Longitudinal Coupling Impedance Structure in Terms of Green's Functions

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I. Introduction

In discussing the longitudinal instability problem for a coasting beam in a conducting ring, an important consideration is the electromagnetic self coupling of a particular frequency component of the beam due to the presence of the conducting walls. This is parametrised by the longitudinal coupling impedance, which is the average voltage change, per unit amplitude of beam current at this frequency, experienced by a beam particle in one turn.

A number of calculations¹⁻³ have been made to determine the value of this impedance under specific circumstances, each calculation involving a specific assumption about the shape of the conducting surface, the location of the beam, etc. Evidently many of the gross features of the results do not depend on the details of the assumptions and some general inferences can be made as discussed by Zotter⁴ in his summary.

The purpose of this note is to point out that by using a more general formulation of the problem in terms of Green's functions instead of electric fields, it is possible to understand the origins of these general features before specifying a particular configuration and to lay a convenient framework for carrying out detailed calculations in some more complicated cases. In particular, the relationship is clarified between the impedance properties of cavities, pipes, and other components in isolation and when combined into a realistic machine.

1. E. Keil and B. Zotter, Particle Accelerators 3, 11 (1972).
2. M. Month and R. Peierls, Nucl. Instrum. Methods 137, 299 (1976).
3. E. Courant, these Proc., R. Gluckstern, these Proc.
4. B. Zotter, these Proc.

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II. Formalism

We make the usual assumption that the curvature of the ring is negligible, and take its closed nature into account by imposing periodic boundary conditions

In other words, the structure we consider is a conducting tubular surface of arbitrary shape surrounding the z axis, with the plane $z = 0$ equated to the plane $z = 2\pi R$. V denotes the interior of the structure and S its surface. Considering a component of the beam current of frequency ω , having current density

$$\vec{j}(\vec{r}) e^{-i\omega t}$$

and resulting in an electric field

$$\vec{E}(\vec{r}) e^{-i\omega t}$$

the longitudinal coupling impedance is

$$Z(\omega) = \frac{1}{I^2} \int_V \vec{j}^*(\vec{r}) \cdot \vec{E}(\vec{r}) d^3\vec{r} \quad (1)$$

I being the magnitude of the beam current at this frequency.

The usual procedure is to expand $\vec{E}$ in terms of an appropriate set of functions. The boundary conditions at the walls of the structure and at the beam impose constraints resulting in a linear set of equations for the components of $\vec{E}$. The resulting matrix equations can be solved straightforwardly, possibly requir-

ing extensive computer analysis; Z can then be calculated. Care has to be exercised near resonances when the equation becomes singular (or nearly singular if the resistivity is nonzero), and the expansion functions have to be chosen with care to get reasonable convergence.

The field $\vec{E}(\vec{r})$ is linearly related to the current density $\vec{j}(\vec{r})$:

$$\vec{E}(\vec{r}) = i\omega\mu \int_V \underline{\vec{G}}(\vec{r}, \vec{r}') \cdot \vec{j}(\vec{r}') d^3\vec{r}' \quad (2)$$

where $\underline{\vec{G}}(\vec{r}, \vec{r}')$ is the Green's function for the field,⁵ satisfying the same boundary conditions as $\vec{E}(\vec{r})$ at the wall of the machine. $\underline{\vec{G}}$ is a dyadic, since it yields a vector when dotted into a vector. If we choose an appropriate set of vector functions $\vec{\psi}(\vec{r})$, orthonormal in a region R which completely includes V ; so that

$$\int_R \vec{\psi}_i(\vec{r}) \cdot \vec{\psi}_j^*(\vec{r}) d^3\vec{r} = \delta_{ij} \quad (3)$$

$\underline{\vec{G}}$ can be expanded:

$$\underline{\vec{G}}(\vec{r}, \vec{r}') = \sum_{ij} G_{ij} \vec{\psi}_i(\vec{r}) \vec{\psi}_j^*(\vec{r}') \quad (4)$$

so that the impedance is given by

$$\left. \begin{aligned} Z(\omega) &= i\omega\mu \sum_{ij} \gamma_i \gamma_j^* G_{ij} \\ \gamma_i &= \int_V \vec{j}(\vec{r}) \cdot \vec{\psi}_i(\vec{r}) d^3\vec{r} \end{aligned} \right\} \quad (5)$$

($\vec{j}$, $\vec{E}$, γ , $\underline{\vec{G}}$ all contain ω as a parameter).

