PARTIAL WAVE EXPANSIONS FOR ARBITRARY SPIN
AND THE ROLE OF NON-CENTRAL FORCES

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ABSTRACT

The partial wave expansion of the amplitudes used by Hooton and Johnson for the scattering of particles of arbitrary spin is derived. A discussion is given of the extent to which effects arising from transition matrix elements that are diagonal and non-diagonal in orbital angular momentum can be distinguished in observables.

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1. Introduction

The scattering of a spin s particle from a spin zero target was discussed in ref. ¹⁾. It was shown there that an expansion of the transition matrix M in terms of the matrix elements of a set of irreducible tensor spin operators provides a particularly useful basis for the study of symmetry properties of the scattering amplitude and the detailed way in which various types of spin-dependent force contribute to quantities measured in experiments with polarized beams.

In sects. 2-6 of this paper the analysis of ref. ¹⁾ is carried a step further through the introduction of a partial wave expansion in terms of amplitudes associated with a definite channel spin transfer, and some of the results of ref. ¹⁾ are generalized to arbitrary spin, s . The forms taken by unitarity, distorted wave expansions, and coupled radial equations in this representation are gathered together in Appendix B.

In sect. 7 the main result of ref. ¹⁾ is rederived in a partial wave context. For the case of elastic scattering by a spin-1 particle explicit expression are given for the partial wave matrix elements to first order in spin-dependent forces.

In sect. 8 the formalism of sect. 2 is used to show how the contributions of transition matrix elements diagonal in orbital angular momentum, ℓ , to certain combinations of observables have a tendency to cancel in the large ℓ limit. Since the same cancellation does not occur in the contributions from off-diagonal elements, this result reinforces the earlier suggestion of ref. ¹⁾ that the same combinations of observables may be particularly useful in the analysis of effects due to tensor forces in elastic deuteron scattering.

2. Partial wave expansion

The reduced transition matrix used here, for an arbitrary process involving an initial two-body state with relative momentum $\underline{K}_1$, channel spin s , and a final two-body state with relative momentum $\underline{K}_2$, channel spin s' , is identical to the quantity defined in eq. (14), Ch. 4 of ref. ²⁾, and is related to differential cross sections by eq. (17), Ch. 4 of ref. ²⁾. The reduced transition matrix in the barycentric system is denoted here by

$$\langle \underline{K}_2, s' \sigma' | M | \underline{K}_1, s \sigma \rangle, \quad (1)$$

and uses the letter M rather than T to avoid confusion with polarization observables. Reference to variables needed to completely specify the internal states of the two channels in addition to channel spin is suppressed.

The plane wave state $|\underline{K}, s \sigma\rangle$ can be expanded

$$|\underline{K}, s \sigma\rangle = \sum_{\ell, j, m, 0}^{\infty} \int dK' K'^2 |K', (\ell s) j m\rangle \langle K', (\ell s) j m | \underline{K}, s \sigma\rangle, \quad (2)$$

where $|K, (\ell s) j m\rangle$ is a simultaneous eigenstate of $\underline{\ell}^2$, $\underline{s}^2$, $\underline{j}^2 (= (\underline{\ell} + \underline{s})^2)$, j_z and the channel Hamiltonian. If $\underline{R}$ is the relative coordinate of the channel fragments and ξ denotes a suitable set of internal variables, then

$$\underline{\ell} = (i)^{-1} \underline{R} \cdot \nabla \underline{R}, \quad (3)$$

and

$$\langle \underline{R}, \xi | K, (\ell s) j m \rangle = i^{\ell} \sqrt{2/\pi} j_{\ell}(KR) \sum_{\lambda, \sigma} (\ell \lambda, s \sigma | j m) Y_{\ell}^{\lambda}(\hat{\underline{R}}) \chi_{s \sigma}(\xi), \quad (4)$$

where $j_{\ell}(KR)$ is the regular spherical Bessel function³⁾, and the notation $\hat{\underline{R}}$ denotes the polar angles of $\underline{R}$. The factor i^{ℓ} included in the normalization in eq. (4) should be noted.

The expansion coefficient in (2) can be obtained from the Rayleigh plane wave expansion, which gives

$$\langle K', (\ell s) j m | \underline{K}, s \sigma \rangle = \frac{\delta(K-K')}{K^2} \sum_{\lambda} (\ell \lambda, s \sigma | j m) Y_{\ell}^{\lambda*}(\hat{\underline{K}}). \quad (5)$$

Rotation invariance and the Wigner-Eckart⁴⁾ theorem imply that

$$\langle K_2, (\ell' s') j' m' | M | K_1, (\ell s) j m \rangle \quad (6)$$

is of the form

$$\delta_{jj'} \delta_{mm'} M_{\ell' s' \ell s}^j \quad (7)$$

where $M_{\ell' s' \ell s}^j$ is independent of m .

Assembling eqs. (2), (5), and (7) gives the result

$$\langle K_2, s' \sigma' | M | K_1, s \sigma \rangle = \sum_{\ell \ell' j m \lambda \lambda'} (\ell' \lambda', s' \sigma' | j m) (\ell \lambda, s \sigma | j m) Y_{\ell'}^{\lambda'}(K_2) Y_{\ell}^{\lambda*}(K_1) M_{\ell' s' \ell s}^j. \quad (8)$$

3. Relation with the work of Hooton and Johnson

Using the relation between Racah coefficients and Clebsch-Gordan coefficients⁴⁾ it is readily shown that[†]

$$\sum_m (\ell' \lambda', s' \sigma' | j m) (\ell \lambda, s \sigma | j m) = \sum_{k q} \hat{k}(s' \sigma', k q | s \sigma) (\ell' \lambda', \ell - \lambda | k q) (-)^{\ell - \lambda} \times \frac{j^2}{s} (-1)^{j - \ell' - s'} W(\ell \ell' s s'; k j). \quad (9)$$

Equation (8) can now be rewritten

$$\langle K_2, s' \sigma' | M | K_1, s \sigma \rangle = \sum_{k q} \hat{k}(s' \sigma', k q | s \sigma) \times \sum_{\ell \lambda \ell' \lambda'} \frac{\hat{s}' \hat{\ell}'}{s} (-1)^{\ell} (\ell' \lambda', \ell \lambda | k q) Y_{\ell'}^{\lambda'}(K_2) Y_{\ell}^{\lambda}(K_1) M_{\ell' s' \ell s}^{(k)}, \quad (10)$$

where

$$M_{\ell' s' \ell s}^{(k)} = \sum_j \frac{j^2}{\hat{s}' \hat{\ell}'} (-1)^{j - \ell' - s'} W(\ell \ell' s s'; k j) M_{\ell' s' \ell s}^j. \quad (11)$$

In deriving eq. (10) the result

$$Y_{\ell}^{\lambda*} = (-1)^{\lambda} Y_{\ell}^{-\lambda}, \quad (12)$$

has been used.

