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**CONSERVATION EQUATIONS FOR VAPOR-DROPLET
FLOWS INCLUDING BOUNDARY-DROPLET EFFECTS**

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Nomenclature

A	area, droplet area
A_p	projected area
b	intensive property of system
B	extensive property of system
c_d	specific heat of droplet
e	specific energy
E	energy
$f_{b,i}$	body force on vapor per unit volume
$f_{c,i}$	nonaerodynamic force on droplet per unit volume
$f_{d,i}$	drag force per unit volume
f_i	force on droplet per unit volume
$F_{c,i}$	nonaerodynamic force on droplet
$F_{d,i}$	drag force on droplet
F_i	total force on droplet
$F_{p,i}$	form drag on droplet
$F_{s,i}$	force at control surface
$F_{\tau,i}$	friction drag on droplet
h	specific enthalpy
h_{fg}	latent heat of vaporization
$\bar{h}_s$	average enthalpy over the droplet surface
i	internal energy
L_i	force on separated surface
m	mass of drop
M	mass of system
n_i	unit outward normal vector
N	number of droplets

p	pressure
p_d	pressure at droplet surface
$\bar{p}_d$	average pressure over droplet surface
$\dot{q}_d$	net heat transfer to droplets per unit volume
$\dot{q}_{d,i}$	heat transfer at droplet surface
$\dot{q}_i$	heat transfer vector
$\dot{q}_{k,i}$	heat transfer due to vapor-droplet bulk temperature difference
$\dot{q}_{s,i}$	heat transfer on separated surface
$\dot{Q}$	heat transfer rate
Re	Reynolds number
s_m	evaporation rate per unit volume
S	control surface
S_m	evaporation rate
t	time
T_d	droplet temperature
u	magnitude of droplet velocity
u_i	droplet velocity
U	speed with respect to inertial reference frame
v	magnitude of vapor velocity
v_i	vapor velocity
V	volume
w	magnitude of efflux velocity
w_i	surface efflux velocity
$\dot{W}$	work rate
$\dot{W}_s$	shaft work rate
x_i	coordinate direction
Z	condensed-phase loading
δ	deviation

ϵ	penetration progress variable
θ	void fraction
ρ	density
ρ'	bulk density
τ_a	relaxation time
$\tau_{a,ij}$	asymmetric shear stress on droplet
τ_c	time between collisions
$\tau_{d,ij}$	shear stress on droplet
τ_{ij}	shear stress tensor
$\tau_{s,ij}$	symmetric shear stress on droplet

SUBSCRIPTS

d	droplet
s	droplet surface
v	vapor

CONSERVATION EQUATIONS FOR VAPOR-DROPLET FLOWS INCLUDING BOUNDARY-DROPLET EFFECTS

Abstract

The governing equations for a flowing vapor with suspended burning, evaporating, or condensing droplets are derived in a straightforward manner using the control volume, or Reynolds transport, approach. The equations demonstrate how mass transfer at the droplet surface contributes to the momentum and energy of a single droplet, a cloud of droplets, and the vapor phase. The inclusion of boundary droplets gives rise to a droplet heat transfer term omitted in previous derivations. These equations provide a starting point for the development of viable numerical and analytic models for vapor-droplet flows.

Introduction

Equations describing the conservation of mass, momentum, and energy of a flowing vapor with suspended particles or droplets have appeared in several books,¹⁻³ papers,⁴⁻¹² and reports.^{13,14} Comparing the various systems of equations that have appeared, one finds many inconsistencies and discrepancies that lead one to question the validity of each. In particular, considerable differences are noted in the equations describing highly loaded, dispersed-phase flows and/or flows with burning or evaporating droplets. It is important that a consistent set of equations be available to model the dispersed-phase flows essential to present and future energy conversion schemes, such as wet-stream turbines, coal combustion, and reactor cooling, to mention a few. The purpose of this paper is to present a straightforward derivation of the governing conservation equations which can be confidently applied to model vapor-droplet flows.

The most significant feature of a vapor-droplet flow is the fact that the droplets do not constitute a continuum. Thus, one cannot define droplet density at a point in the flow. The only meaningful definition of droplet density is bulk density or mass of droplets per unit volume of mixture. Thus, the mixture volume that one considers must contain many droplets to define an

"average." Yet, if differential equations are to be used to express property conservation, the volume must be small enough to be considered a mathematical "point." The same argument applies to the definition of droplet velocity and temperature.

Drew¹¹ has published a very thorough, though complicated, derivation of the differential form of the governing equations for two-phase flow. He defines average properties based on a length ℓ^* and time scale t^* , which are dimensions of a space and time volume just sufficient to establish a "stationary" average; that is, the average will not change by taking even larger values of ℓ^* or t^* . He also takes an average of the averages in the same fashion to insure continuity of the space and time derivatives. He implies that the error in the predicted average properties is the sum of the orders of ℓ^* and t^* . Thus if ℓ^* or t^* are large with reference to the size of the characteristic dimension of the flow system, then differential equations of continua are inadequate to describe the motion of the dispersed phase. Drew's approach is sound but unfamiliar to most applied scientists and engineers.

One feature of highly loaded two-phase flows that is difficult to model analytically is particle-particle collisions. Some researchers have chosen to assume that the particles act like molecules and give rise to a pseudo pressure and stress tensor analogous to that of the flowing gas. It is reasonable to expect, however, that if, between collisions, aerodynamic forces have a large effect on particle motion, then particle-fluid coupling is more important than particle-particle collisions. It seems that the comparative effects can be assessed by the ratio of time between particle collisions τ_c and aerodynamic response time τ_a . Taking the most rudimentary model for particle-particle collisions and using Stokes' drag law for aerodynamic drag, one finds

$$\frac{\tau_a}{\tau_c} \approx Z \frac{Re}{10},$$

where Z is the ratio of particle mass to gas mass in a given volume and Re is the Reynolds number of the particle (or droplet) based on the particle-vapor relative velocity. For most applications, except fluidized beds and similar devices, one finds that this ratio is much less than unity, implying that aerodynamic forces dominate between collisions. The derivation presented in this paper neglects forces due to particle collisions and considers only aerodynamic forces acting on the dispersed phase.

