**The Collisional Plasma Model:
A Velocity-Space Orthogonal-Function
Representation for the Distribution
Function of a Collisional Plasma**

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ORTHOGONAL-FUNCTION REPRESENTATION FOR THE DISTRIBUTION
FUNCTION OF A COLLISIONAL PLASMA

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FUNCTION OF A COLLISIONAL PLASMA*

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ABSTRACT

A velocity-space orthogonal-function representation has been developed for the distribution function of a collisional plasma. The choice of functions is determined by several criteria: 1) the functions must provide an accurate representation of the Coulomb collision operator; 2) the representation must be efficient in that physical plasmas be represented by as few terms as possible in order to minimize both computer time and storage requirements; and 3) the collision and Vlasov operators must lead to simple expressions which can be rapidly and easily calculated. The collision operator in the representation used here is developed both for the nonlinear (general distribution functions) and approximately linear (approximately Maxwellian distribution functions) cases. Some numerical results are discussed. Expressions for macroscopic observables in terms of the expansion amplitudes are also given.

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1. INTRODUCTION

If one is able to calculate the distribution function $f(\vec{x}, \vec{v}, t)$ of a plasma, one has available the maximum amount of information that it is possible to obtain. In particle-pushing models one has to treat an enormous number of particles in order to attain accuracy sufficient to separate useful information from noise in a multi-dimension plasma. This particle-number problem is avoided in models that approximate f by a function that is continuous in $\vec{x}$ and $\vec{v}$. The distribution function in these models is normally expanded in some complete set of orthogonal velocity-space functions which hopefully has rapid convergence properties.¹ The time derivative of any expansion amplitude will, of course, depend on other amplitudes; the fewer of these the better, since dependence on only a few leads to rapid, and hence less expensive, computation. Ideally these functions would be eigenfunctions of the linearized kinetic-equation operator; that would make this operator diagonal in the representation, and the time derivative of an amplitude would be trivial to compute. In practice, since one does not know and cannot easily obtain such eigenfunctions, one expands f in a set of functions which nearly diagonalize the linear operator.

For a "collisional" plasma (that is, one in which Coulomb collisions play a significant role), one can write the Boltzmann equation for the r^{th} species as follows:

$$\frac{\partial f_r}{\partial t} + K_r f_r = \sum_s C_{rs}(f_r, f_s) \quad , \quad (1.1)$$

where K_r represents the Vlasov operator (that is, the operator which describes the particle motion in various electric and magnetic fields), and C_{rs} is the Coulomb collision operator. The characteristics of the

two operators are different. The operator K_r contains multiplicative and derivative operators, and in many representations these give couplings only between neighboring components of the representations. In contrast, C_{rs} involves integration over velocity space; therefore, a choice of representation that restricts coupling to nearby neighbors will inherently be more difficult to achieve.

However, the collision operator does not play an entirely negative role. It also serves a useful role in that highly structured modes are rapidly damped. It therefore provides a convenient aid in preventing or alleviating problems caused by the mathematically artificial truncation of the complete set of functions into a finite set. (Such truncation is a necessary part of any numerical treatment.)

In this paper we describe a velocity space representation which exhibits desirable characteristics in the numerical treatment of Coulomb collisions, hence the name Collisional Plasma Model, or CPM. Expressions for the C_{rs} operator in the representation are derived with complete generality, and a linearized form of C_{rs} (that is, a form in which f is assumed to be approximately Maxwellian) is also derived. The advantages of treating Coulomb collisions in this representation are discussed, and some numerical solutions of Eq. (1.1) with $K_r = 0$ are used to demonstrate them. Finally, expressions for macroscopic observables in terms of the expansion amplitudes are given.

The CPM also provides a practical representation for the Vlasov operator K_r ; details of this representation are discussed in another paper.² Application of the CPM to the calculation of the neoclassical equilibrium of a toroidal plasma bears out the fact that the CPM

does satisfy all the listed criteria; this is discussed in a third paper.³

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2. THE CPM REPRESENTATION

The distribution function for a particular species of particles in a plasma can be expressed as a function of position $\vec{x}$, velocity $\vec{v}$, and time t . In the generalization which is adopted here, we write the distribution function as $f(\vec{x}, \vec{u}, t)$ where $\vec{u}$ is given by

$$\vec{u} = \vec{v} - \vec{V}(\vec{x}, t) , \quad (2.1)$$

with $\vec{V}(\vec{x}, t)$ arbitrary. The motivation for introducing this velocity transformation is that it provides the potential for subtracting the average velocity, a feature that could be important, for example, for injected particles. Following the transformation (2.1), the CPM assumes that f can be represented by the infinite series of orthogonal functions

$$f(\vec{x}, \vec{u}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=0}^{\infty} F_{\ell mn}(\vec{x}, t) \psi_{\ell mn}(\sqrt{\rho(\vec{x}, t)} \vec{u}) , \quad (2.2)$$

where

$$\psi_{\ell mn}(\sqrt{\rho} \vec{u}) = (\sqrt{\rho} u)^{\ell} L_n^{\ell+\frac{1}{2}}(\rho u^2) \exp\left(-\frac{1}{2} \rho u^2\right) Y_{\ell m}(\alpha, \beta) . \quad (2.3)$$

The $Y_{\ell m}$ in Eq. (2.3) are spherical harmonics,⁴

$$Y_{\ell m}(\alpha, \beta) = \frac{1}{2^{\ell} \ell!} \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} \left(-\sin \alpha e^{i\beta}\right)^m \frac{d^{\ell+m}}{d\omega^{\ell+m}} (\omega^2 - 1)^{\ell} ,$$

where

$$\omega = \cos \alpha ,$$

and (u, α, β) are the spherical polar coordinates of $\vec{u}$ in an arbitrary velocity reference frame. The $Y_{\ell m}$ are orthonormal,

$$\int_0^{2\pi} d\beta \int_0^\pi d\alpha \sin \alpha Y_{\ell m}^* (\alpha, \beta) Y_{\ell' m'} (\alpha, \beta) = \delta_{\ell\ell'} \delta_{mm'} \quad , \quad (2.4)$$

and satisfy the relation

$$Y_{\ell m}^* (\alpha, \beta) = (-1)^m Y_{\ell -m} (\alpha, \beta) \quad . \quad (2.5)$$