[†] $\hat{k} \equiv (2k + 1)^{1/2}$.

Equation (10) can be written

$$\langle \underline{K}_2, s' \sigma' | M | \underline{K}_1, s \sigma \rangle = \sum_{kq} \hat{k}(s' \sigma', kq | s \sigma) M_{kq}^{s' s} \quad (13)$$

where

$$M_{kq}^{s' s} = \sum_{\ell \lambda \ell' \lambda'} \frac{\hat{s}' \hat{\ell}'}{\hat{s}} (-)^{\ell} (\ell' \lambda', \ell \lambda | kq) Y_{\ell'}^{\lambda'}(\underline{K}_2) Y_{\ell}^{\lambda}(\underline{K}_1) M_{\ell' s' \ell s}^{(k)} \quad (14)$$

The expansion (13) and (14) is a very general one. It is clear from eq. (13) that the quantum number k is restricted by $|s-s'| \leq k \leq (s+s')$, and can be interpreted as a possible value of channel spin transfer, with q the corresponding projection along the Z -axis.

Emphasis will be placed here on the special case of elastic scattering of a spin s particle by a spin zero target. In this case $s' = s$ only, and eq. (13) can be written

$$\langle \underline{K}_2, s \sigma' | M | \underline{K}_1, s \sigma \rangle = \sum_{kq} \langle s \sigma' | T_{kq}^{\dagger}(s) | s \sigma \rangle M_{kq}^{ss}, \quad (15)$$

where $T_{kq}(s)$ is an irreducible tensor operator of rank k with matrix elements

$$(s \sigma' | T_{kq} | s \sigma) = \hat{k}(s \sigma, kq | s \sigma'), \quad (16)$$

as used in ref. ¹⁾. For $s = 1/2, 1$ these operators are identical to those defined in the Madison Convention ⁵⁾.

The expansion (15) may be compared with the expansion given in eq. (3) of ref. ¹⁾. It will be seen that M_{kq}^{ss} ($0 \leq k \leq 2s$) can be identified with the amplitude $M_{kq}(\underline{K}_2, \underline{K}_1)$ used there. Henceforth the superscripts s will be dropped and the notation of ref. ¹⁾ used in the elastic scattering case.

All observable quantities can be expressed in terms of the M_{kq} . Explicit expression for analyzing powers and polarization tensors are given in eqs. (22a) and (22b) of ref. ¹⁾. Expressions for polarization transfer coefficients are given in Appendix A in terms of the quantities defined by Santos ⁶⁾.

The consequences of parity conservation and symmetries associated with time reversal can be deduced from eqs. (11) and (14) and the results

$$M_{\ell, \ell}^{(k)} = 0, \quad (-1)^{\ell' + \ell} = -1, \quad (17)$$

$$\hat{\ell}' M_{\ell, \ell}^{(k)} = \hat{\ell} M_{\ell \ell'}^{(k)}, \quad (18)$$

which follow from the definition (11) and the analogous properties of the $M_{\ell, \ell}^j$. As in ref. 1) it is found that as consequences of eqs. (17) and (18)[†]

$$\bar{M}_{kq} = \bar{M}_{k-q}, \quad (19)$$

$$\bar{M}_{kq} = 0, \quad k+q \text{ odd}, \quad (20)$$

where quantities with a bar are referred to a coordinate system with Z-axis along

$$\underline{K} = \frac{\underline{K}_1 + \underline{K}_2}{|\underline{K}_1 + \underline{K}_2|}, \quad (21)$$

and Y-axis along

$$\underline{n} = \frac{\underline{K}_1 \wedge \underline{K}_2}{|\underline{K}_1 \wedge \underline{K}_2|}. \quad (22)$$

One advantage of a formulation in terms of the amplitudes M_{kq} is that when the spin-dependent terms in the interaction which generates M are decomposed into a sum of terms each of which transforms as an irreducible tensor of definite rank under rotations in spin space, a term of rank k contributes, to first order in spin-dependent forces, only to the M_{kq} with the same k . In addition, a spin-dependent term of rank k_1 , acting alone, can contribute to M_{kq} in order N only if

$$0 \leq k \leq Nk_1 \quad (23)$$

A general rule which follows from the explicit formula for the observable $\bar{t}_{kq}$ given in eqs. (22) of ref. 1), and the properties (17) and (18), is:

[†] Note that the properties (19) and (20) obtain in the elastic scattering case only, although the expansion (13) can be used in a much more general context.

A term of the form $\bar{M}_{k_1 q_1} \bar{M}_{k_2 q_2}^*$ does not contribute to the observable $\bar{t}_{kq}$ if any of the following are satisfied:

$$q_1 + q_2 \neq q, \quad (24a)$$

$$k < |k_1 - k_2|, \text{ or } k > |k_1 + k_2|, \quad (24b)$$

$$(-1)^{k_1+q_1} = -1, \text{ or } (-1)^{k_2+q_2} = -1. \quad (24c)$$

Some examples of the application of eqs. (23) and (24) are given in ref. ¹⁾ for the case of elastic deuteron scattering ($s = 1$). Analogous results in the case of elastic scattering of a spin-3/2 particle are readily obtained. Thus, for example, if a spin-3/2 particle experiences only central forces and a spin-dependent potential linear in $\underline{l} \cdot \underline{s}$ the observables $\bar{t}_{22}$ and $\bar{t}_{20}$ are second order in the $\underline{l} \cdot \underline{s}$ force whereas $\bar{t}_{21}$ is third order, as in the spin-1 case ¹⁾. Furthermore, the observables $\bar{t}_{33}$ and $\bar{t}_{31}$ are third order in the $\underline{l} \cdot \underline{s}$ force, but $\bar{t}_{32}$ and $\bar{t}_{30}$ are fourth order. In general, for a spin s particle for which $0 \leq k \leq 2s$, and assuming central and linear $\underline{l} \cdot \underline{s}$ forces only, $\bar{t}_{kq}$ is of order k if $k + q$ is even and of order $k + 1$ if $k + q$ is odd.

4. General properties of the elastic scattering parameters $M_{\ell' \ell}^{(k)}$

In the case of elastic scattering from a spin zero target the definition (11) reads, in an obviously abbreviated notation,

$$M_{\ell' \ell}^{(k)} = \sum_j \frac{j^2}{8\ell'} (-1)^{j-\ell'-s} W(\ell \ell' s s; k j) M_{\ell' \ell}^j. \quad (25)$$

The parameters $M_{\ell' \ell}^{(k)}$ characterize the scattering in a way analogous to the $M_{\ell' \ell}^j$, and give a completely equivalent parameterization of the scattering amplitude. Non-vanishing elements $M_{\ell' \ell}^{(k)}$ consistent with parity conservation and time reversal symmetry must satisfy

$$k = 0, 1, 2, \dots, 2s, \quad (26a)$$

$$|\ell - \ell'| \leq k \leq \ell + \ell', \quad (26b)$$

$$(-1)^{\ell+\ell'} = +1, \quad (26c)$$

$$\ell' M_{\ell', \ell}^{(k)} = \ell M_{\ell, \ell'}^{(k)}. \quad (26d)$$

For purely central forces the elements $M_{\ell', \ell}^j$ reduce on the energy shell and in the non-relativistic limit to the familiar (see ref. ²⁾, eq. (51a), Ch. 6)

$$M_{\ell', \ell}^j = \delta_{\ell', \ell} \left(\frac{-\pi^2}{\pi \mu K_1} \right) e^{i\delta_\ell} \sin \delta_\ell, \quad (27)$$

where μ is the reduced mass in the incident channel and δ_ℓ is the phase shift.

