JANUARY 1977

PPPL-1322

THE COUPLING OF TEARING MODES
IN TOKAMAKS

BY

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Printed in the United States of America.

Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151

Price: Printed Copy \$ * ; Microfiche \$3.00

<u>*Pages</u>	<u>NTIS Selling Price</u>
1-50	\$ 4.00
51-150	5.45
151-325	7.60
326-500	10.60
501-1000	13.60

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The Coupling of Tearing Modes in Tokamaks

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ABSTRACT

The simultaneous presence of tearing modes of different helical pitches leads to the destruction of magnetic surfaces, which has been suggested as the mechanism leading to the onset of the disruptive instability in tokamaks. For current profiles in which the $m=2$ mode is unstable, but the $m=3$ is stable, the coupling of the $m=3$ to the $m=2$ through the poloidal variation of the toroidal field can drive the $m=3$ amplitude ψ_3 to order ψ_2 times the inverse aspect ratio. Detailed calculations, both analytical and numerical, have been performed for two models for the equilibrium and $m=2$ mode structure. A slab model and incompressible $m=3$ perturbations are assumed. The $m=3$ amplitude increases with shear, up to a point, showing that as the current channel shrinks, overlap of resonances becomes more likely. Our results also apply qualitatively to other $m, m \pm 1$ interactions.

I. INTRODUCTION

The destruction of magnetic surfaces (magnetic braiding, or stochasticity of field lines) in tokamaks due to the interaction of two helical hydromagnetic modes with different pitches has been hypothesized as a cause of the disruptive instability.^{1,2} This stochasticity is usually envisioned as being due to the overlapping of resonances or magnetic islands.

There seem to be two forms of disruptions commonly observed. The so-called internal disruptions, which have been observed with soft X-ray measurements,³ are well localized within the discharge, apparently inside the $q=1$ surface. These disruptions are universally thought to be associated with some form of $m=n=1$ hydromagnetic activity. (m is the poloidal mode number. Consider the toroidal mode number n to be unity unless noted otherwise.) The possibilities include the $m=1$ internal kink,⁴ the $m=1$ tearing mode,⁵ or a newcomer, the resistive internal kink.⁶

The other type of disruption is the external or major disruption. It is characterized by a sudden increase (up to a factor of two) in minor radius, a decrease in the major radius, and a negative voltage spike in the OH circuits.⁷ (In the context of our model of disruptions, the voltage spike is due to the sudden change in inductance of the plasma column.) It is also accompanied by strong $m=2$ (and often higher mode number) oscillations.^{7,8} At this time there is still controversy over the issue of whether $m=1$ modes typically take part in the major disruptions. However, the work of Karger et al.⁹ shows that disruptions can be triggered

by currents in external $m = 2$ helical coils for otherwise stable tokamak discharges with a toroidal current between 10 and 100 kA. This seems to indicate that major disruptions can occur with the safety factor on the magnetic axis, $q(0)$, greater than 1.

But even if $m = 1$ activity is present during the disruptions, it is unlikely that interaction of the $m = 1$ and $m = 2$ modes alone could cause stochasticity of the field lines outside the $q = 1$ surface. This is because the $m = 1$ helical fields are localized within the $q = 1$ surface in the case of the tearing mode as well as the internal kink.^{4,5}

We are, therefore, led to consider stochasticity caused by the simultaneous presence of the $m = 2$ and $m = 3$ (or higher) modes. This has been done¹ with the result that unstable but nonlinearly saturated $m = 2$ and $m = 3$ modes could produce sufficient destruction of magnetic surfaces to explain the onset of the major disruptions, at least when the current profile is peaked enough to provide the necessary shear. In fact, simultaneous $m = 2$ and $m = 3$ oscillations are observed before some disruptions (see for example, Ref. 10). However, stability diagrams for the $m = 2$ and $m = 3$ tearing modes in a cylindrical tokamak¹¹ show that for realistic current profile models, there is little if any overlap in parameters for instability of the two modes. In fact, for the peaked model of Ref. 11, as the current channel shrinks, the $m = 3$ mode is stabilized before $q(0) = 2$, i.e. before the $m = 2$ mode could possibly appear. Furthermore, these results extend to the nonlinear regime of the tearing modes.¹²

Thus it appears that, within the cylindrical approximation, the $m = 2$ and $m = 3$ modes are not usually unstable simultaneously. A toroidal effect often discussed is coupling of $m = 2$ to $m = 3$ in the field line equations

$$\begin{aligned} dr/d\zeta &= RB_r/B_\zeta, \\ d\theta/d\zeta &= RB_\theta/rB_\zeta. \end{aligned} \quad (1)$$

Here r , θ , ζ are the usual toroidal coordinates (minor radius, poloidal angle, and toroidal angle) and R is the major radius. That is, if $\psi_{m,n} = -iB_r \sim B_\theta \sim \exp(im\theta + in\zeta)$ and $B_\zeta \sim (R + r \cos \theta)^{-1}$, there is an $\exp[i(m \pm 1)\theta + in\zeta]$ component in the motion of the field lines, leading to magnetic islands about the $q = (m \pm 1)/n$ surfaces, as well as at $q = m/n$.¹ This effect, however, does not lead to measurable magnetic fields varying as $\exp[i(m \pm 1)\theta + in\zeta]$. Also, in the case $m = 2$, $n = 1$, Ref. 1 shows that to produce large $m = 3$ islands, large enough to interact with the $m = 2$ islands and cause serious destruction of magnetic surfaces, would require a $m = 2$ mode of considerable strength.

But there is another toroidal effect not usually considered. This is the coupling between the m and $m \pm 1$ modes through the equation of motion of the plasma. Specifically, for $m = 2$, the term $\vec{j}_2 \times \vec{B}_\zeta$ enters the equation of motion for the $m = 3$ mode. This effect should produce islands of width $\sim (\psi_2 r/R)^{1/2}$, the same order of magnitude as the width of the islands caused by

coupling in the field line equations. In this paper we investigate the driving of a $m = 3$ mode by the above mechanism. Although we specialize the calculation to $m = 2 - m = 3$, we use a slab model and therefore the results should be qualitatively correct for any $m, m + 1$ interaction. Our calculations apply less quantitatively to $m = 1$ modes, because of their different mode structure^{4,5} and because our slab model does not include geometric effects.

As in undriven tearing modes, finite conductivity allows magnetic islands to grow. We should, therefore, expect the driven mode to grow initially on a fast (hydrodynamic) time scale but with $\psi_3 \propto B_r = 0$ at the rational surface $q = 3$. This means that the magnetic field lines will develop the $m = 3, n = 1$ kink structure, but without islands. On a longer time scale (i.e. resistive), islands are expected to grow, with fluid being forced into the X-points.