Fundamental to the derivations presented in this paper is the transport theorem which relates the Lagrangian and Eulerian description of fluid motion. This theorem is derived in most texts on fluid mechanics. It states that the rate of change of some extensive property of a closed system (a portion of matter) is equal to the rate of change of the property in a control volume (cv) plus the net efflux of the property across the control surface (cs) at the very instant the system occupies the control volume. Mathematically this can be expressed as

$$\frac{dB}{dt} = \frac{d\left(\int_{cv} \rho b \, dV\right)}{dt} + \int_{cs} \rho b v_{\alpha} n_{\alpha} \, dA, \quad (1)$$

where B is the extensive property of the system and b the corresponding intensive property. The vector v_i is the velocity of matter at the control surface with respect to the control surface, and n_i is the unit outward normal vector. The control surface may move through space with time.

Index notation is used throughout to avoid the unwieldiness of vector notation and to express the equations in a more efficient form. Greek subscripts are used for summed indices; English subscripts are used to indicate coordinate direction.

The first section of this paper presents a derivation of the mass, momentum, and energy equations for a single evaporating, burning, or condensing droplet. These equations are then used to derive expressions for the motion of a cloud of droplets. Vapor phase equations are derived and added to the droplet cloud equations to obtain expressions for the flowing vapor-droplet mixture. Equations are derived in integral and differential form, although no attempt is made to define average properties according to the schemes proposed by Panton¹⁰ or Drew.¹¹

Lagrangian Equations for a Droplet

The transport theorem is now applied to the conservation of mass, momentum, and energy of a droplet. Consider the droplet shown in Fig. 1. The control volume moves with the droplet and the control surface is just outside of and adjacent to the droplet-vapor interface. The velocity w_i is defined as the velocity of the vapor through the control surface with respect to the control surface. The density of the liquid phase is ρ_d (assumed constant

Fig. 1. Control surface for droplet.

throughout the droplet), and the density of the vapor at the control surface is ρ_v . The vapor boundary layer is located outside the control surface.

CONTINUITY EQUATION

By definition, the mass of a closed system is invariant. The intensive property corresponding to mass (mass per unit mass) is unity. Application of the transport theorem yields

$$\begin{aligned} \frac{dM}{dt} &= 0 \\ &= \frac{d\left(\int_{cv} \rho_d dV\right)}{dt} + \int_{cs} \rho_v w_\alpha n_\alpha dA, \end{aligned} \quad (2)$$

where A is the surface area of the droplet. However, by definition, the mass of the droplet is

$$\int_{cv} \rho_d dV = m; \quad (3)$$

so one can write

$$\int_{cs} \rho_v w_\alpha n_\alpha dA = - \frac{dm}{dt}. \quad (4)$$

This equation states simply that the net efflux of mass from the droplet surface is equal to the rate of mass decrease of the droplet. If the mass flux is uniform over the droplet surface, then

$$\int_{cs} \rho_v w_\alpha n_\alpha dA = \rho_v w A = - \frac{dm}{dt} . \quad (5)$$

MOMENTUM EQUATION

Assume that the droplet is not rotating[†] and is moving at a velocity u_i with respect to an inertial reference frame. The velocity of the vapor at the surface of the droplet is given by $u_i + w_i$.^{††} Newton's second law states that the sum of the forces acting on a group of fluid particles (a closed system) is equal to the net rate of change of momentum of the system:

$$F_i = \frac{d(\text{momentum}_i)}{dt} . \quad (6)$$

Momentum is an extensive property of the system; the corresponding intensive property is the velocity with respect to an inertial reference frame. The transport theorem applied to the momentum of the droplet yields

$$F_i = \frac{d\left(\int_{cv} \rho_d u_i dV\right)}{dt} + \int_{cs} (u_i + w_i) \rho_v w_\alpha n_\alpha dA . \quad (7)$$

Because u_i is constant throughout the droplet, the first term simplifies to

$$\frac{d\left(u_i \int_{cv} \rho_d dV\right)}{dt} = u_i \frac{dm}{dt} + m \frac{du_i}{dt} . \quad (8)$$

The second term, the net momentum efflux, can be expanded and combined with the continuity equation, Eq. (4), to yield

$$\int_{cs} (u_i + w_i) \rho_v w_\alpha n_\alpha dA = -u_i \frac{dm}{dt} + \int_{cs} \rho_v w_i w_\alpha n_\alpha dA . \quad (9)$$

[†] Including rotation would be straightforward; the resulting momentum equation would be the same.

^{††} The regression or growth rate of the droplet surface, being generally much smaller than the efflux velocity, is neglected.

If the mass-flux velocity is uniform over the surface of the droplet, which is a reasonably good assumption for most applications, then

$$w_i = w n_i ,$$

and the last term in Eq. (9) reduces to zero:

$$\int_{CS} \rho_v w_i w_\alpha n_\alpha dA = \rho_v w^2 \int_{CS} n_i dA = 0 . \quad (10)$$

Finally, collecting all the terms and substituting them into the transport theorem yields

$$F_i = m \frac{du_i}{dt} . \quad (11)$$

Thus the force acting on a uniformly evaporating droplet is simply equal to the product of its instantaneous mass and acceleration.

Often in the literature, one sees Newton's law for a droplet written as

$$F_i = \frac{d(mu_i)}{dt} \quad (12)$$

and differentiated to give

$$F_i = u_i \frac{dm}{dt} + m \frac{du_i}{dt} . \quad (13)$$

This is incorrect for a droplet exchanging mass because Newton's law as expressed by Eq. (12) is valid only for a fixed quantity of mass (a closed system).

