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Boundary Layer Solutions for Natural Convection in Porous Media

Harry C. Hardee

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BOUNDARY LAYER SOLUTIONS FOR
NATURAL CONVECTION IN POROUS MEDIA

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ABSTRACT

An approximate solution is developed for natural convection in porous media resulting from localized heat sources. The method is applicable for moderate levels of convection. Results of the method compare well with those of other calculational methods and with experimental data. The solution for a spherical source can be used as an indirect method for the determination of permeability in some cases.

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BOUNDARY LAYER SOLUTIONS FOR NATURAL CONVECTION IN POROUS MEDIA

Introduction

There are a number of current problems which involve natural convection in porous media. A few examples are:

1. Geothermal energy recovery from hot porous formations.
2. Fuel debris bed cooling in LMFBR (liquid metal fast breeder reactor) accidents.
3. Radioactive waste disposal in deep ocean sediments.
4. Oil shale recovery problems.

At low levels of heating, the heat transfer will occur primarily by conduction but, at higher heat levels, natural convection will occur in the porous media. Computer codes exist in a few cases which can be applied to some of these problems. There are also a few analytical solutions such as the case of the line source in a porous media discussed by Wooding¹ and Yih.² At moderate heat transfer levels, it is possible to use boundary-layer techniques to generate approximate solutions for various types of heat sources in porous media. This paper describes the use of the boundary-layer technique for the cases of vertical walls or cylinders, horizontal cylinders, and spherical heat sources in porous media. The results of the boundary-layer technique are compared with other calculations and experiments. Some special experiments were run for the spherical heat source geometry. It appears that the spherical solution might be used as an indirect technique for the measurement of permeability.

Natural Convection Adjacent to a Vertical Surface in a Porous Medium

An approximate analysis similar to an integral boundary layer analysis can be used on this type of problem when the Rayleigh number is sufficiently high and when Darcy's law is valid.

The momentum and energy equations are:^{3,4}

$$\frac{d}{dx} \int_0^\delta u^2 dy = g\beta \int_0^\delta \phi dy - \frac{\delta}{\rho} \frac{dP}{dx} - \nu \left(\frac{du}{dy} \right)_{\text{wall}} \quad (1)$$

$$\frac{d}{dx} \int_0^\delta u\phi dy = -\alpha_m \left(\frac{d\phi}{dy} \right)_{\text{wall}} , \quad (2)$$

where ϕ is the dimensionless temperature profile, assumed to be of the form^{2,3}

$$\phi = \phi_w \left(1 - \frac{y}{\delta} \right)^2 \quad \phi_w = T_w - T_F \quad (3)$$

and u is the velocity profile, assumed to be of the form^{4,5}

$$u = 12 \bar{u} \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 , \quad (4)$$

where $\bar{u}$ is the average velocity in the boundary layer. This average velocity is determined by integrating Equation (4) over the boundary-layer thickness, where

$$\int_0^\delta \frac{y}{\delta} \left(1 - \frac{y}{\delta} \right)^2 d\left(\frac{y}{\delta}\right) = 1/12 . \quad (5)$$

For porous natural convection, some of the terms in Equation (1) are negligible, and this equation can be reduced to

$$g\beta \int_0^\delta \phi dy = \frac{\delta}{\rho} \frac{dP}{dx} . \quad (6)$$

From Darcy's law

$$\bar{u} = \frac{\kappa}{\mu} \frac{dP}{dx} , \quad (7)$$

where κ is permeability and μ is the absolute viscosity. Substituting Equations (3) and (7) into Equation (6) gives

$$\frac{1}{3} g \beta \phi_w = \bar{u} \frac{\mu}{\rho \kappa}$$

or

$$\bar{u} = \frac{1}{3} \frac{g \beta \rho \phi_w}{\mu} \kappa \quad (8)$$

Then substituting Equations (3) and (4) into Equation (2) gives

$$\frac{1}{30} \phi_w \frac{d}{dx} (12 \bar{u} \delta) = 2 \alpha_m \frac{\phi_w}{\delta}$$

or

$$\frac{d\delta}{dx} = 5 \frac{\alpha_m}{\delta \bar{u}}$$

The solution is

$$\delta = \sqrt{\frac{10 \alpha_m x}{\bar{u}}} \quad (9)$$

and substituting Equation (8) into Equation (9) gives

$$\delta = \sqrt{\frac{30}{\kappa}} Ra_L^{-1/2} \sqrt{\frac{\alpha_m}{\alpha_f}} \sqrt{L^3 X} \quad (10)$$

where Ra_L is the Rayleigh number

$$Ra_L = \frac{g \beta \rho \phi_w L^3}{\mu \alpha_f} \quad (11)$$

α_f is the thermal diffusivity of the fluid, and α_m is the thermal diffusivity of the combined porous medium.