In the same limit eq. (25) reduces to

$$M_{\ell', \ell}^{(k)} = \delta_{k0} \delta_{\ell', \ell} \left(\frac{-\pi^2}{\pi \mu K_1} \right) e^{i\delta_\ell} \sin \delta_\ell. \quad (28)$$

Thus in this formalism non-vanishing $M_{\ell', \ell}^{(k)}$ with $k \neq 0$ arise purely from non-central forces.

Note also that because of condition (26b) and the parity condition (26c) off-diagonal elements $M_{\ell', \ell}^{(k)}$ with $\ell' \neq \ell$ exist only if $k \geq 2$.

The complete equivalence of the $M_{\ell', \ell}^{(k)}$ and $M_{\ell', \ell}^j$ can be seen by inverting eq. (11) using the unitarity of the Racah coefficients to give in the general case

$$M_{s', \ell', s, \ell}^j = \sum_k \hat{s}' \hat{\ell}' (-1)^{j-\ell'-s'} W(\ell' \ell s' s; k j) k^2 M_{s', \ell', s, \ell}^{(k)}. \quad (29)$$

Radial wavefunctions corresponding to the amplitudes $M_{\ell', \ell}^{(k)}$ and the associated distorted wave expansions, coupled equations, and unitarity relations are discussed in Appendix B.

If all matrix elements with $\ell > \ell_m$ are neglected, and account is taken of the conditions (26), it is found that the maximum number of non-vanishing $M_{\ell', \ell}^{(k)}$ for a given k is

$$\frac{1}{2}(k+2)(\ell_m - \frac{3}{4}k+1), \quad k \text{ even}, \quad (30a)$$

$$\frac{1}{2}(k+1)(\ell_m - \frac{3}{4}k + \frac{3}{4}), \quad k \text{ odd}, \quad (30b)$$

$$0 \leq k \leq 2s, \quad (30c)$$

$$k < \ell_m + 2. \quad (30d)$$

If (30d) is not satisfied the results are more complicated and of limited interest and will not be given here.

As a result of eqs. (30) the maximum total number of non-vanishing matrix elements $M_{\ell', \ell}^{(k)}$ with $k = 0, 1, 2, \dots, 2s$, is

$$(s+1)^2(\ell_m + 1 - s), \quad \text{if } s \text{ is integer}, \quad (31a)$$

$$(s+1/2)(s+3/2)(\ell_m + 1 - s), \quad \text{if } s \text{ is } 1/2 \text{ integer}, \quad (31b)$$

$$2s \leq \ell_m + 1. \quad (31c)$$

Because of the unitary nature of the relations (25) and (29) the results (31) must agree with the values obtained for the maximum number of non-vanishing $M_{\ell', \ell}^j$ with $\ell', \ell \leq \ell_m$.

5. Explicit expressions for $s = 1/2, 1$

Equation (25) can be made more explicit by evaluating the Racah coefficients.

For $s = 1/2$ the results obtained using the tables in ref. ⁴⁾ are given in eqs. (32).

$$M_{\ell', \ell}^{(0)}(1/2) = \delta_{\ell', \ell} \frac{1}{\ell^2} [(\ell+1)M_{\ell \ell}^{\ell+1/2} + \ell M_{\ell \ell}^{\ell-1/2}] \quad (32a)$$

$$M_{\ell', \ell}^{(1)}(1/2) = \delta_{\ell', \ell} \frac{1}{\ell} \left[\frac{\ell(\ell+1)}{3(2\ell+1)} \right]^{1/2} [M_{\ell \ell}^{\ell+1/2} - M_{\ell \ell}^{\ell-1/2}], \quad (32b)$$

In the usual notation

$$M_{\ell \ell}^j = \left(\frac{-\kappa^2}{\pi \mu K_1} \right) e^{i\delta_{\ell j}} \sin \delta_{\ell j}. \quad (32c)$$

With Y-axis along $\underline{n}$, Z-axis along $\underline{K}_1$, eq. (14) gives

$$M_{00}^{\frac{1}{2}\frac{1}{2}} = (4\pi)^{-1} \int \hat{\ell}^2 M_{\ell\ell}^{(0)} P_{\ell}(\cos\theta), \quad (33a)$$

$$M_{11}^{\frac{1}{2}\frac{1}{2}} = M_{1-1}^{\frac{1}{2}\frac{1}{2}} = (4\pi)^{-1} \int_{\ell} \left(\frac{3(2\ell+1)}{2\ell(\ell+1)} \right)^{1/2} \hat{\ell} M_{\ell\ell}^{(1)} P_{\ell}^1(\theta), \quad (33b)$$

$$M_{10}^{\frac{1}{2}\frac{1}{2}} = 0. \quad (33c)$$

It will be seen that $M_{00}^{\frac{1}{2}\frac{1}{2}}$ and $M_{11}^{\frac{1}{2}\frac{1}{2}}$ are proportional to the amplitudes $f(\theta)$ and $g(\theta)$ which arise in the usual formulation of the scattering of a spin-1/2 particle from a spin-zero target (see e.g., ref. ²⁾, eq. (78), Ch. 7).