The aerodynamic force on a droplet is the result of the pressure and shear stress distribution on the droplet's surface. The net pressure force acting on the droplet is given by

$$F_{p,i} = - \int_{CS} n_i p_d dA , \quad (14)$$

where p_d is the pressure at the control surface. It is convenient to express p_d as the sum of the local average pressure of the vapor at the droplet surface and the deviation of p_d from this average pressure:

$$p_d = \bar{p}_d + \delta p . \quad (15)$$

Thus the pressure force on the droplet becomes

$$F_{p,i} = - \int_{CS} n_i \delta p dA , \quad (16)$$

which is the form-drag component of the droplet drag force. This form-drag component includes the force on the droplet due to a pressure gradient in the vapor, or, in other words, buoyancy effects.

The net shear-stress force on the droplet is given by

$$F_{\tau,i} = \int_{CS} \tau_{d,\alpha i} n_{\alpha} dA , \quad (17)$$

where τ_d is the shear-stress tensor at the droplet surface. By definition, the component $\tau_{d,ii}$ (the component normal to the droplet surface) is zero. If the droplet is moving with respect to the vapor and there is no velocity gradient in the vapor, then the shear stress is symmetric about the line in the direction of the relative velocity vector. (This conclusion is based on nonrotating droplets, but the concept can be extended to rotating droplets as well.) If, in addition, the vapor has a local velocity gradient, then an asymmetric stress distribution results. It is convenient to decompose the stress tensor into two parts, one due to the vapor-velocity gradient (asymmetric part) $\tau_{a,ij}$ and the other due to the vapor-droplet relative velocity (symmetric part) $\tau_{s,ij}$:

$$\tau_{d,ij} = \tau_{s,ij} + \tau_{a,ij} . \quad (18)$$

A droplet moving with the local gas velocity in which there is a velocity gradient will experience an aerodynamic force (Saffman lift force) due to an asymmetric pressure distribution, but none due to surface shear stresses.

Thus, one can write

$$\int_{CS} \tau_{a,\alpha i} n_{\alpha} dA = 0 , \quad (19)$$

and the shear stress contribution to the droplet drag force is

$$F_{\tau,i} = \int_{CS} \tau_{s,\alpha i} n_{\alpha} dA . \quad (20)$$

The total aerodynamic force on the droplet is

$$F_{d,i} = F_{p,i} + F_{\tau,i} . \quad (21)$$

Other forces acting on the droplet, such as Coulomb forces on a charged droplet in an electric field, are simply added to the drag force to give the total force; that is,

$$F_i = F_{d,i} + F_{c,i} , \quad (22)$$

where $F_{c,i}$ is all forces other than the aerodynamic force acting on the droplet.

ENERGY EQUATION

The first law of thermodynamics states that

$$\dot{Q}_d - \dot{W}_d = \frac{dE_d}{dt}, \quad (23)$$

where E_d is the sum of the thermal and mechanical energies. Application of the transport theorem yields

$$\frac{dE_d}{dt} = \frac{d\left(\int_{cv} e_d \rho_d dV\right)}{dt} + \int_{cs} \rho_v e_d w_\alpha n_\alpha dA, \quad (24)$$

where the energy per unit mass e_d is given by

$$e_d = i_d + \frac{U^2}{2},$$

where i_d is the internal energy per unit mass and U is the velocity magnitude with respect to an inertial reference frame.

The work done per unit time by the system at the boundary in moving the fluid across the control surface (flow work) is given by[†]

$$\dot{W}_f = \int_{cs} p_d w_\alpha n_\alpha dA. \quad (25)$$

However, expressing the pressure at the droplet surface by Eq. (15) yields

$$\dot{W}_f = \int_{cs} (\bar{p}_d + \delta p) w_\alpha n_\alpha dA \quad (26)$$

$$= \int_{cs} \bar{p}_d w_\alpha n_\alpha dA + \int_{cs} \delta p w_\alpha n_\alpha dA. \quad (27)$$

Under the assumption of a constant efflux velocity over the droplet surface, the last integral is zero. The first integral combines with the efflux term in the transport theorem to give

$$\dot{Q}_d - \dot{W}_s = \frac{d\left[\int_{cv} (i_d + U^2/2) \rho_d dV\right]}{dt} + \int_{cs} \rho_v \left(h_s + \frac{U^2}{2}\right) w_\alpha n_\alpha dA, \quad (28)$$

[†]As before, the regression rate of the droplet surface is neglected.

where $\dot{W}_s$ is the shaft power and h_s is the specific enthalpy of the vapor at the control surface. The shaft power provided by the droplet is given by

$$\dot{W}_s = -F_\alpha u_\alpha . \quad (29)$$

If the force is in the direction of the droplet velocity, then work is being done on the droplet; hence, the minus sign.

Within the droplet, U is equal to the magnitude of u_i , so one can write

$$\frac{d \left[\int_{cv} \rho_v (i_d + U^2/2) dV \right]}{dt} = \frac{d(mi_d + m u_\alpha u_\alpha / 2)}{dt} . \quad (30)$$

The specific kinetic energy of the vapor at the surface of the droplet is

$$\left(\frac{U^2}{2} \right)_{cs} = \frac{u^2}{2} + \frac{w^2}{2} + u_\alpha w_\alpha . \quad (31)$$

Assuming the efflux velocity is uniform over the droplet surface and using the continuity equation, one can write the efflux term as

$$\int_{cs} \rho_v \left(h_s + \frac{U^2}{2} \right) w_\alpha n_\alpha dA = \left(\bar{h}_s + \frac{u^2}{2} + \frac{w^2}{2} \right) \left(- \frac{dm}{dt} \right) , \quad (32)$$

where $\bar{h}_s$ is the average enthalpy over the droplet surface. Using Eq. (10), it can be shown that the efflux term associated with $u_\alpha w_\alpha$ is zero. Combining all the terms in Eq. (28) and using the momentum equation (Eq. (11)) results in

$$\dot{Q}_d = m \frac{di_d}{dt} - \left(\bar{h}_s - i_d + \frac{w^2}{2} \right) \left(\frac{dm}{dt} \right) . \quad (33)$$