The heat flux at the wall is

$$q/A = -K_m \left(\frac{\partial \phi}{\partial y} \right)_{\text{wall}} = \frac{2K_m \phi_w}{\delta} = h \phi_w ,$$

and the convection heat transfer coefficient h is

$$h = \frac{2K_m}{\delta} . \quad (12)$$

The average value of h over a length L is

$$\bar{h} = \int_0^L \frac{2K_m}{\delta} dx ,$$

and substituting in Equation (9) for δ and integrating gives

$$\begin{aligned} \bar{h} &= \frac{2K_m}{\sqrt{\frac{10}{\alpha_m} \bar{u}}} \int_0^L x^{-1/2} dx \\ \bar{h} &= \frac{4K_m}{\delta_L} , \end{aligned} \quad (13)$$

where δ_L is the boundary-layer thickness at the point $x = L$. The average heat flux is then

$$q/A = \bar{h} \phi_w = \frac{4K_m \phi_w}{\delta_L}$$

or

$$q/A = 4K_m \sqrt{\frac{\kappa}{30}} \frac{\phi_w}{L} Ra_L^{1/2} \sqrt{\frac{\alpha_f}{\alpha_m}} . \quad (14)$$

Since

$$\alpha_m = \frac{K_m}{(\rho c)_m}$$

and $(\rho c)_m$ is approximately constant for many materials,⁶ Equation (14) can be written

$$q/A = 0.730 \frac{\sqrt{\kappa} \phi_w}{L^2} \sqrt{K_m K_f Ra_L} . \quad (15)$$

The boundary-layer solution can be used for either a vertical wall or a vertical cylinder. In order for the boundary-layer method to be accurate, however, the final boundary-layer thickness should be smaller than the height of the wall or

$$\delta/L \leq 1 .$$

Using Equation (10) for δ when $x = L$ results in the condition

$$\kappa \geq \frac{30 L^2 K_m}{Ra_L K_f} , \quad (16)$$

and the actual permeability must be greater than this value.

For a horizontal cylinder instead of a vertical wall, we can let

$$L = \pi R ,$$

and the expected heat flux would be

$$q/A = 0.412 \frac{\sqrt{\kappa} \phi_w}{R^2} \sqrt{K_m K_f Ra_R} , \quad (17)$$

where

$$Ra_R = \frac{\rho g \beta \phi_w R^3}{\mu \alpha_f} \quad (18)$$

Natural Convection in a Porous Medium Surrounding a Horizontal Cylindrical Heat Source

An approximate boundary-layer analysis can be made for the horizontal cylinder problem similar to the previous analysis for the vertical surface. In order for this solution to be valid, Darcy's law must apply and the Rayleigh number must be high enough such that the boundary-layer thickness will be small relative to the cylinder radius. The same momentum and energy equations apply as in the previous analysis for the vertical surface except that the buoyant force varies as $\sin \theta$.

$$g\beta \sin \theta \int_0^\delta \phi \, dy = \frac{\delta}{\rho} \frac{dP}{dx} \quad (19)$$

$$\frac{d}{dx} \int_0^\delta u \phi \, dy = -\alpha_m \left(\frac{d\phi}{dy} \right)_{\text{wall}} \quad (20)$$

The temperature and velocity profiles are assumed to be the same form as before in Equations (3) and (4). Substituting Equations (3) and (7) into Equation (19) gives

$$\bar{u} = \frac{1}{3} \frac{g\beta\phi_w}{\mu} \kappa \sin \theta \quad (21)$$

Substituting Equations (3) and (4) into Equation (20) gives

$$\frac{d(u\delta)}{d\theta} = 5 \frac{\alpha_m}{\delta} \frac{dx}{d\theta}, \quad (22)$$

and substituting Equation (21) into Equation (22) gives

$$\frac{d\delta}{d\theta} \sin \theta + \delta \cos \theta = 15 \frac{\alpha_m}{\delta} \frac{\mu}{\rho g \beta \phi_w \kappa} \frac{dx}{d\theta}. \quad (23)$$

Since

$$x = R\theta$$

$$\frac{dx}{d\theta} = R,$$

then Equation (23) becomes

$$\frac{d\delta}{d\theta} + \delta \frac{\cos \theta}{\sin \theta} = 15 \frac{\mu \alpha_m}{\rho g \beta \phi_w \kappa} \frac{R}{\delta \sin \theta}.$$

It can be shown that the solution to this equation is

$$\delta = R^2 \sqrt{\frac{30}{\kappa}} Ra_R^{-1/2} \sqrt{\frac{\alpha_m}{\alpha_f}} \sqrt{\frac{1 - \cos \theta}{\sin \theta}}, \quad (24)$$

where Ra_R is the Rayleigh number based on radius as defined in Equation (18). Again, since $\alpha_m = K_m / (\rho c)_m$ and ρc is approximately constant, Equation (24) can be written

$$\delta = R^2 \sqrt{\frac{30}{\kappa}} Ra_R^{-1/2} \sqrt{K_m / K_f} \sqrt{\frac{1 - \cos \theta}{\sin \theta}}. \quad (25)$$

The convective heat transfer coefficient is given again by Equation (12).

$$h = 2 \frac{K_m}{\delta}, \quad (12)$$

and the average heat transfer coefficient over the entire cylindrical region is

$$\bar{h} = 2K_m \frac{\int_0^\pi \frac{1}{\delta} R d\theta}{\pi R}.$$

Using Equation (25), this becomes

$$\bar{h} = 2K_m \sqrt{\frac{\kappa K_f}{30 K_m}} \frac{Ra_R^{1/2}}{R^2} \frac{\int_0^\pi R \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta}{\pi R}.$$

Since

$$\int_0^\pi \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta = 2 \left[\sqrt{1 - \cos \theta} \right]_0^\pi = 2\sqrt{2},$$

then

$$\bar{h} = 0.329 \frac{\sqrt{\kappa}}{R^2} \sqrt{K_m K_f Ra_R}. \quad (26)$$

The heat flux per unit area is

$$q/A = \bar{h} \phi_w$$

or

$$q/A = 0.329 \frac{\sqrt{\kappa \phi_w}}{R^2} \sqrt{K_m K_f Ra_R}. \quad (27)$$

This result for the horizontal cylinder is the same as the earlier estimate of Equation (17) except for a slightly different coefficient.

