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The Theory and Numerical Calculation of the Acoustic Field Exerted by Eddy-Current Forces

Katsuhiro Kawashima

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THE THEORY AND NUMERICAL CALCULATION OF THE ACOUSTIC
FIELD EXERTED BY EDDY-CURRENT FORCES

Katsuhiko Kawashima

JANUARY 1976

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THE THEORY AND NUMERICAL CALCULATION OF THE ACOUSTIC
FIELD EXERTED BY EDDY-CURRENT FORCES

Katsuhiro Kawashima¹

ABSTRACT

The equations for calculating the acoustic field produced within a nonmagnetic metal by interaction of eddy currents with a static magnetic field were obtained on the assumptions (1) an ultrasonic wave is generated by the electromagnetic force through classical and macroscopic phenomena; (2) the electric, magnetic, and elastic properties of the metal are linear, isotropic, and homogeneous throughout the metal, which occupies semi-infinite space; (3) the whole system is axially symmetric; and (4) eddy currents and elastic waves show a steady-state sinusoidal variation. The acoustic field produced by a specific electromagnetic ultrasonic transducer with axial symmetry was calculated numerically, and the results showed a well-defined ultrasonic wave beam, which was narrower than had been expected from the size of the transducer.

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INTRODUCTION

The method for calculating the acoustic field exerted by the interaction of eddy currents with a stationary magnetic field within a nonferromagnetic metal is based on the following assumptions:

1. Ultrasound is generated by the electromagnetic force through classical and macroscopic phenomena.
2. Electric, magnetic, and elastic properties of the material are linear, isotropic, and homogeneous throughout the material, which occupies semi-infinite space.

Several authors²⁻⁴ have given several methods for treating this subject, all of which are approximate methods with regard to both eddy-current and acoustic field calculations. The rigorous method for calculating the force exerted by eddy currents within a conducting metal has already been obtained by Dodd and Deeds.⁵ To make the best use of the eddy-current calculations, a rigorous way of calculating the acoustic field should also be employed. The simple method of a piston movement source for calculating the acoustic field is apparently inadequate for the problem because it cannot calculate the acoustic field from the actual applied force. The rigorous way of calculating the elastic wave in a solid is given in the theory of elasticity, by Lamb,⁶ Miller and Pursey,⁷ Johnson,⁸ and many others. Some modification,

²R. Turner, K. R. Lyall, and J. F. Cochran, "Generation and Detection of Acoustic Waves in Metals by Means of Electromagnetic Radiation," *Can. J. Phys.* 47: 2293 (1969).

³Yu. M. Shkarlet and N. N. Lokshina, "Noncontact Electromagnetic Transducer for Ultrasonic Thickness Test of Metallic Products," pp. 13-24 in *Preprints 6. ICNT Vol. B, 1970*, Deutsche Gesellschaft für Zerstörungsfreie Prüfverfahren e.V., Berlin.

⁴H. Wüstenberg, "Contactless Electrodynamic Ultrasonic Transducers and Their Application to Ultrasonic Inspection," pp. 37-48 in *Preprints 6. ICNT Vol. B, 1970*, Deutsche Gesellschaft für Zerstörungsfreie Prüfverfahren e.V., Berlin.

⁵C. V. Dodd, W. E. Deeds, J. W. Luquire, and W. G. Spoeri, *Some Eddy-Current Problems and Their Integral Solutions*, ORNL-4384 (April 1969).

⁶H. Lamb, "On the Propagation of Tremors Over the Surface of an Elastic Solid," *Phil. Trans. Roy. Soc. London Ser. A* 203: 1-42 (1904).

⁷F. Miller and H. Pursey, "The Field and Radiation Impedance of Mechanical Radiators on the Free Surface of a Semi-infinite Isotropic Solid," *Proc. Roy. Soc. London Ser. A* 223: 521-41 (1954).

⁸L. R. Johnson, "Green's Function for Lamb's Problem," *Geophys. J. Roy. Astron. Soc.* 37: 99-131 (1974).

however, must be made for the problem since the force exerted by eddy currents characteristically has both radial and vertical components in a cylindrical coordinate system. In this paper, a general method is described first, and then a numerical evaluation is given, followed by discussions.

ELECTROMAGNETIC FORCE

The force exerted by eddy currents has been calculated by Dodd and Deeds,^{5,9} and it agrees very well with measured values. This method is used in this report to calculate the surface force that generates the ultrasound. If the eddy-current penetration depth is much less than the acoustic wavelength, the volume force exerted by the eddy currents can be considered to be a surface force, which acts only on the surface, by integrating it with respect to the depth. The approximate values of the penetration depths and the wavelengths for some metals at a frequency of 1 MHz are shown in Table 1.

Table 1. Eddy-Current Penetration Depths and Acoustic Wavelengths for Selected Metals

Material	Conductivity ($\Omega^{-1} \text{ m}^{-1}$)	Permeability (H/m)	Eddy Current Penetration Depth (μm)	Acoustic Wavelength, mm	
				Longitudinal	Transverse
		$4\pi \times 10^{-7}$			
Steel	6.25×10^6	200	14.2	5.9	3.2
Aluminum	2.4×10^7	1	103	6.3	3.1
Copper	5.8×10^7	1	66	4.7	2.3

In the case of nonmagnetic metals, the force exerted by an electromagnetic field on a unit volume is given, to a very good approximation, by the Lorentz formula

$$\begin{aligned} \vec{F} &= \vec{J} \times \vec{B} \\ &= -\sigma \frac{\partial \vec{A}}{\partial t} \times (\nabla \times \vec{A}), \end{aligned} \quad (1)$$

⁹C. V. Dodd and W. E. Deeds, "Electromagnetic Forces in Conductors," *J. Appl. Phys.* 38(13): 5045-51 (December 1967).

where $\sigma =$ conductivity and $\vec{A} =$ vector potential. If axial symmetry is assumed, A has only an azimuthal component, A_ϕ , and the force has only radial and axial components in a cylindrical coordinate system

$$F_z = -\sigma \frac{\partial A}{\partial t} \frac{\partial A}{\partial z} , \quad (2)$$

$$F_r = -\sigma \frac{\partial A}{\partial t} \frac{1}{r} \frac{\partial}{\partial r}(rA) , \quad (3)$$

where $A \equiv A_\phi$. In general, A has a time-varying part, A_ω , and a steady part, A_0 , so that

$$\begin{aligned} F_z &= -\sigma \frac{\partial}{\partial t}(A_\omega + A_0) \frac{\partial}{\partial z}(A_\omega + A_0) \\ &= -\sigma \frac{\partial}{\partial t} A_\omega \frac{\partial}{\partial z}(A_\omega + A_0) , \end{aligned} \quad (4)$$

$$\begin{aligned} F_r &= -\sigma \frac{\partial}{\partial t}(A_\omega + A_0) \frac{1}{r} \frac{\partial}{\partial r}[r(A_\omega + A_0)] \\ &= -\sigma \frac{\partial}{\partial t} A_\omega \frac{1}{r} \frac{\partial}{\partial r}[r(A_\omega + A_0)] . \end{aligned} \quad (5)$$

Potential

First we consider the case shown in Fig. 1, where A_ω is produced by the time-varying current I_ω in a coil and A_0 is produced by the stationary current I_0 in the same coil.

Fig. 1. Variables in Interaction of Coil with Specimen.

The vector potential A_ω at any point in a semi-infinite conductor produced by I_ω is represented by Eq. (6). (See ref. 5.)

$$A_\omega \equiv A_\omega(r, z) = \mu_1 \frac{NI_\omega}{(h_2 - h_1)(r_2 - r_1)} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) J_1(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{e^{\eta_1 z}}{\frac{\mu_1}{\mu_0} + \frac{\eta_1}{\zeta}} d\zeta, \quad (6)$$

where $\eta_1 = \sqrt{\zeta^2 + j\omega\mu_1\sigma_1}$

$$I(\zeta r_2, \zeta r_1) = I_1(\zeta r_2) - I_1(\zeta r_1)$$

$$I_1(\zeta r_i) = \int_0^{\zeta r_i} p J_1(p) dp$$

We must take the real part of the product of $e^{j\omega t}$ and $A_\omega(r, z)$ to know the real physical value of the vector potential:

$$\tilde{A}_\omega = \text{Real}(A_\omega e^{j\omega t}). \quad (7)$$

Note: $\tilde{X}$ is used to represent the physical value of a complex quantity X in the rest of the report.

The steady part, $A_0(r, z)$, of the vector potential is given as follows:

$$A_0 \equiv A_0(r, z) = \mu_1 \frac{NI_0}{(h_2 - h_1)(r_2 - r_1)} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) J_1(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{e^{\zeta z}}{\frac{\mu_1}{\mu_0} + 1} d\zeta, \quad (8)$$

$$\frac{\partial}{\partial z} A_0 = \mu_1 \frac{NI_0}{(h_2 - h_1)(r_2 - r_1)} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) J_1(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{\zeta e^{\zeta z}}{\frac{\mu_1}{\mu_0} + 1} d\zeta, \quad (9)$$

$$\frac{1}{r} \frac{\partial}{\partial r} r A_0 = \mu_1 \frac{NI_0}{(h_2 - h_1)(r_2 - r_1)} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) \zeta J_0(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{e^{\zeta z}}{\frac{\mu_1}{\mu_0} + 1} d\zeta. \quad (10)$$

When all lengths are divided by an average radius, $r_{12} = (r_1 + r_2)/2$, ζ is multiplied by r_{12} , and η_1 is redefined as

$$\sqrt{\zeta^2 + j\omega\mu_1\sigma_1 r_{12}^2},$$

Eqs. (6), (9), and (10) become

$$A_\omega \equiv A_\omega(r, z) = \mu_1 \frac{NI_\omega}{(h_2 - h_1)(r_2 - r_1)} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) J_1(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{e^{\eta_1 z}}{\frac{\mu_1}{\mu_0} + \frac{\eta_1}{\zeta}} d\zeta, \quad (11)$$

$$\frac{\partial}{\partial z} A_0 = \mu_1 \frac{NI_0}{(h_2 - h_1)(r_2 - r_1)r_{12}} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) J_1(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{\zeta e^{\zeta z}}{\frac{\mu_1}{\mu_0} + 1} d\zeta, \quad (12)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r A_0) = \mu_1 \frac{NI_0}{(h_2 - h_1)(r_2 - r_1)r_{12}} \int_0^\infty \frac{1}{\zeta^3} I(\zeta r_2, \zeta r_1) \zeta J_0(\zeta r) (e^{-\zeta h_1} - e^{-\zeta h_2}) \frac{e^{\zeta z}}{\frac{\mu_1}{\mu_0} + 1} d\zeta. \quad (13)$$

Volume Force Equations

A physical value of the force is represented as

$$\begin{aligned} \tilde{F}_z &= -\sigma \frac{\partial \tilde{A}}{\partial t} \frac{\partial \tilde{A}}{\partial z} \\ &= -\sigma \frac{\partial}{\partial t} \tilde{A}_\omega \frac{\partial}{\partial z} (\tilde{A}_\omega + \tilde{A}_0) \\ &= -\sigma \frac{\partial}{\partial t} \left[\text{Real} \left(A_\omega e^{j\omega t} \right) \right] \frac{\partial}{\partial z} \left[\text{Real} \left(A_\omega e^{j\omega t} \right) + A_0 \right] \\ &= -\sigma \text{Real} \left(A_\omega j\omega e^{j\omega t} \right) \left[\text{Real} \left(\frac{\partial A_\omega}{\partial z} e^{j\omega t} \right) + \frac{\partial A_0}{\partial z} \right] \\ &= -\sigma \omega \left[\text{Real} \left(A_\omega j e^{j\omega t} \right) \right] \left[\text{Real} \left(\frac{\partial A_\omega}{\partial z} e^{j\omega t} \right) \right] - \sigma \omega \left[\text{Real} \left(A_\omega j e^{j\omega t} \right) \right] \frac{\partial A_0}{\partial z}. \end{aligned} \quad (14)$$

The first term of Eq. (14) yields a steady term, $\tilde{F}_{z0}$, and a time harmonic part, $\tilde{F}_{z2\omega}$, which has an angular frequency of 2ω . The second term yields a time-harmonic part, $\tilde{F}_{z1\omega}$, having an angular frequency of ω .

$$\tilde{F}_{z0} = \frac{\sigma \omega}{2} \left[\text{Im}(A_\omega) \text{Real} \left(\frac{\partial A_\omega}{\partial z} \right) - \text{Real}(A_\omega) \text{Im} \left(\frac{\partial A_\omega}{\partial z} \right) \right], \quad (15)$$

$$\begin{aligned} \tilde{F}_{z2\omega} &= \frac{\sigma \omega}{2} \left\{ \left[\text{Real} \left(\frac{\partial A_\omega}{\partial z} \right) \text{Im}(A_\omega) + \text{Im} \left(\frac{\partial A_\omega}{\partial z} \right) \text{Real}(A_\omega) \right] \cos 2\omega t \right. \\ &\quad \left. + \left[\text{Real} \left(\frac{\partial A_\omega}{\partial z} \right) \text{Real}(A_\omega) - \text{Im} \left(\frac{\partial A_\omega}{\partial z} \right) \text{Im}(A_\omega) \right] \sin 2\omega t \right\}, \end{aligned} \quad (16)$$

$$\begin{aligned}
\tilde{F}_{z1\omega} &= -\sigma\omega \left[\text{Real}(A_\omega j e^{j\omega t}) \right] \frac{\partial A_0}{\partial z} \\
&= \sigma\omega \left[\text{Im}(A_\omega) \cos \omega t + \text{Real}(A_\omega) \sin \omega t \right] \frac{\partial A_0}{\partial z} .
\end{aligned} \tag{17}$$

In a similar manner

$$\tilde{F}_r = \tilde{F}_{r0} + \tilde{F}_{r2\omega} + \tilde{F}_{r1\omega} .$$

where

$$\tilde{F}_{r0} = \frac{\sigma\omega}{2} \left[\text{Im}(A_\omega) \text{Real}\left(\frac{1}{r} \frac{\partial}{\partial r} r A_\omega\right) - \text{Real}(A_\omega) \text{Im}\left(\frac{1}{r} \frac{\partial}{\partial r} r A_\omega\right) \right] , \tag{18}$$

$$\begin{aligned}
\tilde{F}_{r2\omega} &= \frac{\sigma\omega}{2} \left\{ \left[\text{Real}\left(\frac{1}{r} \frac{\partial}{\partial r} r A_\omega\right) \text{Im}(A_\omega) + \text{Im}\left(\frac{1}{r} \frac{\partial}{\partial r} r A_\omega\right) \text{Real}(A_\omega) \right] \cos 2\omega t \right. \\
&\quad \left. + \left[\text{Real}\left(\frac{1}{r} \frac{\partial}{\partial r} r A_\omega\right) \text{Real}(A_\omega) - \text{Im}\left(\frac{1}{r} \frac{\partial}{\partial r} r A_\omega\right) \text{Im}(A_\omega) \right] \sin 2\omega t \right\} ,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\tilde{F}_{r1\omega} &= -\sigma\omega \left[\text{Real}(A_\omega j e^{j\omega t}) \right] \left(\frac{1}{r} \frac{\partial}{\partial r} r A_0 \right) \\
&= \sigma\omega \left[\text{Im}(A_\omega) \cos \omega t + \text{Real}(A_\omega) \sin \omega t \right] \left(\frac{1}{r} \frac{\partial}{\partial r} r A_0 \right) .
\end{aligned} \tag{20}$$

If A_0 is much larger than A_ω ; that is, the steady magnetic field is much larger than the time harmonic magnetic field, we can assume the following equations.

$$\begin{aligned}
\tilde{F}_z &\approx \tilde{F}_{z1\omega} \\
&= -\sigma\omega \left[\text{Real}(A_\omega j e^{j\omega t}) \right] \frac{\partial A_0}{\partial z} ,
\end{aligned} \tag{21a}$$

$$\begin{aligned}
\tilde{F}_r &\approx \tilde{F}_{r1\omega} \\
&= -\sigma\omega \left[\text{Real}(A_\omega j e^{j\omega t}) \right] \left(\frac{1}{r} \frac{\partial}{\partial r} r A_0 \right) .
\end{aligned} \tag{21b}$$

We know that both $\tilde{F}_{z1\omega}$ and $\tilde{F}_{r1\omega}$ are proportional to the product of N^2 , I_0 , and I_ω from Eqs. (11), (12), (13), (17), and (20).

Force Calculation for Actual Transducer

Second, we consider the case shown in Fig. 2, where A_ω is produced by time varying currents in a coil and A_0 is produced by an electromagnet located near the coil.

Fig. 2. Arrangement of Electromagnet with Eddy-Current Coil.

In this case we simply have to replace $\partial A_0 / \partial z$ with B_r , which is the horizontal and radial component of the stationary magnetic field:

$$\begin{aligned}
 \tilde{F}_z &= -\sigma\omega \left[\text{Real} \left(A_\omega j e^{j\omega t} \right) \right] B_r \\
 &= \text{Real} \left(J_\omega e^{j\omega t} \right) B_r \\
 &= \tilde{J}_\omega B_r, \tag{22a}
 \end{aligned}$$

where $\tilde{J}_\omega$ is the real physical value of the eddy current density within the conductor. Similarly,

$$\begin{aligned}
 \tilde{F}_r &= -\sigma\omega \left[\text{Real} \left(A_\omega j e^{j\omega t} \right) \right] B_z \\
 &= \text{Real} \left(J_\omega e^{j\omega t} \right) B_z \\
 &= \tilde{J}_\omega B_z, \tag{22b}
 \end{aligned}$$

where B_z is the vertical component of the stationary magnetic field.

We calculate J_ω , F_z , and F_r using Eqs. (11), (22a), and (22b) by numerical quadrature. The electromagnet is neglected for the eddy-current calculation. It is not considered to have a great effect on the eddy-current distribution within a nonmagnetic conductor, because most of the turns of the coil are far from the electromagnet but very near to the conductor. Coil dimensions and the other conditions for calculation are shown in Table 2 and Fig. 1. Table 3 and Fig. 3 show B_z and B_r .

Table 2. Coil and Conductor Characteristics

Dimensions, mm	
r_1	10.2870
r_2	14.7955
h_1	0.3429
h_2	0.5969
N , number of turns	7.91667
I_ω , current, A	1790
Frequency, MHz	1.6
σ_1 , conductivity, $\Omega^{-1}\text{m}^{-1}$	1.351351×10^6
μ_1 , permeability, H/m	$4\pi \times 10^{-7}$

Table 3. Stationary Magnetic Flux at Conductor Surface

Position, r (mm)	Flux, Wb/m ²	
	B_z	B_r
8.0	0.285	0.480
9.0	0.205	0.390
10.0	0.145	0.360
10.5	0.113	0.340
11.0	0.080	0.320
11.5	0.053	0.300
12.0	0.025	0.280
12.5	0.005	0.270
13.0	-0.015	0.260
13.5	-0.055	0.255
14.0	-0.095	0.250
14.5	-0.103	0.248
15.0	-0.110	0.245
15.5	-0.110	0.245
16.0	-0.110	0.245

Fig. 3. Magnetic Flux at the Surface of a Nonmagnetic Sample.

The stationary magnetic flux B_z is assumed to have the same values at the conductor surface and at small depths (e.g., 0.5 mm). The same assumption is also taken for B_r . These conditions for coils, conductors, and magnetic flux are taken from experimental work conducted by the author.¹⁰

The widths of numerical quadrature for the calculation were for the various ranges of ζ : 0–3.5, $d\zeta = 0.01$; 3.5–9.0, $d\zeta = 0.05$; 9.0–80.0, $d\zeta = 0.1$; the upper limit of integration was 80. These conditions for numerical quadrature are enough for accuracy and convergence in evaluating the integral. The Fortran program for a computer is shown in Appendix A.

The results of calculations of the eddy-current density at a sample surface and at a depth of 0.5 mm are shown in Fig. 4. The computer output is shown in Appendix B. The maximum eddy-current density at the surface was about $1.15 \times 10^{10} \text{ A/m}^2$, which is very large but smaller than the current density in the driver coil, about $3.6 \times 10^{10} \text{ A/m}^2$. Notice that the eddy-current density decreases rapidly as the depth increases, because of the skin effect. The force per unit volume was calculated (Appendix B), and the results are shown in Fig. 5. The vertical force, $|F_z|$, has its peak near the outer radius of the driver coil and decreases very rapidly toward the outside and the inside of the driver coil. The horizontal and radial component of the force, $|F_r|$, has zero value and changes its phase at the point where the vertical component of the stationary magnetic flux has zero value (see Figs. 3 and 5).

¹⁰K. Kawashima, *An Electromagnetic Ultrasonic Transducer*, ORNL-TM-5063, August 1975.

Fig. 4. Eddy-Current Density in a Conductor.

Fig. 5. Electromagnetic Force per Unit Volume Produced by the Electromagnetic Transducer.

Calculation of Surface Forces

The variations of the amplitude and the phases of the forces with depth change are shown in Fig. 6. We know from them that the depth dependence of the force is represented approximately by the following equations:

$$F_{z1\omega} = \left(F_{z1\omega}\right)_{z=0} e^{jk_1 z} e^{k_2 z}, \quad (23)$$

$$F_{r1\omega} = \left(F_{r1\omega}\right)_{z=0} e^{jk_3 z} e^{k_4 z}. \quad (24)$$

Fig. 6. Dependence of (a) Amplitude and (b) Phase of Volume Force F_z upon depth.