The problem of finding the matrix coefficients G_{ij} is essentially equivalent to that of finding the coefficients of the field

5. P.M. Morse and H. Feshbach "Methods of Mathematical Physics" McGraw-Hill, New York (1953): Chapters 7 and 13.

$\vec{E}$, and again results in a matrix equation. There are several advantages to this representation.

First, the coefficients G_{ij} depend only on the characteristics of the tube, not the beam, so that once a solution is obtained, the effects of varying the shape or location of the beam are easily seen.

Secondly, there exist a large body of general properties of Green's functions, which allow what would otherwise be tedious manipulations to be carried out more simply. In particular, the resonance structure appears in a very simple way.

We use two properties of the Green's functions. First of all, suppose that G^0 is the Green's function satisfying boundary conditions for a perfect conductor on a surface S_0 surrounding a volume V_0 , and G is the Green's function satisfying the same condition on a surface S surrounding a volume V where V lies within V_0

Then the Green's function G can be expressed* in terms of G^0 by a matrix equation.

*The derivation follows straightforwardly by applying the vector form of Gauss' Theorem, and will be given elsewhere. (No guarantee is made for the accuracy of the formulas in this note: its purpose is to develop the qualitative conclusions, and show the potential advantage of Green's function techniques.)

When V_0 , S_0 are replaced by V_i , S_i (V lying within V_0):

$$\left. \begin{aligned} G_{ij} &= G_{ij}^0 + \sum_{kl} G_{ik} W_{kl} G_{lj}^0 \\ W_{kl} &= \int_{\Delta V} \left\{ \nabla^2 \bar{\psi}_k^* \cdot \bar{\psi}_l - \bar{\nabla} \times \bar{\psi}_k^* \cdot \bar{\nabla} \times \bar{\psi}_l \right\} dV \end{aligned} \right\} \quad (6)$$

where ΔV is the region between S and S_0 .

Similarly, let G be the Green's function for the electric field in a region V bounded by a perfectly conducting surface S , and G' the Green's function for the same region but with a resistive surface: then

$$\left. \begin{aligned} G'_{ij} &= G_{ij} + \sum_{kl} G'_{ik} V_{kl} G_{lj} \\ V_{kl} &= \int_S \left\{ \hat{n} \times \bar{\nabla} \times \bar{\psi}_k^* \cdot \hat{n} \times \bar{\nabla} \times \bar{\psi}_l \right\} dS \\ \xi &= (1-i) \sqrt{\frac{\epsilon_0}{2\omega\sigma}} \end{aligned} \right\} \quad (7)$$

σ being the conductivity of the surface (not necessarily constant), and $\hat{n}$ being the inward normal unit vector to S .

Thus by using (6) and (7) in succession the Green's function, and hence the impedance, of an arbitrarily shaped region can be determined. The choice of function $\psi_i(\vec{r})$ and of the reference region V_0 can be made to greatly simplify the calculation. If ΔV is a small region then perturbation methods, equivalent to those discussed by E. Courant,³ can be used. Since σ is large for cases of interest, ξ is always small and a perturbative solution to (7) is always possible.