The results obtained from eq. (25) for $s = 1$ using the tables in ref. ⁴⁾ are given in eqs. (34).

$$M_{\ell'\ell}^{(0)} = \frac{1}{3} \delta_{\ell'\ell} \hat{\ell}^{-2} [(2\ell+3)M_{\ell\ell}^{\ell+1} + (2\ell+1)M_{\ell\ell}^{\ell} + (2\ell-1)M_{\ell\ell}^{\ell-1}], \quad (34a)$$

$$M_{\ell'\ell}^{(1)} = \frac{1}{3} \delta_{\ell'\ell} \hat{\ell}^{-2} (2\ell(\ell+1))^{-1/2} [\ell(2\ell+3)M_{\ell\ell}^{\ell+1} - (2\ell+1)M_{\ell\ell}^{\ell} - (\ell+1)(2\ell-1)M_{\ell\ell}^{\ell-1}], \quad (34b)$$

$$M_{\ell'\ell}^{(2)} = \frac{1}{3} \delta_{\ell'\ell} \hat{\ell}^{-2} \left(\frac{(2\ell+3)(2\ell-1)}{10\ell(\ell+1)} \right)^{1/2} [\ell M_{\ell\ell}^{\ell+1} - (2\ell+1)M_{\ell\ell}^{\ell} + (\ell+1)M_{\ell\ell}^{\ell-1}] \\ + \delta_{\ell'\ell+2} \left(\frac{2\ell+3}{15(2\ell+5)} \right)^{1/2} M_{\ell+2,\ell}^{\ell+1} + \delta_{\ell'\ell-2} \left(\frac{2\ell-1}{15(2\ell-3)} \right)^{1/2} M_{\ell,\ell-2}^{\ell-1}. \quad (34c)$$

6. Dynamical Properties of the Scattering Parameters $M_{\ell'\ell}^{(k)}$ for $s = 1$

It has already been mentioned (see the discussion following eq. (28)) that if only central forces are acting, and hence that the $M_{\ell\ell}^{(j)}$ are diagonal in ℓ and independent of j , then the particular linear combinations (34b) and (34c) vanish. It is less obvious that these linear combinations also have the property that the first order effects of an $\underline{\ell.s}$ potential cancel out in (34a) and (34c), and the first order effects of a rank-2 tensor force cancel out in (34a) and (34b). In the case of an $\underline{\ell.s}$ force this result is readily seen by recalling

that the eigenvalues of the operator $\underline{l} \cdot \underline{s}$ are

$$\underline{l} \cdot \underline{s} = \begin{cases} l, & j = l + 1, \\ -1, & j = l, \\ -(\ell + 1), & j = l - 1, \end{cases} \quad (35)$$

and that therefore to first order in the $\underline{l} \cdot \underline{s}$ force the elements $M_{\ell\ell}^j$ have this same j -dependence. This produces the required cancellation in (34a) and (34c).

The analogous formulae for the j -dependence associated with the first order effect of a tensor force can also be written down. However it is more instructive to take a more general point of view from which the quoted properties of the linear combinations (34) becomes immediately obvious.

7. Perturbation theory treatment of spin-dependent forces

The usual "two-potential formula" (see ref. ³⁾, Ch. XIX, §11) gives

$$M \approx M_c + \Omega_c^{(-)\dagger} \Delta V \Omega_c^{(+)}, \quad (36)$$

where M_c is the exact transition amplitude associated with the central forces and ΔV is the spin-dependent part of the interaction. The operators $\Omega_c^{(\pm)}$ convert plane waves into waves distorted by the central potential only. Equation (36) is accurate to first order in ΔV . From eqs. (6), (7), and (36) it follows that

$$\begin{aligned} M_{\ell,\ell}^j &= \langle K_2, (\ell's)jm | M | K_1, (\ell s)jm \rangle, \\ &\approx M_\ell(c) \delta_{\ell,\ell} + \langle K_2, (\ell's)jm | \Omega_c^{(-)\dagger} \Delta V \Omega_c^{(+)} | K_1, (\ell s)jm \rangle \end{aligned} \quad (37)$$

where $M_\ell(c)$ has the form (27).

The operators $\Omega_c^{(\pm)}$ are defined so that

$$\begin{aligned} \langle R, (\ell's)j'm' | \Omega_c^{(+)} | (\ell s)jm, K_1 \rangle &= \langle R, (\ell s)jm | \Omega_c^{(-)} | (\ell s)jm, K_1 \rangle^* \\ &= \delta_{\ell\ell} \delta_{jj} \delta_{mm'} \sqrt{2/\pi} f_\ell^{(+)}(R, K_1), \end{aligned} \quad (38)$$

where $f_\ell^{(+)}$ is a radial distorted wave normalized so that

$$f_{\ell}^{(+)}(R, K_1) \xrightarrow{R \rightarrow \infty} \frac{e^{i(\delta_{\ell} + \sigma_{\ell})}}{K_1 R} (\cos \delta_{\ell} F_{\ell}(K_1 R) + \sin \delta_{\ell} G_{\ell}(K_1 R)) \quad (39)$$

where F_{ℓ} and G_{ℓ} are Coulomb functions and σ_{ℓ} is a Coulomb phase shift.

Eq. (37) can now be written

$$M_{\ell', \ell}^j = M_{\ell}(c) \delta_{\ell', \ell} + \frac{2}{\pi} \int_0^{\infty} dR R^2 \int_0^{\infty} dR' R'^2 f_{\ell}^{(+)}(R) \langle R, (\ell' s) j m | \Delta V | R', (\ell s) j m \rangle f_{\ell}^{(+)}(R'), \quad (40)$$

where it has been assumed that ΔV is invariant under rotations so that a statement analogous to eq. (7) holds for the matrix elements of ΔV . For the same reason it can be assumed that ΔV can be written

$$\Delta V = \sum_{i, k_1} \sum_{q_1} T_{k_1 q_1}^{\dagger} \Delta V_{k_1 q_1}^{(i)}, \quad (41)$$

where $T_{k_1 q_1}$ are the spin operators defined in eq. (16) and the $\Delta V_{k_1 q_1}^{(i)}$ are irreducible tensor operators of rank k_1 built out of dynamical variables other than those associated with $\underline{s}$. The label (i) distinguishes different forces of the same rank (e.g., Satchler's ⁷) T_R , T_L and T_P forces all have $k_1 = 2$). An $\underline{\ell}, \underline{s}$ force is associated with $k_1 = 1$.

The expansion (41) can be combined with the general formula given in ref. ⁴ for the matrix element of a product of tensor operators (see ref. ⁴, eq. 5.13) to give

$$\begin{aligned} & \langle R, (\ell' s) j m | \Delta V | R', (\ell s) j m \rangle \\ &= \sum_{k, i} \hat{\ell}' \hat{s}(-)^{j-\ell'-s} W(\ell' \ell s s; k_1 j) \hat{k}_1 \langle R, \ell' | | \Delta V_{k_1}^{(i)} | | R', \ell \rangle, \end{aligned} \quad (42)$$

where the result

$$\langle s | | T_{k_1} | | s \rangle = \hat{k}_1, \quad (43)$$

has also been used.