In order to satisfy the conditions for a boundary-layer-type flow and obtain good accuracy, Equation (27) should be used when

$$\delta/R \leq 1 \quad \text{at } \theta = \pi/2, \quad (28)$$

and using Equation (25) gives

$$\delta/R = R \sqrt{\frac{30}{\kappa}} Ra_R^{-1/2} \sqrt{K_m/K_f} \leq 1$$

$$\kappa \geq \frac{30 R^2 K_m}{Ra_R K_f} . \quad (29)$$

Equation (27) will become increasingly inaccurate as the permeability falls below the limiting value given by Equation (29)!

A thermal plume will develop in the porous media above the heated cylinder. The upward mass flow in this plume per unit length of cylinder is approximately

$$G/L = 2\delta \bar{U} \rho ,$$

and using Equation (21) for $\bar{U}$ and Equation (24) for δ at the point $\theta = \pi/2$ gives

$$G/L = 3.651 (\rho/R) \sqrt{\kappa Ra_R \alpha_m \alpha_f} . \quad (30)$$

The centerline velocity in the plume directly above the source is then approximately

$$U_c = \frac{G}{2RL\rho} ,$$

and using Equation (30) this becomes

$$U_c = \frac{1.826}{R} \sqrt{\kappa Ra_R \alpha_m \alpha_f} . \quad (31)$$

Natural Convection in a Porous Medium Surrounding a Spherical Heat Source

The integral boundary-layer equations for a three-dimensional boundary layer are given by Schlichting.⁷ For a spherical heat source, these equations become

$$\frac{d}{dx} \int_0^\delta U^2 dy + \frac{1}{r} \frac{dr}{dx} \int_0^\delta U^2 dy = g\beta \sin \theta \int_0^\delta \phi dy - \nu \left(\frac{du}{dy} \right)_{\text{wall}} - \frac{\delta}{\rho} \frac{dP}{dx}$$

$$\frac{d}{dx} \int_0^\delta U\phi dy + \frac{1}{r} \frac{dr}{dx} \int_0^\delta U\phi dy = -\alpha_m \left(\frac{d\phi}{dy} \right)_{\text{wall}},$$

where

$$x = R\theta$$

$$r = R \sin \theta$$

For porous flow, the terms

$$\int_0^\delta U^2 dy$$

$$\nu \left(\frac{du}{dy} \right)_{\text{wall}}$$

are assumed to be negligible as before. With the same assumptions as in the previous solutions

$$\phi = \phi_w \left(1 - \frac{y}{\delta}\right)^2$$

$$U = \frac{\bar{U}}{12} \frac{y}{\delta} \left(1 - \frac{y}{\delta}\right)^2$$

$$\bar{U} = \frac{\kappa}{\mu} \frac{dP}{dx}.$$

The boundary-layer equations become

$$g\beta \sin \theta \int_0^\delta \phi \, dy = \frac{\delta}{\rho} \frac{\mu \bar{U}}{\kappa}$$

$$1/3 \, g\beta \phi_w \sin \theta = \bar{U} \frac{\mu}{\rho \kappa}$$

$$\bar{U} = 1/3 \frac{\rho g \beta \phi_w}{\mu} \kappa \sin \theta \quad (32)$$

and

$$\frac{1}{30} \phi_w \frac{d}{dx} (12 U \delta) + \frac{1}{30} \phi_w \frac{\cos \theta}{\sin \theta} 12 \bar{U} \delta = 2 \alpha_m \frac{\phi_w}{\delta}$$

$$x = R\theta \quad dx = R d\theta$$

$$\frac{d(U\delta)}{d\theta} + \frac{\cos \theta}{\sin \theta} \bar{U} \delta = 5 \frac{\alpha_m R}{\delta}$$

$$\frac{d(\delta \sin \theta)}{d\theta} + \frac{\cos \theta}{\sin \theta} \delta \sin \theta = 15 \frac{\mu \alpha_m}{\rho g \beta \phi_w R^3} \frac{R^4}{\kappa \delta}$$

$$\frac{\delta}{R^2} \frac{d\delta}{d\theta} + 2 \frac{\cos \theta}{\sin \theta} \frac{\delta^2}{R^2} = \frac{15}{Ra_R} \frac{R^2}{\kappa \sin \theta} \frac{\alpha_m}{\alpha_f} \quad (33)$$

