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Trapped Particles and Harmonic Generation

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TRAPPED PARTICLES AND HARMONIC GENERATION

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Abstract

A novel fluid-particle algorithm has been used to study the evolution of continuously generated ion acoustic waves numerically. The code permits the simultaneous analysis of the two nonlinearities involved: wave steepening or harmonic generation, a fluid effect; and wave damping and amplitude oscillations, a particle effect. For a temperature ratio $10 < T_e/T_i < 30$, the behavior of low amplitude long wavelength waves is well described by a Korteweg-deVries equation augmented to include a time dependent damping term.

I. INTRODUCTION

During the last decade, there has been extensive examination of nonlinearities in the ion acoustic system. For a uniform field-free plasma of warm electrons and cold ions a fluid model may be applied. In the weakly nonlinear limit the fluid equations reduce to the Korteweg-deVries equation.¹ In the laboratory frame this equation is written

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial x} + \frac{1}{2} \frac{\partial^3 \phi}{\partial x^3} = 0. \quad (1)$$

The independent variables x and t , displacement and time, are scaled to the electron Debye length λ_{De} and the inverse ion plasma frequency ω_{pi}^{-1} respectively. The state variable ϕ , the potential, is scaled to T_e/e , the electron temperature in volts.

Equation (1) is nonlinear through the convective term $\phi(\partial\phi/\partial x)$ which leads to the generation of harmonic content or wave steepening. This is limited by the dispersive term $\partial^3\phi/\partial x^3$. In an infinite system the balance of these terms results in a given disturbance evolving into a series of pulses which are separated by arbitrarily large distances. These pulses are called solitary waves or solitons. The periodic system also exhibits similar pulses with an additional quality. Once the system attains its maximum harmonic content, it will appear to reverse in time, to "de-steepen", and to reproduce the initial condition.²

Ion acoustic waves also exhibit nonlinear amplitude oscillations due to resonant particles. The analysis of this effect³ was originally made for electron plasma waves, but Sato *et al.*⁴

have experimentally verified that it also applies to finite amplitude ion acoustic waves. For larger amplitude waves, the trapped particles are more appropriately called reflected particles. Ion reflection from a steepening wave front performs an important role in various plasma heating schemes.⁵

For high temperature ratios ($T_e/T_i \geq 30$) the effect of the trapped particles becomes negligible. Likewise, for short wavelengths (as $k \rightarrow \lambda_{De}^{-1}$ and $\omega \rightarrow \omega_{pi}$) the effects of harmonic generation may be ignored. In general, however, these two nonlinearities act together to govern the evolution of the ion acoustic system.

In this article we report on our numerical studies of continuously generated ion acoustic waves where both fluid and kinetic nonlinearities are considered. Our analysis was made possible by the development of the new hybrid computer algorithm which we will describe in the next section. We will then describe our results through a model problem. We will consider a situation which is well described by two Fourier modes. Finally, our results will be interpreted in terms of a mathematical model.

II. THE FLUID-PARTICLE ALGORITHM

Several hybrid codes have been written which treat one species as a fluid and the other as an ensemble of particles. For example, Mason⁶ used a code with particle ions and isothermal fluid electrons to study kinetic effects in the ion acoustic system. In this section we describe an algorithm which differs markedly from previous codes. It represents a single species

as a composite of fluid and particles.

The basic properties of ion acoustic waves are well described by a one dimensional, two-fluid model. As a basis for our algorithm we begin with this model. It is summarized by the fluid equations for the ions,

$$\frac{\partial}{\partial t} \begin{bmatrix} n \\ v \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} nv \\ v^2/2 + \phi + 3 \left(\frac{T_i}{T_e} \right) n \phi \end{bmatrix} = 0, \quad (2)$$

and Poisson's equation with Maxwell-Boltzman electrons,

$$\frac{\partial^2 \phi}{\partial x^2} = \exp(\phi) - n. \quad (3)$$

In these equations n is the density scaled to the equilibrium density n_0 , v is the velocity scaled to the ion acoustic velocity c_s , and T_i and T_e are the ion and electron temperatures, respectively.

