RELATIVISTIC KINEMATICS

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This memo is an expansion of an informal memo prepared for the Table on Relativistic Kinematics first appearing in the 1974 edition of the "Review of Particle Properties", Particle Data Group, Phys. Lett. 50B, April 1974. We demonstrate the formulae that appeared in the Table on pages 20-23. New formulae are presented as well. Finally, an updated Table is given.

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I. NOTATION

We define any 4-vector p by its components:

$$p \equiv (p_0, \vec{p}) \quad \text{with} \quad \begin{cases} p_0 = \text{the energy } E \\ \vec{p} = (p_{\parallel}, \vec{p}_{\perp}) = \text{3-vector of} \\ \text{magnitude } P \end{cases} \quad (\text{I-1})$$

where $p_{\parallel}$ is the longitudinal momentum along a beam direction, and $\vec{p}_{\perp}$ the transverse momentum (with two components along two axes perpendicular to the beam), respectively. Then in the laboratory frame

$$p^{\text{lab}} = (E^{\text{lab}}, p_{\parallel}^{\text{lab}}, \vec{p}_{\perp}^{\text{lab}}), \quad (\text{I-2})$$

and in the general center-of-mass frame

$$p^{\text{cm}} = (E^{\text{cm}}, p_{\parallel}^{\text{cm}}, \vec{p}_{\perp}^{\text{cm}}). \quad (\text{I-3})$$

The 4-vector scalar products are defined by

$$A \cdot B = A_0 \cdot B_0 - \vec{A} \cdot \vec{B}, \quad (\text{I-4})$$

so that

$$p^{\text{lab}} \cdot p^{\text{lab}} = p^{\text{cm}} \cdot p^{\text{cm}} = m^2. \quad (\text{I-5})$$

II. GENERAL INVARIANT DEFINITIONS AND RELATED FORMULAE

For two particles i and j , the most commonly used invariants are defined as follows:

A. The invariant mass squared of the combination of particles i and j is

$$s_{ij} \equiv m_{ij}^2 \equiv (p_i + p_j)^2 = m_i^2 + m_j^2 + 2(E_i E_j - \vec{p}_i \cdot \vec{p}_j). \quad (\text{II-1})$$

We will use the function $\Delta(x, y, z)$ [also called $\lambda(x, y, z)$], defined by:

$$\begin{aligned} \Delta(x, y, z) &\equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \\ &= \left[x - (\sqrt{y} + \sqrt{z})^2 \right] \left[x - (\sqrt{y} - \sqrt{z})^2 \right] \end{aligned} \quad (\text{II-2})$$

A. 1. In the particle j rest frame ($\vec{P}_j = 0$), the longitudinal axis being parallel to $\vec{P}_i$, equation (II-1) gives

$$s_{ij} = m_i^2 + m_j^2 + 2E_i^{(j)}m_j, \quad (II-3)$$

so that

$$E_i^{(j)} = \frac{s_{ij} - m_i^2 - m_j^2}{2m_j} \quad (II-4)$$

and

$$P_i^{(j)} = \sqrt{E_i^{(j)2} - m_i^2} = \frac{1}{2m_j} \sqrt{(s_{ij} - (m_i + m_j)^2)(s_{ij} - (m_i - m_j)^2)}. \quad (II-5)$$

Consequently

$$P_i^{(j)} = \left(\frac{s_{ij} - m_i^2 - m_j^2}{2m_j}, \frac{\sqrt{\Delta(s_{ij}, m_i^2, m_j^2)}}{2m_j}, 0 \right). \quad (II-6)$$

while, by definition,

$$P_j^{(j)} = (m_j, 0, 0)$$

A. 2. In the c. m. frame for particles $(i \text{ and } j)$ ($\vec{P}_j = -\vec{P}_i$), the longitudinal axis being along $\vec{P}_i$, equation (II-1) gives

$$s_{ij} = m_i^2 + m_j^2 + 2(E_i^{(ij)}E_j^{(ij)} + P_i^{(ij)2}), \quad (II-7)$$

and

$$\left(\frac{s_{ij} - m_i^2 - m_j^2}{2} - P_i^{(ij)2} \right) = E_i^{(ij)}E_j^{(ij)} = \sqrt{P_i^{(ij)2} + m_i^2} \sqrt{P_i^{(ij)2} + m_j^2}$$

After squaring both sides and simplifying, one gets

$$\underbrace{\left(\frac{s_{ij} - m_i^2 - m_j^2}{2} \right)^2 - m_i^2 m_j^2}_{\frac{1}{4} \Delta(s_{ij}, m_i^2, m_j^2)} = P_i^{(ij)2} \cdot s_{ij},$$

and finally,

$$\left\{ \begin{aligned} P_i^{(ij)} &= \frac{\sqrt{\Delta(s_{ij}, m_i^2, m_j^2)}}{2\sqrt{s_{ij}}} = -P_j^{(ij)} \end{aligned} \right. \quad (II-8)$$

$$\left\{ \begin{aligned} E_i^{(ij)} &= \sqrt{P_i^{(ij)2} + m_i^2} = \frac{s_{ij} + m_i^2 - m_j^2}{2\sqrt{s_{ij}}} \end{aligned} \right. \quad (II-9)$$

$$\left\{ \begin{aligned} E_j^{(ij)} &= \sqrt{P_j^{(ij)2} + m_j^2} = \frac{s_{ij} + m_j^2 - m_i^2}{2\sqrt{s_{ij}}} \end{aligned} \right. \quad (II-10)$$

So that

$$\left\{ \begin{aligned} p_i^{(ij)} &= \left(\frac{s_{ij} + m_i^2 - m_j^2}{2\sqrt{s_{ij}}}, \frac{\sqrt{\Delta(s_{ij}, m_i^2, m_j^2)}}{2\sqrt{s_{ij}}}, 0 \right) \\ p_j^{(ij)} &= \left(\frac{s_{ij} + m_j^2 - m_i^2}{2\sqrt{s_{ij}}}, -\frac{\sqrt{\Delta(s_{ij}, m_j^2, m_i^2)}}{2\sqrt{s_{ij}}}, 0 \right) \end{aligned} \right\} \quad (II-11)$$

and

$$[p_i^{(ij)} + p_j^{(ij)}] = (\sqrt{s_{ij}}, 0, 0) \quad (II-12)$$

Note that

$$p_i^{(ij)} = \frac{m_j}{\sqrt{s_{ij}}} p_i^{(j)} \quad (II-13)$$

A. 3. Applications: Useful relations involving s_{ij} :

- i) In a Dalitz plot for the 3-body decay, $X \rightarrow 1 + 2 + 3$

From

$$p_X^2 = (p_1 + p_2 + p_3)^2 = m_1^2 + m_2^2 + m_3^2 + 2p_1 p_2 + 2p_1 p_3 + 2p_2 p_3$$

one deduces: (with $2p_i p_j = (p_i + p_j)^2 - m_i^2 - m_j^2$)

$$M_X^2 = s_{12} + s_{13} + s_{23} - m_1^2 - m_2^2 - m_3^2,$$

and

$$\sum_{\substack{i=1, 2, 3 \\ j=1, 2, 3 \\ i < j}} s_{ij} = m_{123}^2 + \sum_{i=1, 2, 3} m_i^2 \quad (\text{II-14})$$

ii) In a triangle plot for the 4-body decay, $X \rightarrow 1 + 2 + 3 + 4$

$$\begin{aligned} \text{From } p_X^2 &= (p_1 + p_2 + p_3 + p_4)^2 \\ &= -2 \left(m_1^2 + m_2^2 + m_3^2 + m_4^2 \right) + s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + s_{34} \end{aligned}$$

one gets

$$\sum_{\substack{i=1 \dots 4 \\ j=1 \dots 4 \\ i < j}} s_{ij} = \sum m_{ij}^2 = m_{1234}^2 + 2 \sum_{i=1 \dots 4} m_i^2 \quad (\text{II-15})$$