Equation (40) can now be written

$$M_{\ell', \ell}^j = M_{\ell}(c) \delta_{\ell \ell'} + \frac{2}{\pi} \sum_{k_1, i} \hat{\ell}' \hat{s}(-)^{j-\ell'-s} W(\ell' \ell s s; k_1 j) \hat{k}_1 \beta_{\ell', \ell}^{(k_1)}(i), \quad (44)$$

where

$$\beta_{\ell', \ell}^{(k_1)}(1) = \int_0^\infty dR R^2 \int_0^\infty dR' R'^2 f_{\ell'}^{(+)}(K_1, R') \langle R', \ell' | | \Delta V_{k_1}^{(1)} | | R, \ell \rangle f_{\ell}^{(+)}(K_1, R). \quad (45)$$

A comparison of eqs. (44) and (11) shows that the same Racah coefficient appears in both formulae, and the unitarity of these coefficients gives immediately

$$M_{\ell', \ell}^{(k)} = \delta_{k, 0} \delta_{\ell', \ell} M_{\ell}^{(c)} + \sum_1 \frac{2}{\pi k} \beta_{\ell', \ell}^{(k)}(1). \quad (46)$$

Equation (46) means that to first order a force associated with rank- k_1 spin-dependence contributes to amplitudes $M_{\ell', \ell}^{(k)}$ with $k = k_1$ only. Since the $M_{\ell', \ell}^{(k)}$ determine the $M_{kq}(\theta)$ (see eq. (14)) this is just the result of ref. ¹⁾ expressed in partial wave form.

In the case of an $\underline{\ell.s}$ force of the form

$$V_{so}(R) \underline{\ell.s}, \quad (47)$$

it is readily shown that for $s = 1$

$$\langle R', \ell' | | \Delta V_1^{so} | | R, \ell \rangle = \frac{\delta(R-R')}{R^2} \delta_{\ell \ell'} \left(\frac{2\ell(\ell+1)}{3} \right)^{1/2} V_{so}(R), \quad (48)$$

so that

$$\beta_{\ell', \ell}^{(1)}(so) = \delta_{\ell \ell'} \left(\frac{2\ell(\ell+1)}{3} \right)^{1/2} \int_0^\infty dR R^2 (f_{\ell}^{(+)}(K_1, R))^2 V_{so}(R), \quad (49)$$

The explicit form of the reduced matrix elements of $\Delta V_{k_1 q_1}^{(1)}$ with $k_1 = 2$ depends on the models used for these forces. Formulae for quantities equivalent to the $\beta_{\ell', \ell}^{(2)}(1)$ defined in eq. (45) are given in ref. ⁸⁾ for the three types of tensor force defined by Satchler ⁷⁾. A possible physical origin for a force of the T_p type has been discussed recently in refs. ⁹⁾ and ¹⁰⁾.

The precise relationship between $\beta_{\ell', \ell}^{(2)}(1)$ (eq. (45)) and the quantities $\bar{\beta}_{\ell', \ell}^{(1)}$ defined in eqs. (A9) - (A12) of ref. ⁸⁾ is

$$\beta_{\ell \ell}^{(2)}(1) = \frac{1^{\ell-\ell'} \hat{\ell}'}{\sqrt{3} \hat{\ell}} \bar{\beta}_{\ell \ell}^{(1)}. \quad (50)$$

All non-vanishing elements can be found using the results given in ref. ⁸⁾ together with eq. (50) and the symmetry property

$$\hat{\ell} \beta_{\ell \ell'}^{(2)}(i) = \hat{\ell}' \beta_{\ell' \ell}^{(2)}(i). \quad (51)$$

Equation (51) is valid if the potentials satisfy

$$K \Delta V K^{-1} = (\Delta V)^\dagger, \quad (52)$$

where K is the time-reversal operator.

For completeness the explicit form of eq. (44) is given in eqs. (53) below.

$$\begin{aligned} M_{\ell' \ell}^j - M_{\ell' \ell}^{(c)} \delta_{\ell' \ell} = & \frac{2}{\pi} \frac{\delta_{\ell \ell'}}{\sqrt{2\ell(\ell+1)}} \left\{ \frac{\sqrt{3}}{2} (j(j+1) - \ell(\ell+1) - 2) \beta_{\ell \ell}^{(1)} \right. \\ & \left. + \frac{\sum \beta_{\ell \ell}^{(2)}(i) \lambda_{\ell j}}{\sqrt{(2\ell+3)(2\ell-1)}} + \delta_{\ell', \ell \pm 2} \delta_{j, \ell \pm 1} \frac{2\sqrt{3} \hat{\ell}'}{\pi j} \sum_i \beta_{\ell' \ell}^{(2)}(i) \right\}, \end{aligned} \quad (53a)$$

where

$$\lambda_{\ell j} = \begin{cases} \ell(2\ell-1), & j = \ell + 1, \\ -(2\ell+3)(2\ell-1), & j = \ell, \\ (2\ell+3)(\ell+1), & j = \ell - 1. \end{cases} \quad (53b)$$

The explicit expression (46) and (53) reflect the fact that $M_{\ell' \ell}^j$ involves $\beta_{\ell' \ell}^{(k)}$ with all k values, whereas $M_{\ell' \ell}^{(k)}$ involves the $\beta_{\ell' \ell}^{(k)}$ with a fixed k .

8. Contribution from off-diagonal $M_{\ell' \ell}^j$ elements

The most striking consequence of the presence of tensor forces in the deuteron-nucleus interaction is that non-vanishing values of the off-diagonal transition matrix elements $M_{\ell' \ell}^j$ and $M_{\ell' \ell}^{(k)}$ with $\ell' = \ell + 2$ now become possible. Non-vanishing values of diagonal elements $M_{\ell \ell}^{(2)}$ can be generated by a rank-1 potential, such as a linear ℓ .s., acting in second order and higher. Off-diagonal elements cannot be generated in this way. From the point of view of the determination of tensor forces in the deuteron-nucleus interaction it would clearly be desirable to know which observable quantities are likely to be particularly sensitive to these off-diagonal elements. It will be shown here that some light can be thrown on this question by consideration of the limit when

$$\frac{(\ell \ 2-\lambda, \ell\lambda | 22)}{(\ell \ -\lambda, \ell\lambda | 20)} = -\left(\frac{3}{2}\right)^{1/2} \left(1 + O\left(\frac{1}{\ell}\right)\right), \quad (57a)$$

$$\frac{(\ell+2 \ 2-\lambda, \ell\lambda | 22)}{(\ell+2 \ -\lambda, \ell\lambda | 20)} = \frac{1}{3}\left(\frac{3}{2}\right)^{1/2} \left(1 + O\left(\frac{1}{\ell}\right)\right). \quad (57b)$$

Hence, eq. (56) can be written

$$\begin{aligned} \bar{M}_{22} - \left(\frac{3}{2}\right)^{1/2} \bar{M}_{20} &= \sum_{\ell\lambda} [(\ell-\lambda, \ell\lambda | 20) (Y_{\ell}^{2-\lambda} + Y_{\ell}^{-\lambda} + O\left(\frac{1}{\ell}\right))] \\ &\times \left(\frac{3}{2}\right)^{1/2} \hat{\ell}(-1)^{\ell+\lambda+1} Y_{\ell}^{\lambda} M_{\ell\ell}^{(2)} \\ &+ 2 \sum_{\ell\lambda} [(\ell+2 \ -\lambda, \ell-\lambda | 20) \left(\frac{1}{3} Y_{\ell+2}^{2-\lambda} - Y_{\ell+2}^{-\lambda} + O\left(\frac{1}{\ell}\right)\right)] \\ &\times \left(\frac{3}{2}\right)^{1/2} (2\ell+5)^{1/2} (-1)^{\ell+\lambda} Y_{\ell}^{\lambda} M_{\ell+2,\ell}^{(2)}. \end{aligned} \quad (58)$$