The energy equation generally found in the literature for a droplet is

$$\dot{Q}_d = m c_d \frac{dT_d}{dt} - h_{fg} \frac{dm}{dt} , \quad (34)$$

where c_d is the specific heat of the droplet, T_d is the droplet temperature, and h_{fg} is the latent heat of vaporization. One notes that the two equations are essentially the same in that $w^2/2$ is generally small compared to the thermal energy and i_d is very nearly the enthalpy of the liquid phase. The energy equation derived here agrees with that derived independently by Pantou.¹⁰

Like the drag force on a droplet, the heat transfer to the droplet has to be decomposed into two parts. If the droplet is in a temperature gradient,

there can be heat transfer through the droplet and no net heat transfer to the droplet. Therefore, it is convenient to consider the local heat transfer rate to the droplet as

$$\dot{q}_{d,i} = \dot{q}_{k,i} + \delta \dot{q}_i , \quad (35)$$

where $\dot{q}_{k,i}$ is the heat transfer rate due to a temperature difference between the droplet and average gas temperature and $\delta \dot{q}_i$ is the heat transfer rate due to the local temperature gradient; thus,

$$\int_{cs} \delta \dot{q}_\alpha n_\alpha dA = 0 , \quad (36)$$

$$\dot{Q}_d = - \int_{cs} \dot{q}_{k,\alpha} n_\alpha dA , \quad (37)$$

because the net heat transfer to the droplet due to a vapor temperature gradient is zero.

Eulerian Equations for a Droplet Cloud

The mechanical and thermal properties of a moving droplet are most conveniently described using a Lagrangian approach; that is, following individual droplets with time. However, when one is concerned with a mixture of droplets and vapor, it becomes more convenient to regard the droplets as a pseudo-continuum and define average properties, such as the mass of droplets per unit volume of mixture. Thus the volume over which the averaging is performed must be large enough to yield a stationary average, according to Drew.¹¹ For purposes of analysis here, we will assume that the volumes over which integral equations apply are sufficiently large to define average droplet properties.

Void fraction is defined as the fraction of mixture volume occupied by the vapor, or

$$\theta = \frac{V_v}{V} , \quad (38)$$

where V_v is the volume of the vapor and V is the volume of the mixture. Therefore, the volume of the droplets per unit volume of mixture is

$$\frac{V_d}{V} = 1 - \theta , \quad (39)$$

and the bulk density ρ'_d is given by

$$\rho'_d = \rho_d (1 - \theta) , \quad (40)$$

where ρ_d is the material density of the droplet.

The transport theorem must be viewed in a somewhat different sense to treat motion of a group of droplets through a control volume. When treating a group of droplets, one is unable to distinguish a continuous surface that defines the interface between the system and the surroundings. The transport theorem in this case must be applied to a given identifiable group of droplets and not a fixed portion of matter.

Consider N droplets moving along a channel, as shown in Fig. 2. At time t , ΔN_{in} droplets have yet to cross the control surface into the control volume, and at time $t + \Delta t$, ΔN_{out} droplets have left the control volume. Assume that each droplet has associated with it a property b , such as mass, so that the net change in property b of all the droplets in time Δt is

$$\Delta(Nb)_{\Delta t} = (N_{cv} b)_{t+\Delta t} - (N_{cv} b)_t + (\Delta N_{out} b_{out}) - (\Delta N_{in} b_{in}) , \quad (41)$$

where N_{cv} is the number of droplets in the control volume. When the limit $\Delta t \rightarrow 0$ is taken, the rate of change of property Nb with time is

$$\frac{d(Nb)}{dt} = \frac{d(Nb)_{cv}}{dt} + (\dot{N}b)_{out} - (\dot{N}b)_{in} . \quad (42)$$

This equation is valid even if the droplets disappear (through complete evaporation, for example) because they are assumed to still exist but have a zero value for property b .

Fig. 2. Droplet motion with respect to control volume (cv). (a) Droplets at time t . (b) Droplets at time $t + \Delta t$.

CONTINUITY EQUATION

To derive the continuity equation, one takes b as the mass of a droplet. The $\dot{Nm}$ terms in the transport equation represent the mass efflux of droplets across the control surface. Defining u_i as the velocity of the droplets with respect to the control surface, one can write

$$(\dot{Nm})_{\text{out}} - (\dot{Nm})_{\text{in}} = \int_{\text{cs}} \rho'_d u_\alpha n_\alpha dS, \quad (43)$$

where n_i is the unit outward normal vector. The $d(\dot{Nm})/dt$ term of the transport equation represents the rate of change of mass of the droplets due to condensation, evaporation, or burning as they pass through the control volume and can be regarded as a mass source (or sink) term. Thus,

$$\begin{aligned} \frac{d(\dot{Nm})}{dt} &= -\dot{S}_m \\ &= -s_m V, \end{aligned} \quad (44)$$

where s_m is the rate of vapor mass addition per unit volume of mixture due to evaporation. If evaporation is occurring, s_m is positive; if condensation is occurring, s_m is negative. Finally, the continuity equation is

$$-\dot{S}_m = \frac{d\left(\int_{\text{cv}} \rho'_d dV\right)}{dt} + \int_{\text{cs}} \rho'_d u_\alpha n_\alpha dS, \quad (45)$$

which states that the net efflux of droplet mass from the control volume plus the rate of change of mass within the control volume is equal to the rate of mass change of the droplets due to condensation or evaporation. The corresponding differential equation, assuming once again that average properties are definable, is

$$\frac{\partial \rho'_d}{\partial t} + \frac{\partial(\rho'_d u_\alpha)}{\partial x_\alpha} = -s_m. \quad (46)$$

MOMENTUM EQUATION

To derive the momentum equation, one takes b equal to the momentum of a droplet; that is, $b = mu_i$, where u_i is the droplet velocity with respect to an inertial reference frame. In this report it is assumed that all droplets have

locally the same velocity, although the analysis is easily extendable to a distribution of droplet velocities. The momentum equation for a cloud of droplets becomes

$$\frac{d(Nmu_i)}{dt} = \frac{d\left(\int_{cv} \rho'_d u_i dV\right)}{dt} + \int_{cs} \rho'_d u_i u_\alpha n_\alpha dS, \quad (47)$$

where the term on the left-hand side of the equation is the rate of change of momentum of the droplets (Lagrangian approach) and the first term on the right-hand side is the rate of momentum change in the control volume (Eulerian approach).