To obtain the equivalent surface force, one has to integrate $\tilde{F}_z$ and $\tilde{F}_r$ with respect to depth from zero to infinity:

$$\begin{aligned} \tilde{P}_z &= \int_{-\infty}^0 \tilde{F}_z dz \\ &\approx \int_{-\infty}^0 \tilde{F}_{z1\omega} dz \\ &= \int_{-\infty}^0 \text{Real} \left(F_{z1\omega} e^{j\omega t} \right) dz \\ &= \text{Real} \left[\left(\int_{-\infty}^0 F_{z1\omega} dz \right) e^{j\omega t} \right], \end{aligned} \quad (25)$$

$$\tilde{P}_r \approx \text{Real} \left[\left(\int_{-\infty}^0 F_{r1\omega} dz \right) e^{j\omega t} \right]. \quad (26)$$

It is very time-consuming work to calculate Eqs. (25) and (26) numerically. We can calculate the surface forces, $\tilde{P}_z$ and $\tilde{P}_r$, by a mixture of analytical and numerical methods without losing much computational accuracy by making use of Eqs. (23) and (24).

$$\begin{aligned} P_z &\approx \int_{-\infty}^0 F_{z1\omega} dz \\ &= \int_{-\infty}^0 (F_{z1\omega})_{z=0} e^{jk_1 z} e^{k_2 z} dz \\ &= (F_{z1\omega})_{z=0} \left[\frac{1}{jk_1 + k_2} e^{jk_1 z} e^{k_2 z} \right]_{-\infty}^0 \\ &= (F_{z1\omega})_{z=0} / (jk_1 + k_2). \end{aligned} \quad (27)$$

We can obtain k_1 and k_2 by calculating two values of $F_{z1\omega}$ at two different depths chosen arbitrarily. Substituting $(F_{z1\omega})_{z=0}$ and $(F_{z1\omega})_{z=z_1}$, the forces calculated numerically at depths $z = 0$ and $z = z_1$, respectively, into Eq. (23) gives

$$(F_{z1\omega})_{z=0} = (F_{z1\omega})_{z=0}, \quad (28)$$

$$(F_{z1\omega})_{z=z_1} = (F_{z1\omega})_{z=0} e^{jk_1 z_1} e^{k_2 z_1}. \quad (29)$$

From these two equations

$$jk_1 + k_2 = \frac{1}{z_1} \ln \frac{(F_{z1\omega})_{z=z_1}}{(F_{z1\omega})_{z=0}}. \quad (30)$$

Substituting Eq. (30) into Eq. (27) gives

$$P_z = (F_{z1\omega})_{z=0} \left[\frac{1}{z_1} \ln \frac{(F_{z1\omega})_{z=z_1}}{(F_{z1\omega})_{z=0}} \right]^{-1}. \quad (31)$$

Similarly,

$$P_r = (F_{r1\omega})_{z=0} \left[\frac{1}{z_1} \ln \frac{(F_{r1\omega})_{z=z_1}}{(F_{r1\omega})_{z=0}} \right]^{-1}. \quad (32)$$

The surface forces, P_z and P_r , which are obtained through integration of the volume forces with respect to the depth, are shown in Fig. 7 (computer output in Appendix B). According to Figs. 5 and 7, the distribution of the surface forces is almost the same as that of the volume forces. The phase of the vertical surface force, $\text{Ph}(P_z)$, does not change much in the region where the amplitude, $|P_z|$, is large enough to contribute the main part of the ultrasonic field.

The phase of the horizontal and radial surface force, $\text{Ph}(P_r)$, is the same as $\text{Ph}(P_z)$, if the signs of the directions of both depth increase and radius increase are taken as positive. The value of $\text{Ph}(P_r)$ changes by π at the point where B_z changes its sign (see Figs. 3 and 7).

Figure 8 and Table 4 give approximations for surface forces that are used for calculation of the ultrasonic field.

Fig. 7. Surface Forces Obtained Through Integration of Volume Forces of Fig. 5.

Fig. 8. Approximation of Surface Forces for Ultrasonic Wave Calculation.

Table 4. Approximations to Surface Forces Used in Ultrasonic Wave Calculations

Range of r (mm)	Vertical Amplitude ^a (MN/m ²)	Horizontal Radial	
		Amplitude (MN/m ²)	Phase (rad)
10.3-12	0.5	0.1	$-\pi$
12-13.5	0.63	0	b
13.5-15.5	0.65	0.25	0

^aThe phase of the vertical component is taken as zero everywhere.

^bThe phase of the horizontal radial component changes from $-\pi$ to zero at about 12.7 mm, the point at which the magnetic flux is zero.

Summary of Force Derivation

The forces that give rise to electromagnetically induced ultrasound depend on the vector potential in the specimen; the pertinent variations in this potential are given by Eqs. (11), (12), and (13) on pages 5 and 6. The volume force due to the field from a coil is then given by Eq. (21), p. 7. For the combination of a coil and a particular electromagnet, the volume force shown in Fig. 5 (p. 11) is calculated. These forces fall off rapidly with depth, and can be integrated to give surface forces.

The equivalent surface force exerted by an electromagnetic volume force is

$$\tilde{P}_z = \text{Real} \left(P_z e^{j\omega t} \right) ,$$

$$\tilde{P}_r = \text{Real} \left(P_r e^{j\omega t} \right) ,$$

where

$$\begin{aligned} P_z &= \int_{-\infty}^0 F_z dz \\ &\approx \int_{-\infty}^0 F_{z1\omega} dz \\ &= \int_{-\infty}^0 -j\omega\sigma A(r,z) \frac{\partial A_0(r,z)}{\partial z} dz \end{aligned}$$

and

$$\begin{aligned} P_r &= \int_{-\infty}^0 F_r dz \\ &\approx \int_{-\infty}^0 F_{r1\omega} dz \\ &= \int_{-\infty}^0 -j\omega\sigma A(r,z) \frac{1}{r} \frac{\partial}{\partial r} r A_0(r,z) dz . \end{aligned}$$

Figure 7 gives the surface forces calculated from the volume forces in Fig. 5.

ELASTIC WAVE FROM SURFACE FORCE

Elastic waves that propagate in a semi-infinite elastic space were first investigated by Lamb.¹¹ In his classical paper he investigated the effect of a vibrating point force on the surface. Miller and Pursey¹² replaced the point source with a disk of time-varying uniform pressure. Bycroft¹³ and later Robertson¹⁴ replaced the point force with a forced vertical vibration of a rigid circular disk, which gives rise to a mixed boundary problem. Johnson¹⁵ recently gave a complete and general solution

to the three-dimensional Lamb's problem for a point source by the method of Green's function. In Appendix C, the equations that are suitable for calculation of elastic wave fields produced by eddy-current surface forces are derived by use of a Green's function. A surface force produced by an interaction of a magnetic field and eddy currents has two components. One is a vertical component and the other is a horizontal and radial component. Both forces distribute symmetrically around the axis. (See Fig. 7.)

A displacement of a particle at an arbitrary point in a solid due to the elastic wave from a sinusoidal surface force has four components, S_{zz} , a vertical component from the vertical surface force P_z ; S_{rz} , a radial horizontal component from the vertical surface force P_z ; S_{zr} , a vertical component from the radial horizontal surface force P_r ; and S_{rr} , a radial horizontal component from the radial horizontal surface force P_r .

$$\begin{aligned}
 S_z = S_{zz} + S_{zr} = \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2\sqrt{\alpha^2 - \kappa_c^2}\sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\
 \left. + 2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha J_0(\alpha a) da \\
 + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2\sqrt{\alpha^2 - \kappa_c^2}\sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2}\sqrt{\alpha^2 - \kappa_s^2} \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\
 \left. + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha J_1(\alpha a) da . \quad (33)
 \end{aligned}$$

¹¹H. Lamb, "On the Propagation of Tremors Over the Surface of an Elastic Solid," *Phil. Trans. Roy. Soc. London Ser. A* 203: 1-42 (1904).

¹²F. Miller and H. Pursey, "The Field and Radiation Impedance of Mechanical Radiators on the Free Surface of a Semi-infinite Isotropic Solid," *Proc. Roy. Soc. London Ser. A* 223: 521-41 (1954).

¹³G. N. Bycroft, "Forced Vibrations of Rigid Circular Plate on a Semi-infinite Elastic Space and on an Elastic Stratum," *Phil. Trans. Roy. Soc. London Ser. A* 284: 327-368 (1956).

¹⁴I. A. Robertson, "Forced Vertical Vibration of a Rigid Circular Disk on a Semi-infinite Elastic Solid," *Proc. Cambridge Phil. Soc.* 62: 547-553 (1966).

¹⁵L. R. Johnson, "Green's Function for Lamb's Problem," *Geophys. J. Roy. Astron. Soc.* 37: 99-131 (1974).

$$\begin{aligned}
S_r = S_{rz} + S_{rr} = \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} & \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\
& + 2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \left. \right] J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha J_0(\alpha a) da \\
& + \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_s^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\
& + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \left. \right] J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha J_1(\alpha a) da . \quad (34)
\end{aligned}$$

The average displacement in a circle of radius a is:

$$\tilde{S}_z(\text{av}) = \text{Real}\left(S_z(\text{av}) e^{j\omega t}\right)$$

$$\begin{aligned}
\text{where } S_z(\text{av}) = \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} & \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\
& + 2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \left. \right] \frac{2J_1(\alpha a)}{\alpha a} d\alpha \int_0^\infty P_z(a) \alpha J_0(\alpha a) da \\
& + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\
& + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \left. \right] \frac{2J_1(\alpha a)}{\alpha a} d\alpha \int_0^\infty P_r(a) \alpha J_1(\alpha a) da . \quad (35)
\end{aligned}$$

Equations (33) and (34) were derived for a sinusoidal current in the coil. If the current is not sinusoidal, it can be resolved by a Fourier analysis into sinusoidal components, and the actual displacements can then be obtained by summing these equations with appropriate Fourier coefficients. This analysis is outlined in Appendix C for a current that is a repetition of harmonic pulses.

Appendix C also treats the case in which a constant surface force is exerted over a circle of radius a_1 :

$$P_z(a) = P_r(a) = \begin{cases} 1, & a \leq a_1 \\ 0, & a > a_1 \end{cases} .$$

Then Eqs. (33) and (34) can be resolved into

$$S_{zz} = \frac{\alpha_1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ \left. + 2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_0(\alpha r) J_1(\alpha a_1) d\alpha , \quad (36)$$

$$S_{zr} = \frac{\alpha_1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ \left. + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_0(\alpha r) \frac{I(\alpha a_1)}{\alpha a_1} d\alpha , \quad (37)$$

$$S_{rz} = \frac{\alpha_1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ \left. + 2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_1(\alpha r) J_1(\alpha a_1) d\alpha , \quad (38)$$

$$S_{rr} = \frac{\alpha_1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_s^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ \left. + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_1(\alpha r) \frac{I(\alpha a_1)}{\alpha a_1} d\alpha , \quad (39)$$

where

$$I(\alpha a_1) = \int_0^{\alpha a_1} p J_1(p) dp .$$

NUMERICAL CALCULATION OF ELASTIC WAVES

Method of Numerical Calculation

We assume $P_z(a)$ and $P_p(a)$ are constant between $a = a_i$ and $a = a_{i+1}$; then

$$\begin{aligned}
 \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da &= \sum_{i=0}^\infty \int_{a_i}^{a_{i+1}} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \alpha a J_0(\alpha a) da \\
 &= \sum_{i=0}^\infty P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_{a_i}^{a_{i+1}} \alpha a J_0(\alpha a) da \\
 &= \sum_{i=0}^\infty P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_{\alpha a_i}^{\alpha a_{i+1}} \frac{1}{\alpha} p J_0(p) dp \quad (p = \alpha a) \\
 &= \sum_{i=0}^\infty P_z \left(\frac{a_i + a_{i+1}}{2} \right) \frac{1}{\alpha} \int_{\alpha a_i}^{\alpha a_{i+1}} \frac{d}{dp} [p J_1(p)] dp \\
 &= \sum_{i=0}^\infty P_z \left(\frac{a_i + a_{i+1}}{2} \right) \frac{1}{\alpha} [\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i)] , \quad (40)
 \end{aligned}$$

since

$$\alpha a_i J_1(\alpha a_i) = \int_0^{\alpha a_i} p J_0(p) dp .$$

Similarly,

$$\begin{aligned}
 \int_0^\infty P_p(a) \alpha a J_1(\alpha a) da &= \sum_{i=0}^\infty \int_{a_i}^{a_{i+1}} P_p \left(\frac{a_i + a_{i+1}}{2} \right) \alpha a J_1(\alpha a) da \\
 &= \sum_{i=0}^\infty P_p \left(\frac{a_i + a_{i+1}}{2} \right) \int_{a_i}^{a_{i+1}} \alpha a J_1(\alpha a) da \\
 &= \sum_{i=0}^\infty P_p \left(\frac{a_i + a_{i+1}}{2} \right) \int_{\alpha a_i}^{\alpha a_{i+1}} \frac{1}{\alpha} p J_1(p) dp \\
 &= \sum_{i=0}^\infty P_p \left(\frac{a_i + a_{i+1}}{2} \right) \frac{1}{\alpha} [I(\alpha a_{i+1}) - I(\alpha a_i)] , \quad (41)
 \end{aligned}$$

where

$$I(\alpha a_i) = \int_0^{\alpha a_i} p J_1(p) dp.$$

Substituting Eqs. (40) and (41) into Eqs. (33) and (34), we have

$$\begin{aligned} S_z = & \frac{1}{\mu} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - \kappa_c^2} / \alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ & \left. + 2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\ & + \frac{1}{\mu} \sum_{i=0}^{\infty} P_r \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{1}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \right. \\ & \left. \times \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_0(\alpha r) \left[I(\alpha a_{i+1}) - I(\alpha a_i) \right] d\alpha, \quad (42) \end{aligned}$$

$$\begin{aligned} S_r = & \frac{1}{\mu} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{1}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ & \left. + 2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_1(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\ & + \frac{1}{\mu} \sum_{i=0}^{\infty} P_r \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - \kappa_s^2} / \alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\alpha^2 \exp\left(-\sqrt{\alpha^2 - \kappa_c^2} z\right) \right. \\ & \left. + (\kappa_s^2 - 2\alpha^2) \exp\left(-\sqrt{\alpha^2 - \kappa_s^2} z\right) \right] J_1(\alpha r) \left[I(\alpha a_{i+1}) - I(\alpha a_i) \right] d\alpha. \quad (43) \end{aligned}$$

After all lengths except S_z and S_r are multiplied by κ_c and α is divided by κ_c , Eqs. (42) and (43) become

$$\begin{aligned}
 S_z &= S_{zz} + S_{zr} \\
 &= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
 &\quad \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa^2} z) \right] J_0(\alpha r) \left[\alpha a_{i+1}^{J_1(\alpha a_{i+1})} - \alpha a_i^{J_1(\alpha a_i)} \right] d\alpha \\
 &\quad + \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_r \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{1}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} \left[2\sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2} \right. \\
 &\quad \left. \times \exp(-\sqrt{\alpha^2 - 1} z) + (\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa^2} z) \right] J_0(\alpha r) \left[I(\alpha a_{i+1}) - I(\alpha a_i) \right] d\alpha, \quad (44)
 \end{aligned}$$

$$\begin{aligned}
 S_r &= S_{rz} + S_{rr} \\
 &= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{1}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
 &\quad \left. + 2\sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2} \exp(-\sqrt{\alpha^2 - \kappa^2} z) \right] J_1(\alpha r) \left[\alpha a_{i+1}^{J_1(\alpha a_{i+1})} - \alpha a_i^{J_1(\alpha a_i)} \right] d\alpha \\
 &\quad + \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_r \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - \kappa^2}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} \left[2\alpha^2 \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
 &\quad \left. + (\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa^2} z) \right] J_1(\alpha r) \left[I(\alpha a_{i+1}) - I(\alpha a_i) \right] d\alpha. \quad (45)
 \end{aligned}$$

The first term of Eq. (44) can be resolved into two terms:

$$\begin{aligned}
S_{zz} &= S_{zz\ell} + S_{zzs} \\
&= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2\sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right] \\
&\quad \times J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\
&+ \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2\sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} 2\alpha^2 \exp(-\sqrt{\alpha^2 - 1} z) \\
&\quad \times J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\
&= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \left[\int_0^{\infty} \phi_{zz\ell}(\alpha) d\alpha + \int_0^{\infty} \phi_{zzs}(\alpha) d\alpha \right]. \tag{46}
\end{aligned}$$

The term $S_{zz\ell}$ is considered to come from $\partial\phi_{11}/\partial z$, and S_{zzs} from $(1/r)\partial(rW_{11})/\partial r$. Equation (46) helps us to understand the inseparability of the displacement vector into longitudinal and transverse wave components.

When $z = \text{zero}$, we can easily see that $S_{zz\ell}$ and S_{zzs} do not have finite values. If α is very large, the function to be integrated becomes

$$\begin{aligned}
\left[\phi_{zz\ell}(\alpha) \right]_{z=0} &= \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2\sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} (\kappa^2 - 2\alpha^2) \\
&\quad \times J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right].
\end{aligned}$$

Bessel functions decrease approximately proportional to $1/\sqrt{\alpha}$, and so

$$\begin{aligned}
 \lim_{\alpha \rightarrow \infty} \left[\phi_{zz\ell}(\alpha) \right]_{z=0} &\propto \frac{\alpha(1 - 1/2\alpha^2)/\alpha}{4\alpha^4 + \kappa^4 - 4\alpha^2\kappa^2 - 4\alpha^4(1 - 1/2\alpha^2 - \kappa^2/2\alpha^2)} (\kappa^2 - 2\alpha^2) \\
 &\times 1/\sqrt{\alpha} \propto 1/\sqrt{\alpha} \\
 &= \frac{1}{\kappa^4 - 4\alpha^2\kappa^2 + 2\alpha^2 + 2\alpha^2\kappa^2} (\kappa^2 - 2\alpha^2) \\
 &= \frac{\kappa^2 - 2\alpha^2}{2\alpha^2(1 - \kappa^2) + \kappa^4} \\
 &\approx -1/(1 - \kappa^2) .
 \end{aligned} \tag{47}$$

Equation (47) shows that the integral fluctuates. When both z and r are zero, it is easy to see that the integral diverges. In a similar manner $\phi_{zzs}(\alpha)$ can be shown to have divergent values. However, if we take a sum of $S_{zz\ell}$ and S_{zzs} (that is, S_{zz}), then $(\kappa^2 - 2\alpha^2)$ and $2\alpha^2$ cancel each other, leaving only κ^2 in Eq. (44), and convergence is assured.

Some equations similar to Eqs. (44) and (45) are seen in seismology, and they are considered exceedingly difficult to evaluate numerically.

The distance between a source of elastic waves and an observation point is usually very large in the case of seismology; consequently, Bessel functions and trigonometric functions [which are derived from $\exp(-\sqrt{\alpha^2 - \kappa_c^2} z)$ and $\exp(-\sqrt{\alpha^2 - \kappa_s^2} z)$ when α is smaller than κ_c and κ_s] fluctuate very rapidly to make numerical evaluation difficult. Instead of numerical calculation, the equations are approximately evaluated by the method of complex contour integrals, which is valid for large values of distance. If the distance is as small as several centimeters, the fluctuation is not very rapid and we can evaluate the equations by numerical calculations.

The first term, S_{zz} , of Eq. (44) is considered in detail. All the other terms of Eqs. (44) and (45) are calculated similarly.

$$\begin{aligned}
S_{zz} &= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - 1} \sqrt{\alpha^2 - \kappa^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
&\quad \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa^2} z) \right] J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\
&= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \phi(\alpha) d\alpha \\
&= \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{a_i + a_{i+1}}{2} \right) \int_0^{\infty} \frac{h(\alpha)}{\Delta(\alpha)} d\alpha .
\end{aligned}$$

Only principal values of radicals are used for numerical integration. This is necessary for the wave traveling from the source toward infinity and also absolutely necessary for the convergence of integration:

$$\begin{aligned}
\sqrt{\alpha^2 - 1} &= \begin{cases} j\sqrt{1 - \alpha^2} & 0 \leq \alpha \leq 1 \\ \sqrt{\alpha^2 - 1} & 1 \leq \alpha \end{cases} , \\
\sqrt{\alpha^2 - \kappa^2} &= \begin{cases} j\sqrt{\kappa^2 - \alpha^2} & 0 \leq \alpha \leq \kappa \\ \sqrt{\alpha^2 - \kappa^2} & \kappa \leq \alpha \end{cases} .
\end{aligned}$$

The integration has three singular points, $\alpha = 1$, κ , and p . Obviously, $\alpha = 1$ and κ are the zeroes of $\sqrt{\alpha^2 - 1}$ and $\sqrt{\alpha^2 - \kappa^2}$, respectively, and p is one of the roots of

$$\Delta(\alpha) = (2\alpha^2 - \kappa^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa^2} \sqrt{\alpha^2 - 1} = 0 \quad (\kappa \approx 2) . \quad (48)$$

To ensure the continuity of the integral at the three singular points, the integral is evaluated on a complex plane ($\eta = \alpha + j\beta$). The contour of integration avoids the three points, as shown in Fig. 9.

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Fig. 9. Path of Integration in the Complex Plane.