B. The invariant momentum transfer squared between particles i and k is

$$\begin{aligned} t_{ik} &= (p_i - p_k)^2, \\ t_{ik} &= p_i^2 + p_k^2 - 2p_i p_k = m_i^2 + m_k^2 - 2(E_i E_k - \vec{P}_i \cdot \vec{P}_k). \quad (\text{II-16}) \end{aligned}$$

The minimum value of t_{ik} is obtained when $\vec{P}_i \cdot \vec{P}_k = P_i P_k$ (i.e., particles i and k are collinear).

$$\text{Then } t_{ik}^{\min} = (E_i - E_k)^2 - (P_i - P_k)^2. \quad (\text{II-17})$$

In a similar way $t_{ik}^{\max}$ is defined when i and k are such that $\vec{P}_i \cdot \vec{P}_k = -P_i P_k$

$$t_{ik}^{\max} = (E_i - E_k)^2 - (P_i + P_k)^2. \quad (\text{II-18})$$

The 4-momentum transfer is then

$$t_{ik} = t_{ik}^{\min} + 2(\vec{P}_i \cdot \vec{P}_k - P_i P_k) = t_{ik}^{\min} - 4P_i P_k \sin^2 \frac{\theta_{ik}}{2} \quad (\text{II-19})$$

(θ_{ik} being the angle between $\vec{P}_i$ and $\vec{P}_k$).

Application to the reaction $a + b \rightarrow c + d$:

1. General formulae:

$$t_{ac}^{\min} = (E_a - E_c)^2 - (P_a \mp P_c)^2 \quad (\text{II-20})$$

In the (ab) rest frame

$$p_a^{(ab)} = \left(\frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \frac{\sqrt{\Delta(s, m_a^2, m_b^2)}}{2\sqrt{s}}, 0 \right).$$

In the (cd) rest frame

$$p_c^{(cd)} = \left(\frac{s + m_c^2 - m_d^2}{2\sqrt{s}}, \frac{\sqrt{\Delta(s, m_c^2, m_d^2)}}{2\sqrt{s}}, 0 \right),$$

$$t_{ac}^{\min} = \left[\frac{(m_a^2 - m_b^2) - (m_c^2 - m_d^2)}{2\sqrt{s}} \right]^2 - \left[\frac{\sqrt{\Delta(s, m_a^2, m_b^2)} \mp \sqrt{\Delta(s, m_c^2, m_d^2)}}{2\sqrt{s}} \right]^2 \quad (\text{II-21})$$

which, after expansion in powers of $(1/s)$, gives:

$$t_{ac}^{\min} = - \frac{(m_a^2 - m_c^2)(m_b^2 - m_d^2)}{s} - \frac{(m_a^2 + m_b^2 - m_c^2 - m_d^2)(m_a^2 m_b^2 - m_c^2 m_d^2)}{s^2} + O\left(\frac{1}{s^3}\right) \quad (\text{II-22})$$

In a similar way, defining $u_{ad} = (p_a - p_d)^2$, one finds

$$u_{ad}^{\min} = - \frac{(m_a^2 - m_d^2)(m_b^2 - m_c^2)}{s} - \frac{(m_a^2 + m_b^2 - m_c^2 - m_d^2)(m_a^2 m_b^2 - m_c^2 m_d^2)}{s^2} + O\left(\frac{1}{s^3}\right) \quad (\text{II-23})$$

2. Particular case

i) $m_a = m_c, m_b \neq m_d$:

$$t_{ac}^{\min} \approx - \frac{m_a^2(m_b^2 - m_d^2)^2}{s^2} \quad (II-24)$$

$$u_{ad}^{\min} \approx - \frac{(m_a^2 - m_d^2)(m_b^2 - m_c^2)}{s} \quad (II-25)$$

ii) $m_a = m_c, m_b = m_d$: (elastic scattering $a + b \rightarrow a' + b'$)

$$t_{aa'}^{\min} = 0,$$

and

$$t_{aa'} = -4P_a^2 \sin^2 \left(\frac{\theta_{aa'}}{2} \right),$$

$$dt_{aa'} = 2P_a^2 d(\cos \theta_{aa'}) \quad (II-26)$$

also

$$u_{ab'}^{\min} = \frac{(m_a^2 - m_b^2)^2}{s} \quad (II-27)$$

III. VERY-HIGH ENERGY PHYSICS: ASYMPTOTIC FORMULAE

In this domain of energies, where s is large, the beams, and frequently the outgoing particles, have masses which are negligible compared with their momenta. The formulae of the previous paragraphs become simple. Let us consider the interaction of particle a (beam) with particle b (target).

A. Colliding beams

The lab reference frame coincides with the c.m. frame of the interaction. Formula (II-1) gives:

$$s \equiv s_{ab} \approx 2(E_a E_b - \vec{P}_a \cdot \vec{P}_b) \quad (III-1)$$

If θ is the supplement of the angle between the two beams (and if we assume $E_i \sim P_i$), then:

$$s \sim 4P_a P_b \cos^2 \frac{\theta}{2} \sim 4P_a P_b \left(1 - \frac{\theta^2}{8} + \dots \right) \quad (\text{III-2})$$

B. Interaction of particle a with target b at rest

B. 1. In the lab frame

$$s = s_{ab} = m_a^2 + m_b^2 + 2m_b E_a^{\text{lab}} \sim 2m_b E_a^{\text{lab}} \quad (\text{III-3})$$

and

$$P_{\text{inc}} = \frac{s}{2m_b} \quad (\text{III-4})$$

B. 2. In the center-of-mass frame

$$s = s_{ab} = (E_a^{\text{cm}} + E_b^{\text{cm}})^2 \sim 4(P_a^{\text{cm}})^2 \quad (\text{III-5})$$

and

$$P_a^{\text{cm}} = P_b^{\text{cm}} \sim \frac{\sqrt{s}}{2} \quad (\text{III-6})$$

IV. RELATIONS BETWEEN THE (j) PARTICLE AND (ij) PARTICLE REST FRAMES: LORENTZ TRANSFORMATION

The general Lorentz transformation has the matrix form

$$\begin{pmatrix} E \\ P_{\parallel} \\ \vec{P}_{\perp} \end{pmatrix}^{(ij)} = \begin{pmatrix} \gamma & -\eta & 0 \\ -\eta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ P_{\parallel} \\ \vec{P}_{\perp} \end{pmatrix}^{(j)} \quad (\text{IV-1})$$

If we define, in any frame, the quantity*: $P_{\pm} = E \pm P_{\parallel}$

*We note that

$$P_+ \cdot P_- = E^2 - P_{\parallel}^2 = m^2 + P_{\perp}^2 \equiv m_{\perp}^2;$$

$m_{\perp}$ is called the transverse mass.