Using the substitution

$$Z = \frac{\delta^2}{R^2} ,$$

Equation (33) reduces to

$$dZ + 4 \frac{\cos \theta}{\sin \theta} Z d\theta = \frac{30}{Ra_R} \frac{R^2}{\kappa \sin \theta} \frac{\alpha_m}{\alpha_f} d\theta .$$

Then multiplying by $\sin^4 \theta$ results in

$$d(Z \sin^4 \theta) = \frac{30}{Ra_R} \frac{R^2 \alpha_m}{\kappa \alpha_f} \sin^3 \theta d\theta ,$$

and integrating gives

$$\frac{\delta}{R} = \sqrt{\frac{30 R^2 \alpha_m}{Ra_R \kappa \alpha_f}} \frac{\left[\int_0^\theta \sin^3 \theta d\theta \right]^{1/2}}{\sin^2 \theta} . \quad (34)$$

Since

$$h\phi_w = K_m \left(\frac{d\phi}{dy} \right)_{y=0}$$

$$h = 2 \frac{K_m}{\delta} ,$$

as before, and the average convection coefficient $\bar{h}$ is

$$\bar{h} = \frac{\int_0^\pi 2\pi R^2 h \sin \theta d\theta}{4\pi R^2} = \frac{1}{2} \int_0^\pi h \sin \theta d\theta ,$$

then

$$\bar{h} = \frac{K_m}{R} \sqrt{\frac{Ra_R \kappa \alpha_f}{30 R^2 \alpha_m}} \int_0^\pi \frac{\sin^3 \theta}{\left[\int_0^\theta \sin^3 \theta d\theta \right]^{1/2}} d\theta .$$

Using the substitution

$$Z = \int_0^\theta \sin^3 \theta d\theta \quad dZ = \sin^3 \theta ,$$

the integral becomes

$$I = \int_0^\pi \frac{dZ}{Z^{1/2}} = 2 \left[\sqrt{Z} \right]_0^\pi$$

$$I = 2 \left[\int_0^\pi \sin^3 \theta d\theta \right]^{1/2} = 2 \sqrt{\left[-1/3 \cos \theta (\sin^2 \theta + 2) \right]_0^\pi}$$

$$I = \frac{4}{\sqrt{3}}$$

and, if ρc is approximately constant,

$$\bar{h} = 4/3 \frac{1}{R^2} \sqrt{\frac{Ra_R \kappa K_m K_f}{10}} . \quad (35)$$

The surface heat flux is then

$$q/A = \bar{h} \phi_w$$

$$q/A = 0.4216 \frac{\sqrt{\kappa} \phi_w}{R^2} \sqrt{K_m K_f Ra_R} . \quad (36)$$

For the boundary-layer method to be applicable

$$\delta/R \leq 1 \quad \text{when } \theta = \pi/2 ,$$

and using Equation (34) for δ gives

$$\sqrt{\frac{30 R^2 K_m}{Ra_R \kappa K_f}} \sqrt{2/3} \leq 1$$

or

$$\kappa \geq \frac{20 R^2 K_m}{Ra_R K_f} . \quad (37)$$

The mass flow in the thermal plume above the sphere can be estimated using

$$G = 2\pi R \delta \rho \bar{U}$$

and, using Equations (32) and (34) for $\bar{U}$ and δ at $\theta = \pi/2$, this gives

$$G = 9.366 \rho \sqrt{\kappa Ra_R \alpha_m \alpha_f} . \quad (38)$$

The centerline velocity in the plume directly above the source is then approximately

$$U_c = \frac{G}{\pi R \rho}$$

and, using Equation (39), this becomes

$$U_c = \frac{2.981}{R} \sqrt{\kappa Ra_R \alpha_m \alpha_f} . \quad (39)$$

Limitations on the Boundary-Layer Technique

As the permeability decreases, the boundary-layer thickness grows in relation to its length until it no longer looks like a boundary layer or until the boundary-layer thickness becomes much larger than the heat source, in the case of the sphere or horizontal cylinder. When this happens, the problem tends to approach the ordinary conduction situation. For example, in the spherical problem, the limit for accurate use of the boundary-layer technique was shown from Equation (39) to be

$$\kappa \geq \frac{20 R^2 K_m}{Ra_R K_f} . \quad (39)$$

Assuming $K_m \approx K_f$, this condition becomes

$$\kappa \geq 20 \frac{R^2}{Ra_R}$$

$$\sqrt{\kappa Ra_R} \geq \sqrt{20} R . \quad (40)$$

The heat-flux result for the sphere, from Equation (36), can be written

$$\sqrt{\kappa Ra_R} = q/A \frac{R^2}{(0.4216)K\phi_w} ,$$

and substituting this result into Equation (40) gives

$$q/A \frac{R^2}{(0.4216)K\phi_w} \geq 20R$$

or

$$q/A \geq \frac{1.88 K\phi_w}{R} . \quad (41)$$

Carslaw and Jaeger⁸ show that the ordinary conduction solution for a spherical heat source at steady state in an infinite medium is

$$(q/A)_{\text{cond.}} = \frac{K\phi_w}{R} . \quad (42)$$

The limiting heat-flux condition for the porous media boundary-layer solution is therefore about twice the ordinary conduction value