Then S_{zz} is obtained as the limiting value of the complex contour integral:

$$S_{zz} = \lim_{\rho_1 \rightarrow 0, \rho_k \rightarrow 0, \rho_p \rightarrow 0} \frac{1}{\mu \kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{\alpha_i + \alpha_{i+1}}{2} \right) \int_C \phi(\eta) d\eta ,$$

$$C = C_{01} + C_1 + C_{1K} + C_K + C_{Kp} + C_p + C_{p\infty} .$$

$$\begin{aligned} \text{Then } S_{zz} = & \frac{1}{\mu \kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{\alpha_i + \alpha_{i+1}}{2} \right) \left[\lim_{\rho_1 \rightarrow 0} \int_0^{1-\rho_1} \phi(\alpha) d\alpha + \lim_{\rho_1 \rightarrow 0} \int_{C_1} \phi(\eta) d\eta \right. \\ & + \lim_{\rho_1, \rho_K \rightarrow 0} \int_{1+\rho_1}^{K-\rho_K} \phi(\alpha) d\alpha + \lim_{\rho_K \rightarrow 0} \int_{C_K} \phi(\eta) d\eta \\ & + \lim_{\rho_K, \rho_p \rightarrow 0} \int_{K+\rho_K}^{p-\rho_p} \phi(\alpha) d\alpha + \lim_{\rho_p \rightarrow 0} \int_{C_p} \phi(\eta) d\eta \\ & \left. + \lim_{\rho_p \rightarrow 0} \int_{p+\rho_p}^{\infty} \phi(\alpha) d\alpha \right] . \end{aligned} \quad (49)$$

While the second and fourth integrals of Eq. (49) have zero value as ρ_1 and ρ_K tend to zero, the sixth term approaches a finite value as ρ_p tends to zero. We assume ρ_p to have a small finite value, and α_{∞} is introduced as an upper limit of integration for the numerical calculation. Therefore,

$$\begin{aligned} S_{zz} \approx & \frac{1}{\mu \kappa_c} \sum_{i=0}^{\infty} P_z \left(\frac{\alpha_i + \alpha_{i+1}}{2} \right) \left[\int_0^1 \phi(\alpha) d\alpha + \int_1^K \phi(\alpha) d\alpha + \int_K^{p-\rho_p} \phi(\alpha) d\alpha \right. \\ & \left. + \int_{C_p} \phi(\eta) d\eta + \int_{p+\rho_p}^{\alpha_{\infty}} \phi(\alpha) d\alpha \right] . \end{aligned} \quad (50)$$

The fourth term of Eq. (50) is represented as

$$\begin{aligned}
 S_{zzCp} &= \int_{C_p} \phi(\eta) d\eta \\
 &= \int_{C_p} \frac{h(\eta)}{\Delta(\eta)} d\eta \\
 &= \int_{C_p} \frac{H(\eta)}{(\eta - p)} d\eta .
 \end{aligned} \tag{51}$$

Next, $H(\eta)$ is represented by the Taylor series expansion around the point $\eta = p$:

$$\begin{aligned}
 H(\eta) &= H(p) + H'(p)(\eta - p) + \frac{1}{2!} H''(p)(\eta - p)^2 + \frac{1}{3!} H'''(p)(\eta - p)^3 \\
 &+ \dots
 \end{aligned} \tag{52}$$

Substituting Eq. (52) into Eq. (51) gives

$$\begin{aligned}
 S_{zzCp} &= \int_{C_p} \frac{H(p)}{\eta - p} d\eta + \int_{C_p} H'(p) d\eta + \int_{C_p} \frac{1}{2!} H''(p)(\eta - p) d\eta \\
 &+ \int_{C_p} \frac{1}{3!} H'''(p)(\eta - p)^2 d\eta + \dots
 \end{aligned} \tag{53}$$

Replacing η of Eq. (53) with $p + \rho_p e^{j\theta}$ gives

$$\begin{aligned}
 S_{zzCp} &= \int_{\pi}^0 \frac{H(p)}{\rho_p e^{j\theta}} j \rho_p e^{j\theta} d\theta + \int_{\pi}^0 H'(p) j \rho_p e^{j\theta} d\theta \\
 &+ \int_{\pi}^0 \frac{1}{2!} H''(p) \rho_p e^{j\theta} j \rho_p e^{j\theta} d\theta + \int_{\pi}^0 \frac{1}{3!} H'''(p) \rho_p^2 e^{2j\theta} j \rho_p e^{j\theta} d\theta
 \end{aligned}$$

+

(continued)

$$\begin{aligned}
&= jH(p) \int_{\pi}^0 d\theta + j\rho_p^{H'}(p) \int_{\pi}^0 e^{j\theta} d\theta \\
&+ \frac{1}{2!} j\rho_p^{2H''}(p) \int_{\pi}^0 e^{2j\theta} d\theta + \frac{1}{3!} j\rho_p^{3H'''}(p) \int_{\pi}^0 e^{3j\theta} d\theta \\
&+ \dots \\
&= jH(p) \left[\theta \right]_{\pi}^0 + j\rho_p^{H'}(p) \left[\frac{1}{j} e^{j\theta} \right]_{\pi}^0 \\
&+ \frac{1}{2!} j\rho_p^{2H''}(p) \left[\frac{1}{2j} e^{2j\theta} \right]_{\pi}^0 + \frac{1}{3!} j\rho_p^{3H'''}(p) \left[\frac{1}{3j} e^{3j\theta} \right]_{\pi}^0 \\
&+ \dots \\
&= -j\pi H(p) + 2\rho_p^{H'}(p) + \frac{1}{9}\rho_p^{3H'''}(p) + \dots
\end{aligned} \tag{54}$$

We chose the value of ρ_p small enough to neglect the third term of Eq. (54). Then $H(p)$ and $H'(p)$ are modified as follows:

$$\begin{aligned}
H(\eta) &= (\eta - p)\phi(\eta) \\
&= (\eta - p)h(\eta)/\Delta(\eta) ;
\end{aligned} \tag{55}$$

$\Delta(\eta)$ is represented by the Taylor series expansion around the point $\eta = p$:

$$\begin{aligned}
\Delta(\eta) &= \Delta(p) + \Delta'(p)(\eta - p) + \frac{1}{2!}\Delta''(p)(\eta - p)^2 + \frac{1}{3!}\Delta'''(p)(\eta - p)^3 \\
&+ \dots \\
&= \Delta'(p)(\eta - p) + \frac{1}{2!}\Delta''(p)(\eta - p)^2 + \frac{1}{3!}\Delta'''(p)(\eta - p)^3 + \dots \quad (56)
\end{aligned}$$

Substituting Eq. (56) into Eq. (55) gives

$$\begin{aligned}
H(\eta) &= (\eta - p) \frac{h(\eta)}{\Delta'(p)(\eta - p) + \frac{1}{2!}\Delta''(p)(\eta - p)^2 + \frac{1}{3!}\Delta'''(p)(\eta - p)^3 + \dots} , \\
&= \frac{h(\eta)}{\Delta'(p) + \frac{1}{2!}\Delta''(p)(\eta - p) + \frac{1}{3!}\Delta'''(p)(\eta - p)^2 + \dots} . \quad (57)
\end{aligned}$$

Therefore,

$$H(p) = h(p) / \Delta'(p) \quad (58)$$

Taking the derivative of Eq. (57) gives

$$H'(\eta) = \frac{h'(\eta) \left[\Delta'(p) + \frac{1}{2!}\Delta''(p)(\eta - p) + \frac{1}{3!}\Delta'''(p)(\eta - p)^2 + \dots \right] - h(\eta) \left[\frac{1}{2!}\Delta''(p) + \frac{2}{3!}\Delta'''(p)(\eta - p) + \dots \right]}{\left[\Delta'(p) + \frac{1}{2!}\Delta''(p)(\eta - p) + \frac{1}{3!}\Delta'''(p)(\eta - p)^2 + \dots \right]^2} ,$$

so

$$H'(p) = \frac{h'(p)\Delta'(p) - h(p) \frac{1}{2!}\Delta''(p)}{\left[\Delta'(p) \right]^2} . \quad (59)$$

Substituting $H(p)$ and $H'(p)$ into Eq. (54) gives

$$\begin{aligned}
S_{zzCp} &\approx -j\pi H(p) + 2\rho_p H'(p) \\
&= -j\pi \frac{h(p)}{\Delta'(p)} + 2\rho_p \frac{h'(p)\Delta'(p) - h(p) \frac{1}{2!}\Delta''(p)}{\left[\Delta'(p) \right]^2} . \quad (60)
\end{aligned}$$

Substituting Eq. (60) into Eq. (50) gives

$$\begin{aligned}
S_{zz} = & \frac{1}{\mu\kappa_c} \sum_{i=0}^{\infty} p_z \left(\frac{a_i + a_{i+1}}{2} \right) \left\{ \int_0^1 \frac{j\sqrt{1-\alpha^2}/\alpha}{(2\alpha^2 - \kappa^2)^2 + 4\alpha^2\sqrt{1-\alpha^2}\sqrt{\kappa^2 - \alpha^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-j\sqrt{1-\alpha^2} z) \right. \right. \\
& + 2\alpha^2 \exp(-j\sqrt{\kappa^2 - \alpha^2} z) \left. \right] J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\
& + \int_1^\kappa \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4j\alpha^2\sqrt{\alpha^2 - 1}\sqrt{\kappa^2 - \alpha^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
& + 2\alpha^2 \exp(-j\sqrt{\kappa^2 - \alpha^2} z) \left. \right] J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\
& + \int_\kappa^{p-p_p} \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2\sqrt{\alpha^2 - 1}\sqrt{\alpha^2 - \kappa^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
& + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa^2} z) \left. \right] J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \\
& - j\pi \frac{h(p)}{\Delta'(p)} + 2\rho_p \frac{h'(p)\Delta'(p) - h(p)\Delta''(p)/2!}{[\Delta'(p)]^2} \\
& + \int_{p+p_p}^{\alpha_\infty} \frac{\sqrt{\alpha^2 - 1}/\alpha}{(2\alpha^2 - \kappa^2)^2 - 4\alpha^2\sqrt{\alpha^2 - 1}\sqrt{\alpha^2 - \kappa^2}} \left[(\kappa^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - 1} z) \right. \\
& + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa^2} z) \left. \right] J_0(\alpha r) \left[\alpha a_{i+1} J_1(\alpha a_{i+1}) - \alpha a_i J_1(\alpha a_i) \right] d\alpha \left. \right\}. \quad (61)
\end{aligned}$$

Equation (60) is negligible when z is large, but it increases as z decreases, and it is especially important when z is very small or zero because most of the imaginary part of Eq. (61) comes from Eq. (60).

Throughout the whole integration range, Bessel functions fluctuate at the same rates, which are approximately $2\pi/a_i$ for $J_1(\alpha a_i)$ and $2\pi/r$ for $J_0(\alpha r)$. The exponentials $\exp(-j\sqrt{1-\alpha^2} z)$ and $\exp(-j\sqrt{\kappa^2 - \alpha^2} z)$ fluctuate very rapidly when α is slightly less than 1 and α is slightly less than κ , respectively. Upon evaluating the integrals by numerical quadrature, the width of quadrature, $\Delta\alpha$, is changed according to the change of the fluctuation rates of $\exp(-j\sqrt{1-\alpha^2} z)$ and $\exp(-j\sqrt{\kappa^2 - \alpha^2} z)$. Also $\Delta\alpha$ is kept small enough to be able to follow the fluctuations of the Bessel functions and the other terms in the integration.

Calculation of Elastic Wave Exerted by Uniform Surface Force

Equations (44) and (45) are numerically evaluated for a simple case, in which the surface force has uniform vertical and radial components in a circular area of radius a , as illustrated by Fig. 10. In such a case, Eq. (44) is considered to be the sum of Eqs. (36) and (37), and Eq. (45) is considered to be the sum of Eqs. (38) and (39). The calculation assumes a hypothetical metal resembling aluminum, with properties shown in Table 5 along with other quantities used in the numerical integration.

Fig. 10. Uniform Surface Force Over a Circular Area.

Table 5. Numerical Values Used in Calculation

Quantity	Value
ρ , density, kg/m^3	2.7×10^3
Lamé constants, $\text{kg m}^{-1} \text{sec}^{-2}$	
μ	2.5947×10^{10}
λ	5.1894×10^{10}
Wave velocity m/sec	
Longitudinal, $C_\ell = \sqrt{(\lambda + 2\mu)/\rho}$	6200
Transverse, $C_t = \sqrt{\mu/\rho}$	3100
Surface force, $\tilde{P}_z$ or $\tilde{P}_r$, N/m^2	
Inside circular coil	$\cos \omega t$
Outside circular coil	0
Angular frequency ω , rad/sec	$2\pi \times 10^6$
Real positive root of $\Delta(\alpha) = 0$	$p = 2.1447125$

The integrals are evaluated for the conditions shown in Table 5 by numerical quadrature. Trial numerical calculations were carried out for various values of the upper limit of integration (α_∞) and the width of numerical quadrature ($\Delta\alpha$) to test the convergence of the integrals and to obtain necessary accuracy. The range of integration was divided into several intervals, as shown in Table 6. The width of quadrature at any value of α was taken as the least of the following:

$\Delta\alpha_1 =$ a constant for each interval (values are given in Table 6),

$\Delta\alpha_2 = 2\pi/50(r + \alpha)$,

$$\Delta\alpha_3 = \begin{cases} \sqrt{1 - (2\pi N/50z)^2} - \sqrt{1 - [2\pi(N+1)/50z]^2} & \text{for } 0 \leq \alpha \leq 1 \\ \sqrt{\kappa^2 - (2\pi N/50z)^2} - \sqrt{\kappa^2 - [2\pi(N+1)/50z]^2} & \text{for } 1 \leq \alpha \leq \kappa \\ \text{not defined for } \alpha > \kappa. \end{cases}$$

In the expressions for $\Delta\alpha_3$, N is defined as $50z\sqrt{1 - \alpha^2}/2\pi$ for the first interval and $50z\sqrt{\kappa^2 - \alpha^2}/2\pi$ for the second.

Table 6. Conditions for Numerical Quadrature

Interval	Value of $\Delta\alpha_1$ or ρ_p	
	Case 1	Case 2
$0 \leq \alpha \leq 1$	0.003	0.0006
$1 \leq \alpha \leq \kappa$ ($\kappa = 2$)	0.003	0.0006
$\kappa \leq \alpha \leq 2.144^a$	0.001	0.0002
$2.144 \leq \alpha \leq 2.1445^b$		0.00001
ρ_p	0.0007125	0.0002125
$2.1449250^c \leq \alpha \leq 2.145425$		0.00001
$2.145425^d \leq \alpha \leq 2.289425^e$	0.001	0.0002
$2.289425 \leq \alpha \leq \alpha_\infty$	0.005	0.001

^a $p - \rho_p$ for case 1.

^b $p - \rho_p$ for case 2.

^c $p + \rho_p$ for case 2.

^d $p + \rho_p$ for case 1.

^e $2p - \kappa$.

The calculated integrals for two cases of integration intervals and several upper limits are given in Table 7. For convenience in observing the convergence of the integral the term $-j\pi h(p)/\Delta'(p)$ was eliminated from the calculation. From the results shown in Table 7, the widths of quadrature are chosen to be those of case 1 in Table 6, and the upper limit of integration, α_∞ , is chosen to be 2.5 for $z \geq 5$ mm, to be 7 for $1 \leq z \leq 5$ mm, to be 15 for $z = 0$ and $r = 70$ mm, and to be 50 for $z = 0$ and $r = 0$.

The z component, S_{zz} , of the displacement vector exerted by the vertical surface force is calculated along the z axis and shown in Fig. 11. It is compared with

$$\sin \left[\frac{a}{\lambda} \pi \left(\sqrt{1 + (z/a)^2} - \frac{z}{a} \right) \right],$$

which is obtained by the assumption of a piston movement source.

The envelope of many consecutive peaks of S_{zz} coincides with the sine function fairly well, and the first peak of the former (indicated as P_1 in Fig. 11) coincides very well with the second peak of the latter (indicated as Q_1 in Fig. 11), which shows there is an elemental coincidence between them. However, S_{zz} seems to have infinitely many peaks that cannot be seen in the sine function.

Fig. 11. The z Component of the Displacement Vector Along the z Axis Exerted by a Vertical Surface Force that is Uniformly Distributed Over a Circular Area of 10-mm Radius. The S_{zz} curve is compared with the sine function calculated for a piston movement source and with curves for average S_{zz} values over circular areas about the z axis.

Table 7. Results of Trial Integrations

z, r (mm)	Case	α_∞	$ S_{zz} $	Phase (S_{zz}) (rad)	$ S_{rz} $	Phase (S_{rz}) (rad)	$ S_{zr} $	Phase (S_{zr}) (rad)	$ S_{rr} $	Phase (S_{rr}) (rad)
1, 0	1	5	2.152389×10^{-14}	0.2697456	0	0	1.149228×10^{-14}	0.7056541	0	0
	1	7	2.150337	0.2700094	0	0	1.146818	0.7074461	0	0
	1	10	2.150211	0.2700257	0	0	1.146481	0.7076976	0	0
5, 0	1	2.3	8.426872×10^{-15}	1.218601	0	0	1.622095×10^{-14}	0.4578723	0	0
	1	2.5	8.440092	1.214364	0	0	1.622405	0.4577783	0	0
	1	3	8.439656	1.214502	0	0	1.622424	0.4577724	0	0
70, 0	1	2.3	6.731603×10^{-15}	0.8765144	0	0	3.469332×10^{-15}	-1.519320	0	0
	1	2.5	6.731603	0.8765144	0	0	3.469332	-1.519320	0	0
70 cos 5°, 70 sin 5°	1	2.3	5.893158×10^{-15}	0.9248500	1.970109×10^{-15}	1.028361	2.192425×10^{-15}	0.9503430	1.244738×10^{-14}	0.2338918
	1	2.5	5.893158	0.9248500	1.970109	1.028361	2.192425	0.9503430	1.247738	0.2338918
	2	2.5	5.893172	0.9248465	1.970095	1.028370	2.191920	0.9503527	1.246473	0.2330313
70 cos 45°, 70 sin 45°	1	2.3	3.189158×10^{-16}	-1.270700	3.445511×10^{-16}	1.116582	1.578155×10^{-16}	-0.6709027	3.833751×10^{-16}	-0.5478679
	1	2.5	3.189158	-1.270700	3.445511	1.116582	1.578155	-0.6709027	3.833751	-0.5478679
	2	2.5	3.190211	-1.270525	3.445498	1.116602	1.571987	-0.6756774	3.860736	-0.5564306
70 cos 85°, 70 sin 85°	1	2.5	3.174814×10^{-17}	0.9691312	2.833472×10^{-17}	-0.3999404	2.096557×10^{-17}	-0.1740580	2.092257×10^{-16}	-0.5338810
	1	3	3.174674	0.9691956	2.833197	-0.3999814	2.096399	-0.1740712	2.092284	-0.5338733
	2	2.5	3.180156	0.9690650	2.833453	-0.3999589	2.069394	-0.1671055	2.113009	-0.5386272
70 cos 89.5°, 70 sin 89.5°	1	5	9.633899×10^{-16}	0.001404492	8.703992×10^{-17}	-0.06365836	9.197552×10^{-16}	0.03224713	2.278714×10^{-16}	0.4213050
	1	7	9.631721	0.001404810	8.699993	-0.06368766	9.193993	0.03225962	2.279217	0.4212061
	1	10	9.631207	0.001404885	8.706124	-0.06364275	9.193823	0.03226021	2.279451	0.4211602
	2	7	9.630081	0.001380697	8.700243	-0.06366003	9.211876	0.03185464	2.291131	0.4215584
0, 0	1	30	1.467929×10^{-14}	0.4228726	0	0	9.809952×10^{-15}	0.01372709	0	0
	1	50	1.466540	0.4232986	0	0	9.904608	0.01359590	0	0
	1	70	1.468335	0.4227479	0	0	9.938967	0.01354890	0	0
0, 70	1	10	9.525477×10^{-16}	-0.0005924255	9.678008×10^{-16}	0.004813686	8.878025×10^{-16}	0.04445589	9.038651×10^{-16}	-0.1971114
	1	15	9.523725	-0.0005925345	9.678415	0.004813483	8.879571	0.04444814	9.039475	-0.1970932
	1	20	9.527025	-0.0005923292	9.678458	0.004813462	8.879485	0.04444871	9.038686	-0.1971107

The displacement S_{zz} is composed of the z component of the longitudinal wave and the z component of the transverse wave, and we cannot know each one individually, as explained previously. The fluctuation of S_{zz} can be explained as an interference between the longitudinal and transverse waves. Such a fluctuation has never been detected so far in an ultrasonic technique, because pulse waves are used in the usual ultrasonic technique, making it impossible for the longitudinal wave and transverse wave to coincide except for the very near field (because of the velocity differences).

The following averaging effect also conceals the fluctuation phenomenon. Average displacements in circular areas of radii 2, 5, and 10 mm were calculated with Eq. (35), and the results are also shown in Fig. 11. In the case of a 10-mm radius (similar to common ultrasonic techniques), very little fluctuation can be observed, which may have been unintentionally overlooked in the past. The z component of the displacement, S_{zz} , exerted along the symmetry axis by a horizontal surface force, is shown in Fig. 12.

Fig. 12. The z Component of the Displacement Vector Along the z Axis Exerted by a Horizontal Surface Force that has only a Radial Component and is Distributed Uniformly over a Circular Area of 10-mm Radius Symmetrically Around the z axis.

The z component, S_{zz} , and r component, S_{rz} , of the displacement on a hemisphere of 70-mm radius that are exerted by a vertical surface force are shown in Fig. 13. The first minimum of S_{zz} is at an angle of about

Fig. 13. The r Component (S_{rz}) and z Component (S_{zz}) of the Displacement Vector on a Hemisphere of 70-mm Radius Exerted by a Vertical Surface Force Uniformly Distributed over a Circular Area of 10-mm Radius.

21.5°, which coincides fairly well with the value of 22.2° calculated from $\theta = \sin^{-1}(0.61\lambda_l/a)$ on the assumption of a piston movement source. The angle for S_{rz} is 12.5°, which also coincides well with 11.4° calculated from $\theta = \sin^{-1}(0.61\lambda_t/a)$. Displacements exerted by a horizontal surface force are also shown in Fig. 14. In all four cases, the last peak appears at an angle of 90°, that is, on the surface. These peaks are confined to only small depths, which is characteristic of surface waves.

Fig. 14. The r Component (S_{rr}) and z Component (S_{zr}) of the Displacement Vector on a Hemisphere of 70-mm Radius Exerted by a Horizontal Surface Force that has only a Radial Component and is Uniformly and Symmetrically Distributed in a Circular Area of 10-mm Radius Around the z Axis.