The $L_n^{\ell+\frac{1}{2}}$ in Eq. (2.3) are generalized Laguerre polynomials,⁵

$$L_n^{\ell+\frac{1}{2}}(s) = \frac{1}{n! s^{\ell+\frac{1}{2}}} e^s \frac{d^n}{ds^n} (s^{\ell+n+\frac{1}{2}} e^{-s}) \quad .$$

They satisfy the orthogonality condition

$$\int_0^\infty ds s^{\ell+\frac{1}{2}} e^{-s} L_n^{\ell+\frac{1}{2}}(s) L_{n'}^{\ell+\frac{1}{2}}(s) = \delta_{nn'} \frac{\Gamma(\ell+n+\frac{3}{2})}{n!} \quad . \quad (2.6)$$

It follows from Eqs. (2.4) and (2.6) that the functions (2.3) satisfy the orthogonality condition

$$\int d^3u \psi_{\ell mn}^* (\sqrt{\rho} \vec{u}) \psi_{\ell' m' n'} (\sqrt{\rho} \vec{u}) = \delta_{\ell\ell'} \delta_{mm'} \delta_{nn'} \frac{\Gamma(\ell+n+\frac{3}{2})}{2n! \rho^{\frac{3}{2}}} \quad . \quad (2.7)$$

From the orthogonality condition (2.7) and (2.2), we find

$$F_{\ell mn}(\vec{x}, t) = \frac{2n! \rho^{\frac{3}{2}}}{\Gamma(\ell+n+\frac{3}{2})} \int d^3u \psi_{\ell mn}^* (\sqrt{\rho} \vec{u}) f(\vec{x}, \vec{u}, t) \quad .$$

Because f is real, it follows from Eq. (2.5) that

$$F_{\ell mn}^* = (-1)^m F_{\ell -mn} \quad .$$

The parameter $\rho(\vec{x}, t)$ in Eq. (2.2) has the dimensions of (velocity)⁻² and, like $\vec{V}(\vec{x}, t)$, is arbitrary. However, with $\vec{V}$ as the average velocity and $\rho = M/kT$, where M and T are the species mass and temperature, the $\ell = m = n = 0$ term in Eq. (2.2) is identifiable as the Maxwellian component. This result indicates how, for predominantly Maxwellian particles, the appropriate choice of ρ and $\vec{V}$ can give a

series expansion of f that converges after relatively few terms.

The motivation for choosing (u, α, β) as velocity-space coordinates is based on physical considerations. With $\vec{V} = 0$ and the velocity-space z axis taken to be in the direction of $\vec{B}$ (which it normally is), α and β are the conventional pitch and gyro angles, and u is proportional to the square root of the kinetic energy. For non-zero $\vec{V}$, (u, α, β) still have the same interpretation, but in a frame of reference moving with velocity $\vec{V}$ in the velocity coordinate system.

The choice of spherical harmonics as expansion functions for (α, β) is natural because they form an orthogonal set over the unit sphere. They have the additional desirable property that they diagonalize the large $\vec{u} \times \vec{\Omega} \cdot \vec{\nabla}_u$ ($\vec{\Omega} = Ze\vec{B}/Mc$) term in the Boltzmann equation. (This property is fundamentally what makes it feasible to derive drift-kinetic approximations using the CPM representation.) Also, the linearized collision operator is diagonal in ℓ (see Sect. 4).

There are several restrictions on the choice for the u dependence of the expansion function. First, it is desirable that the function be orthogonal with respect to some weight function over the range $0 \leq u \leq \infty$. It must also have the form $u^\ell F(u^2)$ so that the total function, $u^\ell F(u^2) Y_{\ell m}$, is analytic at $u = 0$. Also, it is desirable that the function contain the factor $\exp(-\frac{1}{2} \rho u^2)$, so that the Maxwellian can be a member of the set. The final choice, defined in Eq. (2.3), satisfies these conditions; the orthogonality weight function is 1, as Eq. (2.7) indicates. In addition to the properties described above, operations such as multiplication by $\vec{u}$ or differentiation with respect to $\vec{u}$ result in expressions involving only neighboring

ℓ , m , and n values. This guarantees that the Vlasov part of the kinetic operator has non-zero matrix elements only near the diagonal.²

The choice of the generalized Laguerre polynomials may seem arbitrary, but was suggested by the use of the closely allied Hermite polynomials in earlier work.¹ Also, the functions (2.3) are the well-known orthogonal eigenfunctions of the quantum mechanical spherical harmonic oscillator problem.⁶

Another function, given by Eq. (2.3) with $L_n^{\ell+\frac{1}{2}}(\rho u^2)$ replaced by $L_n^{\ell+\frac{1}{2}}(\frac{1}{2} \rho u^2)$, has also been used in the past.⁷ This function has the advantage of giving a close relationship between the amplitudes of the distribution function and macroscopic observables, such as number density, etc. In this alternate representation, the number density, for example, is given by

$$N = \pi \sqrt{\frac{2}{\rho^3}} F_{000}(\vec{x}, t),$$

whereas in the representation (2.3) it is given by (see Sect. 5)

$$N = \pi \sqrt{\frac{2}{\rho^3}} \sum_{n=0}^{\infty} F_{00n}(\vec{x}, t) (-1)^n \binom{n + \frac{1}{2}}{n}.$$