In our calculations, we assume that the $m = 2$ mode is saturated, or at least in the nonlinear phase where growth is very slow. In fact, we restrict our current profile so that the $m = 2$ mode is marginally stable ($kr_0 \sim 1$,¹⁶ where r_0 is the radius of the current channel) and thus the undriven $m = 3$ mode is stable. We also neglect the velocity associated with the $m = 2$ mode because it is small and localized about the $q = 2$ surface.¹³ The effect of the $m = 2 - m = 3$ coupling, therefore, is merely to add a time independent force to the equation of motion for the $m = 3$ mode. Note that we are neglecting the $m = 2$ mode rotation.⁸

We assume, in fact, that the $m = 3$ driven mode will lock in and rotate at the same rate while growing.²¹ Furthermore, we make the usual assumption that the $m = 3$ perturbation is incompressible, because the toroidal field is much larger than the poloidal field in tokamaks. In a sense, the aim of our calculations is to find the $m = 2$ toroidal eigenmode. However, this is true only in a rough sense, because the pure $m = 2$ mode we consider is nonlinearly saturated. In addition, we consider the nonlinear saturation of the $m = 3$ component.

As pointed out by Stix,² the phenomenon of anomalous current penetration in the early phase of a tokamak discharge may be due to the interaction of modes of the same helicity (the double tearing mode) or to stochasticity due to the presence of incommensurate modes. Therefore, we study two models for the equilibrium and $m = 2$ current. The first, which we call the skin model, has the equilibrium current peaked at the rational surface $q = 2$. It corresponds to a specific time in the early part of a discharge. The second model, which we call the generic model, has the current well developed in the center of the discharge. This requires quite a different model for the $m = 2$ component of current.

It is well known that for skin current profiles, although tearing modes of higher m numbers are unstable, they have a growth rate considerably smaller than that of the $m = 2$. Also, the higher m number modes are unstable over a narrower range of parameters¹¹ and hence strong coupling to the $m = 2$ could have a large effect. Nevertheless, we study the skin model primarily

to develop the techniques and to compare it to the generic model. The basic difference between the two models is that the quasi-equilibrium formed on a fast (hydromagnetic) time scale, with $\psi_3 = 0$ at the $q = 3$ surface (i.e. no $m = 3$ islands) is stable for the skin model, but not for the generic model. In the generic model, the vortex flow around the $q = 3$ surface has the capacity to force fluid into islands, whereas in the skin model the symmetry of the flow precludes this.

If the safety factor on axis, $q(0)$, drops below unity, $m = 1$ modes should appear. They could, by the mechanism described here, drive the $m = 2$ mode to even larger amplitude. As we have discussed, the simultaneous presence of these two modes would destroy magnetic surfaces only inside the $q = 1$ surface. But the driven $m = 2$ mode would drive the $m = 3$ mode further. The recent results of Hutchinson¹⁴ on LT-3 show that $q(0)$ is considerably less than unity immediately before the disruptions and therefore the $m = 1 - m = 2 - m = 3$ coupling may occur. However, we reiterate that our results do not apply directly to the $m = 1 - m = 2$ interaction. Rather, we take the $m = 2$ mode amplitude to be given, whether driven by the $m = 1$ mode or by its own forces, and calculate its effect upon the $m = 3$ mode.

The resistive time scale on which the islands grow is rather slow. Allowing ourselves to speculate a bit, anomalous processes near the X-points could speed up the growth of islands considerably. One such possibility is that the stochasticity of the field lines, which is initially (i.e., for small islands)

localized near the X-points,¹⁸ acts as an effective resistivity. Under this mechanism, as the islands grow, the effective resistivity increases, leading to an explosive type of growth, for both driven and undriven modes. Because of the highly speculative nature of this process, we have not invoked any such arguments in our subsequent analysis. However, the conductivity is not particularly high near the plasma edge, and thus the driven mode should grow to saturation rather rapidly if its rational surface is in this region. And the results of Ref. 7 show that the $q=3$ surface is in this region at the time of the disruptions.

II. BASIC EQUATIONS

For simplicity we use a slab model, with the variable x playing the role of the radial variable r , y the poloidal angle θ , and z the toroidal angle ζ . We model the $(R_0 + r \cos \theta)^{-1}$ dependence of the toroidal field by assuming

$$B_z = B_T (1 - \epsilon_T \cos k_y y) . \quad (2)$$

In a torus, the poloidal field also has such a dependence but in a slab this cannot be accomplished without violating $\vec{\nabla} \cdot \vec{B} = 0$.

Because of the presence of two modes, namely $m = 2, n = 1$ and $m = 3, n = 1$, the magnetic field does not possess a symmetry. Therefore, it cannot be expressed in terms of a single stream (or flux) function, as is customarily done.¹ In fact, we express the vector potential with the aid of two flux functions as

$$A_x = 0 ,$$

(continued)

$$A_y = \frac{k_z}{2k_y} \psi_2(x, 2k_y y + k_z z) + \frac{k_z}{3k_y} \psi_3(x, 3k_y y + k_z z) ,$$

(3)

$$A_z = -\psi_2(x, 2k_y y + k_z z) - \psi_3(x, 3k_y y + k_z z) .$$

The flux function ψ_2 is of the form $\psi_0(x) + \tilde{\psi}_2(x) \cos(2k_y y + k_z z)$ and represents the equilibrium (i.e. poloidal) field and the $m = 2$ driving mode. The flux function ψ_3 for the $m = 3$ mode is of the form $\tilde{\psi}_3(x) \cos(3k_y y + k_z z)$. It should be noted that the vector potential $\vec{A}$ satisfies the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$. Also, the main toroidal field (2) is not included in the vector potential (3).