It is impossible to divide a displacement into longitudinal and transverse wave components. However, if the surface force is pulsed, both waves should be resolved individually in time at a sufficiently distant point from the source, because they travel with different velocities.

The surface forces are assumed to have the following boundary conditions:

$$\tilde{P}_z = \tilde{P}_r = \begin{cases} \cos \omega_0 t \text{ N/m}^3, & |t - nT| \leq T_0 \\ 0, & |t - nT| > T_0 \end{cases}.$$

The pulse displacement due to a pulsed force is calculated with Eqs. (C46) and (C47) from Appendix C. The conditions are

$$2\pi \times 0.5 \times 10^6 \leq \omega \leq 2\pi \times 1.5 \times 10^6 ,$$

$$T_0 = 3 \times 10^{-6} \text{ sec} ,$$

$$T = 34 \times 10^{-6} \text{ sec} ,$$

$$\Delta t = 0.2 \text{ } \mu\text{sec} ,$$

$$\omega_0 = 2\pi \times 10^6$$

for calculating pulse displacements for the points ($z = 70 \text{ mm}$, $r = 0$), ($z = 70 \cos 5^\circ \text{ mm}$, $r = 70 \sin 5^\circ \text{ mm}$), and ($z = 0$, $r = 70 \text{ mm}$). The pulse displacement for ($z = 25 \text{ mm}$, $r = 0$) was calculated for the same conditions except $T_0 = 1.75 \times 10^{-6} \text{ sec}$. Some of the results are shown in Figs. 15 through 21.

Fig. 15. Components of the Pulse Displacement Vector at a Point ($z = 70 \cos 5^\circ \text{ mm}$, $r = 70 \sin 5^\circ \text{ mm}$) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of radius 10 mm. (a) z component (S_{zz}) and (b) r component (S_{rz}) from vertical surface force.

Fig. 16. Components of the Pulse Displacement Vector at a Point ($z = 70 \cos 5^\circ$ mm, $r = 70 \sin 5^\circ$ mm) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of Radius 10 mm. (a) z component (S_{zr}) and (b) r component (S_{rr}) from horizontal force with only radial component and symmetrical about z axis.

Fig. 17. Components of the Pulse Displacement Vector at a Point ($z = 70$ mm, $r = 0$) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of Radius 10 mm. (a) z component (S_{zz}) from vertical surface force. (b) z component (S_{zr}) from horizontal force with only radial component and symmetrical about z axis.

Fig. 18. Components of the Pulse Displacement Vector at a Point ($z = 25$ mm, $r = 0$) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of Radius 10 mm. (a) z component (S_{zz}) from vertical surface force. (b) z component (S_{zr}) from horizontal force with only radial component and symmetrical about z axis.

Fig. 19. Components of the Pulse Displacement Vector Averaged in a Circular Area of Radius 5 mm Centered at ($z = 70$ mm, $r = 0$) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of Radius 10 mm. (a) z component [S_{zz} (average)] from vertical surface force. (b) z component [S_{zr} (average)] from horizontal force with only radial component symmetrical about z axis.

Fig. 20. Components of the Pulse Displacement Vector Averaged Over a Circular Area of Radius 5 mm Centered at ($z = 25$ mm, $r = 0$) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of Radius 10 mm. (a) z component [S_{zz} (average)] from vertical surface force. (b) z component [S_{zr} (average)] from horizontal force with only radial component symmetrical about z axis.

Fig. 21. Components of the Pulse Displacement Vector at a Point ($z = 0$, $r = 70$ mm) Exerted by a Pulsed Surface Force Uniformly Distributed over a Circular Area of Radius 1 mm. (a) z component (S_{zz}) and (b) r component (S_{rz}) from vertical surface force. (c) z component (S_{zr}) and (d) r component (S_{rr}) from horizontal force with only radial component symmetrical about z axis.

The longitudinal and transverse waves are clearly separated in time, as they should be in a real physical phenomenon. The wave velocities calculated from Fig. 15(a) are 6120 m/sec for the longitudinal wave and 3070 m/sec for the transverse wave. In the case of an infinite solid, the velocities are calculated from Lamé constants as 6200 and 3100 m/sec, respectively, by the simple formulas $\sqrt{(\lambda + 2\mu)/\rho}$ and $\sqrt{\mu/\rho}$. The difference in the velocities is considered to result from the differences in the conditions and an inaccurate estimation of the velocity from Fig. 15(a).

There are some interesting facts in Fig. 17, which may look strange at first. That is, a disturbance in the z direction travels along the z axis not only with the velocity of a longitudinal wave but also with the velocity of a transverse wave. It seems impossible at first, but it can be explained in the following qualitative way, with reference to Fig. 22. The theory of elastic waves assumes that both longitudinal and transverse waves are generated by any type of applied force. We assume that two identical vertical forces work simultaneously on points A and B, and pure longitudinal and pure transverse waves are generated from the point sources. They are not necessarily spherical waves, but they must be symmetric around the vertical lines through A and B, respectively. Two transverse wave fronts travel with the transverse wave velocity to meet at a point P on the z axis. Each displacement vector has the same amplitude and the same phase, but the directions are symmetric about the z axis, as is shown in Fig. 22. While the r components of displacement cancel each other, the z components are not canceled. Consequently, the z direction disturbance travels along the z axis approximately with a velocity of transverse wave. However, such a wave has never been observed by the usual ultrasonic techniques. Pulse displacements averaged in a circular area of 5-mm radius, centered at $z = 70$ and 25 mm, are shown in Figs. 19 and 20. In Fig. 19(a), it is reduced but still remains [$S_{zz}t$ in Fig. 19(a)]. In Fig. 20(a) it almost disappears. The reason for the disturbance remaining in Fig. 19(a) and disappearing in Fig. 20(a) is the difference in distances between the source and observation point. When the distance is small, the procedure of averaging displacements in a

Fig. 22. Production of Longitudinal Displacements by Interaction of Transverse Waves from an Extended Source.

circular area of a certain radius is more critical than it is in the case of large distance, where the path difference is very small. This phenomenon may be used to measure the velocity of transverse waves with a longitudinal wave transducer without attaching it to a material surface. A special transducer that consists of two longitudinal wave transducers, one a large disk having a small hole in it and the other a small coaxial disk, will be able to detect both longitudinal waves and the z component of transverse waves (see Fig. 23).

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Fig. 23. Composite Transducer.

One problem is the difference in the conditions between the calculation and the actual ultrasonic testing; a second problem is the very low efficiency of the small transducer that is used to detect the z component of the transverse wave.

Pulse displacements at a point on the surface ($z = 0$, $r = 70$ mm) exerted by a pulsed surface force on a circular area of 1-mm radius are shown in Fig. 21. It is very interesting to know that the disturbance travels approximately with a speed of 2905 m/sec, which is quite close to the supposed surface wave velocity, 2891 m/sec, derived from c_0/p where p is a particular root of $\Delta(\alpha) = 0$.

Numerical Calculation of the Elastic Wave Exerted by an Electromagnetic Force Caused by the Interaction Between Eddy Currents and a Constant Magnetic Field

The displacements S_z ($= S_{zz} + S_{zr}$) and S_r ($= S_{rz} + S_{rr}$), which are produced in a conductor by an eddy-current force, are calculated by use of the approximate function of surface forces shown in Fig. 8 and Table 4, p. 15. The calculated results are listed in Table 8 and shown in Fig. 24 as a polar distribution (at a distance of 58.2676 mm) near the symmetry axis.

Table 8. Numerical Displacement Data Calculated for
 $z = 58.2676 \cos\theta$ mm, $r = 58.2676 \sin\theta$ mm

θ	$S_z = S_{zz} + S_{zr}$		$S_r = S_{rz} + S_{rr}$	
	Amplitude (Å)	Phase (rad)	Amplitude (Å)	Phase (rad)
0°	32.71046	0.5785691	0.000000	0.000000
1	26.17365	0.4572459	20.96264	-0.2911700
2	17.65618	0.08132202	16.92477	-0.07969033
3	14.71417	0.005480981	2.843768	1.014537
4	10.41641	0.3872855	2.273667	-0.5674052
5	3.877380	0.6561141	5.176582	1.564696
6	0.7847714	0.01685626	9.386882	-0.6882312
7	4.047799	-0.7305414	7.579748	0.3255650
8	8.542795	-0.1426877	3.495181	-0.1187731
10	10.40244	0.3757070	7.253332	-0.01513283
12	7.276527	0.6584039	2.170394	-0.1303417
14	1.811628	1.447477	1.615008	0.9760299

Fig. 24. Polar Distribution (distance is 58.2676 mm) of Particle Displacements, S_z and S_r .

The material constants are:

Density ρ : $7.85 \times 10^3 \text{ kg/m}^3$

Lamé constants μ : $5.680339 \times 10^{10} \text{ kg m}^{-1} \text{ sec}^{-2}$

λ : $1.352139 \times 10^{11} \text{ kg m}^{-1} \text{ sec}^{-2}$

Wave velocities: $C_\ell = \sqrt{(\lambda + 2\mu)/\rho} = 5630 \text{ m/sec}$

$C_t = \sqrt{\mu/\rho} = 2690 \text{ m/sec}$

Frequency: 1.6 MHz

Real positive root of $\Delta(\alpha) = 0$: $p = 2.2678044$

As in the calculation for the hypothetical material given previously, the range of integration was divided into several intervals. The width of quadrature at any value of α was taken as the least of the following:

$\Delta\alpha_1 = \text{a constant for each interval,}$

$\Delta\alpha_2 = 2\pi/50(r + a) ,$

$$\Delta\alpha_3 = \begin{cases} \sqrt{1 - (2\pi N/50z)^2} - \sqrt{1 - [2\pi(N+1)/50z]^2} & \text{for } 0 \leq \alpha \leq 1 \\ \sqrt{\kappa^2 - (2\pi N/50z)^2} - \sqrt{\kappa^2 - [2\pi(N+1)/50z]^2} & \text{for } 1 \leq \alpha \leq \kappa \\ \text{not defined for } \alpha > \kappa . \end{cases}$$

In the expressions for $\Delta\alpha_3$, N is defined as $50z\sqrt{1 - \alpha^2}/2\pi$ for the first interval and $50z\sqrt{\kappa^2 - \alpha^2}/2\pi$ for the second. The intervals and $\Delta\alpha_1$ values were as follows:

<u>Interval</u>	<u>$\Delta\alpha_1$</u>
$0 \leq \alpha \leq 1$	0.003
$1 \leq \alpha \leq \kappa$	0.003
$\kappa \leq \alpha \leq p - \rho_p$	0.001
$p + \rho_p \leq \alpha \leq 2p - \kappa$	0.001
$2p - \kappa \leq \alpha \leq \alpha_\infty$	0.005

In this example, numerical values are $\rho_p = 0.0008676$, $\kappa = 2.0929368$, $p - \rho_p = 2.2669368$, $p + \rho_p = 2.2686720$, $2p - \kappa = 2.4426720$, and $\alpha_\infty = 3$. The Fortran program for carrying out this calculation is given in Appendix D.

Accuracy of Numerical Calculation

The accuracy of the numerical calculations of Eqs. (11), (12), and (13) is considered to be very good for the widths of quadrature used in integrating these equations. Simpson et al.¹⁶ give similar examples. The accuracy of the numerical calculation of Eqs. (44) and (45) is considered to be better than 1% for the following reasons:

1. The width of numerical quadrature is chosen to be small enough to follow the fluctuations of the trigonometric functions and the Bessel functions. To illustrate,

$$\int_0^{\pi} \sin x \, dx$$

is calculated by both analytical [Eq. (62)] and numerical [Eq. (63)] methods, and the results are compared in Table 9.

$$\int_0^{\pi} \sin x \, dx = \left[-\cos x \right]_0^{\pi} = 2, \quad (62)$$

$$\int_0^{\pi} \sin x \, dx = \sum_{i=1}^N \frac{\pi}{N} \sin \left[\frac{\pi}{N} \left(\frac{1}{2} + i - 1 \right) \right]. \quad (63)$$

¹⁶W. A. Simpson, C. V. Dodd, J. W. Luquire, and W. G. Spoeri, *Computer Programs for Some Eddy-Current Problems - 1970*, ORNL-TM-3295 (June 1971).

Table 9. Accuracy of Numerical Integration of $\int_0^{\pi} \sin x \, dx$

<i>N</i> , Number of Quadratures	Value of Integral	Error (%)
Analytical	2.000000	
20	2.008248	0.41
22	2.006814	0.34
24	2.005724	0.29
26	2.004876	0.24
28	2.004202	0.21
30	2.003660	0.18

We know from Table 9 that $\int_0^{\pi} \sin x \, dx$ can be numerically calculated with an accuracy better than 0.3%, if the numerical quadrature width is chosen to be $\pi/24$; that is, $1/48$ of the fluctuation period, 2π . The product of two trigonometric functions can be represented as the sum of other trigonometric functions:

$$\begin{aligned} f(z) &= \sin k_1 z \times \sin k_2 z \\ &= \frac{1}{2} \left[\cos(k_1 + k_2)z - \cos(k_1 - k_2)z \right]. \end{aligned} \quad (64)$$

Equation (64) shows that $f(z)$ fluctuates with a period of approximately $2\pi/(k_1 + k_2)$. We assume the product of two Bessel functions can be treated similarly.

Equations (44) and (45) have three fluctuating terms. One of them is the sum of two trigonometric functions, and the other two are Bessel functions. These two Bessel functions fluctuate with the periods of approximately $2\pi/a_i$ and $2\pi/r$ and so the product of these two Bessel functions can be considered to fluctuate with the period of approximately $2\pi/(a_i + r)$. In the numerical calculation the width of the numerical quadrature is chosen to be the least one of the following three values, $0.02 \times$ the fluctuation period of the trigonometric function, $0.02 \times 2\pi/(a_i + r)$ and a certain small value, as is shown in Table 6. The computational accuracy of the whole fluctuating factor can be considered to be about 0.6%, because the computational accuracy of each fluctuating factor, that is a trigonometric function and two Bessel functions, is considered to be about 0.3%.

2. Several different values of the upper limit of integration were tried for several conditions to assure the convergence of the integration. Also numerical quadrature widths one-fifth of the initial values and a smaller value of ρ_p were tried (see Table 6). These modifications are compared in Table 7, which shows that the calculated values vary no more than 0.5%.

PROBLEMS AND CONSIDERATIONS

1. The final purpose of this study should be not only the calculation of the acoustic field produced by an electromagnetic force but should also be the calculation of the voltage induced in a coil by the interaction of a steady magnetic field with the acoustic field that is reflected and returns from the rear surface of the material.

2. In this paper the attenuation of the ultrasound by the material is not taken into consideration. It should eventually be considered.

3. An actual metal is not strictly linear, isotropic, and homogenous as is treated in this paper.

4. In the case of large ultrasonic amplitudes, the theory of nonlinear elastic waves is necessary.

Other problems for consideration are:

Can we treat the problem completely by macroscopic and classical theory?

Can we consider the eddy-current velocity to be infinite when the volume force is integrated to get surface force?

Can we neglect the static force terms, F_{z0} and F_{r0} ?

Can we neglect the attenuation or amplification of the ultrasonic field by the steady magnetic field?

We may have to develop new differential equations to rigorously treat the interaction of the elastic wave field and the electromagnetic field.

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APPENDIX A

Fortran Program for Calculation of Eddy-Current Force

Fortran Program for Calculation of Eddy-Current Force

```

•TYPE FORCE.F4
00100      COMMON X
00200      REAL N,I1,MU,MU0,MU1,K
00201      1      ,J1ZTR,J1IDR2,J1IDR1,J1K1K2
00300      DIMENSION AF(15),ZZF(2),BK(15),BZ(15),CURNTA(15,2)
00400      1      ,CURNTP(15,2),FZ1A(15,2),FZ1P(15,2),FR1A(15,2)
00500      1      ,FR1P(15,2),FZ1A(15),FZ1P(15),FR1A(15),FR1P(15)
00600      COMPLEX A(15,2),ETA1,ETA1,EXPEIA ,w,CURNT(15,2)
00700      1      ,FZ1(15,2),FR1(15,2),JK1K2,JK3K4,FZ1(15),PF1(15)
00800      DATA(AF(IJ),IJ=1,15)/
00900      1      8.E-3,9.E-3,10.E-3,10.5E-3,11.E-3,11.5E-3,12.E-3
01000      1      ,12.5E-3,13.E-3,13.5E-3,14.E-3,14.5E-3,15.E-3
01100      1      ,15.5E-3,16.E-3/
01200      DATA(ZZF(IJ),IJ=1,2)/
01300      1      0.E-3,-0.5E-3/
01400      DATA(BZ(IJ),IJ=1,15)/
01500      1      -0.285,-0.205,-0.145,-0.113,-0.08,-0.053,-0.025,-0.005
01600      1      ,0.015,0.055,0.095,0.103,0.11,0.11,0.11/
01700      DATA(BK(IJ),IJ=1,15)/
01800      1      0.48,0.39,0.36,0.34,0.32,0.30,0.28,0.27,0.26,0.255
01900      1      ,0.25,0.248,0.245,0.245,0.245/
02400      K11=10.287E-3
02500      K22=14.7955E-3
02600      H11=0.3429E-3
02700      H22=0.5969E-3
02800      N=7.91667
02900      I1=1790.
03100      KBAA=(K11+K22)/2.
03200      K1=K11/KBAA
03300      K2=K22/KBAA
03400      H1=H11/KBAA
03500      H2=H22/KBAA
03700      FI=3.1415926
03800      SIGMA1=1.351351E6
03900      MU=4.*FI*1.E-7
04000      MU0=1.
04100      MU1=1.
04200      FREQ=1.6E6
04300      OMEGA=2.*FI*FREQ
04400      OMSK1=OMEGA*MU1*MU*SIGMA1*KBAA**2
04500      TYPE 87
04600      TYPE 87
04700      TYPE 81
04800      TYPE 5963,K11,K22,H11,H22
04900      TYPE 82
05000      TYPE 5963,N,I1,FREQ,SIGMA1
05100      TYPE 87

```

```

22370      DW 121 II=1,15
22500      DW 1111 JJ=1,2
22600      RH=AR(II)
22700      ZZ=ZZR(JJ)
22800      R=RH/RBAA
22900      Z=ZZ/RBAA
23000      A(II,JJ)=(0.,0.)
23300      DW 11 I=1,1200
23400      IF(I-350) 10,10,20
23500      20 IF(I-460) 30,30,40
23600      10 ZETA=.005+.01*FLUAI(I-1)
23700      DZETA=.01
23800      GW TW 50
23900      30 ZETA=3.525+.05*FLUAI(I-351)
24000      DZETA=.05
24100      GW TW 50
24200      40 ZETA=9.05+.1*FLUAI(I-461)
24300      DZETA=.1
24400      GW TW 50
24500      50 ETAA1=CMPLX(ZETA**2,0MSR1)
24600      ETA1=CSGRT(ETAA1)
24700      EXPETA=CEXP(ETA1*Z)
24800      EXPZET1=EXP(ZETA*Z)
24900      W=1./(MU1/MU0+ETA1/ZETA)
25100      EXPH12=EXP(-ZETA*H1)-EXP(-ZETA*H2)
25200      ZETA3=1./ZETA**3
25300      X=ZETA*R
25400      CALL BESSEL1(P1)
25500      J1Z1R=P1
25900      X=ZETA*R2
26000      CALL BESSEL1(P3)
26100      J1IDR2=P3
26200      X=ZETA*R1
26300      CALL BESSEL1(P4)
26400      J1IDR1=P4
26500      J1R1R2=J1IDR2-J1IDR1
26600      A(II,JJ)=A(II,JJ)+ZETA3*J1R1R2*J1Z1R*EXPH12*EXPETA
26700      1      *w*DZETA
27100      11 CONTINUE

```

```

27200      K=MU1*MU*N*I1/((H2-H1)*(K2-K1)*KBAA)
27300      AK=K*KBAA
27500      A(I1,JJ)=AK*A(I1,JJ)
27600      CURNT(I1,JJ)=-SIGMA1*OMEGA*(0.,1.)*A(I1,JJ)
27700      FZ1(I1,JJ)=CURNT(I1,JJ)*BK(I1)
27800      FR1(I1,JJ)=CURNT(I1,JJ)*BZ(I1)
27850      CURNTA(I1,JJ)=CABS(CURNT(I1,JJ))
27860      CURNTP(I1,JJ)=ATAN(AIMAG(CURNT(I1,JJ))/REAL(CURNT(I1,JJ)))
27870      FZ1A(I1,JJ)=CABS(FZ1(I1,JJ))
27880      FZ1P(I1,JJ)=ATAN(AIMAG(FZ1(I1,JJ))/REAL(FZ1(I1,JJ)))
27890      FR1A(I1,JJ)=CABS(FR1(I1,JJ))
27900      FR1P(I1,JJ)=ATAN(AIMAG(FR1(I1,JJ))/REAL(FR1(I1,JJ)))
27903      TYPE 88
27905      TYPE 5963, KK, ZZ
27907      TYPE 83
27910      TYPE 5963, CURNTA(I1,JJ), CURNTP(I1,JJ), BK(I1), BZ(I1)
27915      TYPE 84
27920      TYPE 5963, FZ1A(I1,JJ), FZ1P(I1,JJ), FR1A(I1,JJ), FR1P(I1,JJ)
27921      TYPE 5963, FZ1(I1,JJ), FR1(I1,JJ)
28000      1111 CONTINUE
28100      JK1K2=CL0G(FZ1(I1,2)/FZ1(I1,1))
28101      JK1K2=REAL(JK1K2)+(0.,1.)*(AIMAG(JK1K2)-2.*PI)
28102      JK1K2=JK1K2/ZZF(2)
28200      JK3K4=CL0G(FR1(I1,2)/FR1(I1,1))
28250      JK3K4=REAL(JK3K4)+(0.,1.)*(AIMAG(JK3K4)-2.*PI)
28251      JK3K4=JK3K4/ZZF(2)
28300      PZ1(I1)=FZ1(I1,1)/JK1K2
28400      PR1(I1)=FR1(I1,1)/JK3K4
28450      PZ1A(I1)=CABS(PZ1(I1))
28460      PZ1P(I1)=ATAN(AIMAG(PZ1(I1))/REAL(PZ1(I1)))
28470      PR1A(I1)=CABS(PR1(I1))
28480      PR1P(I1)=ATAN(AIMAG(PR1(I1))/REAL(PR1(I1)))
28490      TYPE 85
28500      TYPE 5963, PZ1A(I1), PZ1P(I1), PR1A(I1), PR1P(I1)
28505      TYPE 86
28510      TYPE 5963, PZ1(I1), PR1(I1)
28515      TYPE 87
28600      121 CONTINUE