III. Resonance Structure

For any enclosed region there exist a discrete set of resonant modes. The fields corresponding to the modes form a complete

set and the Green's function can be expressed⁵ in terms of them in the form

$$G(\vec{r}|\vec{r}') = \sum_n \frac{\vec{e}_n(\vec{r})\vec{e}_n^*(\vec{r}')}{\omega_n^2 - \omega^2} \quad (8)$$

where $\vec{e}_n$ is the normalised field associated with resonant frequency ω_n . If we use these fields as expansion functions, the matrix G takes the simple form:

$$G_{nm}(\omega) = \frac{\epsilon_{nm}}{\omega_n^2 - \omega^2} \quad (9)$$

Equation (6) has the form

$$G = G^0 + G W G^0 \quad (10)$$

(We have suppressed the explicit matrix indices.) This has the formal solution

$$G = G^0 [1 - W G^0]^{-1} \quad (11)$$

G , G^0 and W all being matrix functions of ω . Resonances will occur at those frequencies ω at which

$$\det[1 - W G^0] = 0 \quad (12)$$

We are interested in the case when W is small; we write $W = \lambda \bar{W}$ and consider what happens when $\lambda \rightarrow 0$. Then writing G^0 in the form (9) we have the condition

$$\det \left[\delta_{nm} - \frac{\lambda \bar{W}_{nm}}{\omega_n^2 - \omega^2} \right] = 0 \quad (13)$$

For small λ , it is well known that

$$\det(1 - \lambda A) \approx 1 - \text{Tr} \lambda A \quad ; \quad (14)$$

thus

$$\det(1 - WG^0) \approx 1 - \sum_n \frac{\lambda}{\omega_n^2 - \omega^2} \cdot \bar{W}_{nn} \quad (15)$$

This expression passes through all real values in the neighborhood of $\omega = \omega_n^0$; and hence must vanish at some point in that neighborhood. To first order in λ , therefore

$$\omega_n^2 = \omega_n^{02} + \lambda \bar{W}_{nn}(\omega_n^0) \quad (16)$$

(The result is true as $\lambda \rightarrow 0$, provided $\bar{W}$ is a sufficiently slowly varying function of ω , even though the expansion (15) breaks down for $|\omega^2 - \omega_n^{02}| \ll \lambda$).

Now write G in the form

$$G_{nm} = \sum_k \frac{\sigma_{nk} \bar{\epsilon}_{km}^*}{\omega_n^2 - \omega_k^2} \quad (17)$$

where $\bar{\epsilon}_{nk}$ is the overlap between the perturbed modes and the original ones:

$$\bar{\epsilon}_{nk} = \int_{V_0} \vec{e}_n^0(\vec{r}) \cdot \vec{e}_k(\vec{r}) dV. \quad (18)$$

Equation (6) becomes

$$G_{nm} = \sum_n \frac{\sigma_{nm} \bar{\epsilon}_{km}^*}{\omega_n^2 - \omega_k^2} = \frac{\delta_{nm}}{\omega_n^2 - \omega_m^2} + \sum_{k \neq n} \frac{\sigma_{nk} \bar{\epsilon}_{km}^*}{(\omega_n^2 - \omega_k^2)} \frac{\lambda \bar{W}_{jk}}{(\omega_n^2 - \omega_n^{02})} \quad (19)$$

Equating the residues of the poles at $\omega^2 = \omega_k^2$, we obtain

$$\bar{\epsilon}_{km}^* = \delta_{km} + (1 - \delta_{km}) \frac{\lambda \bar{W}_{km}}{(\omega_k^{02} - \omega_m^{02})} + O(\lambda^2) \quad (20)$$

Substituting in (19) we obtain, finally

$$G_{nm}(\omega) = \frac{1}{(\omega_n^2 - \omega^2)} \left\{ \frac{\delta_{nm} + \lambda \bar{W}_{nm}}{(\omega_n^2 - \omega_n^2)} + \lambda^2 \sum_{k \neq n} \frac{1}{(\omega_n^2 - \omega_k^2)} \left[\frac{\bar{W}_{nk} \bar{W}_{km}}{\omega_n^2 - \omega_n^2} + \frac{\bar{W}_{nk}^* W_{km}}{\omega_n^2 - \omega_k^2} \right] \right\} \quad (21)$$

Hence we see that the perturbation represented by W has two effects

- (1) there is a shift in the resonant frequency
- (2) G is no longer diagonal in the original representation, the nm element coupling to all resonant frequencies, with a residue of order λ^2 for $k \neq n$.