Use of this more complicated expression would be disadvantageous for analytic work, but for a computer calculation it is not very significant, provided the series converges sufficiently rapidly. The disadvantage of the $L_n^{\ell+\frac{1}{2}}(\frac{1}{2} \rho u^2)$ form is that the factor of $\frac{1}{2}$ in the argument of the polynomial results in high-velocity oscillations of the polynomial being overwhelmed by the exponential. As a result, functions with n -values of n and $n+1$ differ by amounts that are very small compared with the maxima of the functions, and in general this

leads to poor convergence properties of the alternate representation. For the functions (2.3), however, the exponential does not dominate until immediately after pu^2 is greater than the last zero of the polynomial. The oscillatory behavior of the functions can be related to that of spherical Bessel functions through the asymptotic relation, valid for large n ,⁵

$$x^\ell I_n^{\ell+\frac{1}{2}}(x^2) e^{-\frac{1}{2}x^2} \sim \frac{2}{\sqrt{\pi}} \frac{\Gamma(\ell+n+\frac{3}{2})}{n!} \left(\frac{2}{v}\right)^\ell j_\ell(vx),$$

where

$$v = \sqrt{2\ell + 4n + 3} \quad .$$

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3. COLLISION OPERATOR IN THE CPM REPRESENTATION

The Landau form for the Coulomb collision operator, describing the interaction between a species r and all species s , is given by⁸

$$Cf_r = \sum_s C_{rs} (f_r, f_s) ,$$

where

$$C_{rs} = \Gamma_{rs} \left[\frac{M_r}{M_s} D_{rs}^{(1)} + D_{rs}^{(2)} \right] ,$$

$$\Gamma_{rs} = 4\pi \left(\frac{Z_r Z_s e^2}{M_r} \right)^2 \ln \Lambda ,$$

$$\Lambda = \frac{3}{2\sqrt{\pi} N_e e^3} (kT_e)^{3/2} ,$$

$$D_{rs}^{(1)} = - \sum_i \frac{\partial}{\partial v_i} \left[f_r(\vec{v} - \vec{v}_r) \frac{\partial}{\partial v_i} G_s(\vec{v}) \right] ,$$

$$D_{rs}^{(2)} = D_{rs}^{(1)} + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial v_i \partial v_j} \left[f_r(\vec{v} - \vec{v}_r) \frac{\partial^2}{\partial v_i \partial v_j} H_s(\vec{v}) \right] ,$$

$$G_s(\vec{v}) = \int d^3v' |\vec{v} - \vec{v}'|^{-1} f_s(\vec{v}' - \vec{v}_s) ,$$

$$H_s(\vec{v}) = \int d^3v' |\vec{v} - \vec{v}'| f_s(\vec{v}' - \vec{v}_s) .$$

In the equations above, f_r and f_s are written as functions of $\vec{u}$, with the $\vec{x}$ and t dependence suppressed; i and j label rectangular cartesian components in an arbitrary velocity reference frame.

Introducing the transformation (2.1) one obtains

$$D_{rs}^{(1)} = - \sum_i \frac{\partial}{\partial u_i} \left[f_r(\vec{u}) \frac{\partial}{\partial u_i} G_{rs}(\vec{u}) \right] , \quad (3.1)$$

$$D_{rs}^{(2)} = D_{rs}^{(1)} + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial u_i \partial u_j} \left[f_r(\vec{u}) \frac{\partial^2}{\partial u_i \partial u_j} H_{rs}(\vec{u}) \right] , \quad (3.2)$$

where

$$G_{rs}(\vec{u}) = \int d^3u' |\vec{u} - \vec{u}'|^{-1} k_{rs}(\vec{u}') , \quad (3.3)$$

$$H_{rs}(\vec{u}) = \int d^3u' |\vec{u} - \vec{u}'| k_{rs}(\vec{u}') , \quad (3.4)$$

$$k_{rs}(\vec{u}) = f_s(\vec{u} + \vec{V}_r - \vec{V}_s) . \quad (3.5)$$

The $\vec{u}$ formulation, given by Eqs. (3.1)-(3.5), is particularly suited to cases where $\sqrt{\rho_s} |\vec{V}_r - \vec{V}_s| \ll 1$. If this condition is not met but the condition $\sqrt{\rho_r} |\vec{V}_r - \vec{V}_s| \ll 1$ is, the $\vec{w}$ formulation is more suitable:

$$D_{rs}^{(1)} = - \sum_i \frac{\partial}{\partial w_i} \left[k_{sr}(\vec{w}) \frac{\partial}{\partial w_i} G_s(\vec{w}) \right] ,$$

$$D_{rs}^{(2)} = D_{rs}^{(1)} + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial w_i \partial w_j} \left[k_{sr}(\vec{w}) \frac{\partial^2}{\partial w_i \partial w_j} H_s(\vec{w}) \right] ,$$

where

$$\vec{w} = \vec{u} + \vec{V}_r - \vec{V}_s ,$$

$$G_s(\vec{w}) = \int d^3u' |\vec{w} - \vec{u}'|^{-1} f_s(\vec{u}') ,$$

$$H_s(\vec{w}) = \int d^3u' |\vec{w} - \vec{u}'| f_s(\vec{u}') .$$

The $\vec{u}$ and $\vec{w}$ formulations are very similar, and it is sufficient to develop only the $\vec{u}$ formulation. The $\vec{w}$ formulation can then be obtained from the $\vec{u}$ formulation, although a final transformation to the $\vec{u}$ representation also has to be made (see Eqs. (3.17) and (3.18)).

Proceeding with the development of the $\vec{u}$ formulation, expansions of the form (2.2) are substituted for f_r and f_s in the defining equations of the collision operator. One obtains, using the methods outlined in the Appendix, the following expansion for the Coulomb collision operator;

$$C_{rs} = \Gamma_{rs} \sum_{lmn} D_{lmn}^{(rs)} \psi_{lmn}(\sqrt{\rho_r} \vec{u}) , \quad (3.6)$$

where

$$D_{lmn}^{(rs)} = \sum_{\substack{\ell', m', n' \\ \ell'', m'', n''}} Z_{[mm' m'']}^{[\ell \ell' \ell'']} S_{rs}[\ell \ell' \ell'']_{nn' n''} F_{\ell' m' n}^{(r)} K_{\ell'' m'' n''}^{(rs)} , \quad (3.7)$$

$$Z_{[mm' m'']}^{[\ell \ell' \ell'']} = \sqrt{\frac{(2\ell'+1)(2\ell'''+1)}{4\pi(2\ell+1)}} C(\ell' \ell'' \ell, 000) C(\ell' \ell'' \ell, m' m'' m) , \quad (3.8)$$