```

```

28700 5963 FORMAT(4E)
28710 81 FORMAT(T2,'INNER RADIUS'T19,'JUTER RADIUS'
28711 1 T36,'LIFT OFF'T53,'L.O.+COIL THICK')
28720 82 FORMAT(T2,'TURNS'T19,'DRIVING CURRENT'
28721 1 T36,'FREQUENCY'T53,'CONDUCTIVITY')
28730 83 FORMAT(T2,'E.C. AMPLITUDE'T19,'E.C. PHASE'
28731 1 T36,'BK'T53,'BZ')
28740 84 FORMAT(T2,'V0L FZ AMP'T19,'V0L FZ PHASE'
28741 1 T36,'V0L FR AMP'T53,'V0L FR PHASE')
28750 85 FORMAT(T2,'SURFS FZ AMP'T19,'SURFS FZ PHASE'
28751 1 T36,'SURFS FR AMP'T53,'SURFS FR PHASE')
28760 86 FORMAT(T2,'REAL PT OF FZ'T19,'IMAG PT OF FZ'
28761 1 T36,'REAL PT OF FR'T53,'IMAG PT OF FR')
28770 87 FORMAT(T27,'*****')
28780 88 FORMAT(T2,'K'T19,'Z')
28800 END
91900 SUBROUTINE BESEL1(ZZZ)
92000 COMMON X
92100 Y=X
92200 IF(Y-3.) 201,201,202
92300 201 YY=(Y/3.):**2
92400 ZZZ=Y*(0.5 -0.56249985 *YY+0.21093573 *YY**2
92500 1 -0.03954289 *YY**3+0.00443319 *YY**4
92600 1 -0.00031761 *YY**5+0.00001109 *YY**6)
92700 GO TO 204
92800 202 YY=3./Y
92900 F1=0.79788456 +0.00000156 *YY+0.01659667 *YY**2
93000 1 +0.00017105 *YY**3-0.00249511 *YY**4
93100 1 +0.00113653 *YY**5-0.00020033 *YY**6
93200 TETA1=Y-2.35619449
93300 1 +0.12499612 *YY+0.00005650 *YY**2
93400 1 -0.00637879 *YY**3+0.00074348 *YY**4
93500 1 +0.00079824 *YY**5-0.00029166 *YY**6
93600 ZZZ=(1. / SQRT(Y))*F1* COS(TETA1)
93700 204 RETURN
93800 END
93900 SUBROUTINE BESLID(BS)
94000 COMMON X
94100 IF(X-5.) 138,138,139
94200 138 IS=INT(2.*X)+3
94300 AS=(1./6.)*X**3
94400 BS=AS
94500 DO 5000 KK=1,IS
94600 AS=-0.25*AS*X*X*(2.*FL0AT(KK-1)+3.)
94700 1 /(FL0AT(KK)*(FL0AT(KK)+1.)*(2.*FL0AT(KK)+3.))
94800 BS=BS+AS
94900 5000 CONTINUE
95000 GO TO 193
95100 139 Q1=((( -188.1357/X+109.1142)/X-23.79333)/X+2.050931)/X
95200 Q1=((Q1-0.1730503)/X+0.7034845)/X-0.064109E-3
95300 Q2=((( -5.817517/X+2.105874)/X-0.6896196)/X+0.4952024)/X
95400 Q2=(Q2-0.187344E-2)/X+0.7979095
95500 BS=1. -SQRT(X)*(Q2*COS(X-PI/4. )-Q1*SIN(X-PI/4. ))
95600 193 RETURN
95700 END

```


APPENDIX B

Numerical Data for Figs. 4, 5, and 7

Numerical Data for Figs. 4, 5, and 7

```

*****
*****
INNER RADIUS      OUTER RADIUS      LIFT OFF      L.O.+COIL THICK
0.1028700E-01    0.1479550E-01    0.3429000E-03    0.5969000E-03
TURN$            DRIVING CURRENT FREQUENCY      CONDUCTIVITY
0.7916670E+01    0.1790000E+04    0.1600000E+07    0.1351351E+07
*****