Because ion kinetic effects are neglected, these fluid equations are not valid unless the ion-electron temperature ratio is small. However, as long as ions are resonant with the wave only in the tail of the ion distribution, we can divide up the ion distribution into nonresonant (thermal) and resonant (suprathermal) ions. We use fluid equations to follow the dynamics of the thermal ions, and the Vlasov equation to follow the dynamics of the suprathermal ions. Rather than explicitly solving the Vlasov equation, we represent the suprathermal particles as macroparticles. This particle portion is represented mathematically as follows,

$$f(x, v) = \sum_p S_p \delta(x - x_p) \delta(v - v_p). \quad (4)$$

Here, δ is the Dirac delta function and S_G is a rectangular impulse function, the cloud representation of a macroparticle; x_p and v_p are the position and velocity of a particle, which evolve according to Lagrangian equations of motion,

$$\frac{dx_p}{dt} = v_p, \quad \frac{dv_p}{dt} = - \frac{\partial \phi(x_p)}{\partial x} \quad (5)$$

The usual cloud-in-cell algorithm⁷ may be used to calculate the dynamics of the particles.

The two parts of the distribution combine to give the ion charge in a hybrid Poisson's equation,

$$\frac{\partial^2 \phi}{\partial x^2} = \exp(\phi) - p_f n - p_p \int_{-\infty}^{\infty} f dv \quad (6)$$

Here, p_f and p_p are the weights of the fluid and particle portions of the distribution, respectively ($p_f + p_p = 1$).

A typical initial ion distribution is depicted in Fig. 1. The Gaussian distribution is dissected horizontally into fluid and particle components. The thermal part is dominated by the fluid portion. The upper bound in velocity for this fluid component is v_h , which we call the hybrid velocity parameter. It is used for determining the weights of the portions of the distribution and for analyzing other properties of the algorithm.⁸

Since we close the fluid equations using an adiabatic equation of state, v_h is chosen to be greater than the thermal velocity, v_t . To include the dominant wave-particle effects, v_h is chosen

to be less than the lower bound of the resonance region of the wave, $v_{ph} - (\phi_{max} - \phi_{min})^{1/2}$, where v_{ph} is the phase velocity of the wave. Together these constraints form a criterion for the application of our algorithm,

$$v_t < v_h < v_{ph} - (\phi_{max} - \phi_{min})^{1/2} \quad (7)$$

This criterion is often satisfied by ion acoustic waves. The phase velocity of long wavelength ion acoustic waves is near the ion acoustic velocity and the ratio of this to the ion thermal velocity v_{ti} is equal to the square-root of the electron-ion temperature ratio,

$$c_s/v_{ti} = (T_e/T_i)^{1/2} \quad (8)$$

We note that this juxtaposition of fluid and particle models implies the following assumptions: (1) Use of the fluid equations for the central portion of the distribution requires that there be some mechanism, such as collisions, to maintain the form of the distribution there; (2) Use of the particle representation in the resonance region requires that the distribution there be collisionless.

III. THE BEHAVIOR OF SMALL AMPLITUDE DISTURBANCES

A typical result for a small-amplitude ion acoustic wave is summarized in Fig. 2. The initial condition described a monochromatic 4% density wave with a wavelength of $32 \lambda_{De}$. The electron-ion temperature ratio was 16 and the hybrid velocity parameter

was $2.5 v_{ti}$. Throughout the calculation the system was well described by the first two Fourier modes. The third mode had a maximum amplitude of 0.5%.