$$S_{rs}[\ell \ell' \ell'']_{nn' n''} = \frac{2\pi n!}{\Gamma(\ell+n+3/2)} \binom{\ell'' + n'' + 1/2}{n''} \times \left\{ \frac{M_r}{M_s} S_{\lambda}^{(1)}[\ell \ell' \ell'']_{nn' n''} + S_{\lambda}^{(2)}[\ell \ell' \ell'']_{nn' n''} \right\} , \quad (3.9)$$

$$S_{\lambda}^{(1)}[\ell \ell' \ell'']_{nn' n''} = \sum_{q=0}^{n''} (-1)^{n''+q} 2^q \binom{n''}{q} \left\{ Q_{\lambda}(\ell \ell' \ell'')_{nn' q} \right. \\ \left. + (\ell - \ell' + 2n - 2n') [\lambda(\ell'' + q + \frac{1}{2})]^{-1} Q_{\lambda}(\ell \ell' \ell'')_{nn' q-1} \right\} , \quad (3.10)$$

$$S_{\lambda}^{(2)}[\ell \ell' \ell'']_{nn' n''} = \sum_{q=0}^{n''} (-1)^{n''+q} 2^{q-1} \binom{n''}{q} \left\{ Q_{\lambda}(\ell \ell' \ell'')_{nn' q} \right. \\ - [\lambda^2(\ell'' + q + \frac{1}{2})(\ell'' + q - \frac{1}{2})]^{-1} [(\ell - \ell' + 2n - 2n')(\ell - \ell' + 2n - 2n' + 1)] \\ \times Q_{\lambda}(\ell \ell' \ell'')_{nn' q-2} + (2n' + 2) Q_{\lambda}(\ell \ell' \ell'')_{nn' q-1} \\ \left. - (2\ell + 2n + 1) Q_{\lambda}(\ell \ell' \ell'')_{nn' q-2} \right\} , \quad (3.11)$$

$$Q_{\lambda}(\ell \ell' \ell'')_{nn' q} = Q_{\lambda}(\ell' \ell \ell'')_{n' n q}$$

$$\begin{aligned}
&= \lambda^{\frac{1}{2}\ell''} \int_0^\infty ds s^{\frac{1}{2}(\ell + \ell' + \ell'' + 1)} \exp \left[-\left(1 + \frac{1}{2}\lambda\right)s \right] \\
&\quad \times L_n^{\ell + \frac{1}{2}}(s) L_n^{\ell' + \frac{1}{2}}(s) \Phi \left(-\ell, \ell'' + \frac{3}{2}; \frac{1}{2}\lambda s \right), \quad (3.12)
\end{aligned}$$

$$\lambda = \rho_s / \rho_r. \quad (3.13)$$

In Eq. (3.8) the C's are Clebsch-Gordan coefficients,⁴ for which Z vanishes unless $\ell + \ell' + \ell''$ is even. In Eq. (3.12) Φ is the confluent hypergeometric function.⁵ The K amplitudes in Eq. (3.7) are given by

$$K_{\ell mn}^{(rs)} = \sum_{\ell' m' n'} A_{\ell m n}^{\ell' m' n'}(\vec{y}_{rs}) F_{\ell' m' n'}^{(s)}, \quad (3.14)$$

with

$$A_{\ell m n}^{\ell' m' n'}(\vec{y}_{rs}) = \frac{2 n!}{\Gamma(\ell + n + 3/2)}$$

$$\times \int d^3p p^\ell L_n^{\ell + \frac{1}{2}}(p^2) Y_{\ell m}^*(\hat{p}) z^{\ell'} L_n^{\ell' + \frac{1}{2}}(z^2) Y_{\ell' m'}(\hat{z}) \exp \left[-\frac{1}{2}(p^2 + z^2) \right], \quad (3.15)$$

$$\vec{z} = \vec{p} + \vec{y}_{rs} = \vec{p} + \sqrt{\rho_s} (\vec{V}_r - \vec{V}_s). \quad (3.16)$$

In the $\vec{w}$ formulation one obtains Eq. (3.6) with

$$D_{\ell mn}^{(rs)} = \sum_{\ell' m' n'} A_{\ell m n}^{\ell' m' n'} \left(\frac{1}{\sqrt{\lambda}} \vec{y}_{rs} \right) W_{\ell' m' n'}^{(rs)}, \quad (3.17)$$

$$W_{\ell mn}^{(rs)} = \sum_{\substack{\ell' m' n' \\ \ell'' m'' n''}} Z_{\ell m m'}^{\ell \ell' \ell''} S_{rs}^{\ell \ell' \ell''} K_{\ell' m' n'}^{(sr)} F_{\ell'' m'' n''}^{(s)}. \quad (3.18)$$

Equations (3.6)-(3.18) define the CPM expansion for the Coulomb collision operator in terms of the CPM expansions for f_r and f_s . If ρ_r and ρ_s are chosen so that

$$\lambda = \rho_s / \rho_r = M_s T_r / M_r T_s , \quad (3.19)$$

one can show that for $\vec{V}_r = \vec{V}_s$ and $T_r = T_s$

$$S_{rs} \begin{bmatrix} 000 \\ Ln00 \end{bmatrix} = 0 . \quad (3.20)$$

Thus, as is well known, the collision operator plays no role for species having Maxwellian distributions, equal temperatures, and equal average velocities.

