1/2}}{x^{1/2}} \quad (56)$$

$$\delta(x) = .252 \delta_m^{1/2} x^{1/2} \quad (57)$$

where U_0 is the free-stream velocity, U_s is the velocity defect at the centerplane, $l(x)$ is the half-width of the wake based on velocity defect, δ_m is the momentum thickness of the boundary layer at the trailing edge of the plate. Equations (56) and (57) were taken from Tennekes and Lumley (9). The equations used are the following:

$$f_1'(\eta)h^2(\eta) = -\frac{F_y}{(U_s^2/l)} + \frac{M_y - \bar{v}_1 E}{(U_s^2/l)} - \left(\frac{\bar{p}_1}{\rho U_s^2}\right)f_1'(\eta) = \text{MOM} - Y \quad (58)$$

$$f_1(\eta)\{.0797[f_2(\eta) + f_2'(\eta)] - g'(\eta)\} - f_1'(\eta)g(\eta) = \frac{F_x}{(U_s^2/l)} + \frac{M_x - \bar{u}_1 E}{(U_s^2/l)} = \text{MOM} - X \quad (59)$$

The left sides of (58) and (59) were computed from Ali's measurements and the results are plotted in Figures 1 and 2. In Fig. 1 (MOM-X) stands for the right hand side of (59) as a function of η ; pressure and viscous forces as well as average entrainment of x-momentum minus the momentum entrained by the mean velocity $\bar{u}_1$ are included in (MOM-X). The abnormal "bumps" at $\eta_1 = 1.125$ and $\eta_2 = 1.375$ are almost certainly due to error in the differentiation process. Using Tutu's modeling (12) for the right side of (59) with $C_{1u} \approx 1$ yields

$$\text{R.H.S. of (59)} \approx 8.95 \left(\frac{f_y l}{U_0}\right) f_2(\eta)$$

The values of the latter are between one and two orders of magnitude higher than those computed from (59). In Fig. 2 (MOM-Y) stands for the right side of (58). All the values obtained for the sum of mean y-force, average entrainment of y-momentum minus the momentum entrained by $\bar{v}_1$ and the pressure force $\bar{p}_1 \frac{\partial y}{\partial \eta}$ are negative. This may be due to the predominance of the net entrainment of y-momentum with the mean vertical velocity $\bar{v}_1$ which has a negative sign.

Fig. 1. Average entrainment of x-momentum plus direct force for the heated wake as a function of the similarity variable η .

Fig. 2. Average entrainment of y-momentum plus direct force for the heated wake as a function of the similarity variable η .

Libby's modeling of the entrainment of x-momentum has the form

$$\bar{w}u' \approx \frac{|\bar{u}'v'|}{U_s^2}(1-\gamma)\eta \frac{\bar{u}U_s}{\Lambda}$$

The latter expression with a proportionality constant of .188 is compared with the results obtained here in Table 1.

Table 1. Libby's modeling of x-momentum entrainment and results from equation (59).

η	1.0	1.25	1.5	1.75	2.0
$\frac{\bar{w}u'}{(U_s^2/\Lambda)}$	.00201	.00763	.01209	.00972	.00020
MOM-X (59)	.00839	.00981	.01209	.00600	.00356

Round Jet

The following definitions were used

$$\gamma = f_1(\eta) \quad (60)$$

$$\bar{u}_1 = U_m(x)f_2(\eta) \quad (61)$$

$$\bar{v}_1 = U_m(x)f_3(\eta) \quad (62)$$

$$-\bar{u}_1'\bar{v}_1' = U_m^2(x)g(\eta) \quad (63)$$

$$\bar{T}_1 = T_m(x)f_4(\eta) \quad (64)$$

$$\bar{v}_1'\bar{\theta}_1' = U_m(x)T_m(x)h(\eta) \quad (65)$$

$$\eta = \frac{r}{r_1(x)} \quad (66)$$

$$U_m \propto 1/x \quad (67)$$

$$T_m \propto 1/x \quad (67')$$

$$r_{1/2}(x) = .067 x \quad (68)$$

where U_m is the maximum mean velocity at a given section and T_m is the maximum mean temperature, $r_{1/2}$ is the half-radius of the jet, i.e. r for which $\bar{u}/U_m = .5$, and relations (67) and (68) were taken from Tennäkes and Lumley (9).