Special consideration must be given here to the droplets that are intersected by the control surface—a feature generally omitted in derivations of two-phase flow equations. Consider the droplet in Fig. 3 which is penetrating the control surface S . Let ϵ be a progress variable describing the degree of penetration ($\epsilon = 0^+$, penetration commencing; $\epsilon = 1$, penetration complete). If, conceptually, the droplets were physically separated along the control surface, a force would have to be added to the separated surface area A_p to replace the effect of the separated portion (classical free-body approach).

Fig. 3. Droplet penetrating control surface.

Assuming uniform mass flux from the droplet surface, the equation of motion for the portion inside the control surface is

$$m(\epsilon) \frac{du_i}{dt} = (\rho_v w^2)_{s_i} n_i A_p(\epsilon) + F_i(\epsilon) + L_i(\epsilon) , \quad (48)$$

where n_i is the unit outward normal vector to the control surface and L_i is the force on the surface of separation. Correspondingly, for a droplet penetrated $1 - \epsilon$, one can write

$$m(1 - \epsilon) \frac{du_i}{dt} = (\rho_v w^2)_{s_i} n_i A_p(1 - \epsilon) + F_i(1 - \epsilon) + L_i(1 - \epsilon) . \quad (49)$$

Taking the separated surface area A_p and the force L_i as symmetric, that is,

$$A_p(1 - \epsilon) = A_p(\epsilon) , \quad (50a)$$

$$L_i(1 - \epsilon) = L_i(\epsilon) , \quad (50b)$$

and adding Eqs. (48) and (49) yields

$$m \frac{du_i}{dt} = F_i + 2(\rho_v w^2)_{s_i} n_i A_p(\epsilon) + 2L_i(\epsilon) . \quad (51)$$

However, referring back to the equation of motion for an individual droplet [Eq. (11)], one concludes

$$n_i (\rho_v w^2)_{s_i} A_p(\epsilon) + L_i(\epsilon) = 0 . \quad (52)$$

Therefore, the equation of motion for the boundary droplet is

$$m(\epsilon) \frac{du_i}{dt} = F_i(\epsilon) . \quad (53)$$

Summing the momenta for all the droplets partially and completely inside the control volume, one has

$$\sum_{cv} F_i = Nm \frac{du_i}{dt} . \quad (54)$$

The integral form of the momentum equation can now be written as

$$-S_m u_i + Nm \frac{du_i}{dt} = \frac{d \left(\int_{cv} \rho'_d u_i dV \right)}{dt} + \int_{cs} \rho'_d u_i u_\alpha n_\alpha dS , \quad (55)$$

and the corresponding differential equation is

$$-s_m u_i + f_i = \frac{\partial(\rho'_d u_i)}{\partial t} + \frac{\partial(\rho'_d u_\alpha u_i)}{\partial x_\alpha} , \quad (56)$$

where f_i is the force per unit volume acting on the droplets. This force is the sum of aerodynamic and other forces. [See Eq. (22).] The aerodynamic force includes the force due to the pressure gradient, steady-state drag (Stokes' law or equivalent), virtual-mass effects, and the Basset force. Subtracting the continuity equation [Eq. (46)] results in

$$f_i = \rho'_d \frac{\partial u_i}{\partial t} + \rho'_d u_\alpha \frac{\partial u_i}{\partial x_\alpha}, \quad (57)$$

which simply states that the acceleration of the droplets per unit volume is equal to the force per unit volume acting on the droplets. If droplet collisions are incorporated, pseudopressure and viscous stress terms must be incorporated into Eq. (57). Ben Zarty¹⁵ has recently discussed the significance of such terms for vapor-particle flow in circular and rectangular ducts.

ENERGY EQUATION

The total energy of the droplets is the sum of the mechanical and thermal energies. Thus, the energy of a group of droplets is $Nm(i_d + u^2/2)$, where u is the speed of the droplets with respect to an inertial reference frame. As it was in the momentum equation, it is assumed that all droplets have locally the same velocity and internal energy. However, the analysis can be easily extended to a distribution of velocities and internal energies. Substitution of droplet energy into the transport equation yields

$$\frac{d[Nm(i_d + u^2/2)]}{dt} = \frac{d\left[\int_{cv} \rho'_d (i_d + u^2/2) dV\right]}{dt} + \int_{cs} \left(i_d + \frac{u^2}{2}\right) u_\alpha n_\alpha dS. \quad (58)$$

As with the momentum equation, special consideration must be given to the "boundary" droplets. Refer once more to Fig. 3. If the droplet is severed along the control surface, the effect of the separated portion must be replaced by a heat transfer rate $\dot{q}_s(\epsilon)A_p(\epsilon)$, where $\dot{q}_s(\epsilon)$ is the heat transfer rate per unit area. With the use of Eq. (35) for the variation of heat transfer rate over the droplet surface, the energy equation for the portion of the droplet inside the control volume is

$$\begin{aligned} \frac{d[m(\epsilon)i_d(\epsilon)]}{dt} + \int_0^\epsilon (\rho_v w)_s \left(h_s + \frac{w^2}{2}\right) dA \\ = - \int_0^\epsilon \dot{q}_{k,\alpha} n_\alpha dA - \int_0^\epsilon \delta \dot{q}_\alpha n_\alpha dA + \dot{q}_s(\epsilon)A_p(\epsilon). \end{aligned} \quad (59)$$