so that for any particle

$$\left\{ \begin{array}{ll} P_{\pm}^{(j)} = E^{(j)} \pm P_{\parallel}^{(j)} & \text{in the (j) particle frame} \\ P_{\pm}^{(ij)} = E^{(ij)} \pm P_{\parallel}^{(ij)} & \text{in the (ij) particle frame,} \end{array} \right.$$

then, the first two equations of the above Lorentz transformation give

$$E^{(ij)} \pm P_{\parallel}^{(ij)} = (\gamma \mp \eta) \left[E^{(j)} \pm P_{\parallel}^{(j)} \right], \quad (IV-2)$$

and the transformation then becomes

$$\begin{pmatrix} P_+ \\ P_- \\ \vec{P}_\perp \end{pmatrix}^{(ij)} = \begin{pmatrix} (\gamma - \eta) & 0 & 0 \\ 0 & (\gamma + \eta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_+ \\ P_- \\ \vec{P}_\perp \end{pmatrix}^{(j)} \quad (IV-3)$$

From Eqs. (II-6) and (II-11), one finds

$$\gamma \mp \eta = \frac{s_{ij} + m_j^2 - m_i^2 \mp \sqrt{\Delta(s_{ij}, m_i^2, m_j^2)}}{2m_j \sqrt{s_{ij}}}. \quad (IV-4)$$

This quantity is also sometimes set equal to $e^{\mp \xi_{ij}}$. (We will see below the significance of ξ_{ij} .)*

From $\frac{\eta}{\gamma} = \frac{(\gamma + \eta) - (\gamma - \eta)}{(\gamma + \eta) + (\gamma - \eta)}$, one finds

*Note that at high energy (see paragraph III for definition),

$$e^{+\xi_{ij}} \approx \frac{\sqrt{s_{ij}}}{m_j}$$

$$\left\{ \begin{array}{l} \vec{\beta} = \frac{\vec{\eta}}{\gamma} + \tanh \xi_{ij} = \frac{\sqrt{\Delta(s_{ij}, m_i^2, m_j^2)}}{s_{ij} + m_j^2 - m_i^2} = \frac{P_i^{(j)}}{E_i^{(j)} + m_j} \end{array} \right. \quad (IV-5)$$

$$\left\{ \begin{array}{l} \gamma = \frac{1}{\sqrt{1 - \beta^2}} = \cosh \xi_{ij} = \frac{s_{ij} + m_j^2 - m_i^2}{2m_j \sqrt{s_{ij}}} = \frac{E_i^{(j)} + m_j}{\sqrt{s_{ij}}} \end{array} \right. \quad (IV-6)$$

$$\left\{ \begin{array}{l} \vec{\eta} = \vec{\beta} \gamma = \sinh \xi_{ij} = \frac{\sqrt{\Delta(s_{ij}, m_i^2, m_j^2)}}{2m_j \sqrt{s_{ij}}} = \frac{P_i^{(j)}}{\sqrt{s_{ij}}} \end{array} \right. \quad (IV-7)$$

with $\gamma^2 - \eta^2 = 1$, and

$$\left(P_i^{(j)} + p_j^{(j)} \right) = \left(\sqrt{s_{ij}} \cosh \xi_{ij}, \sqrt{s_{ij}} \sinh \xi_{ij}, 0 \right). \quad (IV-8)$$

The spatial part of the matrix in equation (IV-1) above can be written as the set of equations

$$\left\{ \begin{array}{l} P_{||}^{(ij)} = -\eta E^{(j)} + \gamma P_{||}^{(j)} \\ \vec{P}_{\perp}^{(ij)} = \vec{P}_{\perp}^{(j)} \end{array} \right.$$

so that

$$\vec{P}^{(ij)} = \vec{P}^{(j)} - \vec{\eta} E^{(j)} + \frac{\gamma^2 - 1}{\gamma + 1} \vec{P}_{||}^{(j)},$$

and the classical vector form of the Lorentz transform of a vector $\vec{P}^{(j)}$ into a vector $\vec{P}^{(ij)}$ comes out to be

$$\vec{P}^{(ij)} = \vec{P}^{(j)} + \vec{\eta} \left(\frac{\vec{\eta} \cdot \vec{P}}{\gamma + 1} - E^{(j)} \right) \quad (IV-9)$$

A convenient way to write the latter expression, taking into account

$$E^{(ij)} = \gamma E^{(j)} - \vec{\eta} \cdot \vec{P}^{(j)},$$

is

$$\vec{P}^{(ij)} = \vec{P}^{(j)} - \vec{\eta} \left[\frac{E^{(ij)} + E^{(j)}}{\gamma + 1} \right] \quad (\text{IV-10})$$

Application

1. To reaction a (beam) + b (target)

From the above, the transformation from the lab frame into the c.m. frame is given by:

$$\gamma = \frac{E_a^{\text{lab}} + m_b}{\sqrt{s}} \quad \text{and} \quad \eta = \frac{P_{\text{inc}}}{\sqrt{s}}$$

$$P_{\pm}^{\text{cm}} = e^{\pm \xi} P_{\pm}^{\text{lab}}$$

with

$$e^{\pm \xi} = \frac{s + m_b^2 - m_a^2 \mp \sqrt{\Delta(s, m_a^2, m_b^2)}}{2m_b \sqrt{s}} \quad (\text{IV-11})$$

2. The Lorentz transform of any vector $p_1 = (E_1, \vec{P}_1)$, given in a frame containing another vector $p_2 = (E_2, \vec{P}_2)$, into the p_2 rest frame -- where $p_2 = (m_2, 0)$, and $p_1 = (E'_1, \vec{P}'_1)$ -- is to be calculated using:

$$\gamma = \frac{E_2}{m_2} \quad \text{and} \quad \vec{\eta} = \frac{\vec{P}_2}{m_2};$$

then

$$\left. \begin{aligned} E'_1 &= \gamma E_1 - \vec{\eta} \cdot \vec{P}_1 = \frac{E_1 E_2 - \vec{P}_1 \cdot \vec{P}_2}{m_2}, \\ \vec{P}'_1 &= \vec{P}_1 - \frac{\vec{P}_2}{m_2} \left(\frac{E_1 + E'_1}{\gamma + 1} \right) = \vec{P}_1 - \vec{P}_2 \left(\frac{E_1 + E'_1}{E_2 + m_2} \right). \end{aligned} \right\} \quad (\text{IV-12})$$

V. RAPIDITY VARIABLE: DEFINITION AND KINEMATIC RELATIONS

In any frame, the longitudinal rapidity variable is defined by any of the equivalent definitions:*

$$y \equiv \frac{1}{2} \ln \frac{E + P_{\parallel}}{E - P_{\parallel}} = \tanh^{-1} \frac{P_{\parallel}}{E};$$

also (V-1)

$$y \equiv \frac{1}{2} \ln \frac{P_{+}}{P_{-}} = \ln \frac{P_{+}}{m_{\perp}} = -\ln \frac{P_{-}}{m_{\perp}}$$

($m_{\perp}$ being the transverse mass = $\sqrt{m^2 + P_{\perp}^2}$ as defined above).