```

```

R              Z
0.8000000E-02  0.0000000E+00
E.C. AMPLITUDE  E.C. PHASE      BR              BZ
0.9408181E+09  0.7130697E+00    0.4800000E+00  -0.2850000E+00
VØL FZ AMP     VØL FZ PHASE      VØL FR AMP     VØL FR PHASE
0.4515927E+09  0.7130697E+00    0.2681332E+09  0.7130697E+00
-0.3415654E+09 -0.2954133E+09    0.2028045E+09  0.1754016E+09
%FRSAPR FLØATING UNDERFLØW      PC= 3033

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R              Z
0.8000000E-02  -0.5000000E-03
E.C. AMPLITUDE  E.C. PHASE      BR              BZ
0.2142270E+09  -0.7387192E+00    0.4800000E+00  -0.2850000E+00
VØL FZ AMP     VØL FZ PHASE      VØL FR AMP     VØL FR PHASE
0.1028290E+09  -0.7387192E+00    0.6105470E+08  -0.7387192E+00
-0.7602470E+08  0.6923902E+08    0.4513967E+08  -0.4111067E+08
SURFS PZ AMP    SURFS PZ PHASE    SURFS PR AMP    SURFS PR PHASE
0.1089236E+06  -0.6280300E-01    0.6467338E+05  -0.6280301E-01 -  $\pi$ 
REAL PT ØF PZ  IMAG PT ØF PZ  REAL PT ØF PR  IMAG PT ØF PR
-0.1087088E+06  0.6836233E+04    0.6454588E+05  -0.4059014E+04
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R              Z
0.9000000E-02  0.0000000E+00
E.C. AMPLITUDE  E.C. PHASE      BR              BZ
0.6791710E+09  0.4002471E+00    0.3900000E+00  -0.2050000E+00
VØL FZ AMP     VØL FZ PHASE      VØL FR AMP     VØL FR PHASE
0.2648767E+09  0.4002471E+00    0.1392300E+09  0.4002471E+00
-0.2439421E+09 -0.1032081E+09    0.1282260E+09  0.5425042E+08

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R              Z
0.9000000E-02  -0.5000000E-03
E.C. AMPLITUDE  E.C. PHASE      BR              BZ
0.1620044E+09  -0.1126776E+01    0.3900000E+00  -0.2050000E+00
VØL FZ AMP     VØL FZ PHASE      VØL FR AMP     VØL FR PHASE
0.6318170E+08  -0.1126776E+01    0.3321089E+08  -0.1126776E+01
-0.2714119E+08  0.5705509E+08    0.1426652E+08  -0.2999049E+08
SURFS PZ AMP    SURFS PZ PHASE    SURFS PR AMP    SURFS PR PHASE
0.6323816E+05  -0.4168177E+00    0.3324057E+05  -0.4168177E+00 -  $\pi$ 
REAL PT ØF PZ  IMAG PT ØF PZ  REAL PT ØF PR  IMAG PT ØF PR
-0.5782383E+05  0.2560214E+05    0.3039458E+05  -0.1345753E+05
*****

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R	Z		
0.1000000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.1320250E+10	0.2802371E+00	0.3600000E+00	-0.1450000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.4752902E+09	0.2802371E+00	0.1914363E+09	0.2802371E+00
-0.4567490E+09	-0.1314574E+09	0.1839684E+09	0.5294813E+08
R	Z		
0.1000000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.3956966E+09	-0.1119222E+01	0.3600000E+00	-0.1450000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.1424508E+09	-0.1119222E+01	0.5737601E+08	-0.1119222E+01
-0.6216299E+08	0.1281717E+09	0.2503787E+08	-0.5162472E+08
SURFS FZ AMP	SURFS FZ PHASE	SURFS FR AMP	SURFS FR PHASE
0.1286856E+06	-0.5797160E+00	0.5183170E+05	-0.5797160E+00 - π
REAL PT OF FZ	IMAG PT OF FZ	REAL PT OF FR	IMAG PT OF FR
-0.1076607E+06	0.7049222E+05	0.4336335E+05	-0.2839270E+05

R	Z		
0.1050000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.3325357E+10	0.7226446E+00	0.3400000E+00	-0.1130000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.1130621E+10	0.7226446E+00	0.3757653E+09	0.7226446E+00
-0.8480330E+09	-0.7477597E+09	0.2818463E+09	0.2485201E+09
R	Z		
0.1050000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.7984065E+09	-0.7560378E+00	0.3400000E+00	-0.1130000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2714582E+09	-0.7560378E+00	0.9021994E+08	-0.7560378E+00
-0.1975021E+09	0.1862323E+09	0.6564042E+08	-0.6189486E+08
SURFS FZ AMP	SURFS FZ PHASE	SURFS FR AMP	SURFS FR PHASE
0.2751233E+06	-0.8063833E-01	0.9143802E+05	-0.8063833E-01 - π
REAL PT OF FZ	IMAG PT OF FZ	REAL PT OF FR	IMAG PT OF FR
-0.2742292E+06	0.2216144E+05	0.9114089E+05	-0.7365421E+04

R	Z		
0.1100000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.5895483E+10	0.8355801E+00	0.3200000E+00	-0.8000000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.1886555E+10	0.8355801E+00	0.4716387E+09	0.8355801E+00
-0.1265402E+10	-0.1399231E+10	0.3163505E+09	0.3498077E+09
R	Z		
0.1100000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.1327934E+10	-0.6229352E+00	0.3200000E+00	-0.8000000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.4249390E+09	-0.6229352E+00	0.1062348E+09	-0.6229352E+00
-0.3451225E+09	0.2479186E+09	0.8628063E+08	-0.6197965E+08
SURFS FZ AMP	SURFS FZ PHASE	SURFS FR AMP	SURFS FR PHASE
0.4523167E+06	0.6104818E-01	0.1130792E+06	0.6104818E-01 - π
REAL PT OF FZ	IMAG PT OF FZ	REAL PT OF FR	IMAG PT OF FR
-0.4514741E+06	-0.2759597E+05	0.1128685E+06	0.6898992E+04

R	Z		
0.1150000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	ER	BZ
0.7713372E+10	0.8428694E+00	0.3000000E+00	-0.5300000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2314012E+10	0.8428694E+00	0.4088087E+09	0.8428694E+00
-0.1539566E+10	-0.1727538E+10	0.2719900E+09	0.3051983E+09
R	Z		
0.1150000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	ER	BZ
0.1763930E+10	-0.6077985E+00	0.3000000E+00	-0.5300000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.5291791E+09	-0.6077986E+00	0.9348831E+08	-0.6077985E+00
-0.4344069E+09	0.3021939E+09	0.7674523E+08	-0.5338759E+08
SURFS PZ AMP	SURFS PZ PHASE	SURFS PR AMP	SURFS PR PHASE
0.5591766E+06	0.6592716E-01	0.9878787E+05	0.6592717E-01 - π
REAL PT OF PZ	IMAG PT OF PZ	REAL PT OF PR	IMAG PT OF PR
-0.5579618E+06	-0.3683822E+05	0.9857326E+05	0.6508088E+04

R	Z		
0.1200000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	ER	BZ
0.8962537E+10	0.8397448E+00	0.2800000E+00	-0.2500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2509510E+10	0.8397448E+00	0.2240634E+09	0.8397448E+00
-0.1675482E+10	-0.1868262E+10	0.1495966E+09	0.1668091E+09
R	Z		
0.1200000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	ER	BZ
0.2067626E+10	-0.6145154E+00	0.2800000E+00	-0.2500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.5789354E+09	-0.6145154E+00	0.5169066E+08	-0.6145154E+00
-0.4730209E+09	0.3337925E+09	0.4223401E+08	-0.2980290E+08
SURFS PZ AMP	SURFS PZ PHASE	SURFS PR AMP	SURFS PR PHASE
0.6075075E+06	0.5858911E-01	0.5424174E+05	0.5858911E-01 - π
REAL PT OF PZ	IMAG PT OF PZ	REAL PT OF PR	IMAG PT OF PR
-0.6064651E+06	-0.3557296E+05	0.5414867E+05	0.3176157E+04

R	Z		
0.1250000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	ER	BZ
0.9873017E+10	0.8378697E+00	0.2700000E+00	-0.5000000E-02
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2665715E+10	0.8378697E+00	0.4936509E+08	0.8378697E+00
-0.1783490E+10	-0.1981211E+10	0.3302759E+08	0.3668910E+08
R	Z		
0.1250000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	ER	BZ
0.2285406E+10	-0.6190963E+00	0.2700000E+00	-0.5000000E-02
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.6170596E+09	-0.6190963E+00	0.1142703E+08	-0.6190963E+00
-0.5025353E+09	0.3580794E+09	0.9306210E+07	-0.6631099E+07
SURFS PZ AMP	SURFS PZ PHASE	SURFS PR AMP	SURFS PR PHASE
0.6454772E+06	0.5462752E-01	0.1195328E+05	0.5462751E-01 - π
REAL PT OF PZ	IMAG PT OF PZ	REAL PT OF PR	IMAG PT OF PR
-0.6445144E+06	-0.3524328E+05	0.1193545E+05	0.6526533E+03

R	Z		
0.1300000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.1057865E+11	0.8390859E+00	0.2600000E+00	0.1500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2750450E+10	0.8390859E+00	0.1586798E+09	0.8390859E+00
-0.1837694E+10	-0.2046425E+10	-0.1060208E+09	-0.1180630E+09
R	Z		
0.1300000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.2453787E+10	-0.6179268E+00	0.2600000E+00	0.1500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.6379846E+09	-0.6179268E+00	0.3680680E+08	-0.6179268E+00
-0.5200093E+09	0.3696142E+09	-0.3000054E+08	0.2132389E+08
SURFS PZ AMP	SURFS PZ PHASE	SURFS PR AMP	SURFS PR PHASE
0.6664545E+06	0.5512462E-01	0.3844930E+05	0.5512462E-01
REAL PT OF FZ	IMAG PT OF FZ	REAL PT OF PR	IMAG PT OF PR
-0.6654422E+06	-0.3671945E+05	-0.3839090E+05	-0.2118430E+04

R	Z		
0.1350000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.1114641E+11	0.8464962E+00	0.2550000E+00	0.5500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2842334E+10	0.8464962E+00	0.6130525E+09	0.8464962E+00
-0.1883363E+10	-0.2128804E+10	-0.4062156E+09	-0.4591539E+09
R	Z		
0.1350000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.2581842E+10	-0.6075050E+00	0.2550000E+00	0.5500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.6583696E+09	-0.6075050E+00	0.1420013E+09	-0.6075050E+00
-0.5405706E+09	0.3758110E+09	-0.1165937E+09	0.8105727E+08
SURFS PZ AMP	SURFS PZ PHASE	SURFS PR AMP	SURFS PR PHASE
0.6890950E+06	0.6405126E-01	0.1486283E+06	0.6405125E-01
REAL PT OF FZ	IMAG PT OF FZ	REAL PT OF PR	IMAG PT OF PR
-0.6876819E+06	-0.4410722E+05	-0.1483236E+06	-0.9513322E+04

R	Z		
0.1400000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.1145871E+11	0.8615616E+00	0.2500000E+00	0.9500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.2864678E+10	0.8615616E+00	0.1088578E+10	0.8615616E+00
-0.1865631E+10	-0.2173891E+10	-0.7089398E+09	-0.8260786E+09
R	Z		
0.1400000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BR	BZ
0.2626919E+10	-0.5849336E+00	0.2500000E+00	0.9500000E-01
VOL FZ AMP	VOL FZ PHASE	VOL FR AMP	VOL FR PHASE
0.6567297E+09	-0.5849336E+00	0.2495573E+09	-0.5849336E+00
-0.5475476E+09	0.3626094E+09	-0.2080681E+09	0.1377916E+09
SURFS PZ AMP	SURFS PZ PHASE	SURFS PR AMP	SURFS PR PHASE
0.6938162E+06	0.8522099E-01	0.2636501E+06	0.8522099E-01
REAL PT OF FZ	IMAG PT OF FZ	REAL PT OF PR	IMAG PT OF PR
-0.6912982E+06	-0.5905615E+05	-0.2626933E+06	-0.2244134E+05

R	Z		
0.1500000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BK	BZ
0.8869102E+10	0.8538283E+00	0.2450000E+00	0.1100000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FK AMP	VOL FK PHASE
0.2172930E+10	0.8538283E+00	0.9756012E+09	0.8538283E+00
-0.1427837E+10	-0.1637958E+10	-0.6410697E+09	-0.7354097E+09
R	Z		
0.1500000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BK	BZ
0.2029189E+10	-0.5966546E+00	0.2450000E+00	0.1100000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FK AMP	VOL FK PHASE
0.4971514E+09	-0.5966546E+00	0.2232108E+09	-0.5966546E+00
-0.4112536E+09	0.2793386E+09	-0.1346445E+09	0.1254173E+09
SURFS PZ AMP	SURFS PZ PHASE	SURFS PK AMP	SURFS PK PHASE
0.5252032E+06	0.7678920E-01	0.2358055E+06	0.7678920E-01
REAL PT OF PZ	IMAG PT OF PZ	REAL PT OF PK	IMAG PT OF PK
-0.5236555E+06	-0.4029031E+05	-0.2351106E+06	-0.1808953E+05

R	Z		
0.1550000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BK	BZ
0.6198375E+10	0.7982939E+00	0.2450000E+00	0.1100000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FK AMP	VOL FK PHASE
0.1518602E+10	0.7982939E+00	0.6818212E+09	0.7982939E+00
-0.1059877E+10	-0.1087572E+10	-0.4758632E+09	-0.4882974E+09
R	Z		
0.1550000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BK	BZ
0.1477272E+10	-0.6616855E+00	0.2450000E+00	0.1100000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FK AMP	VOL FK PHASE
0.3619315E+09	-0.6616855E+00	0.1624999E+09	-0.6616855E+00
-0.2855487E+09	0.2223879E+09	-0.1282055E+09	0.9984764E+03
SURFS PZ AMP	SURFS PZ PHASE	SURFS PK AMP	SURFS PK PHASE
0.3710245E+06	0.3950428E-02	0.1665824E+06	0.3950429E-02
REAL PT OF PZ	IMAG PT OF PZ	REAL PT OF PK	IMAG PT OF PK
-0.3710216E+06	-0.1465702E+04	-0.1665811E+06	-0.6580703E+03

R	Z		
0.1600000E-01	0.0000000E+00		
E.C. AMPLITUDE	E.C. PHASE	BK	BZ
0.4254915E+10	0.7670198E+00	0.2450000E+00	0.1100000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FK AMP	VOL FK PHASE
0.1042454E+10	0.7670198E+00	0.4680407E+09	0.7670198E+00
-0.7505483E+09	-0.7234556E+09	-0.3369809E+09	-0.3248168E+09
R	Z		
0.1600000E-01	-0.5000000E-03		
E.C. AMPLITUDE	E.C. PHASE	BK	BZ
0.1014256E+10	-0.7083891E+00	0.2450000E+00	0.1100000E+00
VOL FZ AMP	VOL FZ PHASE	VOL FK AMP	VOL FK PHASE
0.2484927E+09	-0.7083891E+00	0.1115681E+09	-0.7083891E+00
-0.1887081E+09	0.1616721E+09	-0.8472607E+03	0.1258749E+08
SURFS PZ AMP	SURFS PZ PHASE	SURFS PK AMP	SURFS PK PHASE
0.2533408E+06	-0.3263808E-01	0.1137448E+06	-0.3263809E-01
REAL PT OF PZ	IMAG PT OF PZ	REAL PT OF PK	IMAG PT OF PK
-0.2532058E+06	0.8267089E+04	-0.1136843E+06	0.3711754E+04

APPENDIX C

Derivation of Elastic Wave Fields Produced by Eddy-Current Surface Forces

Derivation of Elastic Wave Fields Produced by Eddy-Current Surface Forces

In this appendix, the equations that are suitable for calculation of elastic wave fields produced by eddy-current surface forces are derived by use of a Green's function. A surface force produced by interaction of a magnetic field and eddy currents has two components. One is a vertical component and the other is a horizontal and radial component. Both forces distribute symmetrically around the axis (see Fig. 7). In this sense, the equations for vertical force derived here are generalizations of the equations obtained by Miller and Pursey.¹ The equations for the horizontal radial force are also derived by the same method.

Useful references on the theory of elastic waves are by Ewing et al.² and by Morse and Feshbach.³

Elastic Wave Equation

The standard form for the elastic wave equations in the absence of body forces is

$$\rho \frac{\partial^2 \vec{S}}{\partial t^2} = (\lambda + 2\mu) \nabla(\nabla \cdot \vec{S}) - \mu \nabla \times (\nabla \times \vec{S}), \quad (C1)$$

where $\vec{S}$ = displacement vector,
 λ, μ = Lamé constants, and
 ρ = density.

A displacement vector can be represented as the sum of an irrotational vector, $\vec{S}_\ell$, and a solenoidal vector, $\vec{S}_t$:

$$\vec{S} = \vec{S}_\ell + \vec{S}_t \quad (C2)$$

$$\nabla \times \vec{S}_\ell = 0$$

$$\nabla \cdot \vec{S}_t = 0$$

¹F. Miller and H. Pursey, "The Field and Radiation Impedance of Mechanical Radiators on the Free Surface of a Semi-infinite Isotropic Solid," *Proc. Roy. Soc. London Ser. A* 223: 521-41 (1954).

²W. M. Ewing, W. S. Jardetzky, and F. Press, *Elastic Waves in Layered Media*, McGraw-Hill, New York, 1957.

³P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953.

Vectors $\vec{S}_\ell$ and $\vec{S}_t$ can be represented as the gradient of a scalar potential ϕ and the curl of a vector potential $\vec{W}$, respectively.

$$\vec{S}_\ell = \nabla\phi \quad , \quad (C3a)$$

$$\vec{S}_t = \nabla \times \vec{W} \quad . \quad (C3b)$$

Substituting Eq. (C2) into Eq. (C1) gives

$$\rho \frac{\partial^2 \vec{S}_\ell}{\partial t^2} + \rho \frac{\partial^2 \vec{S}_t}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \cdot \vec{S}_\ell) - \mu\nabla \times (\nabla \times \vec{S}_t) \quad (C4)$$

Introducing a separation constant vector $\vec{X}$ divides Eq. (C4) into two equations:

$$\left\{ \begin{array}{l} \rho \frac{\partial^2 \vec{S}_\ell}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \cdot \vec{S}_\ell) + \vec{X} \end{array} \right. \quad (C5a)$$

$$\left\{ \begin{array}{l} \rho \frac{\partial^2 \vec{S}_t}{\partial t^2} = -\mu\nabla \times (\nabla \times \vec{S}_t) - \vec{X} \end{array} \right. \quad (C5b)$$

Substituting Eq. (C3a) into Eq. (C5a) gives

$$\rho \frac{\partial^2}{\partial t^2} \nabla\phi = (\lambda + 2\mu)\nabla(\nabla^2\phi) + \vec{X} \quad (C6)$$

Assuming ϕ is time harmonic, Eq. (C6) becomes

$$-\rho\omega^2\nabla\phi = (\lambda + 2\mu)\nabla(\nabla^2\phi) + \vec{X} \quad (C7)$$

Assuming ϕ can be represented as the sum of ϕ_1 and ϕ_2 , and $\nabla\phi_2 = -\vec{X}/\rho\omega^2$, Eq. (C7) becomes

$$-\rho\omega^2\nabla\phi_1 - \rho\omega^2\nabla\phi_2 = (\lambda + 2\mu)\nabla(\nabla^2\phi_1) + (\lambda + 2\mu)\nabla(\nabla^2\phi_2) + \vec{X} \quad .$$

Therefore,

$$-\rho\omega^2\nabla\phi_1 + \vec{X} = (\lambda + 2\mu)\nabla(\nabla^2\phi_1) + (\lambda + 2\mu)\nabla(-\nabla \cdot \vec{X}/\rho\omega^2) + \vec{X} \quad ,$$

or

$$-\rho\omega^2\nabla\phi_1 = (\lambda + 2\mu)\nabla(\nabla^2\phi_1) ,$$

so

$$\nabla[\rho\omega^2\phi_1 + (\lambda + 2\mu)\nabla^2\phi_1] = 0 .$$

Then

$$\rho\omega^2\phi_1 + (\lambda + 2\mu)\nabla^2\phi_1 = \phi_c \quad (\nabla\phi_c = 0) . \quad (C8)$$

Assuming ϕ_1 can be represented as the sum of ϕ_{11} and ϕ_{12} ($\phi_{12} = \phi_c/\rho\omega^2$), Eq. (C8) becomes

$$\rho\omega^2\phi_{11} + \rho\omega^2\phi_{12} + (\lambda + 2\mu)\nabla^2\phi_{11} + (\lambda + 2\mu)\nabla^2\phi_{12} = \phi_c ,$$

or

$$\rho\omega^2\phi_{11} + \phi_c + (\lambda + 2\mu)\nabla^2\phi_{11} + (\lambda + 2\mu)\nabla^2(\phi_c/\rho\omega^2) = \phi_c ,$$

so

$$\rho\omega^2\phi_{11} + (\lambda + 2\mu)\nabla^2\phi_{11} = 0 . \quad (C9)$$

Summarizing the procedure from Eq. (C5a) to Eq. (C9), we have

$$\begin{aligned} \phi &= \phi_1 + \phi_2 & \nabla\phi_2 &= -\vec{X}/\rho\omega^2 \\ &= \phi_{11} + \phi_{12} + \phi_2 & \phi_{12} &= \phi_c/\rho\omega^2 \end{aligned}$$

$$\rho\omega^2\phi_{11} + (\lambda + 2\mu)\nabla^2\phi_{11} = 0 . \quad (C9)$$

$$\begin{aligned} \vec{S}_\ell &= \nabla\phi \\ &= \nabla(\phi_{11} + \phi_{12} + \phi_2) \\ &= \nabla\phi_{11} + \nabla\phi_{12} + \nabla\phi_2 \\ &= \nabla\phi_{11} + \nabla(\phi_c/\rho\omega^2) - \vec{X}/\rho\omega^2 \\ &= \nabla\phi_{11} - \vec{X}/\rho\omega^2 . \end{aligned} \quad (C10)$$

Vector $\vec{W}$ is analyzed similarly:

$$\vec{W} = \vec{W}_1 + \vec{W}_2 = \vec{W}_{11} + \vec{W}_{12} + \vec{W}_2$$

$$\left(\nabla \times \vec{W}_2 = \vec{X}/\rho\omega^2, \vec{W}_{12} = \vec{W}_c/\rho\omega^2 \right) ;$$

$$\rho\omega^2\vec{W}_{11} - \mu\nabla \times (\nabla \times \vec{W}_{11}) = 0 ; \quad (C11)$$

$$\vec{S}_t = \nabla \times \vec{W}_{11} + \vec{X}/\rho\omega^2 . \quad (C12)$$

From Eqs. (C10) and (C11) we learn that we cannot know $\vec{S}_\ell$ and $\vec{S}_t$ individually because of the unknown vector $\vec{X}$. However, we know the sum of two vectors.

$$\begin{aligned} \vec{S} &= \vec{S}_\ell + \vec{S}_t \\ &= \nabla\phi + \nabla \times \vec{W} \\ &= \nabla\phi_{11} - \vec{X}/\rho\omega^2 + \nabla \times \vec{W}_{11} + \vec{X}/\rho\omega^2 \\ &= \nabla\phi_{11} + \nabla \times \vec{W}_{11} . \end{aligned} \quad (C13)$$

The problem is solved in a cylindrical coordinate system under the assumption of axial symmetry. The force produced by the eddy currents has only radial and vertical components, which produce only the same components of the displacement vector. Consequently, $\vec{W}_{11}$ has only an azimuthal component, W_{11} , and both $\partial W_{11}/\partial\theta$ and $\partial\phi_{11}/\partial\theta$ are equal to zero. On the above mentioned assumptions, Eqs. (C9) and (C11) become

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \phi_{11} + \kappa_c^2 \phi_{11} = 0 , \quad (C14)$$

where

$$\kappa_c^2 = \rho\omega^2/(\lambda + 2\mu) .$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) W_{11} + \kappa_s^2 W_{11} = 0 , \quad (C15)$$

where

$$\kappa_s^2 = \rho\omega^2/\mu .$$

Boundary Conditions

The general boundary conditions are the continuity of the displacement and the stresses at the boundary. If one medium is a vacuum, we need only the continuity of the normal and tangential stresses. The stress equations in cylindrical coordinates are

$$\begin{aligned}\sigma_{zz} &= \lambda \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} \right] + 2\mu \frac{\partial S_z}{\partial z} \\ &= \lambda \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi_{11}}{\partial r} \right) + \frac{\partial^2 \phi_{11}}{\partial z^2} \right] + 2\mu \frac{\partial}{\partial z} \left[\frac{\partial \phi_{11}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r W_{11}) \right] \quad (C16)\end{aligned}$$

$$\begin{aligned}\tau_{zr} &= \mu \left(\frac{\partial S_r}{\partial z} + \frac{\partial S_z}{\partial r} \right) \\ &= \mu \left[\frac{\partial}{\partial z} \left(\frac{\partial \phi_{11}}{\partial r} - \frac{\partial W_{11}}{\partial z} \right) + \frac{\partial}{\partial r} \left(\frac{\partial \phi_{11}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r W_{11}) \right) \right] \quad (C17)\end{aligned}$$

σ_{zz} = normal stress on a plane of constant z ,

τ_{zr} = r component of tangential stress on a plane of constant z .

If an elastic solid occupies an infinite half space, the stresses must satisfy the following equations:

$$\left\{ \begin{array}{l} \sigma_{zz} = P_z \text{ at } z = 0, \\ \tau_{zr} = P_r \text{ at } z = 0, \end{array} \right. \quad (C18)$$

$$\left\{ \begin{array}{l} \tau_{zr} = P_r \text{ at } z = 0, \end{array} \right. \quad (C19)$$

where P_z and P_r are respectively the normal and tangential components of the surface force. Now, we solve the homogenous equations (C14) and (C15) with the inhomogenous boundary conditions (C18) and (C19).

Green's Function

We solve the problem by the method of Green's function. First of all, the surface force is assumed to act only on a ring of radius a . Then we get a final solution by superimposing Green's functions:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \phi_G + \kappa_c^2 \phi_G = 0 , \quad (C20)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) W_G + \kappa_s^2 W_G = 0 , \quad (C21)$$

where ϕ_G and W_G are Green's functions of Eqs. (C14) and (C15), respectively.

The boundary conditions become

$$\begin{cases} \sigma_{zz} = P_z \delta(r - a) & \text{at } z = 0 , \\ \tau_{zr} = P_r \delta(r - a) & \text{at } z = 0 . \end{cases} \quad (C22)$$

$$\tau_{zr} = P_r \delta(r - a) \text{ at } z = 0 . \quad (C23)$$

Setting $\phi_G(r, z) = R(r)Z(z)$ in Eq. (C20) gives

$$\frac{1}{R} \frac{\partial^2 R}{\partial r^2} + \frac{1}{rR} \frac{\partial R}{\partial r} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} + \kappa_c^2 = 0 . \quad (C24)$$

Introducing the separation constant α^2 ,

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} + \kappa_c^2 = \alpha^2 .$$

Then

$$Z = C_1' \exp(-\alpha_c z) + C_2' \exp(\alpha_c z) ,$$

where

$$\alpha_c = \sqrt{\alpha^2 - \kappa_c^2} ;$$

C_2' should be zero for convergence of Z at infinity. Then

$$\frac{1}{R} \frac{\partial^2 R}{\partial r^2} + \frac{1}{rR} \frac{\partial R}{\partial r} = -\alpha^2$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \alpha^2 R = 0 .$$

This is a Bessel equation of order zero, a solution of which is

$$R = C_3' J_0(\alpha r) + C_4' Y_0(\alpha r) .$$

Because of the divergence of Y_0 at the origin, C_4 has to be zero. Thus, the general solution of Eq. (C20) is

$$\begin{aligned}\phi_G(r, z) &= \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) C_3 J_0(\alpha r) d\alpha \\ &= \int_0^\infty C_1 \exp(-\alpha_c z) J_0(\alpha r) d\alpha.\end{aligned}\quad (C25)$$

The general solution for Eq. (C22) is obtained in a similar manner:

$$W_G(r, z) = \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha, \quad (C26)$$

where

$$\alpha_s = \sqrt{\alpha^2 - \kappa_s^2}.$$

Substituting Eqs. (C25) and (C26) into Eqs. (C16) and (C17) gives

$$\begin{aligned}\sigma_{zz} &= \lambda \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \right. \\ &\quad \left. + \frac{\partial^2}{\partial z^2} \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \right] \\ &\quad + 2\mu \left[\frac{\partial^2}{\partial z^2} \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \right. \\ &\quad \left. + \frac{\partial}{\partial z} \frac{1}{r} \frac{\partial}{\partial r} r \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha \right] \\ &= \lambda \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} J_0(\alpha r) d\alpha \\ &\quad + (\lambda + 2\mu) \int_0^\infty \frac{\partial^2}{\partial z^2} C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \\ &\quad + 2\mu \int_0^\infty \frac{\partial}{\partial z} C_3(\alpha) \exp(-\alpha_s z) \frac{1}{r} \frac{\partial}{\partial r} r J_1(\alpha r) d\alpha \\ &= \lambda \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) (-\alpha^2) J_0(\alpha r) d\alpha \\ &\quad + (\lambda + 2\mu) \int_0^\infty \alpha_c^2 C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \\ &\quad + 2\mu \int_0^\infty -\alpha_s C_3(\alpha) \exp(-\alpha_s z) \alpha J_0(\alpha r) d\alpha \\ &= \int_0^\infty \left\{ [(\lambda + 2\mu) \alpha_c^2 - \lambda \alpha^2] C_1(\alpha) \exp(-\alpha_c z) - 2\mu \alpha_s C_3(\alpha) \exp(-\alpha_s z) \right\} J_0(\alpha r) d\alpha \\ &= \int_0^\infty [\mu (2\alpha^2 - \kappa_s^2) C_1(\alpha) \exp(-\alpha_c z) - 2\mu \alpha_s C_3(\alpha) \exp(-\alpha_s z)] J_0(\alpha r) d\alpha \\ &= \mu \int_0^\infty [(2\alpha^2 - \kappa_s^2) C_1(\alpha) \exp(-\alpha_c z) - 2\alpha_s C_3(\alpha) \exp(-\alpha_s z)] J_0(\alpha r) d\alpha.\end{aligned}\quad (C27)$$

$$\begin{aligned}
\tau_{zr} &= \mu \left[\frac{\partial}{\partial z} \frac{\partial}{\partial r} \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \right. \\
&\quad - \frac{\partial^2}{\partial z^2} \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha \\
&\quad + \frac{\partial}{\partial r} \frac{\partial}{\partial z} \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \\
&\quad \left. + \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha \right] \\
&= \mu \left\{ \int_0^\infty \frac{\partial}{\partial z} \left[C_1(\alpha) \exp(-\alpha_c z) \right] \frac{\partial}{\partial r} J_0(\alpha r) d\alpha \right. \\
&\quad - \int_0^\infty \frac{\partial^2}{\partial z^2} C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha \\
&\quad + \int_0^\infty \frac{\partial}{\partial z} C_1(\alpha) \exp(-\alpha_c z) \frac{\partial}{\partial r} J_0(\alpha r) d\alpha \\
&\quad \left. + \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} r J_1(\alpha r) d\alpha \right\} \\
&= \mu \left\{ \int_0^\infty -\alpha_c C_1(\alpha) \exp(-\alpha_c z) \left[-\alpha J_1(\alpha r) \right] d\alpha \right. \\
&\quad - \int_0^\infty \alpha_s^2 C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha \\
&\quad + \int_0^\infty -\alpha_c C_1(\alpha) \exp(-\alpha_c z) \left[-\alpha J_1(\alpha r) \right] d\alpha \\
&\quad \left. + \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) \left[-\alpha^2 J_1(\alpha r) \right] d\alpha \right\} \\
&= \mu \int_0^\infty \left[2\alpha_c C_1(\alpha) \exp(-\alpha_c z) - (\alpha_s^2 + \alpha^2) C_3(\alpha) \exp(-\alpha_s z) \right] J_1(\alpha r) d\alpha \\
&= \mu \int_0^\infty \left[2\alpha_c C_1(\alpha) \exp(-\alpha_c z) - (2\alpha^2 - \kappa_s^2) C_3(\alpha) \exp(-\alpha_s z) \right] J_1(\alpha r) d\alpha . \quad (C28)
\end{aligned}$$

Substituting Eqs. (C27) and (C28) into Eqs. (C22) and (C23):

$$\mu \int_0^\infty \left[(2\alpha^2 - \kappa_s^2) C_1(\alpha) - 2\alpha \alpha_s C_3(\alpha) \right] J_0(\alpha r) d\alpha = P_z \delta(r - a) . \quad (C29)$$

$$\mu \int_0^\infty \left[2\alpha \alpha_c C_1(\alpha) - (2\alpha^2 - \kappa_s^2) C_3(\alpha) \right] J_1(\alpha r) d\alpha = P_r \delta(r - a) . \quad (C30)$$

Multiplying both sides of Eq. (C29) by

$$\int_0^\infty r J_0(\alpha' r) dr$$

and then reversing the order of integration, we obtain

$$\mu \int_0^\infty \frac{1}{\alpha} \left[(2\alpha^2 - \kappa_s^2) C_1(\alpha) - 2\alpha \alpha_s C_3(\alpha) \right] \int_0^\infty J_0(\alpha r) J_0(\alpha' r) \alpha r dr d\alpha = \int_0^\infty P_z \delta(r - a) r J_0(\alpha' r) dr . \quad (C31)$$

By use of the Fourier-Bessel equation, Eq. (C31) becomes

$$\mu \frac{1}{\alpha'^2} \left[(2\alpha'^2 - \kappa_s^2) C_1(\alpha') - 2\alpha' \alpha'_s C_3(\alpha') \right] = P_z \alpha J_0(\alpha' a) , \quad (C32)$$

where $\alpha_c'^2 = \alpha'^2 - \kappa_s^2$ and $\alpha_s'^2 = \alpha'^2 - \kappa_c^2$.

Dropping the primes:

$$(2\alpha^2 - \kappa_s^2) C_1(\alpha) - 2\alpha \alpha_s C_3(\alpha) = P_z \alpha \alpha J_0(\alpha a) / \mu . \quad (C33)$$

Similarly, Eq. (C30) becomes

$$2\alpha \alpha_c C_1(\alpha) - (2\alpha^2 - \kappa_s^2) C_3(\alpha) = P_r \alpha \alpha J_1(\alpha a) / \mu . \quad (C34)$$

From Eqs. (C33) and (C34) we can find the unknown coefficients $C_1(\alpha)$ and $C_3(\alpha)$:

$$C_1(\alpha) = \frac{(2\alpha^2 - \kappa_s^2) P_z \alpha \alpha J_0(\alpha a) / \mu - 2\alpha \alpha_s P_r \alpha \alpha J_1(\alpha a) / \mu}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \alpha_c \alpha_s} ,$$

$$C_3(\alpha) = \frac{2\alpha \alpha_c P_z \alpha \alpha J_0(\alpha a) / \mu - (2\alpha^2 - \kappa_s^2) P_r \alpha \alpha J_1(\alpha a) / \mu}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \alpha_c \alpha_s} .$$

If we let

$$\begin{aligned}\Delta &= (2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \alpha_c \alpha_s \\ &= (2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \quad ,\end{aligned}$$

then

$$\begin{aligned}\phi_G(r, z) &= \int_0^\infty C_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha \\ &= \int_0^\infty \left[\frac{2\alpha^2 - \kappa_s^2}{\Delta} \frac{P_z}{\mu} \alpha \alpha J_0(\alpha) - \frac{2\alpha \alpha_s}{\Delta} \frac{P_r}{\mu} \alpha \alpha J_1(\alpha) \right] \exp(-\alpha_c z) J_0(\alpha r) d\alpha \quad .\end{aligned}$$

ϕ_{11} is obtained by integrating the Green's function:

$$\begin{aligned}\phi_{11}(r, z) &= \int_0^\infty \phi_G da \\ &= \int_0^\infty \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \frac{P_z}{\mu} \alpha \alpha J_0(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha da \\ &= \int_0^\infty \int_0^\infty \frac{2\alpha \alpha_s}{\Delta} \frac{P_r}{\mu} \alpha \alpha J_1(\alpha) \exp(-\alpha_c z) J_0(\alpha r) d\alpha da \quad .\end{aligned}$$

Reversing the order of integration gives

$$\begin{aligned}\phi_{11}(r, z) &= \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_z \alpha \alpha J_0(\alpha) d\alpha \right. \\ &\quad \left. - \int_0^\infty \frac{2\alpha \alpha_s}{\Delta} \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_r \alpha \alpha J_1(\alpha) d\alpha \right] \quad .\end{aligned} \tag{C35}$$

One obtains W_G and W_{11} similarly:

$$\begin{aligned}
 W_G(r, z) &= \int_0^\infty C_3(\alpha) \exp(-\alpha_s z) J_1(\alpha r) d\alpha \\
 &= \int_0^\infty \left[\frac{2\alpha\alpha_c}{\Delta} \frac{P_z}{\mu} \alpha \alpha J_0(\alpha a) - \frac{(2\alpha^2 - \kappa_s^2)}{\Delta} \frac{P_r}{\mu} \alpha \alpha J_1(\alpha a) \right] \exp(-\alpha_s z) J_1(\alpha r) d\alpha .
 \end{aligned}$$

Then

$$\begin{aligned}
 W_{11}(r, z) &= \int_0^\infty W_G da \\
 &= \int_0^\infty \int_0^\infty \frac{2\alpha\alpha_c}{\Delta} \frac{P_z}{\mu} \alpha \alpha J_1(\alpha a) \exp(-\alpha_s z) J_1(\alpha r) d\alpha da \\
 &\quad - \int_0^\infty \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \frac{P_r}{\mu} \alpha \alpha J_1(\alpha a) \exp(-\alpha_s z) J_1(\alpha r) d\alpha da .
 \end{aligned}$$

Reversing the order of integration gives

$$\begin{aligned}
 W_{11}(r, z) &= \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha\alpha_c}{\Delta} \exp(-\alpha_s z) J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha \alpha J_0(\alpha a) da \right. \\
 &\quad \left. - \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_s z) J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha \alpha J_1(\alpha a) da \right] . \quad (C36)
 \end{aligned}$$

Displacement Vector

The displacement is obtained by substituting ϕ_{11} and W_{11} into Eq. (C13):

$$\begin{aligned}
S_z &= \frac{\partial \Phi_{11}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r, W_{11}) \\
&= \frac{\partial}{\partial z} \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha \alpha_s}{\Delta} \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha a J_1(\alpha a) da \right] \\
&\quad + \frac{1}{r} \frac{\partial}{\partial r} r \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha \alpha_c}{\Delta} \exp(-\alpha_s z) J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_s z) J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha a J_1(\alpha a) da \right] \\
&= \frac{1}{\mu} \left[\int_0^\infty -\frac{2\alpha^2 - \kappa_s^2}{\Delta} \alpha_c \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty -\frac{2\alpha \alpha_s}{\Delta} \alpha_c \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha a J_1(\alpha a) da \right] \\
&\quad + \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha \alpha_c}{\Delta} \exp(-\alpha_s z) \alpha J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_s z) \alpha J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha a J_1(\alpha a) da \right] \\
&= \frac{1}{\mu} \int_0^\infty \frac{\alpha_c}{\Delta} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\alpha_c z) + 2\alpha^2 \exp(-\alpha_s z) \right] J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da \\
&\quad + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{\Delta} \left[2\alpha \alpha_c \exp(-\alpha_c z) + (\kappa_s^2 - 2\alpha^2) \exp(-\alpha_s z) \right] J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha a J_1(\alpha a) da .
\end{aligned}$$

$$\begin{aligned}
S_r &= \frac{\partial \phi_{11}}{\partial r} - \frac{\partial W_{11}}{\partial z} \\
&= \frac{\partial}{\partial r} \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha \omega J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha \alpha_s}{\Delta} \exp(-\alpha_c z) J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha \omega J_1(\alpha a) da \right] \\
&\quad - \frac{\partial}{\partial z} \frac{1}{\mu} \left[\int_0^\infty \frac{2\alpha \alpha_c}{\Delta} \exp(-\alpha_s z) J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha \omega J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_s z) J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha \omega J_1(\alpha a) da \right] \\
&= \frac{1}{\mu} \left\{ \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} \exp(-\alpha_c z) [-\alpha J_1(\alpha r)] d\alpha \int_0^\infty P_z(a) \alpha \omega J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha \alpha_s}{\Delta} \exp(-\alpha_c z) [-\alpha J_1(\alpha r)] d\alpha \int_0^\infty P_r(a) \alpha \omega J_1(\alpha a) da \right\} \\
&\quad - \frac{1}{\mu} \left\{ \int_0^\infty \frac{2\alpha \alpha_c}{\Delta} [-\alpha_s \exp(-\alpha_s z)] J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha \omega J_0(\alpha a) da \right. \\
&\quad \left. - \int_0^\infty \frac{2\alpha^2 - \kappa_s^2}{\Delta} [-\alpha_s \exp(-\alpha_s z)] J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha \omega J_1(\alpha a) da \right\} \\
&= \frac{1}{\mu} \int_0^\infty \frac{\alpha}{\Delta} [(\kappa_s^2 - 2\alpha^2) \exp(-\alpha_c z) + 2\alpha \alpha_s \exp(-\alpha_s z)] J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha \omega J_0(\alpha a) da \\
&\quad + \frac{1}{\mu} \int_0^\infty \frac{\alpha_s}{\Delta} [2\alpha^2 \exp(-\alpha_c z) + (\kappa_s^2 - 2\alpha^2) \exp(-\alpha_s z)] J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha \omega J_1(\alpha a) da .
\end{aligned}$$

Therefore:

$$\begin{aligned}
S_z &= \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
&\quad \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_0(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha \omega J_0(\alpha a) da \\
&\quad + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
&\quad \left. + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_0(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha \omega J_1(\alpha a) da .
\end{aligned}$$

(C37)

$$\begin{aligned}
S_r = & \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
& + 2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \left. \right] J_1(\alpha r) d\alpha \int_0^\infty P_z(a) \alpha a J_0(\alpha a) da \\
& + \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_s^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
& + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \left. \right] J_1(\alpha r) d\alpha \int_0^\infty P_r(a) \alpha a J_1(\alpha a) da . \quad (C38)
\end{aligned}$$

The physical value of the displacement is obtained by taking the real parts:

$$\tilde{S}_z = \text{Real} \left[S_z e^{j\omega t} \right] . \quad (C39)$$

$$\tilde{S}_r = \text{Real} \left[S_r e^{j\omega t} \right] . \quad (C40)$$

If the surface forces are constant in a circular area:

$$\begin{aligned}
P_z(a) &= \begin{cases} 1 , & a \leq a_1 \\ 0 , & a > a_1 \end{cases} , \\
P_r(a) &= \begin{cases} 1 , & a \leq a_1 \\ 0 , & a > a_1 \end{cases} ,
\end{aligned}$$

then the first term of Eq. (C37) becomes

$$\begin{aligned}
S_{zz} &= \frac{1}{\mu} \int_0^\infty \frac{\alpha_c}{\Delta} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\alpha_c z) + 2\alpha^2 \exp(-\alpha_s z) \right] J_0(\alpha r) d\alpha \int_0^{a_1} \alpha a J_0(\alpha a) da . \\
\int_0^{a_1} \alpha a J_0(\alpha a) da &= \frac{1}{\alpha} \int_0^{a_1} \alpha a J_0(\alpha a) d(\alpha a) \\
&= \frac{1}{\alpha} \int_0^{\alpha a_1} p J_0(p) dp \\
&= \frac{1}{\alpha} \left[p J_1(p) \right]_0^{\alpha a_1} \\
&= \frac{1}{\alpha} \left[\alpha a_1 J_1(\alpha a_1) \right] .
\end{aligned}$$

Therefore,

$$\begin{aligned}
 S_{zz} &= \frac{1}{\mu} \int_0^\infty \frac{\alpha_c}{\Delta} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\alpha_c z) + 2\alpha^2 \exp(-\alpha_s z) \right] J_0(\alpha r) \frac{1}{\alpha} \alpha_1 J_1(\alpha \alpha_1) d\alpha \\
 &= \frac{\alpha_1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
 &\quad \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_0(\alpha r) J_1(\alpha \alpha_1) d\alpha .
 \end{aligned} \tag{C41}$$

Similarly, the second term of Eq. (C37) and the first and second terms of Eq. (C38) become

$$\begin{aligned}
 S_{zz} &= \frac{\alpha_1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
 &\quad \left. + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_0(\alpha r) \frac{I(\alpha \alpha_1)}{\alpha \alpha_1} d\alpha .
 \end{aligned} \tag{C42}$$

$$\begin{aligned}
 S_{rz} &= \frac{\alpha_1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2) - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
 &\quad \left. + 2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_1(\alpha r) J_1(\alpha \alpha_1) d\alpha .
 \end{aligned} \tag{C43}$$

$$\begin{aligned}
 S_{rr} &= \frac{\alpha_1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_s^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
 &\quad \left. + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_1(\alpha r) \frac{I(\alpha \alpha_1)}{\alpha \alpha_1} d\alpha .
 \end{aligned} \tag{C44}$$

where

$$I(\alpha \alpha_1) = \int_0^{\alpha \alpha_1} p J_1(p) dp .$$

Equations (C41) and (C43) are the same ones derived by Miller and Pursey.⁴

⁴F. Miller and H. Pursey, "The Field and Radiation Impedance of Mechanical Radiators on the Free Surface of a Semi-infinite Isotropic Solid," *Proc. Roy. Soc. London Ser. A* 223: 521-41 (1954).

An average displacement within a circle is obtained by integrating Eq. (C37) with respect to r :

$$\begin{aligned}
\zeta_z(\text{av}) &= \frac{1}{\pi a^2} \int_0^a 2\pi r \zeta_z \, dr = \int_0^a \frac{2}{a^2} r \zeta_z \, dr \\
&= \int_0^a \frac{2}{a^2} r \, dr \left\{ \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \right. \\
&\quad \left. \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_0(\alpha r) \, d\alpha \int_0^\infty P_z(\alpha) \alpha \omega J_0(\alpha a) \, d\alpha \right. \\
&\quad \left. + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \right. \\
&\quad \left. \left. + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] J_0(\alpha r) \, d\alpha \int_0^\infty P_r(\alpha) \alpha \omega J_1(\alpha a) \, d\alpha \right\} \\
&= \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
&\quad \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] \frac{2}{a^2} \int_0^a r J_0(\alpha r) \, dr \, d\alpha \int_0^\infty P_z(\alpha) \alpha \omega J_0(\alpha a) \, d\alpha \\
&\quad + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
&\quad \left. + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] \frac{2}{a^2} \int_0^a r J_0(\alpha r) \, dr \, d\alpha \int_0^\infty P_r(\alpha) \alpha \omega J_1(\alpha a) \, d\alpha \\
&= \frac{1}{\mu} \int_0^\infty \frac{\sqrt{\alpha^2 - \kappa_c^2}}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[(\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
&\quad \left. + 2\alpha^2 \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] \frac{2J_1(\alpha a)}{\alpha a} \, d\alpha \int_0^\infty P_z(\alpha) \alpha \omega J_0(\alpha a) \, d\alpha \\
&\quad + \frac{1}{\mu} \int_0^\infty \frac{\alpha}{(2\alpha^2 - \kappa_s^2)^2 - 4\alpha^2 \sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2}} \left[2\sqrt{\alpha^2 - \kappa_c^2} \sqrt{\alpha^2 - \kappa_s^2} \exp(-\sqrt{\alpha^2 - \kappa_c^2} z) \right. \\
&\quad \left. + (\kappa_s^2 - 2\alpha^2) \exp(-\sqrt{\alpha^2 - \kappa_s^2} z) \right] \frac{2J_1(\alpha a)}{\alpha a} \, d\alpha \int_0^\infty P_r(\alpha) \alpha \omega J_1(\alpha a) \, d\alpha .
\end{aligned} \tag{C45}$$

Nonsinusoidal Surface Force

Equations (C37) and (C38) represent displacements exerted by time-harmonic surface forces. If a nonsinusoidal current flows in a coil, we may represent it as a Fourier series of pure sinusoidal currents:

$$I(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\omega_1 t + b_m \sin m\omega_1 t) ,$$

$$\text{where} \quad a_m = \frac{\omega_1}{\pi} \int_{-T/2}^{T/2} I(t) \cos m\omega_1 t \, dt \quad (m = 0, 1, 2, \dots) ,$$

$$b_m = \frac{\omega_1}{\pi} \int_{-T/2}^{T/2} I(t) \sin m\omega_1 t \, dt \quad (m = 1, 2, 3, \dots) ,$$

$$\omega_1 = \frac{2\pi}{T} , \text{ and } T = \text{the period of repetition.}$$

Assuming $I(t)$ is the repetition of single harmonic train:

$$I(t) = \begin{cases} \cos \omega_0 t & |t - nT| \leq T_0 \\ 0 & |t - nT| > T_0 \end{cases} ,$$

we have

$$a_0 = 0$$

$$a_m = C(\omega)$$

$$\equiv C(m\omega_1)$$

$$= \left(\frac{\omega_1}{\pi} T_0 \right) \left[\frac{\sin(\omega_0 + m\omega_1)T_0}{(\omega_0 + m\omega_1)T_0} + \frac{\sin(\omega_0 - m\omega_1)T_0}{(\omega_0 - m\omega_1)T_0} \right]$$

$$b_m = 0 .$$

Therefore,

$$\begin{aligned}
 I(t) &= \sum_{m=1}^{\infty} C(\omega) \cos \omega t \\
 &\equiv \sum_{m=1}^{\infty} C(m\omega_1) \cos m\omega_1 t \\
 &= \sum_{m=1}^{\infty} \frac{\omega_1 T_0}{\pi} \left[\frac{\sin(\omega_0 + m\omega_1)T_0}{(\omega_0 + m\omega_1)T_0} + \frac{\sin(\omega_0 - m\omega_1)T_0}{(\omega_0 - m\omega_1)T_0} \right].
 \end{aligned}$$

Substituting $I(t)$ for I_ω of Eq. (11) and following through the subsequent operations, we have

$$\tilde{S}_z \text{ (by pulse current)} = \text{Real} \left[\sum_{m=1}^{\infty} C(m\omega_1) S_z(m\omega_1) e^{jm\omega_1 t} \right] ; \quad (\text{C46})$$

$$\tilde{S}_r \text{ (by pulse current)} = \text{Real} \left[\sum_{m=1}^{\infty} C(m\omega_1) S_r(m\omega_1) e^{jm\omega_1 t} \right] ; \quad (\text{C47})$$

$$\begin{aligned}
 &\tilde{S}_z \text{ (average displacement by pulse current)} \\
 &= \text{Real} \left[\sum_{m=1}^{\infty} C(m\omega_1) S_{z(av)}(m\omega_1) e^{jm\omega_1 t} \right] . \quad (\text{C48})
 \end{aligned}$$

APPENDIX D

Fortran Program for Calculation of Elastic Wave

Fortran Program for Calculation of Elastic Wave

