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Di-hedral Symmetry and Charge Conjugation*

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Abstract

The charge conjugation C and the parity P are assumed to generate a finite group without assuming any a priori commutation or anti-commutation relation. The group then generated is the di-hedral group D_n (n being even) which has two dimensional irreducible representations. This implies that we could have in principle, elementary particles with parity and charge-conjugation doublet structure.

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The charge (or particle-antiparticle) conjugation C and the parity P are ordinarily assumed either to commute with each other for the case of bosons or to anti-commute for the case of fermions. Although this semi-theoretical rule is experimentally found to be generally valid, it will be of some interest to investigate more general possibilities.

First, we assume on the physical ground, that C and P are involutions, i.e.,

$$C^2 = P^2 = I \quad (1)$$

where I is the identity operation. Next, for the sake of simplicity, we demand that a group generated by C and P is finite. There is then a minimal positive integer n such that we have

$$(CP)^n = I. \quad (2)$$

When we set

$$a = P, b = CP \quad (3)$$

then Eqs. (1) and (2) are rewritten as

$$a^2 = b^n = I, \quad b a b = a. \quad (4)$$

The group generated by two elements a and b satisfying (4) is precisely¹ the di-hedral group D_n which is one of the crystallo-graphic point group.

We next investigate irreducible representations of the group D_n . We then have to distinguish two possible operations implied by C . We may regard C to directly act on Hilbert space of physical particle states. Then, the operation C is linear. On the other hand, if we are interested in the action of C upon field operator $\phi(x)$, transforming it into its complex conjugate $\phi^*(x)$, then C behaves as being anti-linear.

The second case is more complicated, and we shall first consider the case of linear realizations. All linear irreducible representations of the group D_n are well-known¹ to be either one-dimensional or two-dimensional. Since one-dimensional representations do not present any particular new feature, we restrict ourselves to two-dimensional cases where explicit matrix representations are given by²

$$U(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(b) = \begin{pmatrix} \omega & 0 \\ 0 & 1/\omega \end{pmatrix}. \quad (5)$$

Above, ω is any non-trivial (i.e., $\omega \neq \pm 1$) solution of

$$\omega^n = 1. \quad (6)$$

Noting Eq. (3), this gives

$$U(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U(C) = \begin{pmatrix} 0 & \omega \\ 1/\omega & 0 \end{pmatrix}. \quad (7)$$

It is now apparent that we must have automatically a parity and charge-conjugation doublet. To see its physical significance, it is more convenient to consider a representation in which the parity P is diagonal:

$$\tilde{U}(P) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{U}(C) = \begin{pmatrix} \cos\theta & i \sin\theta \\ -i \sin\theta & -\cos\theta \end{pmatrix} \quad (8)$$

where θ is the phase of ω (i.e., $\omega = \exp i\theta$) and θ can assume any value of the form $\theta = 2\pi p/n$, p being any integer. From (8), we see that by the charge conjugation, two opposite parity states now mix together in a rather complicated fashion in contrast to ordinary bosons or fermions.

Next, let us investigate action of C on field operators. As an illustration, we consider here l second-quantized two-component Weyl spinor fields $\phi_j(x)$ ($j=1,2,\dots,l$). We then demand existence of second-

We may indeed check that this, together with (9), defines the D_4 symmetry. The present solution corresponds to the usual Dirac theory, if we define four-component Dirac spinor $\psi(x)$ and Dirac matrices γ_μ ($\mu=4$ or $\mu=k=1,2,3$) by

$$\psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \quad (12)$$

As a matter of fact, (9) can be rewritten now as:

$$\begin{aligned} C: \quad \psi(x) &\rightarrow C \bar{\psi}(x) \\ P: \quad \psi(\underline{x}, x_0) &\rightarrow \gamma_4 \psi(-\underline{x}, x_0) \end{aligned} \quad (13)$$

where C in the right side of (13) stands for the standard 4×4 charge-conjugation matrix which is given in the Weyl representation by

$$C = i \gamma_2 \gamma_4 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}. \quad (14)$$

Next, we can find some (but perhaps not all) four-dimensional irreducible representations of (10) as follows. Instead of four two-component Weyl fields, we use two Dirac spinors $\psi_1(x)$ and $\psi_2(x)$, and consider the following transformation:

$$C: \quad \begin{cases} \psi_1(x) \rightarrow \omega C \bar{\psi}_1(x) \\ \psi_2(x) \rightarrow -\frac{1}{\omega} C \bar{\psi}_2(x) \end{cases} \quad (15a)$$

$$P: \quad \begin{cases} \psi_1(\underline{x}, x_0) \rightarrow \gamma_4 \psi_2(-\underline{x}, x_0) \\ \psi_2(\underline{x}, x_0) \rightarrow \gamma_4 \psi_1(-\underline{x}, x_0) \end{cases} \quad (15b)$$

where the phase ω satisfies

$$\omega^{2n} = 1. \quad (16)$$

Then, we may verify that (15) defines the D_{2n} symmetry.