From the last of the two equations above, one gets:

$$P_{+} = m_{\perp} e^y, \quad (V-2)$$

$$P_{-} = m_{\perp} e^{-y}, \quad (V-3)$$

so that

$$\begin{cases} E = m_{\perp} \cosh y, \\ P_{\parallel} = m_{\perp} \sinh y, \end{cases} \quad (V-4)$$

$$(V-5)$$

A. Relations between the rapidities of different frames

From

$$P_{\pm}^{(ij)} = e^{\mp \xi_{ij}} P_{\pm}^{(j)}, \quad (V-6)$$

one deduces

$$\frac{1}{2} \ln \left(\frac{P_{+}}{P_{-}} \right)^{(ij)} = -\xi_{ij} + \frac{1}{2} \ln \left(\frac{P_{+}}{P_{-}} \right)^{(j)}$$

*Note that at high energy, in the lab frame, if $E^{\text{lab}} \approx P^{\text{lab}}$, then

$$y^{\text{lab}} \approx \frac{1}{2} \ln \frac{1 + \cos \theta}{1 - \cos \theta} \approx -\ln \left(\tan \frac{\theta}{2} \right),$$

θ being the production angle (vs the incoming beam) of the considered particle.

and
$$y^{(ij)} = y^{(j)} - \xi_{ij} \quad (V-7)$$

with $\xi \approx \ln \frac{\sqrt{s_{ij}}}{m_j}$ at high energy (see footnote on page 8). ξ_{ij} is the boost necessary to bring the (ij) particle rest frame into equivalence with the (j) particle rest frame.

B. Application to a reaction $a + b \rightarrow c + d$ (or $a + b \rightarrow c + X$):

In the lab frame,

$$\left\{ \begin{array}{l} p_a^{\text{lab}} = (m_a \cosh y_a^{\text{lab}}, m_a \sinh y_a^{\text{lab}}, 0) \\ p_b^{\text{lab}} = (m_b, 0, 0) \\ p_c^{\text{lab}} = (m_{lc} \cosh y_c^{\text{lab}}, m_{lc} \sinh y_c^{\text{lab}}, \vec{P}_{lc}) \end{array} \right\} \quad (V-8)$$

(and a similar formula for particle d or X)

In the c. m. frame,

$$\left\{ \begin{array}{l} p_a^{\text{cm}} = (m_a \cosh y_a^{\text{cm}}, m_a \sinh y_a^{\text{cm}}, 0) \\ p_b^{\text{cm}} = (m_b \cosh y_b^{\text{cm}}, m_b \sinh y_b^{\text{cm}}, 0) \\ p_c^{\text{cm}} = (m_{lc} \cosh y_c^{\text{cm}}, m_{lc} \sinh y_c^{\text{cm}}, \vec{P}_{lc}) \end{array} \right\} \quad (V-9)$$

(and a similar formula for particle d or X)

Identifying these expressions with Eq. (II-6) and (II-11), one gets:

$$y_a^{\text{lab}} = \cosh^{-1} \left(\frac{s - m_a^2 - m_b^2}{2m_a m_b} \right) \xrightarrow{\text{at large } s} y_a^{\text{lab}} \approx \ln \frac{s}{m_a m_b} \quad (V-10)$$

$$y_a^{cm} = \cosh^{-1} \left(\frac{s + m_a^2 - m_b^2}{2m_a \sqrt{s}} \right) \xrightarrow{\text{at large } s} y_a^{cm} \simeq \ln \frac{\sqrt{s}}{m_a} \quad (V-11)$$

$$y_b^{cm} = \cosh^{-1} \left(\frac{s + m_b^2 - m_a^2}{2m_b \sqrt{s}} \right) \xrightarrow{\text{at large } s} y_b^{cm} \simeq -\ln \frac{\sqrt{s}}{m_b} \quad (V-12)$$

$$y_c^{cm} = \cosh^{-1} \left(\frac{s + m_c^2 - m_d^2}{2m_{lc} \sqrt{s}} \right) \xrightarrow[\text{and } m_d^2 \text{ (or } m_X^2) \ll s]{\text{at large } s} y_c^{cm} \simeq \ln \frac{\sqrt{s}}{m_{lc}} \quad (V-13)$$

(and a similar formula for particle d or X)

With $\xi \simeq \ln \frac{\sqrt{s}}{m_b}$, one gets, at large s:

$$y_c^{lab} \simeq \ln \frac{s}{m_b m_{lc}} \quad (V-14)$$

The maximum rapidity gap for an experiment at large s, is:

i) for the incoming particles (a, b):

$$y_{ab} = y_a^{lab} = y_a^{cm} - y_b^{cm} \simeq \ln \frac{s}{m_a m_b} \quad (V-15)$$

ii) for the outgoing particles (c, d):

$$y_{cd} = y_c^{lab} - y_d^{lab} = y_c^{cm} - y_d^{cm} \simeq \ln \frac{s}{m_{lc} m_{ld}} \quad (V-16)$$

C. Application to reaction $a + b \rightarrow 1 + 2 + \dots + N$, with the N outgoing particles being ordered by increasing rapidity; the rapidity gap y_{ab} between the incoming particles is as above; the maximum rapidity gap between particles 1 and N is analogous to y_{cd} , so that the value of y_{1N} is approximately [under conditions similar to those for (V.13)]:

$$y_{1N} \approx \ln \frac{s}{m_{11} m_{1N}} \quad (V-17)$$

C. 1. The rapidity gap $(y_i - y_j)$ between the rapidities y_i and y_j of particles i and j , respectively, is related to the invariant mass squared s_{ij} of these particles. Using equation (II-1),

$$s_{ij} = m_i^2 + m_j^2 + 2 \left(E_i E_j - P_{\parallel i} P_{\parallel j} - \vec{P}_{\perp i} \cdot \vec{P}_{\perp j} \right),$$

with (from above),

$$E_i E_j - P_{\parallel i} P_{\parallel j} = m_{\perp i} m_{\perp j} (\cosh y_i \cosh y_j - \sinh y_i \sinh y_j)$$

we get

$$s_{ij} = m_i^2 + m_j^2 + 2 m_{\perp i} \cdot m_{\perp j} \cosh(y_i - y_j) - 2 \vec{P}_{\perp i} \cdot \vec{P}_{\perp j} \quad (V-18)$$

and

$$y_i - y_j = \cosh^{-1} \frac{s_{ij} - m_i^2 - m_j^2 + 2 \vec{P}_{\perp i} \cdot \vec{P}_{\perp j}}{2 m_{\perp i} m_{\perp j}} \quad (V-19)$$

C. 2. Another quantity which might be useful is the rapidity gap between either of the extreme particles (1 or N) and its adjacent neighbor [2 or $(N - 1)$ respectively]. If M_1 (M_N) is the missing mass with respect to particle 1 (N) in the reaction $a + b \rightarrow 1 + M_1$ ($a + b \rightarrow N + M_N$), then on

a statistical basis, and in first approximation:

$$|y_1 - y_2| \approx \ln \frac{s}{M_1^2} \quad (V-20)$$

and

$$|y_{N-1} - y_N| \approx \ln \frac{s}{M_N^2} \quad (V-21)$$

These formulae are particularly useful when particle 2, or (N-1), is neutral and consequently no other way exists to measure these rapidity gaps.