Since we made no use of the properties of the matrix W , the same conclusions apply to Eq. (7) only in this case the frequency shift is complex.

IV. Example: perturbed smooth tube

As an example, let us consider a case discussed in a previous calculation¹²: A circular tube of radius b , widening to radius d over a distance g .

For the initial unperturbed region V_0 we choose a conducting circular cylinder of radius d . Then, using the normal modes of such a region (with periodic boundary conditions as discussed in Sec.

II) we can substitute in (5) and (9) to obtain

$$Z(\omega) = i\omega\mu \sum_a \frac{|\gamma_a^0(\omega)|^2}{\omega^2 - \omega_a^2} \quad (22)$$

where

$$\gamma_a = \frac{1}{\Sigma} \int_{V_0} \vec{j}^*(\vec{r}) \cdot \vec{e}_a(\vec{r}) d^3V \quad (23)$$

Here $\vec{e}_a(\vec{r})$, ω_a are the resonant fields and frequencies for the mode labelled by a .

If the current density $\vec{j}(\vec{r})$ is uniform with radius a , and purely axial, then using cylindrical coordinates (r, φ, Z) :

$$\vec{j}(\vec{r}) = \hat{z} \int \frac{\varphi(a-r)}{2} e^{i \frac{nZ}{R}} \quad (24)$$

We consider a single current harmonic n . The frequency associated with this current component is

$$\omega_n = \frac{\beta c n}{R} \quad (25)$$

βc being the particle velocity. This current component is orthogonal to all modes except the axially symmetric TE modes. These are labelled by two integers: $a \equiv m, \alpha$. The fields and frequencies are

$$\left. \begin{aligned} e_{m\alpha}(\vec{r}) &= N_{m\alpha} e^{\frac{inZ}{R}} \left[-i \left(\frac{md}{Rj_\alpha} \right) J_1(j_\alpha r), 0, J_0(j_\alpha r) \right] \\ N_{m\alpha} &= \frac{\chi_a}{\pi \sqrt{2R} d \omega_{m\alpha}^0 J_1(j_\alpha)} \\ \chi_\alpha &= -\frac{j_\alpha}{d} \\ J_0(j_\alpha) &= 0; \quad \alpha = 1, 2, \dots \\ \omega_{m\alpha}^2 &= m^2/R^2 + \chi_\alpha^2 \end{aligned} \right\} \quad (26)$$

Hence

$$\left. \begin{aligned} \gamma_{n\alpha}(\omega_n) &= \frac{2\pi R}{\lambda_{\alpha}^2} N_{n\alpha} \left(\frac{J_1(\chi_{\alpha} a)}{\chi_{\alpha}^2} \right) \\ \text{and} \\ Z &= i\omega L \sum_{\alpha} \left(\frac{4\pi R^2 N_{n\alpha}^2}{\lambda_{\alpha}^2 + n^2/\gamma^2 R^2} \right) \left(\frac{J_1(\chi_{\alpha} a)}{\chi_{\alpha}^2} \right)^2 \end{aligned} \right\} \quad (27)$$

Thus we see that although the impedance has the appearance in (22) of a sum of resonances, the residues all vanish. The sum over α in (27) gives the usual smooth wall capacitive impedance, though clearly this is not the most convenient representation to evaluate it!

Note that below cutoff, i.e., for $\omega < \chi_{\alpha}$, resonances can not be excited because the denominator does not vanish. Above cutoff they do not appear because the residue vanishes, and this is due to the fact that each mode field $\vec{e}_n$ involves only one harmonic in z . This follows from the uniformity of the cylinder.