As in the O'Neil theory,³ the trapped particles are observed to circulate within the separatrix thereby influencing the wave amplitude. The shape of the separatrix evolves slowly, since the wave is no longer monochromatic. Using the initial values of the bounce frequency and the Landau damping coefficient, we can read the monochromatic amplitude oscillations from a graph presented by Morales and O'Neil.⁹ This is included as the dashed line in the figure. We note that near the points where the second mode has a minimum, the first mode approaches the single wave value. These times at which the system returns to being nearly monochromatic are called recurrences. The wave does not return to its initial amplitude since some of its energy has been absorbed by the particles. After the second recurrence at $\omega_{pi} \approx 700$, the ions have phase-mixed to form a velocity plateau in the vicinity of the ion acoustic velocity. After this the first and second modes continue to exchange energy while the energy content of the particles remains approximately constant.

It was generally believed that nonlinearities constituted the first step in the inexorable increase in entropy. Fermi et al.¹⁰ first attempted to numerically study nonlinearly induced thermalization. They found that their systems did not thermalize and observed the recurrences. In our simulations we see that even when kinetic effects are included, the path to maximum entropy may still include a way station.

IV. THE AUGMENTED KORTWEG-DE VRIES EQUATION

We will now review our assumptions. We have assumed the electrons to be isothermal and that their mass effects are negligible. This requires that the electrons be Maxwellian and that the disturbances have small amplitudes. We wish to apply the Korteweg-de Vries equation and this requires that the waves be weakly nonlinear. All Fourier components which contribute strongly to the shape of the wave must have wavelengths longer than $\sim 10 \lambda_{De}$. This emphasizes the requirement for small amplitude waves ($\delta n/n_0 \approx$ a few percent). In our simulations we have found that the ion acoustic system has a strong propensity to steepen. Given sufficient amplitude very short-wavelength high-frequency components will be driven by the process of harmonic generation. We have assumed that the ion acoustic velocity is somewhat greater than the ion thermal velocity, so that the waves are weakly Landau-damped. This requires that the electron-ion temperature ratio be bounded below ($T_e/T_i \geq 10$). For a small amplitude monochromatic wave, this is a situation to which the O'Neil analysis³ applies. The application of his theory requires that the damping induce changes adiabatically in the potential. This means that the damping only induces a negligible number of particles to be trapped and detrapped.

In addition to damping, in the ion acoustic system we must also consider harmonic generation. For weakly nonlinear ion acoustic waves, this is a very slow process and the potential variations remain adiabatic. For a monochromatic wave the amplitude variation

is expressed as follows, $d\phi/dt = \gamma(t)\phi$, where γ is the time dependent damping coefficient.³ To account for the harmonic generation and the dispersion in the ion acoustic system, we assume that $d\phi/dt$ is expressed as the left-hand side of Eq. (1). This is our augmented Korteweg-deVries equation,

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial x} + \frac{1}{2} \frac{\partial^3 \phi}{\partial x^3} = \gamma(t) \phi. \quad (9)$$

Tappert and Judice¹¹ have studied the recurrence phenomena associated with this equation when there are no resonant particles; i.e. $\gamma = 0$. They note that when the system is well described by two Fourier modes, the recurrence time may be obtained on the basis of mode coupling arguments. We will extend their analysis for this case to include the nonlinear damping term.

The potential is assumed to be well described by a Fourier mode and its first harmonic,

$$\begin{aligned} \phi(x,t) = & \rho_1(t) \exp\{i[kx - \omega_1 t - \theta_1(t)]\} \\ & + \rho_2(t) \exp\{i[2kx - \omega_2 t - \theta_2(t)]\} + \text{c.c.} \end{aligned} \quad (10)$$

Here ρ_1 and ρ_2 are the amplitudes of the two modes; ω_1 and ω_2 are the frequencies which correspond to the wave numbers k and $2k$ respectively; and θ_1 and θ_2 are time-dependent phase angles. For this equation to be justified the mode amplitudes must satisfy the inequalities, $\rho_2 < \rho_1 \ll 1$, or else the third mode must be considered.