```

. TYPE ELSTWV.F4
00100      COMMON X
01000      COMPLEX DISPZZ,DISPZR,DISPRZ,DISPRR,DISPZ,DISPR
01050      1      ,DISPZM,DISPRM
01100      1      ,B1,BP,DELTA,EXPB1,EXPBP,CRZZ,CRZR,CRKZ,CRRR
01200      1      ,CDEZ,CDEK,EXPDMG
01550      1      ,DPZZ,DPZR,DPRZ,DPRR
01650      1      ,CRZZL,CRZZS,CRZRL,CRZRS,CRKZL,CRRZS,CRRRL,CRRRS
01655      1      ,VZZL,VZZS,VZRL,VZRS,VKZL,VRZS,VRRL,VRRS
01660      1      ,AVZZL,AVZZS,AVZRL,AVZRS,AVKZL,AVKZS,AVRRL,AVRRS
02000      REAL JOALPR,J1ALPR,J1ALA2,J1ALA1,J1IDA2,J1IDA1,JOAKR
02100      1      ,J1AKR,J1AKA2,J1DAK2,J1AKA1,J1DAK1
02200      1      ,JOAKRD,J1AKRD,J1A2D,J1A1D,J1AK12,J1A12D
02250      1      ,JOAKA1,JOAKA2
20000      COMPLEX A(2),ETAA1,ETA1,EXPETA,W,JK1K2,JK3K4
20050      1      ,PZ1(15),PR1(15)
20100      DIMENSION ZZF(2),A0DZ(2),A0DR(2)
20110      1      ,AF(18),AZ(50),AN(49)
20160      COMPLEX TSTZZS
20310      DATA(AF(111),111=1,18)/
20311      1      10.3E-3,12.E-3,13.5E-3,15.5E-3,14*1.E-3/
20340      DATA(PZ1(111),111=1,15)/
20341      1      (-0.5E6,0.),(-0.63E6,0.),(-0.65E6,0.),12*(1.,0.)/
20345      DATA(PR1(111),111=1,15)/
20346      1      (0.1E6,0.),(0.,0.),(-0.25E6,0.),12*(1.,0.)/
20350      DATA(AZ(111),111=1,50)/
20351      1      10*1.,2.0929368,2.2669368,2.2686720,2.4426720
20352      1      ,7.4426720,27.4426720,107.4426720,33*1./
20360      DATA(AN(111),111=1,49)/
20361      1      10*1.
20362      1      ,174.,3.,174.,1000.,4000.,16000.,33*1./
22175      PI=3.1415926
22180      FREQ=1.6E6
22200      OMEGA=2.*PI*FREQ
29200      J13579=1
29300      J97531=91
29302      AVZZL=(0.,0.)
29304      AVZZS=(0.,0.)
29306      AVZRL=(0.,0.)
29308      AVZRS=(0.,0.)
29320      AVKZL=(0.,0.)
29330      AVKZS=(0.,0.)
29340      AVRRL=(0.,0.)
29350      AVRRS=(0.,0.)
29360      TYPE 66492,J97531
29370      66492 FORMAT(I)

```

```

29375      D0 111 III=1,3
40000      ZDIS=0.E-3
40100      RDIS=58.2676E-3
40200      AK0N=2.2678044
40300      A2=AF(III+1)
40400      A1=AF(III)
40500      SMALLD=AK0N-2.2669368
40510      VZZL=(0.,0.)
40520      VZZS=(0.,0.)
40530      VZKL=(0.,0.)
40540      VZKS=(0.,0.)
40600      VRZL=(0.,0.)
40700      VRZS=(0.,0.)
40800      VKKL=(0.,0.)
40900      VKKS=(0.,0.)
41000      K0=7.85*((100.**3)/1000.)
41100      CS=2690.
41200      CC=5630.
41300      KMU=K0*CS**2
41400      PS=0MEGA/CS
41500      PC=0MEGA/CC
41600      F=PS/PL
41700      ZDIS=ZDIS*PC
41800      RDIS=RDIS*PC
41900      A2=A2*PC
42000      A1=A1*PC
42002      C0EZ=PZ1(III)/(KMU*PC)
42004      C0EK=-PR1(III)/(KMU*PC)
42055      33447 CONTINUE
42059      DAABB=2.*PI/((RDIS+A2)*50.)
42063      IF(DAABB-0.003) 88298,88298,30108
42067      30108 DAABB=0.003
42071      88298 CONTINUE
42074      D0 14102 NPS=11,16
42075      DICM=(AZ(NPS+1)-AZ(NPS))/AN(NPS)
42076      IF(DICM-DAABB) 56473,56473,11144
42077      56473 AN(NPS)=AN(NPS)
42078      GO TO 14102
42090      11144 NNPFSS=INT((AZ(NPS+1)-AZ(NPS))/DAABB)
42095      AN(NPS)=FL0AT(NNPFSS)
42100      14102 CONTINUE

```



```

42125      QDAAAB=1.
42130      PDAAAB=1.
42150      DAA1=QDAAAB*DAAB
42175      DAA2=DAA1/2.
42200      DBB1=PDAAAB*DAAB
42210      DBB2=DBB1/2.
42212      JFK=1
42214      IJK=1
42216      JKL=1
42218      IFL=1
42220      GQG=.5
42222      IHI=11
42225      D0 10011 I11=1,10000000
42275      67587 G0 T0 (39582,93582),IFL
42300      39582 G0 T0 (3927,3928),JFK
42325      3927 G0 T0 (98,79),JKL
42350      98 IF(ZDIS) 1398,8875,1398
42375      8875 ALPHA=1.+DAA2
42400      G0 T0 79
42425      1398 ALP2=SQRT(1.-((2.*PI/ZDIS)*(FLUAT(I11-1)/50.))**2)
42450      ALP1=SQRT(1.-((2.*PI/ZDIS)*(FLUAT(I11 ) /50.))**2)
42475      DALP21=ALP2-ALP1
42500      IF(DALP21-DAA1) 3776,3776,3876
42525      3776 ALPHA=(ALP2+ALP1)/2.
42550      DALPHA=DALP21
42575      G0 T0 3412
42600      3876 ALPHA=ALP2-DAA2
42625      DALPHA=DAA1
42650      JKL=2
42675      G0 T0 3412
42700      79 ALPHA=ALPHA-DAA1
42725      JKL=2
42750      IF(ALPHA-DAA2) 986,986,572
42775      572 DALPHA=DAA1
42800      G0 T0 3412
42825      986 ALPHA=(ALPHA+DAA2)/2.
42850      DALPHA=2.*ALPHA
42860      JFK=2
42875      M=I11+2
42900      G0 T0 3412

```

```

42925      3928 G0 T0 (9382,33444),IJK
42950      9382 IF(ZDIS) 3289,9692,3289
42975      9692 ALPHA=F+DBB2
43000      G0 T0 33444
43025      3289 ALP2=SQRT(F**2-((2.*PI/ZDIS)*(FL0AT(I11-M+1)/50.))**2)
43050      ALP1=SQRT(F**2-((2.*PI/ZDIS)*(FL0AT(I11-M+2)/50.))**2)
43075      DALP21=ALP2-ALP1
43100      IF(DALP21-DBB1) 26593,26593,39562
43125      26593 ALPHA=(ALP2+ALP1)/2.
43150      DALPHA=DALP21
43175      G0 T0 87484
43200      39562 ALPHA=ALP2-DBB2
43225      DALPHA=DBB1
43250      IJK=2
43275      G0 T0 87484
43300      33444 ALPHA=ALPHA-DBB1
43325      IJK=2
43350      IF(ALPHA-1.-DBB2) 7974,7974,418
43375      418 DALPHA=DBB1
43400      G0 T0 87484
43425      7974 ALPHA=(ALPHA+DBB2+1.)/2.
43450      DALPHA=2.*(ALPHA-1.)
43475      IFL=2
43480      MMNN =I11+2
43500      G0 T0 87484
43550      93582 CONTINUE
43555      DALPHA=(AZ(IHI+1)-AZ(IHI))/AN(IHI)
43560      ALPHA=AZ(IHI)+DALPHA*GGG+DALPHA*FL0AT(I11-MMNN+1)
43570      IF(ALPHA-16.) 31059,31059,3810
43590      31059 IF(ALPHA-AZ(IHI+1)) 32154,93933,93933
43600      93933 IHI=IHI+1
43650      MMNN=I11+2
43700      G0 T0 10011
43750      32154 IF(ALPHA-(AK0N-SMALLD)) 12345,55333,55333
43800      55333 IF(ALPHA-(AK0N+SMALLD)) 10011,10011,12345

```