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4. LINEARIZED COLLISION OPERATOR

Substantial simplification of the expansion (3.7) occurs if y_{rs} is negligibly small (see Eqs. (3.14)-(3.16)) and if, as is true for approximately Maxwellian distributions, $F_{\ell'm'n'}^{(r)}, F_{\ell''m''n''}^{(s)}$ is negligibly small unless $\ell' = m' = n' = 0$ and/or $\ell'' = m'' = n'' = 0$. Applying these conditions to Eq. (3.7) and using the ℓ -diagonalization result

$$Z_{\ell m \ell' m'}^{[\ell \ell' 0]} = Z_{\ell 0 m \ell'}^{[\ell 0 \ell']} = \frac{1}{\sqrt{4\pi}} \delta_{\ell \ell'} \delta_{m m'} ,$$

one obtains

$$Cf_r = \sum_{\substack{\ell m n \\ n'}} \left[C_{\ell n n}^{(rr)}, F_{\ell m n}^{(r)}, + \sum_{s \neq r} C_{\ell n n}^{(rs)}, F_{\ell m n}^{(s)}, \right] \psi_{\ell m n} , \quad (4.1)$$

where

$$C_{\ell n n}^{(rr)} = \frac{1}{\sqrt{4\pi}} \left(1 - \frac{1}{2} \delta_{\ell 0} \delta_{n' 0} \right) \left\{ \sum_{s \neq r} \Gamma_{rs} F_{000}^{(s)} S_{rs} [\ell \ell' 0] \right. \\ \left. + \Gamma_{rr} F_{000}^{(r)} \left(S_{rr} [\ell \ell' 0] + S_{rr} [\ell 0 \ell'] \right) \right\} , \quad (4.2)$$

$$C_{\ell n n}^{(rs)} = \frac{1}{\sqrt{4\pi}} \left(1 - \frac{1}{2} \delta_{\ell 0} \delta_{n' 0} \right) \Gamma_{rs} F_{000}^{(r)} S_{rs} [\ell 0 \ell'] .$$

If λ is given by Eq. (3.19) and $T_r = T_s$, Eq. (3.20) gives

$$C_{0n0}^{(rr)} = C_{0n0}^{(rs)} = 0 .$$

For self-collisions only, Eqs. (4.1) and (4.2) reduce to

$$Cf_r = \sum_{\substack{\ell m n \\ n'}} C_{\ell n n}^{(rr)}, F_{\ell m n}^{(r)}, \psi_{\ell m n} , \quad (4.3)$$

$$C_{\ell n n}^{(rr)} = \frac{1}{\sqrt{4\pi}} \Gamma_{rr} F_{000}^{(r)} \left(S_{rr} [\ell \ell' 0] + S_{rr} [\ell 0 \ell'] \right) ,$$

with

$$C_{On0}^{(rr)} = C_{On1}^{(rr)} = C_{ln0}^{(rr)} = 0 \quad . \quad (4.4)$$

The notable feature of these results is the ℓ and m diagonalization that is obtained in the linear approximation. This is a consequence of using the spherical harmonics in the CPM representation.

The representation used in the CPM has been tested numerically with regard to self-collisions by setting K_r equal to zero in Eq. (1.1), replacing the collision term by (4.3), and truncating the n' summation at $n' = NMAX$. Thus the equation for the (ℓmn) amplitude is

$$\frac{\partial}{\partial t} F_{\ell mn}^{(r)} = \sum_{n'=0}^{NMAX} C_{\ell nn'}^{(rr)} F_{\ell mn'}^{(r)} \quad . \quad (4.5)$$

For the first test the initial value of f consisted of a Maxwellian component f_M plus a single non-Maxwellian component $\tilde{f}$ with $(\ell, m, n) = (5, 0, 5)$. Equations (4.4) and (4.5) show that f_M is constant in time, so the test was conducted by following the time evolution of $\tilde{f}$ with various values of $NMAX$ (see (4.5)), and comparing the results. In general one gets good results if $NMAX = N_0 + 1$, where N_0 is the initial n (here $N_0 = 5$). After .25 Spitzer collision times, it is seen (in Fig. 1) that while there is a considerable difference between $NMAX = 5$ and $NMAX = 6$, going to $NMAX = 15$ has relatively little additional effect. This remains true at later times. The further time evolution of this distribution with an $NMAX = 15$ is shown in Fig. 2. Any truncation with $NMAX \geq 6$ will produce nearly the same curves.

A second test consisted of taking a "bump on tail" as an initial value:

$$\tilde{f} \propto (\sqrt{\rho_r} u)^{\ell + 2N_0} \exp(-\frac{1}{2} \rho_r u^2) Y_{\ell 0} \quad .$$

This results initially in a set of nonzero amplitudes, $F_{\ell 0 n}^{(r)}$ with $0 \leq n \leq N_0$. This time it was found that one needs only $NMAX = N_0$ to obtain good accuracy. Figure 3 shows the results at .5 collision times and Fig. 4 at 2 collision times for $NMAX = 5$ and $NMAX = 15$. In the example $(\ell, m, N_0) = (5, 0, 5)$ once again. In neither case is a difference discernible. The time evolution of this "bump on tail" initial distribution is shown in Fig. 5, where we have used $NMAX = 5$; $NMAX = 15$ is identical in appearance.

Many such tests led to the conclusion that the representation used in the CPM is indeed capable of representing the Coulomb collision operator efficiently and accurately.

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5. MACROSCOPIC OBSERVABLES IN THE CPM REPRESENTATION

Given the representation (2.2) for the distribution function f , it is a relatively simple matter to calculate macroscopic observables such as the number density, particle current density, etc., in terms of the CPM amplitudes $F_{\ell mn}$. All the observables of interest involve integrals of the type

$$\begin{aligned} I_s(\ell, m, q) &= \sqrt{\frac{4\pi}{2\ell+1}} \int d^3u \, u^{\ell+2q} Y_{\ell m}^*(\alpha, \beta) f_s \\ &= \sqrt{\frac{\pi}{\ell+\frac{1}{2}}} \left(\frac{1}{\sqrt{\rho_s}} \right)^{\ell+2q+3} 2^{\ell+q+1} \Gamma(\ell + q + \frac{3}{2}) \\ &\times \sum_{n=0}^{\infty} F_{\ell mn}^{(s)} (-1)^n \binom{\ell+n+\frac{1}{2}}{n} {}_2F_1(-n, -q; \ell + \frac{3}{2}; 2) \quad , \end{aligned} \quad (5.1)$$

where s denotes a particular species and ${}_2F_1$ is the Gauss hypergeometric function. It is now a straightforward exercise to determine the expressions for the macroscopic observables in terms of the integrals (5.1).