A consideration of the asymptotic form of the spherical harmonics (see eq. 8.10.7 in ref. ¹²) and Clebsch-Gordan coefficients for large ℓ shows that ($\theta \neq 0$)

$$Y_{\ell}^{2-\lambda} + Y_{\ell}^{-\lambda} = O\left(\frac{1}{\ell}\right), \quad (59a)$$

$$Y_{\ell}^{\lambda} = O(1), \quad (59b)$$

$$\left. \begin{aligned} (\ell \ -\lambda, \ell\lambda | 20) \\ (\ell+2 \ -\lambda, \ell\lambda | 20) \end{aligned} \right\} = O(\ell^{-1/2}). \quad (59c)$$

From eqs. (59) and (58) it can be seen that as far as the contributions from large ℓ are concerned the contribution to the combination

$$\bar{M}_{22} - \left(\frac{3}{2}\right)^{1/2} \bar{M}_{20}, \quad (60)$$

from diagonal matrix elements $M_{\ell\ell}^{(2)}$ is strongly suppressed. It was shown in ref. ¹) that the observables t_{21} and $(t_{22} - (3/2)^{1/2} t_{20})$ are determined[†] by products of the combination (60) with $\bar{M}_{11}$ and $\bar{M}_{00}$ (see eqs. (35) - (38) of ref. ¹)). The contribution from off-diagonal elements to the latter amplitudes is rigorously zero (see the discussion following eq. (28)), and there-

[†]Contributions of the form $\bar{M}_{2q_1} \bar{M}_{2q_2}^*$ are ignored for the purposes of this discussion.

fore contributions to the observables $\sigma(\theta)t_{21}$ and $\sigma(\theta)(t_{22} - (3/2)^{1/2}t_{20})$ from high partial waves are expected to be particularly sensitive to the presence of off-diagonal elements. Experience with deuteron optical model calculations¹³⁾ which include tensor forces suggest that these ideas are useful even at energies as low as 10 MeV, where the contributing l -values are certainly not very large.

In conclusion it should be noted that although some emphasis has been placed on the use of perturbation theory ideas as an aid to qualitative thinking about the formalism, the existence of the expansions (13), (14), and (15) and the existence of the parameters $M_{\ell',\ell}^{(k)}$ in no way depend on the validity of perturbation theory.

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Appendix

A. Polarization transfer coefficients

Santos⁶⁾ has given detailed formulae for polarization transfer coefficients, $X_{k_a q_a k_b q_b}$, in terms of quantities associated with definite spin transfer.

In the notation used here, the definition given in eq. (7) of ref. 6) becomes

$$X_{k_a q_a k_b q_b} = \text{Tr}(M T_{k_a q_a}^\dagger(s) M^\dagger T_{k_b q_b}(s')). \quad (\text{A.1})$$

Santos defines irreducible tensors of rank k associated with spin transfers s_1 and s_2 through the relation

$$A_{kq}^{s_1 s_2} = \sum_{k_a q_a k_b q_b} \hat{s}_1 \hat{s}_2 \hat{k}_a \hat{k}_b (-1)^{k_a - q_a} (k_b q_b, k_a - q_a | kq) \times \left\{ \begin{matrix} s & s & k_a \\ s' & s' & k_b \\ s_1 & s_2 & k \end{matrix} \right\} X_{k_a q_a k_b q_b}. \quad (\text{A.2})$$

The quantity $A_{kq}^{s_1 s_2}$ is identical to the quantity $M_{kq}^{s_1 s_2}$ defined in eq. (17) of ref. 6) with an appropriate change in the notation used for projectile spins.

Using the expansion (13) and the definition (A.2) it is found that

$$A_{kq}^{s_1 s_2} = (-1)^{2s'+2s+k} \hat{s}_1^3 \hat{s}_2 \sum_{\sigma_1 \sigma_2} ()^{s_2 + \sigma_2} (s_1 \sigma_1, s_2 - \sigma_2 | kq) M_{s_1 \sigma_1}^{s' s} M_{s_2 \sigma_2}^{s' s*} \quad (\text{A.3})$$

where $M_{s_1 \sigma_1}^{s' s}$ and $M_{s_2 \sigma_2}^{s' s}$ are given by eq. (14) with (k, q) replaced by (s_1, σ_1) and (s_2, σ_2) respectively. Thus statements in the text concerning the index k in the amplitudes $M_{\ell' s' \ell s}^{(k)}$ can also be stated in terms of possible values of the spin transfer in the reaction.

The inverse of (A.2), which expresses the transfer coefficients in terms

of the $A_{kq}^{s_1 s_2}$, is given in eq. (19) of ref. 6). In the present notation this reads

$$X_{k_a q_a k_b q_b} = \sum_{s_1 s_2 k q} \hat{s}_1 \hat{s}_2 \hat{k} \hat{k}_b (-)^{k_a - q_a} (k_b q_b, k_a - q_a | k q) \times \begin{Bmatrix} s & s & k \\ s' & s' & k_b \\ s_1 & s_2 & k \end{Bmatrix} A_{kq}^{s_1 s_2} \quad (\text{A.4})$$

B. Distorted wave expansion and unitarity relations

B.1 Distorted wave expansions

Distorted waves are defined in terms of matrix elements of the wave operator

$$\Omega^{(+)}(E_1) = 1 + \frac{1}{E_1 - H_0 - V + i\epsilon}, \quad (\text{B.1})$$

through the relation

$$\chi_{\underline{k}_1, s\sigma}^{(+)}(\underline{R}, \xi) = \langle \underline{R}, \xi | \Omega^{(+)}(E_1) | \underline{k}_1, s\sigma \rangle \quad (\text{B.2})$$

where the notation of sect. 1 is used, and

$$E_1 = \frac{\hbar^2 k_1^2}{2\mu}. \quad (\text{B.3})$$

In eq. (B.1) H_0 is the channel Hamiltonian and V is the corresponding interaction. It will be assumed that all rearrangement channels, and possibly some inelastic channels, have been taken into account through the non-Hermitean interaction, V , which, therefore, only explicitly couples channels with the same two channel fragments as in the incident channel, but possibly with different channel spin, energy and other internal quantum numbers. Thus, in (B.2), $\underline{R}$ denotes the relative coordinate of the fragments in any channel, and ξ denotes the corresponding internal coordinates. As in the text, for reasons of notational simplicity, there is no reference in the notation to channel labels other than channel spin and energy.