```

44100      3412 B11=SQRT(1.-ALPHA**2)
44150      B1=CMPLX(0.,B11)
44200      BPP=SQRT(P**2-ALPHA**2)
44250      BP=CMPLX(0.,BPP)
44300      GO TO 54321
44400      87484 B11=SQRT(ALPHA**2-1.)
44500      B1=CMPLX(B11,0.)
44600      BPP=SQRT(P**2-ALPHA**2)
44700      BP=CMPLX(0.,BPP)
44800      GO TO 54321
44900      12345 B11=SQRT(ALPHA**2-1.)
45000      B1=CMPLX(B11,0.)
45100      BPP=SQRT(ALPHA**2-P**2)
45200      BP=CMPLX(BPP,0.)
45300      GO TO 54321
45400      54321 D1=2.*ALPHA**2-P**2
45500      DELTA=D1**2-4.*ALPHA**2*BP*B1
45600      EXPB1=CEXP(-B1*ZDIS)
45700      EXPBP=CEXP(-BP*ZDIS)
45800      ALPHA2=ALPHA**2
45810      CRZS=2.*ALPHA2*EXPBP
45820      CRZL=-D1*EXPB1
45830      CRRS=-D1*EXPBP
45840      CRZRL=2.*B1*BP*EXPB1
45900      CRRS=2.*B1*BP*EXPBP
46000      CRRZL=-D1*EXPB1
46100      CRRRS=-D1*EXPBP/ALPHA
46200      CRRRL=2.*ALPHA*EXPB1
46300      X=ALPHA*KDIS
46400      CALL BESEL0(P11)
46500      JOALPR=P11
46600      CALL BESEL1(P12)
46700      J1ALPR=P12
46800      X=ALPHA*A2
46900      CALL BESEL1(P13)
47000      J1ALA2=P13
47100      CALL BESLID(P14)
47200      J1IDA2=P14
47300      X=ALPHA*A1
47400      CALL BESEL1(P15)
47500      J1ALA1=P15
47600      CALL BESLID(P16)
47700      J1IDA1=P16

```

```

47710 48392 CONTINUE
47718     TSTZZS=
47720     1    B1*CRZZS*JOALPR*(A2*J1ALA2-A1*J1ALA1)*DALPHA/DELTA
47722     VZZS=VZZS+TSTZZS
47730     VKRS=VKRS+
47740     1    1.*CKZKS*JOALPR*(J1IDA2-J1IDA1)*DALPHA/DELTA
47750     VKRS=VKRS+
47760     1    ALPHA*CRKZS*J1ALPR*(A2*J1ALA2-A1*J1ALA1)*DALPHA/DELTA
47770     VKRS=VKRS+
47780     1    BP*CKRRS*J1ALPR*(J1IDA2-J1IDA1)*DALPHA/DELTA
47800     VZZL=VZZL+
47900     1    B1*CRZZL*JOALPR*(A2*J1ALA2-A1*J1ALA1)*DALPHA/DELTA
48000     VZRL=VZRL+
48100     1    1.*CKZKL*JOALPR*(J1IDA2-J1IDA1)*DALPHA/DELTA
48200     VKZL=VKZL+
48300     1    ALPHA*CKRZL*J1ALPR*(A2*J1ALA2-A1*J1ALA1)*DALPHA/DELTA
48400     VRKL=VRKL+
48500     1    BP*CKRRL*J1ALPR*(J1IDA2-J1IDA1)*DALPHA/DELTA
48553
48600 10011 CONTINUE
48625
48650 3810 CONTINUE
48700     AQ=2.*AK0N**2-P**2
48800     BQ=SQRT(AK0N**2-1.)
48900     CQ=SQRT(AK0N**2-F**2)
49000     AQDSH=4.*AK0N
49100     BQDSH=AK0N/BQ
49200     CQDSH=AK0N/CQ
49300     AQDDSH=4.
49400     BQDDSH=(BQ-BQDSH*AK0N)/BQ**2
49500     CQDDSH=(CQ-CQDSH*AK0N)/CQ**2
49600     EXPBQ=EXP(-BQ*ZDIS)
49700     EXPCQ=EXP(-CQ*ZDIS)
49720     EQS=2.*AK0N**2*EXPCQ
49740     EQL=-AQ*EXPBQ
49760     FQS=-AQ*EXPCQ
49780     FQL=2.*BQ*CQ*EXPBQ
49800     GQS=2.*BQ*CQ*EXPCQ
49900     GQL=-AQ*EXPBQ
50000     HQS=-AQ*EXPCQ/AK0N
50100     HQL=2.*AK0N*EXPBQ
50200     EQLD=(-AQDSH+AQ*BQDSH*ZDIS)*EXPBQ
50300     EQSD=(4.*AK0N-2.*AK0N**2*CQDSH*ZDIS)*EXPCQ
50400     FQLD=(2.*BQDSH*CQ+2.*BQ*CQDSH-2.*BQ*CQ*BQDSH*ZDIS)*EXPBQ
50500     FQSD=(-AQDSH+AQ*CQDSH*ZDIS)*EXPCQ
50600     GQLD=(-AQDSH+AQ*BQDSH*ZDIS)*EXPBQ
50700     GQSD=(2.*BQDSH*CQ+2.*BQ*CQDSH-2.*BQ*CQ*CQDSH*ZDIS)*EXPCQ
50800     HQLD=(2.-2.*AK0N*BQDSH*ZDIS)*EXPBQ
50900     HQSD=(-AQDSH+AQ*CQDSH*ZDIS+AQ/AK0N)*(EXPCQ/AK0N)
51000     DLTID=2.*AQ*AQDSH-4.*AK0N**2*(BQDSH*CQ+BQ*CQDSH)
51100     1    -8.*AK0N*BQ*CQ
51200     DLTDD=2.*AQDSH**2+2.*AQ*AQDDSH
51300     1    -16.*AK0N*(BQDSH*CQ+BQ*CQDSH)
51400     1    -4.*AK0N**2*(BQDDSH*CQ+BQ*CQDDSH)
51500     1    -8.*AK0N**2*BQDSH*CQDSH
51600     1    -8.*BQ*CQ

```

53816 SG1DL=BQDSH*EQL*JOAKR*(A2*J1AKA2-A1*J1AKA1)
 53818 1 +BQ*EQLD*JOAKR*(A2*J1AKA2-A1*J1AKA1)
 53820 1 +BQ*EQL*JOAKRD*(A2*J1AKA2-A1*J1AKA1)
 53822 1 +BQ*EQL*JOAKR*(A2*J1A2D-A1*J1A1D)
 53824 SG2DL=FQLD*JOAKR*J1AK12
 53826 1 +FQL*JOAKRD*J1AK12
 53828 1 +FQL*JOAKR*J1A12D
 53830 SG3DL=1. *GQL*J1AKR*(A2*J1AKA2-A1*J1AKA1)
 53832 1 +AK0N*GQLD*J1AKR*(A2*J1AKA2-A1*J1AKA1)
 53834 1 +AK0N*GQL*J1AKRD*(A2*J1AKA2-A1*J1AKA1)
 53836 1 +AK0N*GQL*J1AKR*(A2*J1A2D-A1*J1A1D)
 53838 SG4DL=CQDSH*HQL*J1AKR*J1AK12
 53840 1 +CQ*HQLD*J1AKR*J1AK12
 53842 1 +CQ*HQL*J1AKRD*J1AK12
 53844 1 +CQ*HQL*J1AKR*J1A12D
 53846 SG1DS=BQDSH*EQS*JOAKR*(A2*J1AKA2-A1*J1AKA1)
 53848 1 +BQ*EQSD*JOAKR*(A2*J1AKA2-A1*J1AKA1)
 53850 1 +BQ*EQS*JOAKRD*(A2*J1AKA2-A1*J1AKA1)
 53852 1 +BQ*EQS*JOAKR*(A2*J1A2D-A1*J1A1D)
 53854 SG2DS=FQSD*JOAKR*J1AK12
 53856 1 +FQS*JOAKRD*J1AK12
 53858 1 +FQS*JOAKR*J1A12D
 53860 SG3DS=1. *GQS*J1AKR*(A2*J1AKA2-A1*J1AKA1)
 53862 1 +AK0N*GQSD*J1AKR*(A2*J1AKA2-A1*J1AKA1)
 53864 1 +AK0N*GQS*J1AKRD*(A2*J1AKA2-A1*J1AKA1)
 53866 1 +AK0N*GQS*J1AKR*(A2*J1A2D-A1*J1A1D)
 54100 SG4DS=CQDSH*HQS*J1AKR*J1AK12
 54200 1 +CQ*HQSD*J1AKR*J1AK12
 54300 1 +CQ*HQS*J1AKRD*J1AK12
 54301 1 +CQ*HQS*J1AKR*J1A12D
 54302 G1DL=(SG1DL*DLTD-SG1L*DLTDD/2.)/DLTD**2
 54400 G2DL=(SG2DL*DLTD-SG2L*DLTDD/2.)/DLTD**2
 54401 G3DL=(SG3DL*DLTD-SG3L*DLTDD/2.)/DLTD**2
 54402 G4DL=(SG4DL*DLTD-SG4L*DLTDD/2.)/DLTD**2
 54500 G1DS=(SG1DS*DLTD-SG1S*DLTDD/2.)/DLTD**2
 54600 G2DS=(SG2DS*DLTD-SG2S*DLTDD/2.)/DLTD**2
 54601 G3DS=(SG3DS*DLTD-SG3S*DLTDD/2.)/DLTD**2
 54602 G4DS=(SG4DS*DLTD-SG4S*DLTDD/2.)/DLTD**2
 54700 VZZL=VZZL+2.*SMALLD*G1DL
 54701 VZZS=VZZS+2.*SMALLD*G1DS
 54702 VZRL=VZRL+2.*SMALLD*G2DL
 54703 VZRS=VZRS+2.*SMALLD*G2DS
 54800 VRZL=VRZL+2.*SMALLD*G3DL
 54848 VRZS=VRZS+2.*SMALLD*G3DS
 54850 VRRL=VRRL+2.*SMALLD*G4DL
 54852 VRRS=VRRS+2.*SMALLD*G4DS
 54854

```

51700      X=AK0N*RDIS
51800      CALL BESEL0(P111)
51900      JOAKR=P111
52000      CALL BESEL1(P112)
52100      J1AKR=P112
52200      X=AK0N*A2
52300      CALL BESEL1(P113)
52400      J1AKA2=P113
52500      CALL BESLID(P114)
52600      JIDAK2=P114
52650      CALL BESEL0(P117)
52651      JOAKA2=P117
52700      X=AK0N*A1
52800      CALL BESEL1(P115)
52900      J1AKA1=P115
53000      CALL BESLID(P116)
53100      JIDAK1=P116
53150      CALL BESEL0(P118)
53151      JOAKA1=P118
53200      JOAKRD=-RDIS*J1AKR
53300      J1AKRD=-J1AKR/AK0N+RDIS*JOAKR
53400      J1A2D=-J1AKA2/AK0N+A2*JOAKA2
53500      J1A1D=-J1AKA1/AK0N+A1*JOAKA1
53510      834 CONTINUE
53600      J1AK12=JIDAK2-JIDAK1
53700      J1A12D=AK0N*(A2**2*J1AKA2-A1**2*J1AKA1)
53705      G0 T0 29384
53710      JOAKR=2.*J1AKR/(AK0N*RDIS)
53720      JOAKRD=2.*AK0N*(-2.*J1AKR/(AK0N*RDIS))**2
53721      1      +JOAKR/(AK0N*RDIS))
53750      29384 CONTINUE
53800      SG1L=BQ*EQL*JOAKR*(A2*J1AKA2-A1*J1AKA1)
53802      SG2L=FQL*JOAKR*J1AK12
53804      SG3L=AK0N*GQL*J1AKR*(A2*J1AKA2-A1*J1AKA1)
53806      SG4L=CQ*HQL*J1AKR*J1AK12
53808      SG1S=BQ*EQS*JOAKR*(A2*J1AKA2-A1*J1AKA1)
53810      SG2S=FQS*JOAKR*J1AK12
53812      SG3S=AK0N*GQS*J1AKR*(A2*J1AKA2-A1*J1AKA1)
53814      SG4S=CQ*HQS*J1AKR*J1AK12

```

```

54856      VZZL=VZZL-PI*(0.,1.)*(SG1L/DLTD)
54858      VZZS=VZZS-PI*(0.,1.)*(SG1S/DLTD)
54860      VZKL=VZKL-PI*(0.,1.)*(SG2L/DLTD)
54862      VZKS=VZKS-PI*(0.,1.)*(SG2S/DLTD)
54864      VRZL=VRZL-PI*(0.,1.)*(SG3L/DLTD)
54866      VRZS=VRZS-PI*(0.,1.)*(SG3S/DLTD)
54868      VRRL=VRRL-PI*(0.,1.)*(SG4L/DLTD)
54900      VRKS=VRKS-PI*(0.,1.)*(SG4S/DLTD)
55150      14129  CONTINUE
55175      29129  CONTINUE
55200      VZZL=C0EZ*VZZL
55300      VZZS=C0EZ*VZZS
55400      VZKL=C0EK*VZKL
55500      VZKS=C0EK*VZKS
55600      VRZL=C0EZ*VRZL
55650      VRZS=C0EZ*VRZS
55700
55800      VRRL=C0EK*VRRL
55900      VRKS=C0EK*VRKS
55902      AVZZL=AVZZL+VZZL
55904      AVZZS=AVZZS+VZZS
55906      AVZKL=AVZKL+VZKL
55908      AVZKS=AVZKS+VZKS
55910      AVRZL=AVRZL+VRZL
55912      AVRZS=AVRZS+VRZS
55914      AVRRL=AVRRL+VRRL
55916      AVRKS=AVRKS+VRKS
55920      111  CONTINUE
55940      DPZZ=AVZZL+AVZZS
55960      DPZR=AVZKL+AVZKS
55980      DPRZ=AVRZL+AVRZS
55982      DPRR=AVRKL+AVRKS
55984      DISPZ=DPZZ+DPZR
55986      DISPR=DPRZ+DPRR
55988      DISPZM=-DPZZ+DPZR
56000      DISPRM=-DPRZ+DPRR

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```

56010      149 AVZZLA=CAbs(AVZZL)
56012      IF(AVZZLA) 150,151,150
56014      150 AVZZLP=AIMAG(AVZZL)/REAL(AVZZL)
56016      AVZZLP=ATAN(AVZZLP)
56018      151 AVZZSA=CAbs(AVZZS)
56020      IF(AVZZSA) 152,153,152
56022      152 AVZZSP=AIMAG(AVZZS)/REAL(AVZZS)
56024      AVZZSP=ATAN(AVZZSP)
56026      153 AVZRLA=CAbs(AVZRL)
56028      IF(AVZRLA) 154,155,154
56030      154 AVZRLP=AIMAG(AVZRL)/REAL(AVZRL)
56032      AVZRLP=ATAN(AVZRLP)
56034      155 AVZRSA=CAbs(AVZRS)
56036      IF(AVZRSA) 156,157,156
56038      156 AVZKSP=AIMAG(AVZRS)/REAL(AVZRS)
56040      AVZKSP=ATAN(AVZKSP)
56042      157 AVKZLA=CAbs(AVKZL)
56044      IF(AVKZLA) 158,159,158
56046      158 AVKZLP=AIMAG(AVKZL)/REAL(AVKZL)
56048      AVKZLP=ATAN(AVKZLP)
56050      159 AVKZSA=CAbs(AVKZS)
56052      IF(AVKZSA) 160,161,160
56054      160 AVKZSP=AIMAG(AVKZS)/REAL(AVKZS)
56056      AVKZSP=ATAN(AVKZSP)
56058      161 AVKRLA=CAbs(AVKRL)
56060      IF(AVKRLA) 162,163,162
56062      162 AVKRLP=AIMAG(AVKRL)/REAL(AVKRL)
56064      AVKRLP=ATAN(AVKRLP)
56066      163 AVRRSA=CAbs(AVRRS)
56068      IF(AVRRSA) 164,165,164
56070      164 AVRRSP=AIMAG(AVRRS)/REAL(AVRRS)
56072      AVRRSP=ATAN(AVRRSP)
56134      165 DSPZMA=CAbs(DISPZM)
56136      IF(DSPZMA) 166,167,166
56138      166 DSPZMP=AIMAG(DISPZM)/REAL(DISPZM)
56140      DSPZMP=ATAN(DSPZMP)
56142      167 DSPRMA=CAbs(DISPRM)
56144      IF(DSPRMA) 168,169,168

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56146      168 DSPRMP=AIMAG(DISPXM)/REAL(DISPXM)
56148      DSPRMP=ATAN(DSPRMP)
56150      169 DPZZA=CABS(DPZZ)
56152      IF(DPZZA) 170,171,170
56154      170 DPZZP=AIMAG(DPZZ)/REAL(DPZZ)
56156      DPZZP=ATAN(DPZZP)
56158      171 DPZRA=CABS(DPZR)
56160      IF(DPZRA) 172,173,172
56162      172 DPZRP=AIMAG(DPZR)/REAL(DPZR)
56164      DPZRP=ATAN(DPZRP)
56166      173 DPRZA=CABS(DPRZ)
56168      IF(DPRZA) 174,175,174
56170      174 DPRZP=AIMAG(DPRZ)/REAL(DPRZ)
56172      DPRZP=ATAN(DPRZP)
56174      175 DPRRA=CABS(DPRR)
56176      IF(DPRRA) 176,177,176
56178      176 DPRRP=AIMAG(DPRR)/REAL(DPRR)
56180      DPRRP=ATAN(DPRRP)
56200      177 DISPZA=CABS(DISPZ)
56251      IF(DISPZA) 178,179,178
56252      178 DISPZP=AIMAG(DISPZ)/REAL(DISPZ)
56253      DISPZP=ATAN(DISPZP)
56254      179 DISPRA=CABS(DISPR)
56255      IF(DISPRA) 781,782,781
56260      781 DISPRP=AIMAG(DISPR)/REAL(DISPR)
56262      DISPRP=ATAN(DISPRP)
56268      782 CONTINUE
56270      TYPE 5963,AVZZLA,AVZZLP,AVZZSA,AVZZSP
56280      TYPE 5963,AVZRLA,AVZRLP,AVZRSA,AVZRSP
56300      TYPE 5963,AVRZLA,AVRZLP,AVRZSA,AVRZSP
56301      TYPE 5963,AVRRLA,AVRRLP,AVRRSA,AVRRSP
56302      791 TYPE 5963,DPZZA,DPZZP,DPZRA,DPZRP
56303      TYPE 5963,DPRZA,DPRZP,DPRRA,DPRRP
56304      TYPE 5963,DISPZA,DISPZP,DISPRA,DISPRP
56305      TYPE 5963,DSPZMA,DSPZMP,DSPRMA,DSPRMP
57675      97531 CONTINUE
57680      13579 CONTINUE
57700      87111 CONTINUE
58075      5963 FORMAT(4E)
58100      END
58946

```

```

90000      SUBROUTINE BESELO(ZZZZ)
90100      COMMON X
90200      IF(X-3.) 101,101,102
90300      101 XX=(X/3.):**2
90400          ZZZZ=1. -2.2499997 *XX+1.2656208 *XX**2
90500          1 -0.3163866 *XX**3+0.0444479 *XX**4
90600          1 -0.0039444 *XX**5+0.00021 *XX**6
90700      GO TO 103
90800      102 XX=3./X
90900          F0=0.79788456 -0.00000077 *XX-0.00552740 *XX**2
91000          1 -0.00009512 *XX**3+0.00137237 *XX**4
91100          1 -0.00072805 *XX**5+0.00014476 *XX**6
91200          TETA0=X-0.78539816
91300          1 -0.04166397 *XX -0.00003954 *XX**2
91400          1 +0.00262573 *XX**3-0.00054125 *XX**4
91500          1 -0.00029333 *XX**5+0.00013558 *XX**6
91600          ZZZZ=(1. / SQRT(X))*F0* COS(TETA0)
91700      103 RETURN
91800      END
91900      SUBROUTINE BESEL1(ZZZ)
92000      COMMON X
92100      Y=X
92200      IF(Y-3.) 201,201,202
92300      201 YY=(Y/3.):**2
92400          ZZZ=Y*(0.5 -0.56249985 *YY+0.21093573 *YY**2
92500          1 -0.03954289 *YY**3+0.00443319 *YY**4
92600          1 -0.00031761 *YY**5+0.00001109 *YY**6)
92700      GO TO 204
92800      202 YY=3./Y
92900          F1=0.79788456 +0.00000156 *YY+0.01659667 *YY**2
93000          1 +0.00017105 *YY**3-0.00249511 *YY**4
93100          1 +0.00113653 *YY**5-0.00020033 *YY**6
93200          TETA1=Y-2.35619449
93300          1 +0.12499612 *YY+0.00005650 *YY**2
93400          1 -0.00637879 *YY**3+0.00074348 *YY**4
93500          1 +0.00079824 *YY**5-0.00029166 *YY**6
93600          ZZZ=(1. / SQRT(Y))*F1* COS(TETA1)
93700      204 RETURN
93800      END
93900      SUBROUTINE BESLID(BS)
94000      COMMON X
94100      IF(X-5.) 138,138,139
94200      138 IS=INT(2.*X)+3
94300          AS=(1./6.)*X**3
94400          BS=AS
94500          DO 5000 KK=1,IS
94600              AS=-0.25*AS*X*X*(2.*FLOAT(KK-1)+3.)
94700              1 /(FLOAT(KK)*(FLOAT(KK)+1.)*(2.*FLOAT(KK)+3.))
94800              BS=BS+AS
94900      5000 CONTINUE
95000      GO TO 193
95100      139 Q1=((( -188.1357/X+109.1142)/X-23.79333)/X+2.050931)/X
95200          Q1=((Q1-0.1730503)/X+0.7034845)/X-0.064109E-3
95300          Q2=((( -5.817517/X+2.105874)/X-0.6896196)/X+0.4952024)/X
95400          Q2=(Q2-0.187344E-2)/X+0.7979095
95500          BS=1. -SQRT(X)*(Q2*COS(X-PI/4. )-Q1*SIN(X-PI/4. ))
95600      193 RETURN
95700      END

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