5.1 Number Density

The number density for a species s is given by

$$N_s = \int d^3v \, f_s = I_s(0, 0, 0) \quad .$$

The plasma number density and average mass are given by

$$N = \sum_s N_s \quad ,$$

$$M = \sum_s M_s N_s / N \quad .$$

5.2 Particle Current Density

The particle current density for a species s is given by

$$\vec{J}_s = \int d^3v \vec{v} f_s = N_s \vec{V}_s + \vec{j}_s$$

where

$$\vec{j}_s = \int d^3u \vec{u} f_s .$$

The rectangular Cartesian components of $\vec{j}_s$ in the arbitrary velocity reference frame are given by

$$j_1^{(s)} + i j_2^{(s)} = \sqrt{2} I_s (1, -1, 0) ,$$

$$j_3^{(s)} = I_s (1, 0, 0) .$$

The species average velocity is given by

$$\vec{U}_s = \vec{J}_s / N_s = \vec{V}_s + \vec{j}_s / N_s ,$$

and the plasma average velocity is given by

$$\vec{U} = \sum_s M_s N_s \vec{U}_s / MN .$$

5.3 Energy Density

The energy density for a species s is given by

$$\begin{aligned} W_s &= \frac{1}{2} M_s \int d^3v v^2 f_s \\ &= -\frac{1}{2} M_s N_s (V_s^2 - 2 \vec{V}_s \cdot \vec{U}_s) + w_s , \end{aligned}$$

where

$$w_s = \frac{1}{2} M_s \int d^3u u^2 f_s = \frac{1}{2} M_s I_s (0, 0, 1) .$$

The plasma energy density is given by

$$W = \sum_s W_s .$$

5.4 Scalar Pressure and Temperature

The partial scalar pressure for a species s is given by

$$\begin{aligned} P_s &= \frac{1}{3} M_s \int d^3v (\vec{v} - \vec{U})^2 f_s \\ &= \frac{2}{3} W_s + \frac{1}{3} M_s N_s (U^2 - 2 \vec{U} \cdot \vec{U}_s) . \end{aligned}$$

The plasma scalar pressure is given by

$$P = \sum_s P_s = \frac{2}{3} W - \frac{1}{3} M N U^2 .$$

Species and plasma temperatures are given by

$$kT_s = P_s / N_s , \quad kT = P / N .$$

5.5 Tensor Pressure

The plasma tensor pressure is given by

$$\begin{aligned} \underline{\underline{P}} &= \sum_s M_s \int d^3v (\vec{v} - \vec{U})(\vec{v} - \vec{U}) f_s - \frac{1}{3} P \\ &= \sum_s \left\{ \underline{\underline{p}}_s - \frac{2}{3} \frac{1}{3} W_s - M_s N_s \left[\vec{V}_s \vec{V}_s + \vec{U} \vec{U} \right. \right. \\ &\quad \left. \left. - \vec{V}_s \vec{U}_s - \vec{U}_s \vec{V}_s - \frac{1}{3} \frac{1}{3} (V_s^2 + U^2 - 2 \vec{V}_s \cdot \vec{U}_s) \right] \right\} , \end{aligned}$$

where

$$\underline{\underline{p}}_s = M_s \int d^3u \vec{u} \vec{u} f_s .$$

The components of the tensor $\underline{\underline{p}}_s$ are given by

$$\begin{aligned} p_{11}^{(s)} &= \frac{2}{3} W_s - \frac{1}{3} M_s \left\{ I_s(2, 0, 0) - \sqrt{6} \operatorname{Re}[I_s(2, -2, 0)] \right\} , \\ p_{22}^{(s)} &= \frac{2}{3} W_s - \frac{1}{3} M_s \left\{ I_s(2, 0, 0) + \sqrt{6} \operatorname{Re}[I_s(2, -2, 0)] \right\} , \end{aligned}$$

$$p_{33}^{(s)} = 2 w_s - p_{11}^{(s)} - p_{22}^{(s)} ,$$

$$p_{12}^{(s)} = \sqrt{\frac{2}{3}} M_s \operatorname{Im}[I_s(2, -2, 0)]$$

$$p_{13}^{(s)} + i p_{23}^{(s)} = \sqrt{\frac{2}{3}} M_s I_s(2, -1, 0) .$$

5.6 Heat Flux Vector

The plasma heat flux vector is given by

$$\begin{aligned} \vec{Q} &= \frac{1}{2} \int d^3v (\vec{v} - \vec{U})^2 (\vec{v} - \vec{U}) f_s \\ &= - (\underline{P} + \underline{1} W) \cdot \vec{U} + \sum_s [(\underline{p}_s + \underline{1} W_s) \cdot \vec{V}_s \\ &\quad + \vec{q}_s + \frac{1}{2} M_s N_s V_s^2 (\vec{U}_s - \vec{V}_s)] , \end{aligned}$$

where

$$\vec{q}_s = \frac{1}{2} M_s \int d^3u \vec{u} f_s .$$

The components of $\vec{q}_s$ are given by

$$q_1^{(s)} + i q_2^{(s)} = \frac{1}{\sqrt{2}} M_s I_s(1, -1, 1) ,$$

$$q_3^{(s)} = \frac{1}{2} M_s I_s(1, 0, 1) .$$

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Fig. 1. Comparison at $t = .25$ collision times of an initial $(\ell, m, n) = (5, 0, 5)$ perturbation of a Maxwellian (normalized to a maximum of 1 and shown by the solid curve). Values of $N_{MAX} = 5, 6$, and 15 are compared.

Fig. 2. The time evolution of the perturbation defined in Fig. 1 is displayed for various $t \leq 2$ collision times.

Fig. 3. Comparison at $t = .5$ collision times of an initial "bump on tail" perturbation with $\ell = 5$, $N_0 = 5$. Values of $N_{MAX} = 5$ and 15 produce the same curve.

Fig. 4. The same as Fig. 3 except $t = 2$ collision times.

Fig. 5. The time evolution of the "bump on tail" perturbation for various $t \leq 2$ collision times, with $NMAX = 5$.

