The Physical Interpretation of Quantum Theories of Gravitation

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The Physical Interpretation of Quantum Theories of Gravitation

Prolegomena to a preface to an introduction
to a quantum theory of gravitation

H. Ekstein

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An essential part of gravitational theory is the manner in which observations are influenced by the state of motion (acceleration) of the observer (live or inanimate). Einstein's point of departure was the equivalence principle of Newtonian mechanics. The general theory of relativity can be considered as a synthesis of the Newtonian equivalence principle with special relativity. On the other hand, the efforts to construct a quantum theory of general relativity start from classical general relativity. It may be fruitful to explore an alternative approach to a quantum theory of gravitation in which a quantum version of the Newtonian equivalence principle is married to special relativistic quantum field theory. The following chart shows the proposed scheme of exploration.

It is surprising and somewhat shocking to find that no clear statement of the Galilei-Newton equivalence principle exists in quantum mechanics. The purpose of this note is to take the first step on the proposed

programmatic path by stating and sketching the proof of this nonrelativistic equivalence principle of quantum mechanics. Instead of postulating a formal system to be interpreted later, we will proceed from operational statements to their theoretical expressions.)

In classical physics, the transition from an inertial to an accelerated frame is accomplished simply by the transformation

$$\begin{aligned}\vec{x} &\mapsto \vec{x} + \frac{1}{2}\vec{a}t^2 \\ t &\mapsto t,\end{aligned}\tag{1}$$

and one deduces immediately that the form of Newton's second law in an accelerated frame is

$$m \frac{d^2 \vec{x}}{dt^2} = \vec{F} \quad \mapsto \quad m \frac{d^2 \vec{x}}{dt^2} + m\vec{a} = \vec{F},\tag{2}$$

which is the symbolic statement of the equivalence principle.

To state this result in a more general framework, consider the set W of all world lines. Let $W(H)$ be the subset of world lines that are solutions of the canonical equations with a Hamiltonian H . Then, the equivalence principle states that the acceleration transformation (1) induces a map

$$W(H) \rightarrow W(H + m\vec{a} \cdot \vec{x})$$

from the world lines of $W(H)$ to those generated by the altered Hamiltonian¹
 $H + m\vec{a} \cdot \vec{x}$.

¹Equation (1) should not be considered as a coordinate transformation, but as a mapping of space-time points that induces (characterizes) a bodily motion of observation instruments: the act of putting them on an accelerated vehicle. Coordinate transformations (the art of saying the

In the usual formulation of nonrelativistic quantum mechanics, there is no equivalent to Eq. (1). Other space-time transformations, such as rotations

$$\vec{x} \mapsto R\vec{x}, \quad (3)$$

induce simple transformations in quantum mechanics, such as

$$A \mapsto U(R) A U(R)^{-1}, \quad (4)$$

where A is an operator on Hilbert space, and $U(R)$ a unitary operator induced by a rotation R . No sensible transformation of the algebra *or* of observables (linear operators on the Hilbert space $\mathcal{H}$) can be associated to an acceleration $\vec{a}$. To see this, consider a free particle or a free many-particle system. As an operator equation for the time derivative $\dot{\vec{P}}$ of the momentum, we have

$$i[\vec{P}, H] = \dot{\vec{P}} = 0,$$

and we would like to find a transformation that carries $\dot{\vec{P}}$ into $\dot{\vec{P}} + m\vec{a}$ and (for any real number c) carries $c\dot{\vec{P}}$ into $c(\dot{\vec{P}} + m\vec{a})$, since the choice of units on the dial of an instrument should be irrelevant. Clearly, there exists no unitary operator U that accomplishes

$$U \cdot 0 \cdot U^{-1} \neq 0$$

same thing in different ways) are Characterbuilding And All That, but they frequently lead to confusion in basic considerations. In this note, no coordinates are used. In particular, the usual coordinate transformations (such as the transformation from Cartesian to polar coordinates) have no relevance to this discussion since they do not induce a bodily motion of rigid instruments.

and, more generally, there exists no transformation of the algebra that can do what is required.² Thus, the first step toward any formulation of the equivalence principle is the construction of a framework in which acceleration induces a reasonable transformation.