We directly calculate the magnetic field, excluding the main toroidal field, to be

$$\vec{B} = \vec{e}_2 \times \vec{\nabla} \psi_2 + \vec{e}_3 \times \vec{\nabla} \psi_3 ,$$

(4)

where $\vec{e}_m = \hat{e}_z - \frac{k_z}{mk_y} \hat{e}_y$. The current is given by

$$\vec{J} = \frac{c}{4\pi} \nabla^2 \psi_2 \vec{e}_2 + \frac{c}{4\pi} \nabla^2 \psi_3 \vec{e}_3 .$$

(5)

Note that we have not included that part of the current due to the inhomogeneity in B_z , namely $(c/4\pi) \partial B_z / \partial y$, because in toroidal geometry it would not be present. As discussed in the Introduction, we assume that the $m = 3$ mode is incompressible. Because we are neglecting the $m = 2$ component of the velocity, we assume that the velocity can be written in the form

$$\vec{v} = \vec{e}_3 \times \vec{\nabla} \phi_3 ,$$

(6)

where the stream function or velocity potential ϕ_3 satisfies

$$\phi_3 = \tilde{\phi}_3(x) \sin(3k_y y + k_z z).$$

Using (4) - (6), we write the $m = 3, n = 1$ Fourier component of the linearized form of Ohm's law $\vec{E} + \vec{v} \times \vec{B}/c = \eta \vec{j}$ as

$$\frac{\partial \tilde{\psi}_3}{\partial t} - [3k_y \psi_0'(x) + k_z B_T] \tilde{\phi}_3(x) = \frac{\eta c^2}{4\pi} [\tilde{\psi}_3''(x) - 9k_y^2 \tilde{\psi}_3(x)]. \quad (7)$$

Here and below, we use the large aspect ratio assumption

$$k_z^2/k_y^2 \ll 1.$$

Next we take the curl of the linearized equation of motion to eliminate the pressure term. Also, because the singular layer around the $q=3$ surface is narrow, we assume that the density is constant. We therefore have

$$\rho \frac{\partial}{\partial t} [\tilde{\phi}_3''(x) - 9k_y^2 \tilde{\phi}_3(x)] = \nabla x (\vec{j} \times \vec{B}) \quad (8)$$

The important terms in the equation of motion are from $\vec{j}_2 \times \vec{B}_T$, $\vec{j}_3 \times \vec{B}_0$ and $\vec{j}_0 \times \vec{B}_3$. (As before, the subscripts 2 and 3 refer to the poloidal (y) mode number m , the subscript 0 refers to the equilibrium--poloidal magnetic field and toroidal current--and $\vec{B}_T$ is the main toroidal magnetic field.) The first of these four terms represents the coupling of the $m = 3$ mode to the $m = 2$ mode. Taking the $m = 3, n = 1$ Fourier component and the vector component parallel to $\vec{e}_3$ we find that (8) reduces, after some

algebra, to

$$\begin{aligned} \rho \frac{\partial}{\partial t} (\tilde{\phi}_3'' - 9k_y^2 \tilde{\phi}_3) &= \frac{3B_T \epsilon_T k_z}{16\pi} (\tilde{\psi}_2'' - 4k_y^2 \tilde{\psi}_2) \\ &- \frac{1}{4\pi} (3k_y \psi_0' + k_z B_T) (\tilde{\psi}_3'' - 9k_y^2 \tilde{\psi}_3) \\ &+ \frac{1}{4\pi} 3k_y \psi_0''' \tilde{\psi}_3. \end{aligned} \quad (9)$$

The first term on the right-hand side of (9) is the coupling term. We can see immediately that $\tilde{\psi}_3$ is of order $\epsilon_T \tilde{\psi}_2$, so that the effect we are considering is of the same order of magnitude as the coupling in the field line equations. The other terms, and the terms in Ohm's law (7), are standard in the treatment of the linear tearing mode.¹⁶ We pick the profile $\psi_0(x)$ and the wave number k_y so that the $m = 2$ mode is marginally stable, i.e. $2k_y \approx |\psi_0''' / \psi_0'|^{1/2}$.¹⁶ For these parameters, the $m = 3$ mode is stable. The coupling term thus acts as a time-independent driving force on a stable oscillator system.

We study two distinct models for the current profiles. The first, so-called skin model, is given by

$$\psi_2 = B_p k_y^{-1} \ln[\cosh 2k_y x + \epsilon_2 \cos(2k_y y + k_z z)] \quad (10)$$

Here B_p sets the scale for the poloidal field and ϵ_2 gives the relative amplitude of the $m = 2$ mode. This is in fact an exact hydromagnetic equilibrium with $m = 2$, $n = 1$ symmetry (and with

magnetic islands.)¹⁷ In fact we have

$$\begin{aligned}
 j_2 &= \frac{c}{4\pi} \nabla^2 \psi_2 \\
 &= c\pi^{-1} k_Y B_p (1 - \epsilon_2^2) \exp(-2k_Y \psi_2 / B_p) \\
 &= c\pi^{-1} k_Y B_p \cosh(2k_Y x)^{-2} \\
 &\quad - 2c\pi^{-1} \epsilon_2 k_Y B_p \cosh(2k_Y x)^{-3} \cos(2k_Y y + k_Z z).
 \end{aligned} \tag{11}$$

This is a model which describes the early phase of a tokamak discharge. The equilibrium (i.e. $m = n = 0$) current is not peaked at the center of the discharge but at the $q = 2$ rational surface, where the $m = 2$ part of the current, $\tilde{j}_2$, is also peaked. This is the "symmetric case" of Ref. 16.

Our second model, which we call the generic model, has

$$\psi_0 = B_p k_Y^{-1} (\ln \cosh 2k_Y x + b k_Y x). \tag{12}$$

The $q = 3$ rational surface is located at the point $x = x_3$ where the factor $(3k_Y \psi_0'(x) + k_Z B_T)$ present in both (7) and (9) vanishes. Thus

$$x_3 = -(2k_Y)^{-1} \tanh^{-1}(b/2 + \bar{q}/6), \tag{13}$$

where $\bar{q} = k_Z B_T / k_Y B_p$. Similarly the $q = 2$ surface is located at x_2 , where the factor $(2k_Y \psi_0'(x) + k_Z B_T)$ is zero. We find

$$x_2 = -(2k_Y)^{-1} \tanh^{-1}(b/2 + \bar{q}/4). \tag{14}$$

This equilibrium has two adjustable parameters, b and $\bar{q}$, or alternatively x_2 and x_3 . This gives us the freedom to adjust both the safety factor q and the shear.

To calculate the $m = 2$ current $\tilde{j}_2$ consistent with ψ_0 , we write $j_0 + \tilde{j}_2 = j_0(\psi_0 + \tilde{\psi}_2)$, to first order, or

$$\tilde{j}_2(x) = j_0'(x) \tilde{\psi}_2 / (\psi_0'(x) + \frac{k_z B_T}{2k_y}) \quad (15)$$

for $|k_y x| \gg \epsilon_2^{1/2}$ (i.e. outside the magnetic island) and $\tilde{j}_2$ linear in x inside the magnetic island (i.e. $|k_y x| < \epsilon_2^{1/2}$.)