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APPENDIX - DERIVATION OF THE COLLISION MATRIX

In this appendix we present some of the details of the derivation of Eqs. (3.6)-(3.18). The starting point is to rewrite Eqs. (3.1) and (3.2) in the forms

$$D_{rs}^{(1)} = \frac{1}{2} [4\pi f_r k_{rs} + G_{rs} \Delta f_r - \Delta(f_r G_{rs})] \quad (A1)$$

and

$$D_{rs}^{(2)} = \frac{1}{8} [8\pi f_r k_{rs} + \Delta^2(f_r H_{rs}) + H_{rs} \Delta^2 f_r - 2\Delta(H_{rs} \Delta f_r)] , \quad (A2)$$

where

$$\Delta = \sum_i \frac{\partial^2}{\partial u_i^2}$$

is the Laplacian.

A1. Expansions of k_{rs} , G_{rs} , H_{rs}

The modified distribution function k_{rs} , defined by Eq. (3.5), can be expanded as in Eq. (2.2),

$$k_{rs}(\vec{u}) = \sum_{\ell mn} K_{\ell mn}^{(rs)} \psi_{\ell mn}(\sqrt{\rho_s} \vec{u}) , \quad (A3)$$

the amplitudes $K_{\ell mn}^{(rs)}$ being given by Eqs. (3.14)-(3.16). The functions G_{rs} and H_{rs} , defined by Eqs. (3.3) and (3.4), have expansions^{4,5}

$$G_{rs}(\vec{u}) = \frac{4\pi}{\rho_s} \sum_{\ell mn} K_{\ell mn}^{(rs)} G_{\ell n}(\sqrt{\rho_s} u) Y_{\ell m}(\alpha, \beta) , \quad (A4)$$

$$H_{rs}(\vec{u}) = \frac{4\pi}{\rho_s^2} \sum_{\ell mn} K_{\ell mn}^{(rs)} H_{\ell n}(\sqrt{\rho_s} u) Y_{\ell m}(\alpha, \beta) ,$$

where

$$G_{ln}(x) = \frac{1}{2^{l+1}} \int_0^\infty dx' x'^2 \frac{x_{<}^l}{x_{>}^{l+1}} x'^l L_n^{\frac{l+1}{2}}(x'^2) \exp\left(-\frac{1}{2} x'^2\right), \quad (A5)$$

$$H_{ln}(x) = \frac{1}{2^{l+1}} \int_0^\infty dx' x'^2 \left[(x^2 + x'^2) \frac{x_{<}^l}{x_{>}^{l+1}} - xx' \left(\frac{2l}{2l-1} \frac{x_{<}^{l-1}}{x_{>}^l} + \frac{2l+2}{2l+3} \frac{x_{<}^{l+1}}{x_{>}^{l+2}} \right) \right] x'^l L_n^{\frac{l+1}{2}}(x'^2) \exp\left(-\frac{1}{2} x'^2\right), \quad (A6)$$

$$x_{<} = \min(x, x') \quad , \quad x_{>} = \max(x, x') \quad .$$

Using various properties of the generalized Laguerre polynomials,⁵ one can reduce Eqs. (A5) and (A6) to

$$G_{ln}(x) = \binom{l+n+\frac{1}{2}}{n} x^l \exp\left(-\frac{1}{2} x^2\right) \sum_{q=0}^n (-1)^{n+q} 2^{q-1} x \binom{n}{q} \left(l+q+\frac{1}{2}\right)^{-1} \Phi\left(1-q, l+\frac{3}{2}; \frac{1}{2} x^2\right) \quad , \quad (A7)$$

$$H_{ln}(x) = \binom{l+n+\frac{1}{2}}{n} x^l \exp\left(-\frac{1}{2} x^2\right) \sum_{q=0}^n (-1)^{n+q+1} 2^{q-1} x \binom{n}{q} \left[\left(l+q+\frac{1}{2}\right)\left(l+q-\frac{1}{2}\right)\right]^{-1} \Phi\left(2-q, l+\frac{3}{2}; \frac{1}{2} x^2\right) \quad ,$$

where Φ is the confluent hypergeometric function.⁵

A2. Expansion of $D_{rs}^{(1)}$

The function $D_{rs}^{(1)}$ of Eq. (A1) can be expanded,

$$D_{rs}^{(1)} = \sum_{lmn} D_{lmn}^{(1)} \psi_{lmn}(\sqrt{\rho_r} \vec{u}) \quad , \quad (A8)$$

Eq. (2.7) and Green's theorem giving

$$D_{lmn}^{(1)} = \frac{n! \rho_r^{3/2}}{\Gamma(\ell + n + 3/2)} \int d^3n \left\{ 4\pi f_r k_{rs} \right. \\ \left. + G_{rs} \left[(\Delta f_r) - f_r \Delta \right] \right\} \psi_{lmn}^* (\sqrt{\rho_r} \vec{u}) . \quad (A9)$$

The first term in Eq. (A9) is evaluated using the relations^{4,5}

$$\int_0^{2\pi} d\beta \int_0^\pi d\alpha \sin \alpha Y_{lm}^* (\alpha, \beta) Y_{\ell' m'} (\alpha, \beta) Y_{\ell'' m''} (\alpha, \beta) = Z_{[mm' m'']}^{\ell \ell' \ell''}, \quad (A10)$$

$$L_n^{\ell + \frac{1}{2}}(x) = \binom{\ell + n + \frac{1}{2}}{n} \sum_{q=0}^n (-1)^{n+q} 2^q \binom{n}{q}$$

$$\times \Phi \left(-q, \ell + \frac{3}{2}; \frac{1}{2} x \right) ,$$

after one substitutes the expansions (2.2) and (A3) for f_r and k_{rs} .