The enthalpy varies over the droplet surface because of pressure and temperature variations, so it is convenient to write

$$h_s = \bar{h}_s + \delta h_s, \quad (60)$$

where $\bar{h}_s$ is the average enthalpy and δh_s the deviation therefrom. Equation (59) can now be rewritten as

$$\begin{aligned} \frac{d[m(\epsilon)i_d(\epsilon)]}{dt} + (\rho_v w)_s \left(\bar{h}_s + \frac{w^2}{2} \right) A(\epsilon) + (\rho_v w)_s \int_0^\epsilon \delta h_s dA \\ = - \int_0^\epsilon \dot{q}_{k,\alpha} n_\alpha dA - \int_0^\epsilon \delta \dot{q}_\alpha n_\alpha dA + \dot{q}_s(\epsilon) A_p(\epsilon). \end{aligned} \quad (61)$$

Taking $\dot{q}_s(\epsilon)$ equal to $\dot{q}_s(1 - \epsilon)$ along with δq_i and δh_s as asymmetric functions and adding two boundary droplets penetrated ϵ and $1 - \epsilon$ gives

$$\begin{aligned} \frac{d(mi_d)}{dt} - \left(\bar{h}_s + \frac{w^2}{2} \right) \frac{dm}{dt} + 2(\rho_v w)_s \int_0^\epsilon \delta h_s dA \\ = \dot{Q}_d - 2 \int_0^\epsilon \delta \dot{q}_\alpha n_\alpha dA + 2\dot{q}_s(\epsilon) A_p(\epsilon). \end{aligned} \quad (62)$$

Comparing Eq. (62) with that for a single droplet, Eq. (33), one concludes

$$\dot{q}_s(\epsilon) A_p(\epsilon) = (\rho_v w)_s \int_0^\epsilon \delta h_s dA + \int_0^\epsilon \delta q_\alpha n_\alpha dA, \quad (63)$$

so the energy equation for the boundary droplet is

$$\frac{d[m(\epsilon)i_d(\epsilon)]}{dt} + \left(\bar{h}_s + \frac{w^2}{2} \right) (\rho_v w)_s A(\epsilon) = \dot{Q}_d(\epsilon). \quad (64)$$

Summing the energy for all the droplets inside (completely and partially) the control volume gives

$$\frac{d(Nmi_d)}{dt} = V\dot{q}_d - \left(\bar{h}_s + \frac{w^2}{2} \right) S_m, \quad (65)$$

where $\dot{q}_d$ is the heat transfer rate to the droplets per unit volume. Substituting Eq. (65) into the transport equation and expressing the resulting equation in differential form, assuming as always that the averages are definable and the derivatives continuous, results in

$$\begin{aligned} \frac{d[(Nm/V)(u^2/2)]}{dt} - s_m \left(\bar{h}_s + \frac{w^2}{2} \right) + \dot{q}_d \\ = \frac{\partial[\rho'_d(i_d + u^2/2)]}{\partial t} + \frac{\partial[\rho'_d u_\alpha (i_d + u^2/2)]}{\partial x_\alpha}. \end{aligned} \quad (66)$$

Subtracting out the continuity equation [Eq. (46)] and the momentum equation [Eq. (56)] leaves the thermal energy equation in the form

$$\dot{q}_d - s_m \left(\bar{h}_s + \frac{w^2}{2} \right) - i_d = \rho'_d \frac{\partial i_d}{\partial t} + \rho'_d u_\alpha \frac{\partial i_d}{\partial x_\alpha} . \quad (67)$$

One notes that the rate of energy change of the droplets per unit volume is equal to the rate of heat transfer to the droplets minus the energy convected away because of evaporation. It is important to note here that the enthalpy of the droplet matter does not appear in the energy equation because the droplets do not perform flow work; that is, one droplet is incapable of doing flow work on another. Several derivations falsely include a flow work for the droplets—a fallacy arising from regarding the discrete droplets as a continuum. Of course, if droplet collisions are important, a pressure could be associated with the droplets, and a flow work term could arise.

The regression rate of the droplet surface has also been neglected so no term appears relating to the work done because of change in droplet volume.

It is important to note that no conservation principles were applied in the derivation of the above equations. The Lagrangian description of droplet property variations was simply converted to an Eulerian description for a cloud of droplets and written in the form of differential equations.

Eulerian Equations for Vapor Phase

Consider a group of droplets enclosed by an arbitrary surface S , which may or may not intersect some droplets, as shown in Fig. 4. The portion of S which intersects the boundary droplets is identified as δS . The surface surrounding and adjacent to every interior droplet is S_p , and the portion of surface surrounding the boundary droplets interior to S is defined as δS_p . Thus the complete control surface for the vapor phase is $S + S_p - \delta S + \delta S_p$. To shorten the notation, the portion of the control surface through the vapor, $S - \delta S$, will be referred to as S' and that adjacent to the droplet surface, $S_p + \delta S_p$, as S'_p .

Void fraction can be used to relate the mass of the vapor per unit volume of mixture to the actual vapor density:

$$\rho'_v = \rho_v \theta , \quad (68)$$

where ρ'_v is the vapor density of the mixture and ρ_v is the actual vapor density, a thermodynamic property.

Fig. 4. Control surface for vapor phase.