Equation (15) is derived by dropping the inertia term in the $m = 2$ (undriven) analog of (9). In the range $\epsilon_2^{1/2} \ll |k_y x| \ll 1$ we can expand (15), using $\tilde{\psi}_2 \approx \tilde{\psi}_2(x_2) \equiv B_p \epsilon_2 / k_y$ to find

$$\tilde{j}_2 \approx c B_p \epsilon_2 (\bar{q} + 2b) / 4\pi (x - x_2). \quad (16)$$

We pick a model which matches the linear behavior for $|k_y x| \ll \epsilon_2^{1/2}$ and matches (16) for $|k_y x| \gg \epsilon_2^{1/2}$, namely

$$\tilde{j}_2 = c k_y^2 B_p (\bar{q} + 2b) (x - x_2) / 4\pi [1 + k_y^2 (x - x_2)^2 / \epsilon_2] \quad (17)$$

It should be noted that in the skin model the perturbed current $\tilde{j}_2$ does not change sign, whereas in the generic model it changes sign at the $q = 2$ surface.

III. SKIN MODEL

Before proceeding to solve (7) and (9), we introduce the hydromagnetic time scale τ_A by $\tau_A^2 = 4\pi\rho/k_Y^2 B_p^2$ and the dimensionless variables $\bar{t} = t/\tau_A$, $\bar{x} = k_Y x$, $\bar{\psi} = k_Y \psi_3/B_p$, $\bar{\psi}_0 = k_Y \psi_0/B_p$, $\bar{\phi} = k_Y^2 \tau_A \tilde{\phi}_3$, $\bar{j}_2 = (d^2 \tilde{\psi}_2/dx^2 - 4k_Y^2 \tilde{\psi}_2)/k_Y B_p$ and $\bar{\eta} = \eta c^2 k_Y^2 \tau_A/4\pi$. The dimensionless resistivity $\bar{\eta}$ ($=1/S$ in the notation of Ref.16. S is often called the magnetic Reynolds number) is the ratio of the hydromagnetic time scale to the resistive time scale, and is of order 10^{-5} for typical tokamak parameters. Writing (7) and (9) in terms of these variables and dropping the bars for convenience, we arrive at

$$\begin{aligned} \frac{\partial}{\partial t}(\phi'' - 9\phi) &= 0.75 \epsilon_T \bar{q} j_2(x) \\ &- (3\psi_0' + \bar{q})(\psi'' - 9\psi) \\ &+ 3\psi_0''' \psi \end{aligned} \quad (18)$$

and

$$\frac{\partial \psi}{\partial t} = (3\psi_0' + \bar{q})\phi + \bar{\eta}(\psi'' - 9\psi). \quad (19)$$

We approach this system of equations by two independent methods. The first, which was developed primarily to study the nonlinear evolution of the driven $m = 3$ mode for the generic model (Section 4), is strictly numerical. We use (18) and (19) to compute $\omega = \phi'' - 9\phi$ and ψ at each subsequent time step. This

method requires a rather large value of η , because ϕ and $j = \psi'' - 9\psi$ turn out to be localized within the singular layer. See Appendix A for details.

The second method, which is analytic in nature, and which uses η as a small parameter, differs for the two models we consider. For the skin model, we rely upon the results of the numerical method which show that $\psi(x_3) = 0$ for all t . (These results are discussed in more detail later in this section.) This means that the commonly used "constant- ψ approximation" is not valid.

We begin by Laplace transforming Eqs. (18) and (19) with respect to time and ordering $|x - x_3| \sim \eta^{1/3}$, $\partial/\partial x \sim \eta^{-1/3}$. We arrive at

$$p\phi'' = -R_1/p - \alpha z\psi'', \quad (20)$$

$$p\psi = \alpha z\phi + \eta\psi'', \quad (21)$$

where $R_1 = -0.75 \epsilon_T \bar{q} j_2(x_3)$, $z = x - x_3$ and $\alpha = 3\psi_0''(x_3)$. We have set the initial values of ϕ and ψ equal to zero, since the behavior as $t \rightarrow \infty$ of a driven mode should be independent of initial conditions. (Unlike an unstable tearing mode, our stable driven mode can saturate without the influence of nonlinear terms.) We also consider $p \leq \eta^{1/3}$, consistent with $t \rightarrow \infty$.

Now substitute ψ'' from (21) into (20) and divide by pz to find

$$\phi''/z + \alpha\psi/\eta - \alpha^2 z\phi/p\eta = -R_1/p^2 z. \quad (22)$$

The constant- ψ approximation consists in setting ψ here equal to $\psi(x_3)$, thereby obtaining a second order equation in ϕ . Instead, we take two derivatives and substitute from (20) to obtain

$$(\phi''/z)'' - p\phi''/\eta z - \alpha^2 (z\phi)''/p\eta = R_1/\eta p z - 2R_1/p^2 z^3. \quad (23)$$

Note that this equation is essentially third order because we can expand $(z\phi)''$ to give $z\phi'' + 2\phi'$. Next, substitute $\xi = z\eta^{-1/3}$, $\beta = p\eta^{-1/3}$, and $W = d\phi/d\xi$. The result is

$$\frac{d^2}{d\xi^2} \left(\frac{1}{\xi} \frac{dW}{d\xi} \right) - \frac{\beta}{\xi} \frac{dW}{d\xi} - \frac{\alpha^2}{\beta} \left(\xi \frac{dW}{d\xi} + 2W \right) = \frac{R_1}{\beta\xi} - \frac{2R_1}{\beta^2 \xi^3}. \quad (24)$$

Upon multiplying by ξ it will be seen that each term can be integrated exactly. Thus we have

$$\frac{d^2 W}{d\xi^2} - \frac{2}{\xi} \frac{dW}{d\xi} - \left(\beta + \frac{\alpha^2 \xi^2}{\beta} \right) W = \frac{R_1 \xi}{\beta} + \frac{2R_1}{\beta^2 \xi} + C, \quad (25)$$

where C is an integration constant. With the assumption $\beta \ll 2\alpha^{3/2}$, the solution to this equation is

$$\begin{aligned}
 W = & -\frac{1}{2} \xi^{3/2} I_{3/4}(\alpha \xi^2 / 2\beta^{1/2}) \int_{\xi}^{\infty} \left(\frac{\zeta R_1}{\beta} + \frac{2R_1}{\beta^2 \zeta} + C \right) \zeta^{-1/2} K_{3/4}(\alpha \zeta^2 / 2\beta^{1/2}) d\zeta \\
 & - \frac{1}{2} \xi^{3/2} K_{3/4}(\alpha \xi^2 / 2\beta^{1/2}) \int_0^{\xi} \left(\frac{\zeta R_1}{\beta} + \frac{2R_1}{\beta^2 \zeta} + C \right) \zeta^{-1/2} I_{3/4}(\alpha \zeta^2 / 2\beta^{1/2}) d\zeta .
 \end{aligned}
 \tag{26}$$

For further details see Appendix B. There we show that for $\xi \rightarrow \infty$ ($\eta^{1/3} \ll |x - x_3| \ll 1$), there is one particular solution

$$\psi_p = -R_1 p^{-1} \alpha^{-1} (x - x_3) \ln |x - x_3| \tag{27}$$

and two solutions to the homogeneous Eq. (23) which have the correct behavior as $\xi \rightarrow \infty$,

$$\psi_1 = |x - x_3| \tag{28}$$

and

$$\psi_2 = x - x_3 . \tag{29}$$

From (27) - (29), we see that $\psi(x_3) = 0$.