One obtains

$$\rho_r^{3/2} \int d^3u f_r k_{rs} \psi_{lmn}^* = \sum_{\substack{\ell', m', n' \\ \ell'', m'', n''}} F_{\ell' m' n'}^{(r)} K_{\ell'' m'' n''}^{(rs)} \\ \times Z_{[mm' m'']}^{\ell \ell' \ell''} \left(\ell'' + \frac{n''}{2} + \frac{1}{2} \right) \sum_{q=0}^{n''} (-1)^{n''+q} 2^{q-1} \binom{n''}{q} \\ \times Q_\lambda \left(\ell \ell' \ell'' \right)_{nn_q} , \quad (A11)$$

where Z , Q_λ and λ are defined by Eqs. (3.8), (3.12), and (3.13). To calculate the remaining part of the integral (A9) one uses the relation⁶

$$\Delta \psi_{lmn} (\sqrt{\rho_r} \vec{u}) = \rho_r \left[\rho_r u^2 - 2 \left(\ell + 2n + \frac{3}{2} \right) \right] \psi_{lmn} , \quad (A12)$$

so that

$$\begin{aligned}
[(\Delta f_r) - f_r \Delta] \psi_{\ell mn}^* &= 2\rho_r \sum_{\ell', m', n'} F_{\ell', m', n'}^{(r)} \\
&\times (\ell - \ell' + 2n - 2n') \psi_{\ell', m', n'} \psi_{\ell mn}^* . \quad (A13)
\end{aligned}$$

Substituting from Eqs. (A4), (A7), and (A13) into Eq. (A9), using Eq. (A10), and combining the result with that of Eq. (A11), one obtains

$$\begin{aligned}
D_{\ell mn}^{(1)} &= \frac{2\pi n!}{\Gamma(\ell + n + 3/2)} \sum_{\substack{\ell', m', n' \\ \ell'', m'', n''}} F_{\ell', m', n'}^{(r)} K_{\ell'', m'', n''}^{(rs)} \\
&\times Z_{\left[\begin{smallmatrix} \ell \ell' \ell'' \\ mn m'' \end{smallmatrix} \right]} \left(\ell'' + \frac{n''}{n} + \frac{1}{2} \right) \sum_{q=0}^{n''} (-1)^{n''+q} 2^q \binom{n''}{q} \\
&\times \left[Q_\lambda \left(\begin{smallmatrix} \ell \ell' \ell'' \\ nn q \end{smallmatrix} \right) + (\ell - \ell' + 2n - 2n') \left[\lambda \left(\ell'' + q + \frac{1}{2} \right) \right]^{-1} \right. \\
&\quad \left. \times Q_\lambda \left(\begin{smallmatrix} \ell \ell' \ell'' \\ nn q-1 \end{smallmatrix} \right) \right] . \quad (A14)
\end{aligned}$$

A3. Expansion of $D_{rs}^{(2)}$

The function of $D_{rs}^{(2)}$ of Eq. (A2) can be expanded,

$$D_{rs}^{(2)} = \sum_{\ell mn} D_{\ell mn}^{(2)} \psi_{\ell mn}(\sqrt{\rho_r} \vec{u}) , \quad (A15)$$

Eq. (2.7) and Green's theorem giving

$$\begin{aligned}
D_{\ell mn}^{(2)} &= \frac{n! \rho_r^{3/2}}{4\Gamma(\ell + n + 3/2)} \int d^3u \{ 8\pi f_r k_{rs} \\
&+ \bar{H}_{rs} [f_r \Delta^2 + (\Delta^2 f_r) - 2(\Delta f_r) \Delta] \} \psi_{\ell mn}^* (\sqrt{\rho_r} \vec{u}) . \quad (A16)
\end{aligned}$$

The first term in Eq. (A16) is given by Eq. (A11). Using Eq. (A12) one can show that

$$\begin{aligned}
& [f_r \Delta^2 + \Delta^2 f_r - 2(\Delta f_r) \Delta] \psi_{\ell m n}^* = 4\rho_r^2 \sum_{\ell' m' n'} \\
& \times F_{\ell' m' n'}^{(r)} [(\ell - \ell' + 2n - 2n') \psi_{\ell' m' n'} \psi_{\ell m n}^* + (n' + 1) \\
& \times \psi_{\ell' m' n'+1} \psi_{\ell m n}^* + (n + 1) \psi_{\ell' m' n'} \psi_{\ell m n+1}^* \\
& - (\ell' + n' + \frac{1}{2}) \psi_{\ell' m' n'-1} \psi_{\ell m n}^* - (\ell + n + \frac{1}{2}) \psi_{\ell' m' n'} \psi_{\ell m n-1}^*] .
\end{aligned}$$

Proceeding as in Sect. A2, one obtains

$$\begin{aligned}
D_{\ell m n}^{(2)} &= \frac{\pi n!}{\Gamma(\ell + n + 3/2)} \sum_{\substack{\ell' m' n' \\ \ell'' m'' n''}} F_{\ell' m' n'}^{(r)} K_{\ell'' m'' n''}^{(rs)} \\
&\times Z_{\ell m n}^{[\ell \ell' \ell'', m' m'']} \left(\ell'' + n'' + \frac{1}{2} \right) \sum_{q=0}^{n''} (-1)^{n''+q} 2^q \binom{n''}{q} \\
&\times \left\{ Q_{\lambda} \left(\ell \ell' \ell'', n n' q \right) - \left[\lambda^2 \left(\ell'' + q - \frac{1}{2} \right) \left(\ell'' + q - \frac{1}{2} \right) \right]^{-1} \right. \\
&\times \left[(\ell - \ell' + 2n - 2n') (\ell - \ell' + 2n - 2n' + 1) Q_{\lambda} \left(\ell \ell' \ell'', n n' q-2 \right) \right. \\
&\left. \left. + (2n' + 2) Q_{\lambda} \left(\ell \ell' \ell'', n n'+1 q-2 \right) - (2\ell + 2n + 1) Q_{\lambda} \left(\ell \ell' \ell'', n-1 n' q-2 \right) \right] \right\} . (A17)
\end{aligned}$$

A4. Collected Results

It is a straightforward exercise to collect the results of Eqs. (A8), (A14), (A15), and (A17) to obtain Eqs. (3.6)-(3.16). The $\vec{u}$ formulation given by Eqs. (3.17) and (3.18) involves only an additional transformation of the type defined by Eqs. (3.14)-(3.16) in order to obtain the final expression in the $\vec{u}$ representation.

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