C. 3. The rapidity gap y_{a1} between the incoming particle a and the outgoing particle closest to a, 1, is related to the transverse momentum $t_{a1} = (p_a - p_1)^2$:

$$t_{a1} = m_a^2 + m_1^2 - 2m_a m_{11} \cosh(y_a - y_1) \quad (V-22)$$

and

$$y_a - y_1 = \cosh^{-1} \left(\frac{t_{a1} - m_a^2 - m_1^2}{2m_a m_{11}} \right) \quad (V-23)$$

VI. INCLUSIVE REACTIONS AND SCALING VARIABLES: DEFINITIONS AND KINEMATIC RELATIONS:

In an inclusive process $a + b \rightarrow c + X$ — where X denotes an unknown system of particles — all the information is obtained from the measurement of the momentum and type of particle c. One thus measures the quantity

$$\frac{d^3\sigma_c}{dp_c^3} = \frac{d^3\sigma_c}{P_c^2 dP_c d\Omega_c} \quad (VI-1)$$

in the laboratory frame. This one particle distribution is not Lorentz invariant and instead, one uses the invariant distribution function

$$f(p_c, s) = E_c \frac{d^3 \sigma_c}{dp_c^3} \quad (\text{VI-2})$$

In all that follows, we will omit index c of the variables related to the 4-momentum p_c of particle c (m_c , E_c , P_c^{lab} , y_c , etc.). In experiments with unpolarized incident particles, the distribution $f(p, s)$ does not depend on rotations around the beam axis. Consequently, the invariant cross-section being integrated over the azimuth angle will only depend on two remaining variables.

A. The following sets of variables are generally used

- A.1. P_a^{lab} and θ^{lab} , the absolute value and polar angle of the momentum $\vec{P}_c$ of particle c .
- A.2. $P_{\parallel}$ and $P_{\perp}$, the longitudinal and transverse components of $\vec{P}_c$.
- A.3. The Feynman variable x and the transverse momentum $P_{\perp}$:

The variable x is defined by: $x \equiv \frac{P_{\parallel}^{\text{cm}}}{P_{\text{max}}^{\text{cm}}} \quad (\text{VI-3})$

where [see expression (II-11) above]

$$P_{\text{max}}^{\text{cm}} = \frac{\sqrt{\Delta(s, m_c^2, m_{X, \text{min}}^2)}}{2\sqrt{s}} \quad (\text{VI-4})$$

is the center-of-mass momentum of particle c and the lightest possible particle(s) $m_{X, \text{min}}$ which, consistent with conservation laws, could recoil against particle c .

At high energy, $P_{\text{max}}^{\text{cm}} \approx \frac{\sqrt{s}}{2}$ and $x \approx \frac{2P_{\parallel}^{\text{cm}}}{\sqrt{s}} \quad (\text{VI-5})$

On the other hand, due to the fact that

$$m_X^2 = (p_a + p_b - p_c)^2 = (p_a + p_b)^2 + p_c^2 - 2p_c(p_a + p_b)$$

and
$$m_X^2 - m_c^2 = s - 2E_c \sqrt{s} \approx s \left(1 - \frac{2P_c}{\sqrt{s}} \right),$$

one gets

$$x = 1 - \frac{m_X^2 - m_c^2}{s} \approx 1 - \frac{m_X^2}{s} \quad (\text{for } \pi_X^2 \gg m_c^2) \quad (\text{VI-6})$$

Note: an alternative definition sometimes used is

$$x' \equiv \frac{p_{||}^{\text{cm}}}{p_{|| \text{ max}}^{\text{cm}}}, \quad (\text{VI-7})$$

where

$$p_{|| \text{ max}}^{\text{cm}} = \sqrt{(P_{\text{max}}^{\text{cm}})^2 - p_{\perp}^2}.$$

At high energy, $p_{|| \text{ max}}^{\text{cm}} \approx p_{\text{max}}^{\text{cm}} \approx \frac{\sqrt{s}}{2}$ and $x' \approx x$.

A.4. The longitudinal rapidity y and transverse momentum $\vec{P}_{\perp}$ (or transverse mass $m_{\perp}$): The relation between x and y is trivial for very high energy (s large); using expression (V-5):

$$x \approx 2 \frac{p_{||}^{\text{cm}}}{\sqrt{s}} \approx \frac{2m_{\perp}}{\sqrt{s}} \sinh y^{\text{cm}} \quad (\text{VI-8})$$

and, for y also large:

$$y^{\text{cm}} \approx \ln \frac{x\sqrt{s}}{m_{\perp}}. \quad (\text{VI-9})$$

A.5. The invariant momentum transfer, t , from particle a to particle c, and m_X^2 , the square of the missing mass:

$$\begin{cases} t = (p_a - p_c)^2 \\ m_X^2 = (p_a + p_b - p_c)^2 \end{cases} .$$

B. The invariant differential cross section may then be written in the following forms using the different expressions of the phase space volume $\frac{d^3 p}{E}$:

B. 1. For variables P^{lab} and θ^{lab}

$$\frac{d^3 p}{E} = \frac{P^2 dP d\Omega}{E} = \frac{P^2 dP (\cos \theta) d\phi}{E} ,$$

so that the invariant function $f(p)$ is given by

$$f(p) = E \frac{d^3 \sigma}{dp^3} = \frac{E^{\text{lab}}}{(P^{\text{lab}})^2} \cdot \frac{d^3 \sigma}{dP^{\text{lab}} \cdot d\Omega^{\text{lab}}} ,$$

and

$$f(p) = E \frac{d^3 \sigma}{dp^3} = \frac{E^{\text{lab}}}{2\pi (P^{\text{lab}})^2} \cdot \frac{d^2 \sigma}{dP^{\text{lab}} d(\cos \theta^{\text{lab}})} \quad (\text{VI-10})$$

(averaged over azimuth).

B. 2. For variables $P_{\parallel}$ and $P_{\perp}$

$$\frac{d^3 p}{E} = \frac{P_{\perp} dP_{\perp} dP_{\parallel} d\phi}{E} = \frac{d(P_{\perp}^2) dP_{\parallel} d\phi}{2E} ,$$

so that the invariant function $f(p)$ is

$$f(p) = E \frac{d^3 \sigma}{dp^3} = \frac{E}{\pi} \cdot \frac{d^2 \sigma}{dP_{\parallel} \cdot dP_{\perp}^2} \quad (\text{VI-11})$$

(averaged over azimuth).

B. 3. For variables x and $P_{\perp}$, with

$$x = \frac{P_{\parallel}^{\text{cm}}}{P_{\text{max}}^{\text{cm}}} \approx \frac{2P_{\parallel}^{\text{cm}}}{\sqrt{s}},$$

$$f(p) = E \frac{d^3\sigma}{dp^3} = \frac{1}{\pi} \cdot \frac{E^{\text{cm}}}{P_{\text{max}}^{\text{cm}}} \cdot \frac{d^2\sigma}{dx \cdot dP_{\perp}^2} \approx \frac{2}{\pi} \cdot \frac{E^{\text{cm}}}{\sqrt{s}} \frac{d^2\sigma}{dx \cdot dP_{\perp}^2} \quad (\text{VI-12})$$

(averaged over azimuth).

B. 4. For variables y and $P_{\perp}$ (or $m_{\perp}$)From $P_{\parallel} = m_{\perp} \sinh y$, and $m_{\perp}^2 = m^2 + P_{\perp}^2$,

$$dP_{\parallel} = m_{\perp} \cosh y \, dy = E dy, \text{ and } dm_{\perp}^2 = dP_{\perp}^2,$$

and one deduces that the invariant function $f(p)$ is

$$f(p) = E \frac{d^3\sigma}{dp^3} = \frac{1}{\pi} \frac{d^2\sigma}{dy \cdot dP_{\perp}^2} = \frac{1}{\pi} \frac{d^2\sigma}{dy \cdot dm_{\perp}^2} \quad (\text{VI-13})$$

(averaged over azimuth).