Let us consider the physical interpretation of the quantum-mechanical formalism more closely. According to von Neumann, an observable is first a procedure, and is then associated to a self-adjoint operator on Hilbert space. While this association between laboratory operations and mathematical objects is always implicit in the thinking and writing of physicists, we find it necessary to consider explicitly a set of observation procedures with a mapping $\bar{\Phi}: \bar{\mathcal{O}} \rightarrow \bar{\mathcal{O}}\mathcal{L}$ of $\bar{\mathcal{O}}$ onto a set $\bar{\mathcal{O}}\mathcal{L}$ of self-adjoint operators on $\mathcal{H}$. This is a many-one mapping. For instance, the statement that for free systems the momentum $\bar{P}$ is time-independent is expressed by the equation

$$\bar{P}(t_1) = \bar{P}(t_2)$$

for pairs of Heisenberg operators $\bar{P}(t_i)$ in $\bar{\mathcal{O}}\mathcal{L}$. In our more explicit language, we consider laboratory procedures $\bar{p}(t_i)$ which are, of course, distinct if $t_1 \neq t_2$. However, for some mappings $\bar{\Phi}$, we have

$$\bar{\Phi}[\bar{p}(t_1)] = \bar{\Phi}[\bar{p}(t_2)] = \bar{P}(t_1).$$

²One may propose a rule of association of operators, such as

$$\bar{Q}(t) \mapsto \bar{Q}(t) + \frac{1}{2} \bar{a} t^2,$$

$$\bar{P}(t) \mapsto \bar{P}(t) + \bar{a} t.$$

However, for a single spinless particle, every set of the operators $\{\bar{Q}(t), \bar{P}(t)\}$ (for any fixed t) generates $\bar{\mathcal{O}}\mathcal{L}$. Hence, the above rule associates infinitely many operators B to a given $A \in \bar{\mathcal{O}}\mathcal{L}$, according to the choice of t . Hence, the above rule is not a transformation or a map, which, by definition, associates a single image to every element $A \in \bar{\mathcal{O}}\mathcal{L}$. This remark is the result of a discussion with P. G. Bergmann.

The observable implication of this equation is that the expectation values of results of these two observation procedures are numerically equal with respect to any state-preparing procedure. That is,

$$\mathcal{E}[s, \vec{p}(t_1)] = \mathcal{E}[s, \vec{p}(t_2)] \quad \text{for all } s, \quad (5)$$

where

$$\mathcal{E}(s, a) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n(s) \quad (6)$$

is the mean value of individual dial readings a_n obtained by the procedure $a \in \bar{\mathcal{O}}$ on a state prepared by a procedure s .

Now consider space-time transformations. Primarily, in its operational meaning, such a transformation is a motion of laboratory instruments, e.g., the act of putting an electrometer on a moving vehicle. Some motions of instruments will permute the existing supply of laboratory instruments, e.g., a 180° rotation will interchange two telescopes pointing in opposite directions. A permutation V_g of the set $\bar{\mathcal{O}}$ of observation procedures,

$$V_g: \bar{\mathcal{O}} \rightarrow \bar{\mathcal{O}},$$

has an effect on $\bar{\mathcal{O}}$ through the mapping $\bar{\Phi}$. To see what this effect can be, consider some purely mathematical objects: the sets $\bar{\mathcal{O}}$ and $\bar{\mathcal{O}}$, a many-one mapping $\bar{\Phi}$ on $\bar{\mathcal{O}}$ onto $\bar{\mathcal{O}}$, and a permutation V_g of the set $\bar{\mathcal{O}}$. The permutation V_g induces a transformation Q_g of $\bar{\mathcal{O}}$ if for every $A \in \bar{\mathcal{O}}$ there exists a mapping $A \mapsto Q_g A \in \bar{\mathcal{O}}$ such that

$$\bar{\Phi} V_g a = Q_g \bar{\Phi} a. \quad (7)$$

Clearly, such a mapping Q_g can exist only if the statement $\bar{\Phi} a = \bar{\Phi} \beta$ implies that $\bar{\Phi} V_g a = \bar{\Phi} V_g \beta$, i.e., if two points $a, \beta \in \bar{\mathcal{O}}$ with coincident images in $\bar{\mathcal{O}}$ still have identical images after being moved by the permutation $a \mapsto V_g a, \beta \mapsto V_g \beta$. Otherwise the operator Q_g would have

to carry the single point $\bar{\Phi}\alpha = \bar{\Phi}\beta$ of $\bar{\mathcal{H}}$ into two distinct points $\bar{\Phi}V_g\alpha$ and $\bar{\Phi}V_g\beta$. In this case the mapping $\bar{\Phi}$ and the permutation V_g are said to be incompatible, and V_g induces no transformation in $\bar{\mathcal{H}}$.