$$(q/A)_{\text{limit}} = 1.88 (q/A)_{\text{Cond.}} . \quad (43)$$

At very high flow rates, the flow will become nonlinear-laminar and eventually turbulent as pointed out by Bear.⁹ When this happens, Darcy's law no longer applies. This limiting condition for Darcy's law can be specified in terms of a Reynold's number limit.⁹

$$\text{Re} = \frac{\bar{U}d}{\nu} \leq 10. \quad (44)$$

Using Equation (8) for $\bar{U}$ and the definition for Rayleigh number from Equation (18), this condition becomes

$$\frac{\text{Ra}_R \kappa}{R^2} \leq 30 (\text{Pr } R/d), \quad (43)$$

where Pr is the Prandtl number of the fluid

$$\text{Pr} = \frac{\mu c}{K_f} . \quad (44)$$

Comparing Equation (43) with Equations (29) and (37), we see that the boundary-layer method is generally valid when

$$1 \leq \frac{\text{Ra}_R \kappa}{30 R^2} \leq \text{Pr } R/d. \quad (45)$$

At the lower limit, the boundary layer becomes too thick and the boundary-layer approximation breaks down. At the upper limit, the flow becomes turbulent and Darcy's law no longer applies. For liquids, the Prandtl number will usually be greater than one. The ratio of the source radius R to particle diameter d will generally be greater than 1000 for typical situations. Equation (45) then gives a range of at least 1000 for $\text{Ra}_R \kappa / R^2$ for which the boundary-layer solution is valid.

Some authors mention a lower limit for Darcy's law.⁹ This is attributed to the non-Newtonian behavior of water and occurs at small values of hydraulic gradient. Apparently the worse case occurs for clay soils with very low permeability. Bear⁹ indicates that, for clay soils, the minimum value of the hydraulic gradient may be as high as 30 or

$$\frac{1}{\rho} \frac{dP}{dx} \geq 30 \quad (46)$$

and for Darcy's law to hold, Equation (46) must be satisfied. Elimination of $\bar{U}$ from Equations (7) and (8) gives

$$\frac{dP}{dx} = \frac{1}{3} g \rho \beta \phi_w ,$$

and substituting this into Equation (46) gives

$$\beta \phi_w \geq 90 \quad (47)$$

as the corresponding condition for natural convection in fine clay soils.

Since $\beta \phi_w$ is generally much less than 90 for typical situations, this suggests that Darcy's law would not apply for very fine clay soils. Such soils have very low values of permeability, and the earlier condition of Equation (29) or (37) has already excluded such flows from the boundary-layer solutions anyway.

Comparison of the Boundary-Layer Results with Experiments and Other Analytical Techniques

The SCHAFF code developed at Lawrence Berkeley Lab is a numerical technique for solving general heat transfer problems in porous media. This code is an extension of the TRUMP code and was used by Witherspoon¹⁰ to calculate heat extraction rates for a geothermal downhole heat exchanger design. A comparison of Witherspoon's result and those using the boundary-layer solution is shown in Table I. (The boundary-layer calculations are included in the appendix.) The agreement between the boundary-layer solution and Witherspoon's calculations is good when the permeability is above the lower-limit value of Equation (29). As the permeability drops below this limit, as in the case when $\kappa = 1$ Darcy, the boundary-layer method begins to give answers that are too low.

TABLE I
Comparison of Boundary-Layer Method
with Numerical Calculations
for a Horizontal Cylinder in a Porous Media

	κ Lower Limit Eq (29)	Average Heat Flux q/A Eq (27)	Witherspoon's Calculated ⁵ q/A
Downhole Heat Exchanger $\kappa = 1$ darcys $\phi_w = 170^\circ\text{F}$	4 darcys	$0.693 \frac{\text{kW}}{\text{m}^2}$	$1.1 \frac{\text{kW}}{\text{m}^2}$
Downhole Heat Exchanger $\kappa = 10$ darcys $\phi_w = 170^\circ\text{F}$	4 darcys	$2.19 \frac{\text{kW}}{\text{m}^2}$	$1.9 \frac{\text{kW}}{\text{m}^2}$

Schrock et al.¹¹ ran a series of experiments on heated cylinders in a porous medium. A comparison of some of their experimental results with the boundary-layer method is shown in Table II. (An example boundary-layer calculation is included in the appendix.) The agreement between the boundary-layer method and the test data is again good when the permeability is higher than the limiting value of Equation (29) as in the case of Monterey Sand with $\phi_w = 95^\circ\text{C}$. Again, when the permeability is lower than the limit value, the boundary-layer method predicts heat flux values that are too low. It should be pointed out that the experiments of Schrock et al.¹¹ were run for finite length cylinders, but Equation (27) was developed for a horizontal cylinder of infinite length. The end effects may cause some of the differences between the theoretical and experimental results.