If we now use the results of Sec. III we see from Eq. (21) that perturbing the boundary shape so that the cross section varies mixes in other modes. From Eq. (21) we can see that all the resonances can now be excited: that corresponding to the mode $k\beta$ giving a contribution to Z of the form

$$Z_n^{(k\beta)} \approx \frac{i\omega L}{\omega^2 - \omega_{k\beta}^2} \sum_{\alpha} \frac{|\gamma_{n\alpha}|^2 |W_{n\alpha, k\beta}|^2}{(\omega^2 - \omega_{n\alpha}^0)^2 (\omega_{n\alpha}^0 - \omega_{k\beta}^0)^2}$$

where $\omega_{n\alpha}^0$ is given by (26) and

$$\omega_{k\beta}^0 \approx \omega_{k\beta}^0 + W_{k\beta, k\beta}(\omega_{k\beta}^0)$$

The matrix W is just that given by (6) with the function $\vec{\psi}$ replaced by the mode function $\vec{e}_n$ defined in (27) and the integration region

is the shaded annular region shown in the figure.

Hence a perturbed, perfectly conducting closed ring has an impedance which exhibits a number of closely spaced resonances whose frequencies are given by

$$\omega = c \sqrt{\frac{m^2}{R^2} + \frac{j\alpha^2}{d^2}} + W_{m\alpha} \quad \begin{matrix} m = 1, 2, \dots \\ \alpha = 1, 2, \dots \end{matrix}$$

When the walls become resistive, the major effect is that each resonance frequency acquires an imaginary part. If the imaginary part of the resonant frequency is much larger than the resonance spacing then no resonant structure will be visible but the real part of the impedance will be dominated by the value of the residue at the nearest pole. This is in qualitative agreement with the results calculated previously and will be spelled out in detail elsewhere.

V. Coupling of cavities due to the tube: effects of attenuation.

In the last section, we discussed the case of very small cavities regarded as a perturbation on the tube. In this section we consider the opposite case: cavities of appreciable size, where the tube acts as the perturbation which shifts the resonant frequency of the cavity and couples different cavities together

For a single cavity, the problem is straightforward: the perturbation is treated essentially as in the previous section but now the expansion functions are those of a conducting cavity and the region ΔV needed to construct the perturbation matrix W is the shaded

part of the tube. Once again, the frequency shift can be explicitly calculated.

Now suppose that we have two identical cavities separated by a distance S

Using the same general approach as before, the unperturbed problem consists of two independent cavities: the unshaded area in the figure. The effect of the tube perturbation is to pick some particular linear combination of the independent resonances

$$\vec{e}_m(\vec{r}) \approx \theta_1(\vec{r})e_m^0(\vec{r}) + e^{ik_m s} \theta_2(\vec{r})e_m^0(\vec{r})$$

where $\theta_1(\vec{r})$, $\theta_2(\vec{r})$ vanish unless $\vec{r}$ is in the first or second cavity respectively, and k_m is the propagation constant for the m^{th} cavity mode in the Z direction. Inserting this into (5) and (9) we see that

$$Z = 2Z_1[1 + \cos k_m s]$$

where Z_1 would be the impedance of a single cavity leading to the natural conclusion that the cavities interfere constructively or destructively according to their separation. Note that this mixing of levels and thus the interference is possible only if the cavities are identical so that their resonances overlap.

The effect of multiple cavities can be taken into account in just the same way, the impedance acquiring a factor

$$\left| \sum_r e^{ik_m S_r} \right|^2$$

S_r being the distance of the r^{th} cavity from the first.

In practice, attenuation will prevent the coupling of more than a few adjacent cavities. Putting resistive losses in the walls gives the resonance frequencies an imaginary part, leading in turn to an imaginary part for k_m , and a behaviour

$$e^{-\text{Im}k_m |Z-Z'|}$$

for the Green's function, so that there is little coupling between current elements at distances large compared with $(\text{Im}k_m)^{-1}$.

VI. Conclusions

In this note we have sketched a description of coupling impedance in terms of Green's functions. The problem has been defined as a stationary one, dealing with fixed frequency components of a coasting beam. Exactly the same kind of treatment can be given to a nonstationary situation by using a time dependent Green's function

$$\vec{G}(r,t|r',t')$$

which may be of use in understanding the way instabilities grow.