The equations obtained by applying the previous assumptions to the conditioned equations for the turbulent fluid in a jet are

$$f_1'(\eta) [-0.067 \eta f_2(\eta) + f_3(\eta)] + f_1(\eta) \left\{ -0.067 [f_2(\eta) + \eta f_2'(\eta)] + f_3'(\eta) + \frac{f_3(\eta)}{\eta} \right\} = \frac{E}{U_m/r_{1/2}} \quad (69)$$

$$\begin{aligned} -0.067 f_1(\eta) f_2(\eta) [f_2(\eta) + \eta f_2'(\eta)] + f_1 \left\{ f_2'(\eta) f_3(\eta) - g'(\eta) \right. \\ \left. - \frac{g(\eta)}{\eta} \right\} - g(\eta) f_1'(\eta) = - \frac{F_x}{(U_m^2/r_{1/2})} \\ + \frac{M_x - \bar{u}_1 E}{(U_m^2/r_{1/2})} = \text{MOM-X} \quad (70) \end{aligned}$$

$$\begin{aligned} \frac{r_{1/2}}{T_m} \frac{dT_m}{dx} f_1(\eta) f_2(\eta) f_4(\eta) - \frac{dr_{1/2}}{dx} f_1(\eta) f_2(\eta) \eta f_4'(\eta) \\ + f_1(\eta) f_3(\eta) f_4'(\eta) + h(\eta) f_1'(\eta) + f_1(\eta) h'(\eta) \\ + \frac{f_1(\eta) h(\eta)}{\eta} = \frac{C}{(U_m T_m/r_{1/2})} + \frac{\Theta - \bar{T}_1 E}{(U_m T_m/r_{1/2})} = \text{HEAT} \quad (71) \end{aligned}$$

The left sides of (69)-(71) can be computed from Tutu's experimental data and the results appear on Figures 3-6. In Fig. 3 the dimensionless entrainment from (69) is compared with modeling in Reference (12) of that term. A significant difference in the location of the maxima as well as in the values of these maxima is observed. This may be due to an oversimplification of the actual physical phenomenon in the above reference. Libby's modeling while possibly more realistic includes unconditioned variables and do not serve for predictive purposes here. In Fig. 4 the marked difference between the computed right side of (70) and the modeling in (12) is more noticeable. The latter completely misses a negative region for $\eta \leq 1.00$. We ascribe the existence of this negative region to the large positive values of $\bar{u}_1 E$ which are plotted in dimensionless form in Fig. 5. While the total entrainment of x-momentum

$$\rho M_x = \lim_{V \rightarrow 0} \frac{1}{V} \int_{S(x,t)} \rho u v^e dS$$

is positive, the difference

$$M_x - \bar{u}_1 E = \lim_{V \rightarrow 0} \frac{1}{V} \int_{S(x,t)} (u - \bar{u}_1) v^e dS$$

may well be negative.

Fig. 6 is a plot of the right side of (71) versus η . Tutu's model multiplied by a factor of 10^{-1} is plotted for comparison. The agreement is better than for momentum entrainment.

Fig. 3. Mass entrainment per unit mass for the heated jet. The dashed line is the result of taking for v_1/U_m the smooth curve in Reference (12). —○— Tutu's modeling, i.e.,

$$\frac{2f\gamma}{(U_m/r_{1/2})}$$

Fig. 4. Average entrainment of x-momentum for the heated jet. —○— Tutu's modeling, i.e.

$$\frac{f\gamma\bar{u}_1}{(U_m^2/r_{1/2})}$$

Fig. 5. x-momentum entrained by the mean mass entrainment with the mean velocity for the heated jet.

Fig. 6. Average heat entrainment for the heated jet.

—○— Tutu's modeling, i.e.,

$$\frac{f_y \bar{T}_1}{\eta(U_m T_m / r_{1/2})}$$

multiplied by 10^{-1} .

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