quantized unitary operators U_C and U_P satisfying the following transformation properties:

$$C: \quad \phi_j(x) \rightarrow \phi'_j(x) = U_C \phi_j(x) U_C^{-1} = \sum_{k=1}^{\ell} \sigma_2 \phi_k^+(x) U_{kj}(C) \quad (9a)$$

$$P: \quad \phi_j(\underline{x}, x_0) \rightarrow \phi'_j(\underline{x}, x_0) = U_P \phi_j(\underline{x}, x_0) U_P^{-1} = \sum_{k=1}^{\ell} \phi_k(-\underline{x}, x_0) U_{kj}(P) \quad (9b)$$

where $U_{kj}(C)$ and $U_{kj}(P)$ are $\ell \times \ell$ matrices to be determined, and where σ_2 is the standard 2×2 Pauli matrix operating in the two-component spinor $\phi_k(x)$. As before, we require that C and P operations form D_{2n} (not D_n now) group with $C^2 = P^2 = (CP)^{2n} = I$. Note that we changed n into $2n$ here since the odd n case is unphysical. Then, $\ell \times \ell$ matrices $U(C)$ and $U(P)$ must satisfy modified relations

$$\begin{aligned} -U^*(C)U(C) &= U(P)U(P) = E, \\ [U^*(CP)U(CP)]^n &= (-1)^n E, \end{aligned} \quad (10)$$

Here, E stands for the $\ell \times \ell$ unit matrix and $U^*(C)$ and $U^*(CP)$ are complex conjugate (but not hermitian conjugate) matrices of $U(C)$ and $U(CP)$, respectively. Equation (10) determines then anti-linear realizations³ of the D_{2n} group.

Unfortunately, it is quite difficult to determine general solutions of (10) and we have to be content with special cases. First, let us investigate two-dimensional irreducible representation (i.e., $\ell=2$). Then, we find after some calculations that only possible solution is for n being an even integer with

$$U(C) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (11)$$

So far, we have not discussed other symmetry groups such as $SU(3)$ or baryonic gauge group $U_B(1)$. In order to include such a group into our consideration, we must know the action of C and P upon $SU(3)$ and $U_B(1)$. With respect to the parity P , it is reasonable to suppose that it commutes with all these internal groups. However, for C , it is more complicated. A physically reasonable ansatz is to assume⁴ that C acts as an automorphism of the group $SU(3) \otimes U_B(1)$ under the operation of C . Then, the physical group is the semi-direct product

$$G = [SU(3) \otimes U_B(1)] \overline{S} D_n. \quad (17)$$

Linear representations of G are easily obtained as follows. Let us label state vectors of irreducible representations of the group $SU(3) \otimes U_B(1)$ as

$$\alpha = \{(p,q), I, I_3, Y, B\}, \quad (18)$$

where p and q specify the standard representation $D(p,q)$ of the $SU(3)$. Then the charge conjugation C will change signs of all additive quantum numbers, such as I_3 , the third component of the isospin I , the hypercharge Y , and the baryonic number B . This implies⁴ that C acts as the outer automorphism of the $SU(3) \otimes U_B(1)$ so that C interchanges $p \leftrightarrow q$ also. Therefore, we have

$$C: \alpha \rightarrow \tilde{\alpha} = \{(q,p), I, -I_3, -Y, -B\}. \quad (19)$$

The linear induced representation of G is now given by

$$\begin{aligned} C|\alpha, j\rangle &= \sum_{k=1}^2 |\tilde{\alpha}, k\rangle U_{kj}(C), \\ P|\alpha, j\rangle &= \sum_{k=1}^2 |\alpha, k\rangle U_{kj}(P) \end{aligned} \quad (20)$$

where $U_{kj}(C)$ and $U_{kj}(P)$ ($k,j=1,2$) are the two-dimensional matrices given by (7) or (8).

In the above, we discussed true representations of the group G . But strictly speaking, we must consider more general projective representations. This problem is complicated and the ready answer does not seem to be available in literature.⁵ However, we notice that the projective irreducible representations of the di-hedral group D_n are known⁶ to be still one or two dimensional. Also, if n is odd, the projective representations of the D_n are equivalent to true representations although this is not so for more interesting cases of even integral n .

In conclusion, we could have in principle a new type of particle with parity and charge conjugation doublet structures. So far, no such particles are experimentally discovered. Also, we did not discuss the implication of adding the time-reversal T into our scheme. However, if the TCP theorem is valid, then the time-reversal will not lead to any new complication.

Finally, we simply remark that some other crystallographic groups such as octa-hedral group have been previously discussed by Case, Karplus and Yang⁷ some years ago in a different context of strangeness quantum number.

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