In the outer region $|x - x_3| \gg \eta^{1/3}$ we can neglect the resistivity and inertia terms. Therefore, (18) becomes

$$[3\psi'_0(x) + \bar{q}](\psi'' - 9\psi) - 3\psi'''_0(x)\psi = 0.75 \epsilon_T \bar{q} j_2(x)/p . \tag{30}$$

From (11) we have $j_2(x) = -8\epsilon_2 \cosh(2x)^{-3}$ for the skin model. We continue the three solutions (27) - (29) into the outer region by

integrating (30) numerically. We then pick the right linear combination of the two solutions to the homogeneous equation in order that ψ satisfy the boundary conditions $\psi(a_1) = \psi(a_2) = 0$, with $a_1 < x_3 < a_2$. The left end point a_1 corresponds to the magnetic axis, whereas a_2 corresponds to the radius of the conducting shell. In practice, a_1 and a_2 are picked to be large enough so that the final results are approximately independent of their values. (Essentially we are letting $a_1 \rightarrow -\infty$, $a_2 \rightarrow +\infty$.) In the outer region ψ should become independent of t as $t \rightarrow \infty$ [c.f., Eq. (30)], so we can invert the Laplace transform trivially:

$$\psi(x,t) = p\psi(x,p) .$$

The results, for $\epsilon_2 = 2 \times 10^{-2}$, $\eta = 10^{-5}$, $\epsilon_T = 0.33$, and $\bar{q} = 0.5, 2.0, 4.0$ are shown in Fig. 1. As mentioned, the value of ψ is approximately zero at the $q = 3$ rational surface $x = x_3$. Also $[\psi']$, the jump discontinuity in ψ' [i.e., the coefficient of (28) in ψ] is small. This indicates that ϕ is nearly an even function about x_3 . Since this requires that v_y be odd in x , islands cannot grow.

The numerical results, shown in Fig. 2, show the same behavior for ψ , and in addition show that the velocity potential ϕ is localized near the $q = 3$ surface and even about x_3 . In addition, the current j is seen to have a behavior not unlike that in (17). Although ψ appears to be independent of time in the limit as $t \rightarrow \infty$, our results show that ϕ and j increase

linearly in time in that limit. (This behavior continues as long as the nonlinear terms are negligible. A discussion of the nonlinear evolution of this mode, in the context of the generic model, is given in Sec. 4.) In order to use a fairly coarse mesh, we used $\eta = 10^{-2}$. We also performed computations with $\eta = 10^{-1}$ and 10^{-3} with the result that the limiting value of ψ is independent of η . [This is consistent with the fact that the driver is centered away from the $q = 3$ surface, and with the weak dependence upon η shown in (B5). The time dependence of ψ and j are, of course, strongly affected by the value of η .]

However, the result $\psi(x_3) \approx 0$ indicates that the effect considered here does not increase the width of the $m = 3$ islands that are present due to the coupling in the field line equations (1). Instead, the major influence is to make the existing islands highly asymmetric. Using the parameters above and Eqs. (22) and (23) of Ref. 1, we find that the original islands (i.e., caused only by coupling in the field line equations) extend to $x - x_3 = \pm 0.078$. Taking into account the coupling calculated here, we find $x - x_3 = -0.052, +0.108$. Thus, in spite of the fact that this effect does not increase the size of the $m = 3$ islands in the skin model, the asymmetry produced can influence the island overlap, by an amount of order 50%.

IV. GENERIC MODEL

The analytic treatment described in Sec. 3, when applied to the generic model, gives results similar to those described for the skin model. In particular, Fig. 3 is a typical plot of ψ against x , for $\bar{q} = 2$, $b = -2$ ($x_2 = 0.274$, $x_3 = 0.402$), $\eta = 10^{-5}$, $\epsilon_2 = 2 \times 10^{-2}$, $\epsilon_T = 0.33$. Again $\psi(x_3) \approx 0$, but contrary to the results of the skin model, $[\psi']$ is large. This difference is due to the fact that the generic model current given by (17) changes sign at x_2 , whereas the skin model current (11) is negative for all x . This means that $V_y = \partial\phi/\partial x \sin y$ has a component which is even in x ; and which could, therefore drive fluid into an island if one were to form. Thus, this quasi-equilibrium would appear to be unstable to the formation of islands.

Indeed, the results of the numerical method indicate that the saturated state found analytically using the ordering of Sec. 2 is not stable in the generic case. Consider Fig. 4, where we have $\psi_{\max}$, the maximum value of ψ for a fixed value of t , and also $\psi(x_3)$ plotted as functions of t . For $4.0 < t < 8.0$, $\psi_{\max}$ is near a minimum and $\psi(x_3)$ is small. Figure 5 shows ψ as well as ϕ and j for $t = 4.0$. For this value of t , ψ compares well with that in Fig. 3. Nevertheless, for larger values of t , ψ increases considerably. Figure 6 shows ψ , ϕ , and j for $t = 32$. Note that ϕ and j are much more localized than at $t = 4.0$, and that $\psi(x_3)$ is no longer small. In fact, we determine empirically from our numerical results:

$$\psi_{\max} \approx 5 \times 10^{-3} t^{1/4} \quad \text{for} \quad 20 < t < 40$$

$$\psi_{\max} \rightarrow 1.5 \times 10^{-2} \quad \text{as} \quad t \rightarrow \infty$$

$$\psi(x_3) \approx 0.67 \psi_{\max} . \quad (31)$$

Also, $\phi_{\max}$ increases linearly with t for large t , and the asymmetry in ϕ , which allows islands to grow, disappears as $t \rightarrow \infty$.

The main difference between our results for the two models is that $m = 3$ magnetic islands grow in the generic model, whereas in the skin model the islands are distorted considerably, but do not grow. This can be understood in terms of the $m = 2$ current models (11) and (17). Since $\tilde{j}_2$ in (17) changes sign across $x = x_2$ (and since x_3 is not far from x_2 when the shear is large), the coupling term in (18) can force fluid toward the rational surface from both sides. Since $\tilde{j}_2$ in (11) is negative for all x , it merely pushes fluid across the $q = 3$ surface toward the $q = 2$ surface.