TABLE II

Comparison of Boundary-Layer Method with Experiments
on Heated Horizontal Cylinders in Porous Media

	κ Lower Limit Eq (29)	Average Heat Flux q/A Eq (27)	Schrock, et al. Experimental ⁶ q/A
Monterey Crystal Amber Sand $\kappa = 373$ darcys $\phi_w = 95^\circ\text{C}$	157 darcys	$1.15 \frac{w}{\text{cm}^2}$	$1.0 \frac{w}{\text{cm}^2}$
Monterey Crystal Amber Sand $\kappa = 373$ darcys $\phi_w = 50^\circ\text{C}$	293 darcys	$0.22 \frac{w}{\text{cm}^2}$	$0.40 \frac{w}{\text{cm}^2}$
Ottawa Silica Sand $\kappa = 13.7$ darcys $\phi_w = 80^\circ\text{C}$	344 darcys	$0.234 \frac{w}{\text{cm}^2}$	$0.50 \frac{w}{\text{cm}^2}$

Wooding^{1,2} solved the problem of a line heat source in a porous medium using an analytical technique. It is not possible to compare the surface temperature of the line source as a function of heat flux with the boundary-layer method because the line source would have an infinite temperature at its origin. It is possible, however, to compare plume velocities calculated by the two methods. An example problem was chosen where the properties were:

$$K = 0.00191 \frac{\text{cal}}{\text{cm-s-}^\circ\text{C}}$$

$$\alpha = 0.0023 \frac{\text{cm}^2}{\text{s}}$$

$$\mu = 0.018 \frac{\text{gm}}{\text{cm-s}}$$

$$\rho = 1.019 \frac{\text{gm}}{\text{cm}^3}$$

$$\beta = 2.55 \times 10^{-4} \text{ } 1/^\circ\text{C}$$

$$\kappa = 150 \text{ darcys } (1.48 \times 10^{-6} \text{ cm}^2)$$

A comparison of the plume velocities for two cases is shown in Table III. The approximate plume velocity determined from the boundary-layer method agrees reasonably well with the exact value determined from the line source solution.

TABLE III
Comparison of Plume Velocity for a Horizontal
Cylindrical Heat Source in a Porous Media

	κ Lower Limit Eq (29)	Approximate Plume Velocity Eq (31)	Plume Velocity From Line Source Solution ^{1, 2}
$q/L = 3 \frac{\text{kW}}{\text{m}}$ $R = 16.7 \text{ cm}$ $\kappa = 150 \text{ darcys}$	106 darcys	5.87 cm/hr	$6.59 \frac{\text{cm}}{\text{hr}}$ 2 m. above source $1.80 \frac{\text{cm}}{\text{hr}}$ 100 m above source
$q/L = 1 \frac{\text{kW}}{\text{m}}$ $R = 25 \text{ cm}$ $\kappa = 150 \text{ darcys}$	117 darcys	3.10 cm/hr	$2.88 \frac{\text{cm}}{\text{hr}}$ 2 m above source $1.08 \frac{\text{cm}}{\text{hr}}$ 100 m above source

An experiment was run with a spherical heater in a porous medium of fine abrasive sand saturated with water. The grain diameter of the sand was measured to be around 200 microns. Using the empirical correlation of Bear,¹² the permeability should be

$$\kappa = 0.617 \times 10^{-11} d^2 \text{ cm}^2, \text{ where } d \text{ is in microns} \quad (48)$$

$$\kappa = 2.468 \times 10^{-7} \text{ cm}^2 (25 \text{ darcys}) . \quad (49)$$

The results of the experiment are compared with the boundary-layer calculations in Table IV. The calculated values agree roughly with the experimental values, although they are slightly low. The measured values agree even closer with the ordinary conduction prediction of Equation (42). This result is expected since the actual permeability (25 darcys) is below the lower limit for the boundary-layer solution.

TABLE IV

Comparison of Boundary-Layer Method with Experimental Results
for a Heated Sphere in Fine Abrasive Sand

	κ Lower Limit Eq (29)	Average Heat Flux, q/A Eq (27)	Experimental Heat Flux	Conduction Heat Flux Eq (42)
$Q = 20$ watts $\phi_w = 55^\circ\text{C}$ $\kappa = 25$ darcys $R = 1.90$ cm	150 darcys	$0.912 \frac{\text{cal}}{\text{s-cm}^2}$	$0.1047 \frac{\text{cal}}{\text{s-cm}^2}$	$0.115 \frac{\text{cal}}{\text{s-cm}^2}$
$Q = 30$ watts $\phi_w = 66^\circ\text{C}$ $\kappa = 25$ darcys $R = 1.90$ cm	120 darcys	$0.120 \frac{\text{cal}}{\text{s-cm}^2}$	$0.157 \frac{\text{cal}}{\text{s-cm}^2}$	$0.140 \frac{\text{cal}}{\text{s-cm}^2}$

The spherical heater experiment was repeated using a coarser sand* with an average grain diameter of 450 microns. This allowed the actual permeability and the permeability limit of Equation (29) to be in better agreement than in the previous experiment. The experimental and measured heat fluxes for the water-saturated coarse sand are shown in Table V. The predicted heat fluxes based on the boundary-layer solution are in fairly good agreement with the measured values and in considerably better agreement than the conduction prediction. At the higher values of permeability in this experiment, the effect of natural convection has become significant and the heat-flux values are as high as twice the conduction prediction.