CONTINUITY EQUATION

The conservation of mass given by the transport theorem is

$$\frac{d\left(\int_{cv} \rho'_v dV\right)}{dt} + \int_{cs} \rho_v v_\alpha n_\alpha dS = 0, \quad (69)$$

where v_i is the vapor velocity relative to the control surface. For the control surface of the vapor, as depicted in Fig. 4, one has

$$\frac{d\left(\int_V \rho'_v dV\right)}{dt} + \int_{S'} \rho_v v_\alpha n_\alpha dS + \int_{S'_p} \rho_v w_\alpha n_\alpha dS = 0, \quad (70)$$

where n_i is the unit outward normal vector, which on the surface S'_p is directed toward the interior of the droplets, and V is the volume enclosed by the surface S . By definition of vapor mixture density one can write

$$\int_{S'} \rho_v v_\alpha n_\alpha dS = \int_S \rho'_v v_\alpha n_\alpha dS. \quad (71)$$

Also, from the continuity equation for the droplets, one realizes that

$$\int_{S'_p} \rho_v w_\alpha n_\alpha dS = -S_m, \quad (72)$$

where S_m is the rate of mass flow from the droplets. If the droplets are evaporating, S_m is a positive number.

The final form of the integral equation is

$$\frac{d\left(\int_V \rho'_v dV\right)}{dt} + \int_S \rho'_v v_\alpha n_\alpha dS = S_m, \quad (73)$$

which simply expresses the fact that the rate of increase of vapor mass inside V plus the net efflux of vapor through S is equal to the rate of vapor mass addition by the droplets. The corresponding differential equation is

$$\frac{\partial \rho'_v}{\partial t} + \frac{\partial (\rho'_v v_\alpha)}{\partial x_\alpha} = s_m, \quad (74)$$

where, as before, it is assumed that averages can be defined and derivatives are continuous.

MOMENTUM EQUATION

Define u_i as the local velocity of the droplets with respect to an inertial reference frame. Thus the velocity of the vapor at the droplet surface with respect to an inertial reference frame is $u_i + w_i$. Applying the transport equation for the momentum of the vapor phase yields

$$F_i = \frac{d\left(\int_V \rho'_v v_i dV\right)}{dt} + \int_{S'} \rho'_v v_i v_\alpha n_\alpha dS + \int_{S'_p} \rho'_v (w_i + u_i) w_\alpha n_\alpha dS, \quad (75)$$

where it is assumed that the surface S is an inertial reference frame and v_i is measured with respect to S . Because u_i is constant over the surface of the droplet, one can write from the droplet continuity equation that

$$\int_{S'_p} u_i \rho'_v w_\alpha n_\alpha dS = -u_i S_m. \quad (76)$$

Also, if the mass flux is uniform over the surface of a droplet, one has

$$\int_{S'_p} \rho'_v w_i w_\alpha n_\alpha dS = \int_{\delta S_p} \rho'_v w_i^2 n_i dS, \quad (77)$$

because the integral over each interior droplet S_p is zero and w_i has the opposite sense to n_i . Using the definition of void fraction, the above integral can be written as

$$\int_{\delta S_p} \rho'_v w_i^2 n_i dS = \int_S (\rho'_v w^2)_s (1 - \theta) n_i dS. \quad (78)$$

The forces acting on the vapor are the pressure and shear forces on the surface S' and the aerodynamic forces on the surface S'_p . On the surface elements S'_p one can write

$$\begin{aligned} \int_{S'_p} -p_d n_i dS + \int_{S'_p} \tau_{d,\alpha i} n_\alpha dS \\ = -\int_{S'_p} \bar{p}_d n_i dS + \int_{S'_p} \tau_{a,\alpha i} n_\alpha dS - \int_{S'_p} \delta \bar{p} n_i dS + \int_{S'_p} \tau_{s,\alpha i} n_\alpha dS . \end{aligned} \quad (79)$$

The last two terms are recognized as the negative sum of the aerodynamic drag force on the droplets (because n_i is the unit inward normal vector to the droplet). The first two terms can be written as

$$-\int_{S'_p} \bar{p}_d n_i dS + \int_{S'_p} \tau_{a,\alpha i} n_\alpha dS = -\int_S (1 - \theta) \bar{p}_d n_i dS + \int_S (1 - \theta) \tau_{a,\alpha i} n_\alpha dS \quad (80)$$

because the integrals over the interior droplets are zero. Finally, summing all the integrals for the forces acting over the control surface gives

$$\begin{aligned} F_{s,i} &= -\int_S \theta p n_i dS + \int_S \theta \tau_{\alpha i} n_\alpha dS + \int_S (1 - \theta) \tau_{a,\alpha i} n_\alpha dS \\ &\quad - \int_S (1 - \theta) \bar{p}_d n_i dS - F_{d,i} \\ &= -\int_S p n_i dS + \int_S (1 - \theta) (p - \bar{p}_d) n_i dS + \int_{S'} \theta \tau_{\alpha i} n_\alpha dS \\ &\quad + \int_S (1 - \theta) \tau_{a,\alpha i} n_\alpha dS - F_{d,i} , \end{aligned} \quad (81)$$

where $F_{d,i}$ is the cumulative aerodynamic drag force on the droplets in the volume V , τ_{ij} is the shear stress on the vapor in the j direction due to the surface represented by the normal n_i , p is the pressure in the vapor, $\bar{p}_d$ is the average pressure of the droplet surface, and $\tau_{a,ij}$ is the asymmetric shear stress on the droplet due to a vapor velocity gradient. The void fraction represents the fraction of S that passes through the vapor phase.

The pressure difference $p - p_d$ is the pressure change across the droplet boundary layer. Assuming a thin boundary layer consistent with conventional

boundary layer assumptions, one concludes by application of the momentum equation across the boundary layer that

$$p - \bar{p}_d = (\rho_v w^2)_s . \quad (82)$$

This term will cancel the boundary droplet momentum efflux term [Eq. (77)].