To understand the results for large t , and to determine whether the numerical results hold for more realistic values of η , we look at the inner region equations derived from (18) and (19) with $\partial\psi/\partial t$ set equal to zero. Similar to (20) and (21), we have

$$\partial/\partial t \psi'' = -\alpha z \psi'' - R_1 - R_2 z, \quad (32)$$

$$0 = \alpha z \phi + \eta \psi'' \quad (33)$$

where $R_1 + R_2 x$ represents the first two terms in the Taylor series expansion of $-0.75 \epsilon_T \bar{q} j_2(x)$ about $x = x_3$, $\alpha = 3\psi_0''(x_3)$ and $z = x - x_3$, as before. Now integrate (32) with respect to t and substitute ψ'' from (33), neglecting the integration constant because $t \rightarrow \infty$. Making the substitution $\xi = (\alpha^2 t / \eta)^{1/4} z$, we find

$$d^2 \phi / d\xi^2 - \xi^2 \phi = -\eta^{1/2} R_1 t^{1/2} / \alpha - \eta^{3/4} R_2 t^{1/4} \xi / \alpha^{3/2}. \quad (34)$$

The even part of ϕ approaches $R_1 \eta^{1/2} t^{1/2} / \alpha \xi^2$ or $\eta R_1 / \alpha^2 z^2$ as $\xi \rightarrow \infty$. By (33) this corresponds to ψ of the same form as (27). The odd part of ϕ leads to an even particular solution for ψ with a jump in the first derivative across $x = x_3$. To estimate the value of this discontinuity, introduce $v = -\alpha^{3/2} \phi_{\text{odd}} / \eta^{3/4} R_2 t^{1/4}$. Then we find

$$d^2 v / d\xi^2 - \xi^2 v = \xi. \quad (35)$$

From (32) we deduce that the jump in ψ' is

$$\begin{aligned} [\psi'] &= \int_{-z_s}^{z_s} \psi'' dz \\ &= \frac{\eta}{\partial t} R_2 \eta^{1/4} t^{3/4} \alpha^{-3/2} \int_{\xi s}^{\xi s} d^2 v / d\xi^2 d\xi / \xi - 2R_1 z_s / \alpha - R_2 z_s^2 / \alpha \end{aligned} \quad (36a)$$

$$\approx 3\pi \eta^{1/4} \alpha^{-3/2} R_2 \Gamma(3/4) / 8 \Gamma(5/4) t^{1/4} - 2R_1 z_s / \alpha \quad (36b)$$

We have neglected the last term in (36a) because the width of the singular layer, $2z_s$, is small (ϕ is, of course, localized within the singular layer, so we let the limits of integration $\pm \xi_s = \pm \alpha^{1/2} (t/\eta)^{1/4} z_s$ go to $\pm \infty$, as usual.) The integral is performed as in the usual treatment of the tearing mode.¹⁶ We note that both terms in (36b) are of order $(\eta/t)^{1/4}$ [c.f. (31)] and thus $[\psi'] \rightarrow 0$ as $t \rightarrow \infty$. The solutions to the homogeneous equation, too, are smooth across the singular surface. As in Sec. 3, we match these solutions to the numerical solutions of the exterior region Eq. (30) and apply the boundary conditions at $x = a_1, a_2$. For $\bar{q} = 2$, $b = -2$, these results check with those of the numerical computations. Figure 7 shows the dependence of $\psi(x_2)$ upon safety factor and shear, expressed in terms of x_3 and $\delta = x_3 - x_2$, again for $\epsilon_2 = 2 \times 10^{-2}$, $\epsilon_T = 0.33$. (Note that the results are independent of η as $t \rightarrow \infty$.) The reason for plotting $\psi(x_2)$, instead of $\psi(x_3)$ or $\psi_{\max}$ for example, is that it is this quantity that determines the breakup of magnetic surfaces, which for small ψ is localized around the $q = 2$ surface.¹⁸ Or, using Eqs. (22) and (23) of Ref. 1, we find that the approximate width of the $m = 3$ islands is found by solving for x in

$$2(1 - q^{*2}/36)(x - x_3)^2 = [\psi_3(x) + 2\epsilon_T\epsilon_2/3], \quad (37)$$

where $q^* = \bar{q} + 3b$. The term $2\epsilon_T\epsilon_2/3$ is the contribution due to coupling in the field line equations. For the parameters we have been considering this is approximately 4×10^{-3} .

From Fig. 7 it is apparent that for $x_3 > 0.25$, the coupling of the $m = 3$ mode to the $m = 2$ in the force balance equation is a more important effect than the coupling in the field line equations. (In fact since $\psi(x_3) > \psi(x_2)$, in general, this conclusion is true for x_3 somewhat smaller than 0.25.) As an example, for $\epsilon_2 = 2 \times 10^{-2}$, $\bar{q} = 2$, $b = -2$, we find $m = 3$ islands as wide as 0.1, whereas the $m = 2$ islands have a width ≈ 0.15 . The largest $m = 3$ islands, with width 0.13, occur for $x_3 = 2$, $\delta = 0.1$.

Since the width of the singular layer is decreasing and the width of the islands increasing (up to a point), these quantities will become comparable after a point in time, if ϵ_2 is large enough. As in other theories,¹² this is the criterion for the nonlinear effects to become appreciable. To study the nonlinear properties of this mode, we use a quasilinear model, as discussed by Rutherford¹² and as pursued in greater detail by White et al.¹⁹ for the undriven tearing mode. We write $\psi_3 = \tilde{\psi}(x, y) + \delta\psi_0(x)$ and $\phi_3 = \tilde{\phi}(x, y) + \delta\phi_0(x)$, with $\tilde{\psi} = \psi(x) \cos(3k_y y + k_z z)$ and $\phi = \phi(x) \sin(3k_y y + k_z z)$. As in (3), the flux function for the equilibrium is contained in ψ_2 . Keeping second order terms, the D.C. component of the equation of motion is (using $k_z^2/k_y^2 \ll 1$)

$$\partial/\partial t \delta\phi_0''(x) = -[\tilde{\phi}, \nabla^2 \tilde{\phi}]_0 + [\tilde{\psi}, \nabla^2 \tilde{\psi}]_0, \quad (38)$$

where $[f, g]_0$ is the D.C. component of the Poisson bracket $k_y^{-1}(\partial f/\partial x \partial g/\partial y - \partial f/\partial y \partial g/\partial x)$. Both terms on the right hand side of (38) are zero. By applying the initial conditions

and the boundary condition $\delta\phi_0 \rightarrow 0$ as $x \rightarrow \pm \infty$, we see that $\delta\phi_0 = 0$. From Ohm's law we have

$$\partial/\partial t \delta\psi_0 + [\tilde{\phi}, \tilde{\psi}]_0 = \eta \delta\psi_0''(x). \quad (39)$$