Assuming that, j , the total angular momentum, is a good quantum number, the usual expansion of the distorted wave (B.2) is

$$\chi_{\underline{K}_1 s}^{(+)}(\underline{R}, \xi) = \sum (\ell \lambda, s \sigma | j m) (\ell \lambda', s' \sigma' | j m) Y_{\ell}^{\lambda*}(\underline{\hat{K}}_1) Y_{\ell}^{\lambda'}(\underline{\hat{R}}) \times i^{\ell'} \chi_{s' \sigma'}^{\ell'}(\xi) \Omega_{\ell' s' \ell s}^j(K_1, R), \quad (\text{B.4})$$

where $\chi_{s' \sigma'}^{\ell'}$ is constructed from internal wavefunctions of the channel fragments, and the summation is over $\ell, \lambda, \ell', \lambda', s', \sigma', j, m$. The radial wavefunction in (B.4) is defined as

$$\Omega_{\ell' s' \ell s}^j(K_1, R) = \langle R, (\ell' s) j m | \Omega^{(+)} | K_1, (\ell s) j m \rangle, \quad (\text{B.5})$$

and satisfies the boundary condition

$$\Omega_{\ell' s' \ell s}^j \xrightarrow{R \rightarrow \infty} (2/\pi)^{1/2} \left[\delta_{\ell \ell'} \delta_{s s'} j_{\ell}(K_1 R) + h_{\ell'}^{(+)}(K_2 R) \left(-\frac{\mu \pi K_2}{\hbar^2} M_{\ell' s' \ell s}^j \right) \right] \quad (\text{B.6})$$

where j_{ℓ} and $h_{\ell}^{(+)}$ are spherical Bessel functions as defined in Appendix B of ref. 3), and $M_{\ell' s' \ell s}^j$ is the amplitude defined in eqs. (6) and (7) of sect. 1.

In eq. (B.6) unscreened Coulomb forces have been ignored. They can be treated in the following by a straightforward generalization of the usual approach.

The point-charge Rutherford amplitude contributes to the spin transfer amplitude with $k = 0$ only.

The identity given in eq. (9) of the text converts (B.4) into

$$\chi_{\underline{K}_1 s \sigma}^{(+)}(\underline{R}, \xi) = \sum_{k q s' \sigma'} \hat{k}(s' \sigma', k q | s \sigma) \chi_{s' \sigma'}^{\ell'}(\xi) \chi_{k q}^{s' s}(\underline{R}, \underline{K}_1), \quad (\text{B.7})$$

where

$$\chi_{k q}^{s' s}(\underline{R}, \underline{K}_1) = \sum_{\ell' \lambda' \ell \lambda} \frac{\hat{s}' \hat{\ell}'}{\hat{s}} (\ell \lambda', \ell - \lambda | k q) (-1)^{\ell - \lambda} Y_{\ell}^{\lambda*}(\underline{\hat{K}}_1) Y_{\ell'}^{\lambda'}(\underline{\hat{R}}) i^{\ell'} \Omega_{\ell' s' \ell s}^{(k)}(R) \quad (\text{B.8})$$

and (c.f., eq. (11))

$$\Omega_{\ell' s' \ell s}^{(k)} = \sum_j \frac{j^2}{\ell' \ell s} (-1)^{j - \ell' - s'} W(\ell' \ell s' s; k j) \Omega_{\ell' s' \ell s}^j(R). \quad (\text{B.9})$$

The boundary condition corresponding to (B.6) is

$$\Omega_{\ell' s' \ell s}^{(k)}(R) \xrightarrow{R \rightarrow \infty} \left[\delta_{k 0} \delta_{\ell' \ell} \delta_{s' s} j_{\ell}(K_1 R) + h_{\ell'}^{(+)}(K_2 R) \left(-\frac{\mu \pi K_2}{\hbar^2} M_{\ell' s' \ell s}^{(k)} \right) \right] \quad (\text{B.10})$$

where the amplitudes $M_{\ell' s' \ell s}^{(k)}$ and $M_{\ell' s' \ell s}^j$ are related by eq. (11).

Distorted wave expansions of the form (B.7) have been found useful in ref. 13), and more recently in ref. 14).

B.2 Coupled radial equations

The radial wavefunctions $\Omega_{\ell's'\ell_s}^{(k)}$ are characterized by a definite value of the channel-spin transfer, k , which in general is not a good quantum number. As a result radial wavefunctions with different values of k are coupled in the presence of spin-dependent forces, whereas in the j -representation different j -values are not coupled if H conserves angular momentum. In both representations different values of orbital angular momentum are coupled in general.

The detailed form of the coupled equations can be derived by transforming the standard coupled equations for the $\Omega_{\ell's'\ell_s}^j$, or directly from the expansion (B.7) and (B.8), together with the Schrödinger equation. The result is of the form

$$(E_s - H_0(\ell')) \Omega_{\ell's'\ell_s}^{(k)} = \sum_{k'\ell''s''} H_{\ell's'\ell''s''}^{kk'}(\ell, s) \Omega_{\ell''s''\ell_s}^{(k')}, \quad (\text{B.11})$$

where

$$H_0(\ell') = -\frac{\hbar^2}{2\mu} \left(\frac{d^2}{dR^2} + \frac{2}{R} \frac{d}{dR} - \frac{\ell'(\ell'+1)}{R^2} \right) + V_s(R). \quad (\text{B.12})$$

Expressions for $V_s(R)$ and $H_{\ell's'\ell''s''}^{kk'}(\ell, s)$ can be obtained by assuming that the Hamiltonian has an expansion of the form

$$\int d\xi \chi_{s'\sigma'}^*(\xi) H \chi_{s\sigma}(\xi) = \sum_{k_1 q_1} \hat{k}_1(s\sigma, k_1 q_1 | s'\sigma') H_{k_1 q_1}^{s's} (-1)^{q_1}, \quad (\text{B.13})$$

where $H_{k_1 q_1}^{s's}$ is an irreducible tensor of rank k_1 constructed from the dynamical variables, $\underline{R}$, $\underline{p}_R$. The terms in eq. (B.13) with $k_1 = 0$ can be absorbed into the definition of $H_0(\ell')$ of eq. (B.12) and the matrix $H_{\ell's'\ell''s''}^{kk'}$ is then given by

$$H_{\ell's',\ell''s''}^{kk'}(\ell,s) = \sum_{k_1(\neq 0),q_1} (-1)^{k_1} \hat{k}_1 \hat{k} \hat{k}' W(k'k_1\ell\ell';k\ell'') W(kk_1ss'';k's') \times \langle \ell's' | \ell' | H_{k_1}^{s's''} | \ell'' \rangle, \quad (B.14)$$

If the Hamiltonian, H , satisfies

$$K H K^\dagger = H^\dagger, \quad (B.15)$$

where K is the time reversal operator for the system, the reduced matrix elements in eq. (B.14) satisfy

$$(-1)^{\ell'-s'} \langle \ell's' | \ell' | H_{k_1}^{s's''} | \ell'' \rangle = (-1)^{\ell''-s''} \langle \ell''s'' | \ell' | H_{k_1}^{s's'} | \ell' \rangle, \quad (B.16)$$

and as a result

$$H_{\ell's',\ell''s''}^{kk'}(\ell,s) = H_{\ell''s'',\ell's'}^{k'k}(\ell,s). \quad (B.17)$$

It should be noted that the coupling matrix given in eq. (B.14) involves the values of ℓ, s . It can be seen from eqs. (B.10) and (B.11) that there can be interpreted as possible values of the orbital angular momentum and channel spin in the incident channel. This is quite different from the situation which occurs in the coupled equations for the $\Omega_{\ell's',\ell s}^j$. In the latter case the values of ℓ and s do not influence the structure of the coupled equations, and appear only in the boundary condition, eq. (B.6). This difference reflects the fact that the definition of the quantum number k involves both the initial and final channels in a fundamental way.