I assert that this is just what happens for accelerations, considered as physical motions of hardware. They induce transformations of the hardware space $\bar{\mathcal{O}}$, but not of $\bar{\mathcal{H}}$. To make this heuristic statement more precise, some more structure will be assumed for the set $\bar{\mathcal{O}}$ and the mappings $\bar{\Phi}$. The set $\bar{\mathcal{H}}$ of self-adjoint operators is a real linear space, and the operation of squaring

$$A \mapsto A^2 \in \bar{\mathcal{H}}$$

is defined in it. We assume naturally that this structure is inherited from the set $\bar{\mathcal{O}}$. Explicitly, we assume (1) that $\bar{\mathcal{O}}$ is real linear and that squaring is defined in it, and (2) that $\bar{\Phi}$ is a morphism, i. e., it preserves the structure of $\bar{\mathcal{O}}$ so that

$$\bar{\Phi}(a\alpha + b\beta) = a\bar{\Phi}\alpha + b\bar{\Phi}\beta \quad (\alpha, \beta \in \bar{\mathcal{O}}, a, b, \text{ real numbers})$$

and

$$\bar{\Phi}\alpha^2 = (\bar{\Phi}\alpha)^2.$$

Furthermore, the set $\bar{\mathcal{H}}$ of self-adjoint operators is conveniently imbedded into the star-algebra $\mathcal{H}$ of operators, which is generated in the natural way by complex linear combinations of self-adjoint operators. A similar imbedding of $\bar{\mathcal{O}}$ in a star-algebra $\mathcal{O}$ is assumed, and the original morphism $\bar{\Phi}: \bar{\mathcal{O}} \rightarrow \bar{\mathcal{H}}$ is extended to a morphism

$$\Phi: \mathcal{O} \rightarrow \mathcal{H}$$

that preserves complex linear combinations, products, and involutions.

An observation procedure is specified by a blueprint for the construction of an instrument and by instructions for the positioning of marks on the instrument in space and for the timing of certain operations (e.g., pressing a "start" button). Every procedure $\alpha = \alpha^\dagger \in \mathcal{O}$ can be considered as a pair

$$\alpha = (b, \{P_n\}),$$

where b is a blueprint, and the sequence $\{P_n\}$ is a set of space-time events at points $P_n \in V_4$. A space-time transformation consists in leaving all blueprints unchanged, and changing the reference events $P_1 \dots P_n$ in accordance with a transformation $g: V_4 \rightarrow V_4$ of space-time. One can formally define an operator V_g

$$V_g \alpha \equiv (b, \{gP_n\}) \quad (8)$$

but in general the changed instructions will not lead to a new observation procedure. For instance, a scale transformation

$$\vec{x} \rightarrow \alpha \vec{x}$$

of V_4 does not induce a permutation in $\mathcal{O}$ (a crushed ammeter is no ammeter). Consider the group of transformations of V_4 that constitute the usual definition of the Galilei group G . Experience shows not only that this group induces permutations in $\mathcal{O}$ (i.e., $V_g \alpha \in \mathcal{O}$, if $g \in G$) but that these permutations preserve algebraic relations between procedures. In other words, the group $V_g (g \in G)$ induced by G according to Eq. (8) is a group of automorphisms of $\mathcal{O}$. As a further reasonable extrapolation from experience, the acceleration group

$$A: \begin{pmatrix} \vec{x} \mapsto \vec{x} + \sum_n \vec{a}_n t^n \\ t \mapsto t \end{pmatrix} \quad (9)$$

can be assumed to induce automorphisms V_g of $\mathcal{O}$. The group $G + A$ generated by the Galilei group G and the acceleration group is considered as the presymmetry group. With this added structure, the nature of a transformation Q_g in $\mathcal{U}$ (if it exists) is more restricted: it is an automorphism of $\mathcal{U}$. For if there exists a mapping $Q_g: \mathcal{U} \rightarrow \mathcal{U}$ such that

$$\Phi V_g a = Q_g \Phi a, \quad (10)$$

and Φ is a morphism onto $\mathcal{U}$, then Q_g must be an automorphism.