Substituting Eq. (10) into (9) and letting $\rho_1 = \rho_{10}$ and $\rho_2 = 0$ at time $t = 0$, we find a relation for the mode amplitudes

$$\rho_1^2 + \rho_2^2 = (\rho_{10}^2 / \Gamma), \quad (11)$$

where $\Gamma = \exp(-\int_0^t \gamma dt')$. This is a statement of constant energy which is consistent with our observations of the previous section.

After some manipulation (the procedure parallels that of Watanabe¹²), we find that the amplitude of the second mode satisfies a first order ordinary differential equation,

$$\frac{d(\Gamma \rho_2)}{dt} = \pm k \left\{ \left[\frac{\rho_{10}^2 - (\Gamma \rho_2)^2}{\Gamma} \right]^2 - \left(\frac{\Delta\omega}{2k_1} \right)^2 (\Gamma \rho_2)^2 \right\}^{1/2} \quad (12)$$

where $\Delta\omega = 2\omega_1 - \omega_2$. The sign of the right hand side is that of the sine of the relative phase angle between the two modes, $\sin[2\theta_1(t) - \theta_2(t) + \Delta\omega t]$. When Γ assumes its asymptotic value Γ_∞ , the solution for ρ_2 is a Jacobian elliptic function. The solution is then the same as that for a wave with an initial amplitude of ρ_{10}/Γ_∞ when damping is neglected. A numerical solution of Eq. (11) and (12) is shown in Fig. 3 [we have used an approximate function for $\Gamma(t)$]. The close agreement between these results and the hybrid results (Fig. 2) should be noted. The time interval shown in Fig. 3 corresponds to $t\omega_{pi} \approx 1050$, somewhat less than the $t\omega_{pi} = 1250$ value of Fig. 2. The discrepancy is partially explained by noting the the Korteweg-deVries equation which we have used assumes cold ions. Tappert¹³ has derived an improved Korteweg-deVries equation which includes

temperature fluid effects. Using it Tappert and Judice¹¹ have obtained formulae for the recurrence time of ion acoustic waves. Substituting the parameters of our model problem into their small-amplitude (two-mode) formula, we find a recurrence period equal to $276\omega_{pi}^{-1}$ for cold ions and $302\omega_{pi}^{-1}$ for ions with $T_i/T_e = 16$. This lengthening of the recurrence period brings the time intervals of the two figures into more satisfactory correspondence. Further improvement is expected if we take into account the long-wavelength expansion used in the Korteweg-deVries equation.¹¹

V. SUMMARY

With our fluid-particle algorithm we were able to successfully study recurrence phenomena with kinetic effects. The recurrence of ion acoustic waves had been studied numerically by Tappert and Judice¹¹ and had been experimentally observed by Ikezi.¹⁴ We considered a situation which corresponds approximately to Ikezi's laboratory conditions where two Fourier modes dominate the system. We observed the circulation of the trapped ions within the separatrix and noted that the level of our fundamental waves at the recurrences was approximately equal to that of the O'Neil solution.³ This led us to assume that an augmented Korteweg-deVries equation could be used to model our system in restricted situations. The solution of this equation compared satisfactorily with our simulation results.

We have performed numerical computations involving larger amplitude waves of longer wavelengths. Our results are consistent with the theory of the previous section when the latter is

applicable. Trapped particles take energy from the wave, but the harmonic generation continues.

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Fig. 1. The initial fluid-particle distribution. The upper bound of the fluid portion is v_h , the hybrid velocity parameter.

Fig. 2. The evolution of a small amplitude ion acoustic wave. The initial condition is a monochromatic density wave $\delta n/n_0 = 0.04 \sin(\pi x/15 \lambda_{De})$. The dashed line is the O'Neil solution for a single Fourier mode.

Fig. 3. Solution of the augmented Korteweg-deVries equation. The constraints are the same as those for Fig. 2. The dashed line is ρ_{10}/I , the mock solution for a single Fourier mode.

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Figure 1.

Figure 2

Figure 3