B. 5. For variables t and m_X^2 ,

$$\text{from } m_X^2 = (p_a + p_b - p_c)^2 = s + m_c^2 - 2\sqrt{s} E_c^{\text{cm}},$$

$$\left\{ \begin{array}{l} dm_X^2 = -2\sqrt{s} \, dE_c^{\text{cm}} = -2\sqrt{s} \, P^{\text{cm}} \frac{dP^{\text{cm}}}{E^{\text{cm}}} \\ dt = 2 \underbrace{P_a^{\text{cm}} P_c^{\text{cm}}}_{P_{\perp}} \sin \theta^{\text{cm}} d\theta^{\text{cm}}, \end{array} \right.$$

and

$$\begin{aligned} \frac{d^3 p}{E} &= \frac{1}{E} P^2 \sin \theta \, dP \, d\theta \, d\phi = \frac{1}{4P_a \sqrt{s}} \, dt \, dm_X^2 \, d\phi \\ &= \frac{1}{2 \sqrt{\Delta(s, m_a^2, m_b^2)}} \, dt \, dm_X^2 \, d\phi, \end{aligned}$$

so that the invariant function is

$$f(p) = E \frac{d^3 \sigma}{dp^3} = \frac{1}{\pi} \cdot \sqrt{\Delta(s, m_a^2, m_b^2)} \frac{d^2 \sigma}{dt \, dm_X^2} \quad (\text{VI-14})$$

(averaged over azimuth).

Table I summarizes these results.

Acknowledgments:

I wish to thank Art Rosenfeld for his having suggested this memo and Alan Rittenberg for a careful reading of the manuscript.

RELATIVISTIC KINEMATICS (Cont'd)

Application to the relation $a+b=c+d$

1. General formulae

$$t_{ac} - t_{bd} = \frac{t_{ac}^{\min} - t_{bd}^{\min}}{c} = 4p_a p_c \sin^2\left(\frac{\theta_{ac}}{2}\right),$$

$$\text{with } t_{ac}^{\min} = (E_a - E_c)^2 - (p_a - p_c)^2 \quad (II-20)$$

which, after expansion in powers of $(1/s)$, gives

$$t_{ac}^{\min} = -\frac{(m_a^2 - m_c^2)(m_b^2 - m_d^2)}{s} - \frac{(m_a^2 + m_b^2 - m_c^2 - m_d^2)(m_a^2 m_b^2 - m_c^2 m_d^2)}{s^2} + O\left(\frac{1}{s^3}\right) \quad (II-21)$$

In a similar way, defining $u_{ad} = (p_a - p_d)^2$, one finds

$$u_{ad} = t_{bc} = u_{ad}^{\min} - 4p_a p_d \sin^2\left(\frac{\theta_{ad}}{2}\right),$$

$$u_{ad}^{\min} = -\frac{(m_b^2 - m_d^2)(m_c^2 - m_e^2)}{s} - \frac{(m_b^2 + m_c^2 - m_d^2 - m_e^2)(m_b^2 m_c^2 - m_d^2 m_e^2)}{s^2} + O\left(\frac{1}{s^3}\right) \quad (II-23)$$

A general relation between the invariants is

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2.$$

IV. RELATIONS BETWEEN THE (1) PARTICLE AND (12) PARTICLE REST FRAMES; LORENTZ TRANSFORMATION

The general Lorentz transformation has the matrix form

$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}^{(12)} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}^{(1)} \quad (IV-1)$$

If we define, in any frame, the quantity: $P_1 = E + p_z$,

$$\begin{pmatrix} P_1 \\ p_x \\ p_y \\ p_z \end{pmatrix}^{(12)} = \begin{pmatrix} (1+\gamma) & 0 & 0 & 0 \\ 0 & (\gamma+\gamma v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ p_x \\ p_y \\ p_z \end{pmatrix}^{(1)} \quad (IV-2)$$

$$\gamma + \gamma v = \frac{s_{12} + m_1^2 - m_2^2 + \sqrt{\Delta(s_{12}, m_1^2, m_2^2)}}{2m_1 \sqrt{s_{12}}} = v \tau_{12} \quad (IV-4)$$

$$\left\{ \begin{aligned} |\vec{s}| = \frac{|\vec{n}|}{\gamma} = \tanh \epsilon_{12} = \frac{\sqrt{\Delta(s_{12}, m_1^2, m_2^2)}}{s_{12} + m_1^2 - m_2^2} = \frac{p_1^{(1)}}{E_1^{(1)} + m_1} \end{aligned} \right. \quad (IV-5)$$

$$\left\{ \begin{aligned} \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh \epsilon_{12} = \frac{s_{12} + m_1^2 - m_2^2}{2s_{12}} = \frac{E_1^{(1)} + m_1}{\sqrt{s_{12}}} \end{aligned} \right. \quad (IV-6)$$

$$\left\{ \begin{aligned} |\vec{n}| = |\vec{s}| \gamma = \sinh \epsilon_{12} = \frac{\sqrt{\Delta(s_{12}, m_1^2, m_2^2)}}{2s_{12} \sqrt{s_{12}}} = \frac{p_1^{(1)}}{\sqrt{s_{12}}} \end{aligned} \right. \quad (IV-7)$$

The spatial part of the matrix in Eq. (IV-1) above can be written with the classical vector form

$$\vec{p}^{(12)} = \vec{p}^{(1)} + \vec{n} \left(\frac{\vec{n} \cdot \vec{p}^{(1)}}{\gamma + 1} - E^{(1)} \right) \quad (IV-9)$$

A convenient way to write the latter expression is

$$\vec{p}^{(12)} = \vec{p}^{(1)} - \vec{n} \left[\frac{\vec{p}^{(1)} + E^{(1)} \vec{n}}{\gamma + 1} \right] \quad (IV-10)$$

2. Elastic scattering

In the c.m.: $dt = 2p_a^2 d(\cos \theta_{ad}^{\text{c.m.}})$.

i) $m_a = m_c$, $m_b = m_d$:

$$t_{ac}^{\min} = -\frac{m_a^2(m_c^2 - m_d^2)}{s} \quad (II-24)$$

$$u_{ad}^{\min} = -\frac{(m_a^2 - m_c^2)(m_b^2 - m_d^2)}{s} \quad (II-25)$$

ii) For elastic scattering $(a+b = a'+b')$,

$$t_{aa'}^{\min} = 0,$$

$$\text{and } t_{aa'} = -4p_a^2 \sin^2\left(\frac{\theta_{aa'}}{2}\right),$$

$$dt_{aa'} = 2p_a^2 d(\cos \theta_{aa'}); \quad (II-26)$$

$$\text{also } u_{ab'}^{\min} = -\frac{(m_a^2 - m_b^2)}{s} \quad (II-27)$$

At high energy:

$$\epsilon^{(1)} \sim \frac{\sqrt{s_{12}}}{m_1} \quad \text{and} \quad \epsilon^{(12)} \sim \ln \frac{\sqrt{s_{12}}}{m_1}.$$

ϵ_{12} is called the logarithm of the Lorentz transformation that connects the particle 1) rest frame to the particle 12) rest frame.