In the saturation phase we can assume $\partial/\partial t \ll \eta/z_s^2$. We obtain

$$\delta\psi_0'' = -3(\phi\psi)'/2\eta. \quad (40)$$

We now get a quasilinear description of the driven mode by substituting $\psi_0 \rightarrow \psi_0 + \delta\psi_0$ in (18) and (19). By (40), these effects are localized in the interior region. By comparing $3\psi_0'\psi''$ with $3\delta\psi_0'''\psi$ and $3\psi_0'\phi$ with $\eta\psi''$ (we order $\psi' \sim \psi$ and $\psi'' \sim \psi/z_s$, as is appropriate when using the constant- ψ approximation) we determine that the width of the inner region $2z_s$ scales as $\psi^{1/2}$, i.e. the same order as the $m = 3$ island width. In addition, we see that ϕ is of order η . Using this ordering, we find that $|\delta\psi_0'| \ll |\psi_0'|$, so that the only important nonlinear effect is obtained by adding the term $-9(\phi\psi)''\psi/2\eta$ to the right hand side of (18). We have performed our numerical computations with this term included and the results indicate that as $t \rightarrow \infty$, ψ approaches a value nearly equal to the value obtained in the linear case, but on a somewhat slower time scale. As pointed out by Rutherford,¹² the nonlinear term above replaces the inertia term $\partial/\partial t(\phi'' - 9\phi)$ in the nonlinear phase of the mode. In order that this term dominate inertia within relatively few time steps, and since we use $\eta = 10^{-2}$ in our computations, we use a large value of ϵ_2 , namely $\epsilon_2 = 0.32$.

To extend these results to more realistic values of η and ϵ_2 , we suppose that the mode is saturated and use the constant- ψ ordering to find, similar to (32) and (33)

$$9\psi_0^2 \phi'' / 2\eta = -\alpha z \psi'' - R_1 - R_2 z, \quad (41)$$

$$0 = \alpha z \phi + \eta \psi'' \quad (42)$$

In (41), $\psi_0 \equiv \psi(x_3)$. The odd part of ϕ again gives rise to a jump discontinuity in ψ' across the singular layer. Writing $x = 3^{1/2} \psi_0^{1/2} \xi / 2^{1/4} \alpha^{1/2}$ and $\phi_{\text{odd}} = 2^{1/4} \eta R_2 v / 3^{1/2} \psi_0^{1/2} \alpha^{3/2}$, we calculate, in a manner essentially identical to that in (36),

$$[\psi'] = 3^{1/2} \psi_0^{1/2} R_2 \pi \Gamma(3/4) / 2^{5/4} \alpha^{3/2} \Gamma(5/4) \quad (43)$$

The formal method to proceed is to pick an initial guess for ψ_0 , of order ϵ_2 , use (43) to match the solutions on either side of the rational surface, and from the outer solution obtain a refined value of ψ_0 . We would then proceed until the process converged. However, the jump discontinuity (43) is small (the scaled value $[\psi']/\psi_0$, often called Δ' , is of order $\epsilon_2^{1/2}$) and hence the results are approximately the same as those of the linear analysis (within 10% for $\epsilon_2 \sim 2 \times 10^{-2}$.)

V. CONCLUSIONS

Our results provide insight between tearing modes and Petschek's model of magnetic field line merging.¹⁵ Petschek's model assumes that in the outer region inertia, but not resistivity, is important and highly localized standing hydromagnetic waves take part in the merging process. The inner, or diffusion region, where the current must be included in Ohm's law, covers a very small area around the X-point. As a result, merging rates in the Petschek model have an upper limit nearly equal to the Alfvén speed. (Purely resistive processes, with no waves attached to the diffusion region, have merging rates intermediate between Alfvén speeds and magnetic diffusion rates.) Our results for the generic model show a steepening of the velocity and current profiles. [Recall that the inner region scales as $(\eta/t)^{1/4}$.] However, the merging rate is characteristic of purely resistive models. The reason for this apparent discrepancy is that the electric field which drives the merging, $\partial\psi/\partial t$, decays to zero on the same time scale that is required to set up the steady flow. A coupling to the $m=2$ mode through Ohm's law instead of the equation of motion, however, would act as an effective externally applied electric field, and would presumably lead to flows similar to those in Petschek's model.

Summarizing, the essential attribute of the $m=2$ component of current in the generic model -- that it can force fluid from both sides of the $q=3$ surface toward the X-points -- is

responsible for its efficacy in driving a stable $m=3$ mode to a large amplitude. The coupling is strong enough that it is possible that more than two modes will be involved, especially if a $m=1$ mode is present. For example, a third mode would interact via the second order term in $\vec{j}_2 \times \vec{B}_T$ as well as by a cascade through the second mode.

As shown in Fig. 7, the amplitude of the driven mode increases with increasing shear, up to a point (shear $\sim 1/\delta$) and increases as its rational surface moves out. Thus, as the current channel peaks, the driven modes should steadily increase in amplitude. This substantiates the observation made in Ref. 1, namely that the destruction of magnetic surfaces becomes a more dominant process as the current channel peaks.

ACKNOWLEDGMENTS

I wish to thank J. M. Greene, P. K. Kaw, D. A. Monticello, and C. Oberman for valuable discussions.

This work was supported by U. S. Energy Research and Development Administration Contract E(11-1)-3073.

APPENDIX A

NUMERICAL METHOD

We first write $\psi_{n,m} = \psi(n\Delta x, m\Delta t)$, and similarly for ϕ , j and ω . The simplest finite difference scheme we can propose to solve (18) and (19) is

$$\omega_{n,m+1} = \omega_{n,m} + \Delta t(-f_{n,m} j_{n,m} + g_{n,m} \psi_{n,m} + h_{n,m}) \quad (A1)$$

$$\equiv A_{n,m} ,$$

$$\psi_{n,m+1} = \psi_{n,m} + \Delta t(f_{n,m} \phi_{n,m} + n j_{n,m})$$

$$(A2)$$

$$\equiv B_{n,m} .$$

Here, f, g and h represent quantized forms of $3\psi_0' + \bar{q}$, ψ_0''' and $0.75\epsilon_T \bar{q} j_2$, respectively. From ω we compute ϕ by numerical integration of the Green's function relation $\phi(x) = \int_{a_1}^{a_2} G(x, x') \omega(x') dx'$. The function G , the Green's function for the operator $\phi'' - 9\phi$, vanishes at the endpoints a_1 and a_2 . The current is obtained by finite differencing ψ , i.e. $j_{n,m} = (\psi_{n+1} + \psi_{n-1} - 11\psi_n)/\Delta x^2$.