In the experiment with coarse sand, an attempt was made to measure the plume velocity. A dye capsule was placed in the sand on top of the spherical heater. At the end of the test, the height of penetration by the dye was measured and an approximate velocity was determined. For a test at 36 watts, the average plume velocity was found to be approximately 15 cm/hr. Calculations based on Equation (39) predict a plume velocity of 46 cm/hr. An analytical solution for a point source does not currently exist as in the case of the line source and it is not possible to compare the result of Equation (39) with an exact solution. A crude comparison can be made with the line source solution of Wooding¹ by assuming that the 36 watts of heat was spread out in a line source of a length equal to the diameter of the spherical heater. Using this assumption, the line source solution predicts a plume velocity of about 11 cm/hr which compares reasonably well with the experimental result, although this value is lower than the result from Equation (39).

* Whedron White Silica Sand

TABLE V

Comparison of Boundary-Layer Method with Experimental Results
for a Heated Sphere in Coarse Sand

	κ Lower Limit Eq (29)	Average Heat Flux, q/A Eq (27)	Experimental Heat Flux	Conduction Heat Flux Eq (42)
$Q = 18$ watts $\phi_w = 32^\circ\text{C}$ $\kappa = 125$ darcys	250 darcys	$0.908 \frac{\text{cal}}{\text{s-cm}^2}$	$0.943 \frac{\text{cal}}{\text{s-cm}^2}$	$0.676 \frac{\text{cal}}{\text{s-cm}^2}$
$Q = 45$ watts $\phi_w = 56^\circ\text{C}$ $\kappa = 125$ darcys	150 darcys	$0.206 \frac{\text{cal}}{\text{s-cm}^2}$	$0.236 \frac{\text{cal}}{\text{s-cm}^2}$	$0.117 \frac{\text{cal}}{\text{s-cm}^2}$

One difficulty with this type of test is that there is a significant transient period during which the sand is heating up and during which the flow pattern is being established. This transient period is neglected in Equation (39), which assumes that a steady flow situation exists during the entire test. If the experimental plume velocity were based only on the steady-state portion of the test, the experimental velocity would be closer to 26 cm/hr, which is in better agreement with Equation (39). In any event, Equation (39) is a rough approximation, and agreement with experimental data to within a factor of 2 or 3 is about the expected accuracy.

Permeability Measurement Using a Spherical Heat Source

The spherical source heater described earlier can also be used to measure permeability in various porous media which might be difficult to handle using conventional permeability measurement techniques. In the coarse sand test described in Table V earlier, the measured heat flux and surface temperature could have been used to determine permeability. Using Equation (36) and a thermal conductivity of $0.004 \text{ cal/cm-s-}^\circ\text{C}$ and the measured heat flux (45 watts) and surface temperature (56°C) for the 1.9-cm-radius sphere, the permeability would be $\kappa = 165$ darcys. McAdams¹³ lists thermal conductivities for typical coarse sands ranging from 0.004 to $0.0075 \text{ cal/cm-s-}^\circ\text{C}$. At the upper end of the thermal conductivity range ($0.0075 \text{ cal/cm-s-}^\circ\text{C}$), the permeability determined from the spherical experiment would be $\kappa = 47$ darcys. The estimate of permeability based on a grain size of 450 microns using Equation (48) would be $\kappa = 125$ darcys. The permeability of this coarse sand was measured directly using a constant-head permeameter¹⁴ and found to be $\kappa = 80$ darcys. These values are all summarized in Table VI. The range of permeability based on the sphere experiment bracket the values obtained by the conventional approaches.

It appears that the sphere method might be useful as an indirect permeability measurement technique in situations where the conventional method is difficult to use, such as with hazardous materials. The accuracy of the sphere method depends strongly on the value of thermal conductivity used for the porous media.

TABLE VI
Permeability Determination for Coarse Sand

Spherical Heater Method, Equation (36)		Constant-Head, Permeameter ¹⁴	Estimated Based on Grain Diameter ¹¹
$K = 0.0075 \frac{\text{cal}}{\text{cm-s-}^\circ\text{C}}$	$K = 0.004 \frac{\text{cal}}{\text{cm-s-}^\circ\text{C}}$		
47 darcys	165 darcys	80 darcys	125 darcys

Conclusions

Comparisons of the results of the boundary-layer method have been made with numerical computer code calculations, exact analytical solutions, and experimental data. The boundary-layer solutions appear to give very good results for heat flux as a function of surface temperature as long as the permeability criteria of Equation (16), (29), or (37) is satisfied. The calculation for plume velocity also checked well, at least for the horizontal cylinder case. It appears that the spherical source solution can be used as a technique for measuring permeability provided that the thermal conductivity of the porous media is known to a reasonable degree. This permeability measurement technique might be very useful in determining the permeability of hazardous materials which are difficult to handle in conventional test devices such as the constant-head permeameter.