Disintegration

1. $a + b \rightarrow c + d$ (beam) + b (target)

The transformation from the lab frame into the c.m. frame is given by:

$$\gamma = \frac{E_a^{\text{lab}} + m_b}{\sqrt{s}} \quad \text{and} \quad v = \frac{p_a^{\text{lab}}}{\sqrt{s}};$$

$$\text{so } p_1^{\text{c.m.}} = \gamma \tau_1^{\text{lab}}$$

with

$$\tau_1^{\text{lab}} = \frac{s + m_1^2 - m_2^2 + \sqrt{\Delta(s, m_1^2, m_2^2)}}{2m_1 \sqrt{s}} \quad (IV-11)$$

2. The inverse transformation of the beam

$\vec{p}_1 = (E_1, \vec{p}_1)$ given in a frame containing another vector $\vec{p}_2 = (E_2, \vec{p}_2)$ into the $\vec{p}_2$ rest frame - where $\vec{p}_2 = (m_2, 0)$, and $\vec{p}_1 = (E_1, \vec{p}_1)$ - may be calculated using:

$$\gamma = \frac{E_2}{m_2} \quad \text{and} \quad \vec{n} = \frac{\vec{p}_2}{m_2};$$

$$\left. \begin{aligned} \vec{E}_1' &= \gamma E_1 - \vec{n} \cdot \vec{p}_1 = \frac{E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2}{m_2}, \\ \vec{p}_1' &= \vec{p}_1 - \frac{\vec{p}_2}{m_2} \left(\frac{E_1 + E_1'}{\gamma + 1} \right) = \vec{p}_1 - \frac{\vec{p}_2}{m_2} \left(\frac{E_1 + E_1'}{\gamma + 1} \right) \end{aligned} \right\} \quad (IV-12)$$

RELATIVISTIC KINEMATICS (Cont'd)

V. RAPIDITY VARIABLE: DEFINITION AND KINEMATIC RELATIONS

In addition, in any system, for a particle of energy E and longitudinal momentum p_L , the rapidity is defined by

$$y = \frac{1}{2} \ln \left(\frac{E + p_L}{E - p_L} \right) = \tanh^{-1} \left(\frac{p_L}{E} \right) \quad (V-1)$$

$$= \frac{1}{2} \ln \left(\frac{p_+}{p_-} \right) = \ln \left(\frac{p_+}{p_-} \right)^{1/2} = \ln \left(\frac{p}{m_T} \right)$$

with the transverse mass $m_T = (m^2 + p_T^2)^{1/2}$.
From the last of the two equations above, one gets

$$p_+ = m_T e^y \quad (V-2)$$

$$p_- = m_T e^{-y} \quad (V-3)$$

$$\text{so that } \begin{cases} E = m_T \cosh y \\ p_L = m_T \sinh y \end{cases} \quad (V-4)$$

$$p_T = m_T \sinh y \quad (V-5)$$

A. $\frac{p_{\pm}}{p_{\pm}'} = \frac{E \pm p_L}{E' \pm p_L'} = \frac{m_T e^{\pm y}}{m_T' e^{\pm y'}} = \frac{m_T}{m_T'} e^{\pm(y-y')}$

$$\frac{p_{\pm}}{p_{\pm}'} = \frac{m_T}{m_T'} e^{\pm(y-y')} \quad (V-7)$$

B. $\frac{p_{\pm}}{p_{\pm}'} = \frac{E \pm p_L}{E' \pm p_L'} = \frac{m_T e^{\pm y}}{m_T' e^{\pm y'}} = \frac{m_T}{m_T'} e^{\pm(y-y')}$

In the lab frame,

$$\begin{cases} p_a^{\text{lab}} = (m_a \cosh y_a^{\text{lab}}, m_a \sinh y_a^{\text{lab}}, 0) \\ p_b^{\text{lab}} = (m_b, 0, 0) \\ p_c^{\text{lab}} = (m_c \cosh y_c^{\text{lab}}, m_c \sinh y_c^{\text{lab}}, p_{Tc}^{\text{lab}}) \end{cases} \quad (V-8)$$

(and a similar formula for particle d or X).

In the c.m. frame,

$$\begin{cases} p_a^{\text{cm}} = (m_a \cosh y_a^{\text{cm}}, m_a \sinh y_a^{\text{cm}}, 0) \\ p_b^{\text{cm}} = (m_b \cosh y_b^{\text{cm}}, m_b \sinh y_b^{\text{cm}}, 0) \\ p_c^{\text{cm}} = (m_c \cosh y_c^{\text{cm}}, m_c \sinh y_c^{\text{cm}}, p_{Tc}^{\text{cm}}) \end{cases} \quad (V-9)$$

(and a similar formula for particle d or X).

$$y_a^{\text{lab}} - \cosh^{-1} \left(\frac{s - m_b^2 - m_a^2}{2m_a m_b} \right) \xrightarrow{\text{at large } s} y_a^{\text{lab}} \approx \ln \frac{\sqrt{s}}{m_a} \quad (V-10)$$

$$y_a^{\text{cm}} - \cosh^{-1} \left(\frac{s + m_b^2 - m_a^2}{2m_a \sqrt{s}} \right) \xrightarrow{\text{at large } s} y_a^{\text{cm}} \approx \ln \frac{\sqrt{s}}{m_a} \quad (V-11)$$

$$y_b^{\text{cm}} - \cosh^{-1} \left(\frac{s + m_b^2 - m_a^2}{2m_b \sqrt{s}} \right) \xrightarrow{\text{at large } s} y_b^{\text{cm}} \approx -\ln \frac{\sqrt{s}}{m_b} \quad (V-12)$$

$$y_c^{\text{cm}} - \cosh^{-1} \left(\frac{s + m_c^2 - m_a^2}{2m_c \sqrt{s}} \right) \xrightarrow{\text{at large } s} y_c^{\text{cm}} \approx \ln \frac{\sqrt{s}}{m_{1c}} \quad (V-13)$$

and $m_{1c}^2 = (m_a^2 + m_b^2) \dots s$

With $\xi = \ln \frac{\sqrt{s}}{m_b}$, one gets, at large s :

$$y_c^{\text{lab}} \approx \ln \frac{\sqrt{s}}{m_{1c}} \quad (V-14)$$

$$y^{\text{lab}} = \frac{1}{2} \ln \left(\frac{p_+^{\text{lab}} + p_-^{\text{lab}}}{p_+^{\text{lab}} - p_-^{\text{lab}}} \right) = \ln \left(\frac{E}{p_L} \right)$$

The maximum rapidity gap for the reaction $a + b \rightarrow c + d$ at large s is:

i) for the incoming particles (a,b):

$$y_{ab} = y_a^{\text{lab}} - y_b^{\text{cm}} = y_b^{\text{cm}} \approx \ln \frac{\sqrt{s}}{m_a m_b} \quad (V-15)$$

ii) for the outgoing particles (c,d):

$$y_{cd} = y_c^{\text{lab}} - y_d^{\text{lab}} = y_c^{\text{cm}} - y_d^{\text{cm}} \approx \ln \frac{\sqrt{s}}{m_c m_d} \quad (V-16)$$

Application to reaction $a + b \rightarrow 1 + 2 + \dots + N$, with the N outgoing particles being ordered by increasing rapidity.