We also iterate Eqs. (A1), (A2) at each time step. Specifically, writing a superscript (k) for the k th iterate at $t = n\Delta t$, we compute

$$\omega_{n,m+1}^{(k+1)} = \bar{A}_{n,m}^{(k)} , \quad (A3)$$

$$\psi_{n,m+1}^{(k+1)} = \bar{B}_{n,m}^{(k)} , \quad (A4)$$

where $A_{n,m}^{-(k)} = (A_{n,m+1}^{(k)} + A_{n,m}^{(k)})/2$ and similarly for $\bar{B}_{n,m}$. This procedure improves the accuracy and stability of the numerical scheme, especially when nonlinear terms are added.

Finally, we should mention that shear Alfvén waves are permitted by Eqs. (18) and (19). Thus we expect a stability criterion $\Delta t \lesssim \Delta x/v_A \approx 0.12$. The quantity $v_A = (B_p^2/4\pi\rho)^{1/2}$ is the poloidal Alfvén speed. Empirically this turns out to be correct and hence we use $\Delta t = 0.08$ for all our computations.

APPENDIX B
SKIN MODEL -- INNER REGION

If we make the substitution $Z = W/\xi$, the left hand side of (25) becomes

$$Z'' - (2/\xi^2 + \beta + \alpha^2 \xi^2/\beta)Z = 0. \quad (B1)$$

This clearly shows that for $\beta \ll 2\alpha^{2/3}$ ($=7.1$ for $\bar{q} = 4$), the middle term in the coefficient of Z is negligible. Therefore, the solutions to the homogeneous equation can be written in terms of the Bessel functions $I_{3/4}$, $K_{3/4}$:

$$W_1 = \xi^{3/2} I_{3/4}(\alpha \xi^2/2\beta^{1/2}), \quad (B2)$$

$$W_2 = \xi^{3/2} K_{3/4}(\alpha \xi^2/2\beta^{1/2}). \quad (B3)$$

Both solutions are regular as $\xi \rightarrow 0$; in fact, we have $W_1 \sim \xi^3$, $W_2 \sim 1$.

The Wronskian of W_1 and W_2 is $-2\xi^2$, and therefore variation of parameters leads to the particular solution W given by (26). The odd part of W is obtained by setting $C = 0$, and approaches $-R_1/\alpha^2 \xi$ as $\xi \rightarrow \infty$. This corresponds to the function ψ_p given by (27). The even part of W is obtained by setting $R_1 = 0$. [This is a solution to the homogeneous fourth order Eq. (23), but is considered here as a part of the particular solution of (25).] As $\xi \rightarrow \infty$, we have $W_{\text{even}} \rightarrow -C\beta/\alpha^2 \xi^2$. The two homogeneous solutions W_1 and W_2 cannot be used -- W_1 because it blows up like

$\exp(\alpha\xi^2/2\beta^{1/2})$ as $\xi \rightarrow \pm\infty$ and W_2 , which behaves as $\exp(-\alpha\xi^2/2\beta^{1/2})$ as $\xi \rightarrow \pm\infty$, because it cannot contribute in the matching.

To W_{even} correspond the functions ϕ_{odd} and ψ_{even} , the first of which is given by

$$\phi_{\text{odd}}(\xi) = \int_0^\xi W_{\text{even}}(\xi) d\xi . \quad (\text{B4})$$

As $\xi \rightarrow +\infty$, ϕ_{odd} approaches $\phi_{\text{odd}}^{(\infty)} + C\beta/\alpha^2\xi$, and thus (21) shows

$$\psi_{\text{even}} \sim \alpha\phi_{\text{odd}}^{(\infty)}|x|/p + C/\alpha . \quad (\text{B5})$$

Since $\phi_{\text{odd}}^{(\infty)}$ is of order $C\beta^{3/4} \approx C$, the first term in ψ_{even} dominates the second, and we obtain one homogeneous solution (28).

The second solution to the homogeneous equation in (23) is

$\phi = \text{const.}$ This leads directly to ψ_2 , given by (29).

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Fig. 1. Results of the analytical method for the skin model ψ as a function of x for $\varepsilon_2 = 0.02$. The locations of the $q = 3$ surfaces for $\bar{q} = 4, 2$, and 0.5 are marked on the x -axis.

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 Fig. 2. Typical results of the numerical method for the skin model. ψ (————), j (-----), and ϕ (— · — · —) are scaled to fit on the same graph, but $\psi_{\max} = 6.2 \times 10^{-3}$, $j_{\max} = 1.05 \times 10^{-1}$, and $\phi_{\max} = 7.7 \times 10^{-3}$. Parameters are $\epsilon_2 = 0.02$, $\eta = 0.01$, $t = 32$, and $\bar{q} = 4$. The rational surface is located at $x = -0.402$.

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Fig. 3. Results obtained by using the analytical method of Sec. 3 with the generic model. Parameters are $\epsilon_2 = 0.02$, $\bar{q} = 2$, $b = -2$. Because of the form of the driving term, $[\psi']$ is much larger than in Fig. 1.

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 Fig. 4. $\psi_{\max}$ and $\psi(x_3)$ as functions of time. Whereas $\psi(x_3)$ rises steadily, $\psi_{\max}$ has a minimum near $t = 5$. Same parameters as in Fig. 3.

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Fig. 5. Numerical results with parameters equal to those in Fig. 3, and with $\eta = 10^{-2}$, $t = 4$. ψ (—), j (-----), and ϕ (-·-·-) are scaled as in Fig. 1, with $\psi_{\max} = 4.2 \times 10^{-3}$, $j_{\max} = 4.8 \times 10^{-2}$, and $\phi_{\max} = 1.41 \times 10^{-3}$. Note the similarity of ψ here to that in Fig. 3.

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Fig. 6. Same as Fig. 5, for $t = 32$. $\psi_{\max} = 1.13 \times 10^{-2}$,
 $j_{\max} = 1.18 \times 10^{-1}$, $\phi_{\max} = 7.4 \times 10^{-3}$.

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Fig. 7. Results of the analytical method for the generic model, using Eq. (36) for $t \rightarrow \infty$. $\psi_3(x_2)$ is shown as a function of $\delta = x_3 - x_2$, a shear parameter, for various values of x_3 , and for $\varepsilon_2 = 0.02$. The results of the nonlinear analysis, which uses (43), give slightly smaller values of ψ .