APPENDIX

EXAMPLE PROBLEM NO. 1

For the case of a Horizontal Downhole Heat Exchanger when the permeability is assumed to be 10 darcys:

$$\kappa = 10 \text{ darcys } (1.06 \times 10^{-10} \text{ ft}^2)$$

$$K_f = 9.67 \times 10^{-5} \frac{\text{Btu}}{\text{ft-sec-}^\circ\text{F}}$$

$$\alpha = 1.66 \times 10^{-6} \frac{\text{ft}^2}{\text{sec}}$$

$$\rho = 1.52 \frac{\text{Lb-sec}^2}{\text{ft}^4} \left(49 \frac{\text{Lb}}{\text{ft}^3} \right)$$

$$K = 53.2 \times 10^{-5} \frac{\text{Btu}}{\text{ft-sec-}^\circ\text{F}}$$

$$\mu = 0.418 \times 10^{-5} \frac{\text{Lb-sec}}{\text{ft}^2}$$

$$\beta = 1 \times 10^{-3} \text{ } 1/^\circ\text{F}$$

$$\phi_w = 170^\circ\text{F}$$

$$R = \frac{6.5}{12} \text{ ft}$$

$$Ra_R = \frac{\left(32.2 \frac{\text{ft}}{\text{sec}^2} \right) \left(1 \times 10^{-3} \frac{1}{^\circ\text{F}} \right) \left(1.52 \frac{\text{Lb-sec}^2}{\text{ft}^4} \right) \left(170^\circ\text{F} \right) \left(\frac{6.5}{12} \right)^3 \text{ ft}^3}{\left(0.418 \times 10^{-5} \frac{\text{Lb-sec}}{\text{ft}^2} \right) \left(1.66 \times 10^{-6} \frac{\text{ft}^2}{\text{sec}} \right)}$$

$$Ra_R = 1.91 \times 10^{11}$$

From Equation (29),

$$\kappa \geq \frac{30 R^2}{Ra_R}$$

$$\kappa \geq \frac{30 \left(\frac{6.5}{12}\right)^2}{1.91 \times 10^{11}} = 0.460 \times 10^{-10} \text{ ft}^2$$

$$\kappa \geq 4 \text{ darcys.}$$

Since the permeability in this example is 10 darcys, Equation (27) can be used to determine the heat extraction rate.

$$q/A = 0.329 \frac{\sqrt{\kappa} \phi_w}{R^2} \sqrt{K_m K_f} \sqrt{Ra_R}$$

$$q/A = 0.329 \frac{\sqrt{1.06 \times 10^{-10} \text{ ft}^2 (170^\circ \text{F})}}{\left(\frac{6.5}{12} \text{ ft}\right)^2} \sqrt{53.2 \times 10^{-5} \times 9.67 \times 10^{-5}}$$

$$\frac{\text{Btu}}{\text{ft} \cdot \text{sec}^\circ \text{F}} \sqrt{1.91 \times 10^{11}}$$

$$q/A = 0.194 \frac{\text{Btu}}{\text{sec ft}^2}$$

or

$$q/A = 2.19 \frac{\text{kW}}{\text{m}^2} .$$

Witherspoon (5) obtains the answer $q/A = 1.90 \frac{\text{kW}}{\text{m}^2}$ for this same problem using a modified version of the TRUMP code.

EXAMPLE PROBLEM NO. 2

Schrock et al.¹¹, ran a series of experiments on heated cylinders in a porous medium.
For Monterey crystal amber sand with the following conditions:

$$K_f = 0.394 \frac{\text{Btu}}{\text{hr ft}^2 \text{ } ^\circ\text{F}}$$

$$\kappa = 373 \text{ darcys } (39.5 \times 10^{-10} \text{ ft}^2)$$

$$K_m = 0.00331 \frac{\text{cal}}{\text{sec-cm-}^\circ\text{C}} (0.800 \frac{\text{Btu}}{\text{hr-ft-}^\circ\text{F}})$$

$$R = 0.104 \text{ ft.}$$

At 210°F,

$$Ra_R = \frac{g \rho^2 \beta}{\mu} \text{Pr} R^3 \phi_w = (11.1 \times 10^8) (1.88) (.104)^3 \phi_w$$

$$Ra_R = 2.35 \times 10^6 \phi_w$$

and

$$q/A = 0.328 \frac{\sqrt{\kappa}}{R^2} \sqrt{K_m K_f Ra_R} \phi_w$$

$$q/A = 0.328 \frac{\sqrt{39.5 \times 10^{-10}}}{(.104)^2} \sqrt{(0.800) (0.394) (2.35 \times 10^6)} \phi_w^{3/2}$$

$$q/A = 1.64 \phi_w^{3/2} \frac{\text{Btu}}{\text{hr ft}^2 \text{ } ^\circ\text{F}}$$

when $\phi_w = 170^\circ\text{F}$

$$q/A = 3635 \frac{\text{Btu}}{\text{hr ft}^2}$$

or

$$q/A = 1.15 \frac{w}{\text{cm}^2} .$$

Schrock et al.¹¹ got an experimental result of 1.0 w/cm^2 for this case.

The limiting value of permeability for the boundary-layer calculation is:

$$\kappa \geq \frac{30 R^2 K_m}{Ra_R K_f}$$

$$\kappa \geq \frac{30 (0.104)^2 (0.800)}{2.35 \times 10^6 \times 170 (0.394)}$$

$$\kappa \geq 16.4 \times 10^{10} \text{ ft}^2$$

or

$$\kappa \geq 157 \text{ darcys}$$

and since $\kappa = 373$ darcys in this example, the boundary-layer method is applicable. A comparison of Equation (27) and data from Schrock and calculations from Witherspoon are shown in Table I and Table II.

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