In the nonrelativistic case, one can show that the automorphisms Q_g induced by space-time motions are spatial, i.e., there exists a unitary operator $U \in \mathcal{U}$ such that

$$Q_g A = U A U^{-1} \quad (A \in \mathcal{U}).$$

One can show that every morphism Φ corresponds to a one-parameter family $H(t)$ of (in general time-dependent) Hamiltonians. The analog of our previous empirical statement concerning accelerations can now be stated in this framework: There exists no physical morphism Φ that is compatible with an automorphism V_g ($g \in A$) of the hardware set $\mathcal{O}$. Accelerations induce automorphisms of $\mathcal{O}$, but not of $\mathcal{U}$. The derivation of this empirical fact from prime principles will be sketched below.

The set of laboratory procedures that are used to prepare samples for measurements—the set $\mathcal{J}$ of state-preparing procedures—is assumed to have the structure of a convex linear set familiar from the theory of density matrices. The samples produced by two procedures $s, r \in \mathcal{J}$ can be mixed in some ratio $\lambda/(1-\lambda)$, where $0 \leq \lambda \leq 1$. Specifically, to any two procedures s, r with expectation values

$$\mathcal{E}(s, a) = \lim \frac{1}{N} \sum_{n=1}^N a_n(s),$$

$$\mathcal{E}(r, a) = \lim \frac{1}{N} \sum_{n=1}^N a_n(r)$$

one can associate a limit

$$\sum_1^{n\lambda} a_n(s) + \sum_{n\lambda+1}^{n\lambda+1+n(1-\lambda)} a_n(r) + \dots$$

by alternating samples prepared by the two procedures in the ratio $\lambda/(1-\lambda)$, and define it as

$$\mathcal{E}[\lambda s + (1-\lambda)r, a].$$

This convex linear set $\mathcal{J}$ is mapped by a morphism $\Psi: \mathcal{J} \rightarrow S$ onto the set S of "states" on $\mathcal{A}$, i.e., onto the convex linear set of linear, positive, and normalized forms on the algebra $\mathcal{A}$. (In the simplest case without superselection, one can equate S with the space of density operators.) As before, the physical space-time motions of the group $(G+A)$ are assumed to induce automorphisms W_g of $\mathcal{J}$ which for some morphisms Ψ induce automorphisms of S .

With this assumption, the morphisms Φ and Ψ can be connected to the expectation $\mathcal{E}$. This can be considered as a mapping

$$\mathcal{E}: \mathcal{J} \times \mathcal{O} \rightarrow \mathbb{C}$$

from the Cartesian product $\mathcal{J} \times \mathcal{O}$ into the complex numbers. The kernel

$$K(\mathcal{E}) = \{a \mid \mathcal{E}(s, a) = 0, \quad \forall s \in \mathcal{J}\} \quad (11)$$

defines a morphism

$$\Phi(\mathcal{E}): \mathcal{O} \rightarrow \mathcal{O}/K(\mathcal{E}) \cong \mathcal{A}$$

from $\mathcal{O}$ to the quotient set $\mathcal{O}/K(\mathcal{E})$ which is isomorphic to the algebra $\mathcal{A}$, assumed to be the same for all expectations $\mathcal{E}$. [Note that "expectation," a function of $\mathcal{J} \times \mathcal{O}$, is distinguished from "expectation value," a number $\mathcal{E}(s, a)$.]

Similarly, $\mathcal{E}$ defines an equivalence E in the set $\mathcal{S}$, and an isomorphism between the quotient $\mathcal{S}/E$ and the set S of states.