1. The maximum rapidity gap between the outgoing particles has an approximate value

$$y_{1N} \approx \ln \frac{\sqrt{s}}{m_1 m_N} \quad (V-17)$$

2. The rapidity gap ($y_i - y_j$) between the rapidities y_i and y_j of particles i and j , respectively, is related to the invariant mass squared of particles i and j

$$s_{ij} = m_i^2 + m_j^2 + 2m_i m_j \cosh(y_i - y_j) = 2E_{i1}^2 \frac{p_{Tij}^2}{m_i^2} \quad (V-18)$$

$$y_i - y_j = \cosh^{-1} \left(\frac{E_{i1}^2 - m_i^2 - m_j^2 + 2E_{i1}^2 \frac{p_{Tij}^2}{m_i^2}}{2m_i m_j} \right) \quad (V-19)$$

3. The rapidity gap between either of the extreme particles (1 or N) and the adjacent neighbor (2 or (N-1) respectively) is, on a statistical basis, and in first approximation

$$|y_1 - y_2| \approx \ln \frac{\sqrt{s}}{m_1} \quad (V-20)$$

and

$$|y_{N-1} - y_N| \approx \ln \frac{\sqrt{s}}{m_N} \quad (V-21)$$

M_1 (M_N) being the missing mass with respect to particle 1 (N) in the reaction $a + b \rightarrow 1 + M_1$ ($a + b \rightarrow N + M_N$).

4. The rapidity gap y_{a1} between the incoming particle a and the outgoing particle closest to a, 1, is related to the transverse momentum $p_{a1} = (p_a - p_1)^2$:

$$y_a - y_1 = \cosh^{-1} \left(\frac{E_{a1}^2 - m_a^2 - m_1^2}{2m_a m_1} \right) \quad (V-22)$$

RELATIVISTIC KINEMATICS (Cont'd)

VI. INCLUSIVE REACTIONS AND SCALING VARIABLES:
DEFINITIONS AND KINEMATIC RELATIONS

A. Scaling Variable: Definitions

- For the inclusive reaction $a + b \rightarrow c + X$

- 1) Feynman's definition:

$$x = P_{||}^{cm} / P_{||max}^{cm} \quad (VI-3)$$

$$\text{where } P_{||max}^{cm} = \frac{\sqrt{\Delta(s, m_c^2, m_X^2)} m_X}{2\sqrt{s}} \quad (VI-4)$$

is the center-of-mass momentum of particle c and the lightest possible particle(s) $m_{X,min}$ which, consistent with conservation laws, could recoil against particle c.

A relationship between x and m_X^2 which becomes useful at high energy [see (VI-6)] derives from

$$m_X^2 = (p_a + p_b - p_c)^2 = s + m_c^2 - 2p_c \cdot (p_a + p_b)$$

- 2) An alternative definition:

$$x' = p_{||}^{cm} / p_{||max}^{cm} \quad (VI-7)$$

$$\text{with } P_{||max}^{cm} = \sqrt{\left(P_{||}^{cm}\right)^2 - P_{\perp}^2}$$

- For electro- and photoproduction:

- 1) Bjorken's definition:

$$\omega = \frac{2m_p v}{q^2}$$

$$\text{where } \begin{cases} v = e - e' = \text{the difference between the} \\ \text{incident and the outgoing energies} \\ m_p = \text{proton mass} \\ q^2 = \text{momentum transfer squared.} \end{cases}$$

(NOTE: ω^{-1} is sometimes also called x.)

- 2) Bloom-Gilman's definition:

$$\omega' = \omega + \frac{m^2 - q^2}{q^2}$$

B. For the inclusive reaction $a + b \rightarrow c + X$, the invariant differential cross section $E(d\sigma/dp_c^2)$ may be written in the following forms using various expressions of the phase space volume d^3p_c/E_c (and omitting the subscript c):

Longitudinal variable of particle c	Transverse variable of particle c	Corresponding invariant (differential) cross section $E(d\sigma/dp_c^2)$
$p_{ }^{lab}$	0^{lab}	$\frac{E^{lab}}{(P_{ }^{lab})^2} \frac{d^2\sigma}{dp_{ }^{lab} d_{\perp}^{lab}} \quad (VI-10)$
$P_{ }$	$P_{\perp}$	$\frac{E}{P_{ }} \frac{d^2\sigma}{dp_{ } dp_{\perp}^2} \quad (\text{averaged over azimuth}) \quad (VI-11)$
x	$P_{\perp}$	$\frac{P_{ }^{cm}}{P_{ max}^{cm}} \cdot \frac{1}{\pi} \frac{d^2\sigma}{dx dp_{\perp}^2} \approx \frac{2P_{ }^{cm}}{\pi\sqrt{s}} \frac{d^2\sigma}{dx dp_{\perp}^2} \quad (\text{averaged over azimuth}) \quad (VI-12)$
y	$P_{\perp}$ (or $m_{\perp}$)	$\frac{1}{\pi} \frac{d^2\sigma}{dy dp_{\perp}^2} = \frac{1}{\pi} \frac{d^2\sigma}{dy dm_{\perp}^2} \quad (\text{averaged over azimuth}) \quad (VI-13)$
$m_A^2 = (p_a + p_b - p_c)^2$	$\tau = (p_a - p_c)^2$	$\frac{\sqrt{\Delta(s, m_A^2, m_c^2)}}{\pi} \frac{d^2\sigma}{dm_A^2 d\tau} \quad (\text{averaged over azimuth}) \quad (VI-14)$

For more formulae and details, see references in this memo.

Revised and expanded April 1976.

- At high energy,

$$P_{||max}^{cm} \approx \frac{\sqrt{s}}{2} \quad \text{and} \quad x \approx \frac{2P_{||}^{cm}}{\sqrt{s}} \quad (VI-5)$$

The relation between x and y is trivial:

$$x \approx 2 \frac{P_{||}^{cm}}{\sqrt{s}} = \frac{2m_{\perp}}{\sqrt{s}} \sinh y^{cm} \quad (VI-8)$$

and, for y also large:

$$y^{cm} \approx 2 \ln \frac{\sqrt{s}}{m_{\perp}} \quad (VI-9)$$

$$x \approx 1 - \frac{m_{\perp}^2 - m_c^2}{s} \approx 1 - \frac{m_{\perp}^2}{s} \quad (VI-10)$$

(for $m_{\perp}^2 \gg m_c^2$).

For $P_{\perp}$ fixed as $s \rightarrow \infty$,

$$x' \approx \frac{2P_{||}^{cm}}{\sqrt{s}} \approx x$$

For q^2 and v large (Bjorken limit),

$$\omega' \approx \omega - 1$$

REFERENCES

- Byckling, E., and Kajantie, K., Particle Kinematics (John Wiley and Sons, New York, 1973).
- Chew, D. M., and Shankar, R., LBL-2467 (February 1974) and LBL-3012 (May 1974); Phys. Rev. D 11, 1036 (1975).
- Frazer, W. R., Ingber, L., Mehta, C. H., Poon, C. H. Silverman, D. J., Stowe, K., Ting, P. D., and Yesian, H. J., Rev. Mod. Phys. 44, 284 (1972).
- Hagedorn, R., Relativistic Kinematics (W. A. Benjamin, Inc., New York, 1964).
- Kasman, J., LBL-80 (1972), Part III.
- Pilkuhn, H., The Interactions of Hadrons (North-Holland, Amsterdam, 1967).