B.3 Unitarity relations

If the Hamiltonian H which generates the distorted wave (B.2) is Hermitean, the S^j -matrix defined by

$$S_{\ell's',\ell s}^j = \delta_{\ell\ell'} \delta_{ss'} - 2\pi i \frac{\mu}{\hbar} \sqrt{K_{s'}} M_{\ell's',\ell s}^j \sqrt{K_s} \quad (B.18)$$

satisfies the unitarity relation

large values of the orbital angular momentum ℓ dominate the transition amplitude.

In the coordinate system defined in eqs. (21), (22) the polar angles of $\underline{K}_1$ and $\underline{K}_2$ are

$$\theta_{\underline{K}_1} = \frac{1}{2}\theta, \quad \phi_{\underline{K}_1} = \pi, \quad (54a)$$

$$\theta_{\underline{K}_2} = \frac{1}{2}\theta, \quad \phi_{\underline{K}_2} = 0, \quad (54b)$$

where θ is the scattering angle. The formulae (14) now reduces to

$$\begin{aligned} \bar{M}_{2q} &= \sum_{\ell\ell'\lambda} (\ell'q-\lambda, \ell\lambda | 2q) \hat{\ell}' (-1)^{\ell+\lambda} Y_{\ell'}^{q-\lambda}(\frac{1}{2}\theta, 0) Y_{\ell}^{\lambda}(\frac{1}{2}\theta, 0) M_{\ell'\ell}^{(2)} \\ &= \sum_{\ell, \lambda} (\ell q-\lambda, \ell\lambda | 2q) \hat{\ell} (-1)^{\ell+\lambda} Y_{\ell}^{q-\lambda}(\frac{1}{2}\theta, 0) Y_{\ell}^{\lambda}(\frac{1}{2}\theta, 0) M_{\ell\ell}^{(2)} \\ &\quad + 2 \sum_{\ell, \lambda} (\ell+2 q-\lambda, \ell\lambda | 2q) (2\ell+5)^{1/2} (-1)^{\ell+\lambda} Y_{\ell+2}^{q-\lambda}(\frac{1}{2}\theta, 0) Y_{\ell}^{\lambda}(\frac{1}{2}\theta, 0) M_{\ell+2, \ell}^{(2)} \end{aligned} \quad (55)$$

where use has been made of the result eq. (12), and the symmetry given in eq. (18).

From eq. (55) it follows that

$$\begin{aligned} \bar{M}_{22} - \left(\frac{3}{2}\right)^{1/2} \bar{M}_{20} &= \sum_{\ell\lambda} [(\ell 2-\lambda, \ell\lambda | 22) Y_{\ell}^{2-\lambda} - \left(\frac{3}{2}\right)^{1/2} (\ell-\lambda, \ell\lambda | 20) Y_{\ell}^{-\lambda}] \\ &\quad \times \hat{\ell} (-1)^{\ell+\lambda} Y_{\ell}^{\lambda} M_{\ell\ell}^{(2)} + 2 \sum_{\ell\lambda} [\ell+2 2-\lambda, \ell\lambda | 22) Y_{\ell+2}^{2-\lambda} \\ &\quad - \left(\frac{3}{2}\right)^{1/2} (\ell+2-\lambda, \ell\lambda | 20) Y_{\ell+2}^{-\lambda}] (2\ell+5)^{1/2} (-1)^{\ell+\lambda} Y_{\ell}^{\lambda} M_{\ell+2, \ell}^{(2)}, \end{aligned} \quad (56)$$

where the arguments $(\frac{1}{2}\theta, 0)$ of the spherical harmonics have been omitted for clarity.

From the tables of Clebsch-Gordan coefficients given in ref. ⁴⁾ it is found that non-vanishing coefficients satisfy

$$\sum_{\ell''s''} S_{\ell s \ell'' s''}^j S_{\ell' s' \ell'' s''}^{j*} = \delta_{\ell \ell'} \delta_{s s'} \quad (B.19)$$

A matrix $S^{(k)}$ may be defined by

$$S_{\ell' s' \ell s}^{(k)} = \sum_j \frac{j^2}{\hat{\ell} \hat{s}} (-)^{j-\ell'-s'} W(\ell' \ell s' s; k j) S_{\ell' s' \ell s}^j \quad (B.20)$$

The identity (ref. 4), p. 117)

$$\sum_j j^2 W(j g f a; c i) W(j d i b; h f) W(j g d e; c h) = W(a b c d; e f) W(a b g h; e i), \quad (B.21)$$

can be used to show that the definition (B.20) implies that

$$\sum_{\ell s} S_{\ell' s' \ell s}^j S_{\ell'' s'' \ell s}^{j*} = \sum_{k_1} \hat{k}_1^2 W(s' s'' \ell' \ell''; k_1 j) (1)^{j-\ell'-s'+k_1} X_{k_1}(\ell' s'; \ell'' s''), \quad (B.22)$$

where the quantity X_{k_1} is defined by

$$X_{k_1}(\ell' s'; \ell'' s'') = \sum_{k, k'} W(k' k_1 \ell \ell'; k \ell'') W(k k_1 s s''; k' s') \\ \times S_{\ell' s' \ell s}^{(k)} S_{\ell'' s'' \ell s}^{(k')} \hat{\ell} \hat{s} \hat{\ell}'' \hat{s}'' \hat{k}^2 \hat{k}'^2 \quad (B.23)$$

It is readily shown that the relation (B.19) is completely equivalent to the result

$$X_{k_1}(\ell' s'; \ell'' s'') = \hat{\ell} \hat{s} \delta_{s' s''} \delta_{\ell' \ell''} \delta_{k_1 0}, \quad (B.24)$$

and in particular, putting $k_1 = 0$ in (B.23) and (B.24)

$$\sum_{k \ell' s'} \hat{k}^2 |S_{\ell' s' \ell'' s''}^{(k)}|^2 = 1. \quad (B.25)$$

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