The operational definition of space-time symmetry is: If observation procedures and state-preparing procedures are subjected to a motion g , then all expectation values remain invariant. More formally, the equation

$$\mathcal{E}_{s, g}(s, a) = \mathcal{E}_{s, g}(W_g s, V_g a) \quad (12)$$

for every $s \in \mathcal{S}$ and every $a \in \mathcal{O}$ characterizes the particular class of expectations $\mathcal{E}_{s, g}$ that exhibit symmetry³ with respect to the motion g . A more general statement refers to cases (such as acceleration) in which Eq. (12) does not hold. If, in addition to moving (e.g., accelerating) observation and state-preparing procedures, one changes the expectation (i.e., the Hamiltonian) in an appropriate way, all expectation values remain unchanged. More precisely, I assume that for every expectation $\mathcal{E}$ and every motion $g \in G + A$ there exists another expectation $\mathcal{E}_g$ such that

$$\mathcal{E}_g(W_g s, V_g a) = \mathcal{E}(s, a) \quad (13)$$

for every $s \in \mathcal{S}$ and $a \in \mathcal{O}$. Intuitively, Eq. (13) asserts that one can change or create an external field to compensate for the effects of simultaneous motions of state-preparing and observation instruments. An example is a particle placed in an external field and subjected to a rotation. If observation instruments, state-preparing procedures, and the sources of the external field are simultaneously subjected to the same rotation, all expectation values are unchanged. Equation (13) is the articulate expression of an assumption made implicitly by all those who discuss external fields.

³The use of the word "symmetry" here is in agreement with the language of J. L. Anderson, Principles of Relativity-Physics (Academic Press, New York, 1967). "Presymmetry" is a special case of what Anderson calls "covariance."

The addition of the acceleration group A to the Galilei group (resulting in the presymmetry group) implies the first part of the equivalence principle: Given any dynamical system with a given Hamiltonian, a "change of frame" (i. e., the physical act of putting both observation and state-preparing instruments on an accelerated vehicle) can be compensated by an external field. One can also say: There exists no privileged inertial system of reference provided that there are no means other than observations on particle motions for studying external fields (excluding the possibility that external fields can be determined, for example, by looking at distant stars).

The nontrivial part of the equivalence principle is that the external force to be added in the change from $\mathcal{E}$ to $\mathcal{E}_g$ in Eq. (13) is proportional to the mass of each particle. Explicitly, if $H(\mathcal{E})$ is the Hamiltonian associated to the expectation $\mathcal{E}$, and $H(\mathcal{E}_a)$ that associated to the expectation $\mathcal{E}_a$, where $\vec{a}$ is the acceleration implied by the motion g , then

$$H(\mathcal{E}_a) = H(\mathcal{E}) + \vec{a} \cdot \vec{K}, \quad (14)$$

where $\vec{K}$, the generator of Galilei transformations, is given by

$$\vec{K} = M\vec{Q} = \sum m_i \vec{q}_i. \quad (15)$$

Here, M is the total mass, $\vec{Q}$ the position of the center of mass, and m_i and $\vec{q}_i$ are the masses and positions of the particles.

The proof of Eq. (14) requires some more assumptions on the structure of the expectation $\mathcal{E}$ and a bit of group theory. Most of the assumptions are just explicit and directly verifiable counterparts of the usual (sometimes implicit) ones, so only one will be listed. According to this postulate, the displacement d produced by a translation $\sigma \mapsto V_d \sigma$, no matter how small, can be ascertained by some measurement. Stated more formally: For every distance d (no matter how small) and for every

expectation value $\mathcal{E}$, there exists a state-preparing procedure $s \in \mathcal{S}$ and an observation procedure $a \in \mathcal{O}$ such that the function

$$F(d) = |\mathcal{E}(s, V_d a)|$$

is discontinuous at $d = 0$.

The details of the assumptions and the derivation are contained in a forthcoming paper.⁴

At the cost of introducing a rather complicated conceptual framework, we have made a small gain: the derivation, from prime principles, of the quantum-mechanical analog of the nonrelativistic equivalence principle, which is trivial in classical mechanics. This enlargement of the framework of quantum mechanics is necessary, because the concept of world lines is absent in quantum mechanics, the reason being that t becomes a parameter while x becomes an operator. The conclusion is that any quantum theory of gravitation with consistent physical interpretation will probably have to use a similarly enlarged conceptual framework.

⁴H. Ekstein, Presymmetry II (to be published). This is a sequel to H. Ekstein, Phys. Rev. 153, 1397 (1967).