Finally, the integral form of the vapor phase momentum equation can be written as

$$\begin{aligned} -F_{d,i} + \int_S [-pn_i + \theta \tau_{\alpha i} n_\alpha + (1 - \theta) \tau_{a,\alpha i} n_\alpha] dS + \int_V f_{b,i} \rho'_v dV \\ = \frac{d \left(\int_V \rho'_v v_i dV \right)}{dt} + \int_S \rho'_v v_i v_\alpha n_\alpha dS - u_i s_m , \end{aligned} \quad (83)$$

where $f_{b,i}$ is the body force vector per unit mass acting on the vapor.

The corresponding differential form is

$$\begin{aligned} \frac{\partial(\rho'_v v_i)}{\partial t} + \frac{\partial(\rho'_v v_i v_\alpha)}{\partial x_\alpha} \\ = - \frac{\partial p}{\partial x_i} + \frac{\partial[(1 - \theta) \tau_{a,\alpha i}]}{\partial x_\alpha} + \frac{\partial(\theta \tau_{\alpha i})}{\partial x_\alpha} - f_{d,i} + \rho'_v f_{b,i} + u_i s_m , \end{aligned} \quad (84)$$

where $f_{d,i}$ and s_m are, respectively, the drag force on and the vapor mass addition by the droplets per unit volume. The term relating to the shear stress on the droplet due to a vapor velocity gradient is generally overlooked or not readily apparent in other derivations.

Subtracting the continuity equation from the momentum equation results in the following differential equation:

$$\begin{aligned} \rho'_v \frac{\partial v_i}{\partial t} + \rho'_v v_\alpha \frac{\partial v_i}{\partial x_\alpha} \\ = - \frac{\partial p}{\partial x_i} + \frac{\partial(\theta \tau_{\alpha i})}{\partial x_\alpha} + \frac{\partial[(1 - \theta) \tau_{a,\alpha i}]}{\partial x_\alpha} - f_{d,i} + \rho'_v f_{b,i} + (u_i - v_i) s_m , \end{aligned} \quad (85)$$

in which one notes that the momentum addition term due to evaporation or condensation is proportional to the velocity difference between the vapor and the droplet.

ENERGY EQUATION

The energy of the vapor phase consists of thermal and mechanical energies. The energy per unit mass is $i_v + v^2/2$, where v is the vapor speed with

respect to an inertial reference frame. Applying the first law of thermodynamics to the system, one has

$$\dot{Q} - \dot{W} = \frac{d(Me)}{dt} . \quad (86)$$

Using the transport theorem for the rate of change of energy yields

$$\dot{Q} - \dot{W} = \frac{d\left(\int_V \rho'_v e \, dV\right)}{dt} + \int_{S'} \rho'_v e v_\alpha n_\alpha \, dS + \int_{S'_p} \rho'_v e w_\alpha n_\alpha \, dS . \quad (87)$$

The shaft work-rate term involves work that is done against pressure and shear stresses on the boundaries S' and S'_p :

$$\begin{aligned} \dot{W}_s = \int_{S'} p v_\alpha n_\alpha \, dA - \int_{S'} \tau_{\alpha\beta} v_\alpha n_\beta \, dS + \int_{S'_p} p(w_\alpha + u_\alpha) n_\alpha \, dS \\ - \int_{S'_p} \tau_{\alpha\beta} (w_\alpha + u_\alpha) n_\beta \, dS . \end{aligned} \quad (88)$$

The integrals over the droplet surface require special consideration. The integral of the pressure-velocity product can be written as

$$\int_{S'_p} p(w_\alpha + u_\alpha) n_\alpha \, dS = \int_{S_p} p_d (w_\alpha + u_\alpha) n_\alpha \, dS + \int_{\delta S_p} p_d (w_\alpha + u_\alpha) n_\alpha \, dS . \quad (89)$$

Considering the integral over S_p and substituting Eq. (15) for p_d , one has

$$\int_{S_p} p_d (w_\alpha + u_\alpha) n_\alpha \, dS = \int_{S_p} \bar{p}_d (w_\alpha + u_\alpha) n_\alpha \, dS + \int_{S_p} \delta p (w_\alpha + u_\alpha) n_\alpha \, dS . \quad (90)$$

Because u_i is constant over a droplet surface and the magnitude of w_i is assumed constant, the above integral reduces to

$$\int_{S_p} p_d (w_\alpha + u_\alpha) n_\alpha \, dS = u_\alpha \int_{S_p} \delta p n_\alpha \, dS + \int_{S_p} \bar{p}_d w_\alpha n_\alpha \, dS . \quad (91)$$

The integral over δS_p can be written out in the form

$$\begin{aligned} \int_{\delta S_p} p_d (w_\alpha + u_\alpha) n_\alpha \, dS = u_\alpha \int_{\delta S_p} \delta p n_\alpha \, dS + \int_{\delta S_p} \bar{p}_d w_\alpha n_\alpha \, dS \\ + \int_{\delta S_p} \bar{p}_d u_\alpha n_\alpha \, dS + \int_{\delta S_p} \delta p w_\alpha n_\alpha \, dS . \end{aligned} \quad (92)$$

By definition of void fraction, the third term on the right-hand side can be rewritten as

$$\int_{\delta S_p} \bar{p}_d u_\alpha n_\alpha dS = \int_S (1 - \theta) u_\alpha \bar{p}_d n_\alpha dS . \quad (93)$$

Finally, the last integral can be reformulated to give

$$\int_{\delta S_p} \delta p w_\alpha n_\alpha dS = \int_{\delta S_p} \frac{\delta p}{\rho_v} \rho_v w_\alpha n_\alpha dS , \quad (94)$$

which will be combined with the energy efflux at the boundary-droplet surfaces to yield the enthalpy.

The integral of the shear stress-velocity term over the droplet surface reduces to

$$\int_{S'_p} \tau_{\alpha\beta} (w_\alpha + u_\alpha) n_\beta dS = \int_{S'_p} \tau_{d,\alpha\beta} u_\alpha n_\beta dS \quad (95)$$

because, by definition, the shear stress is normal to the velocity efflux vector. Expressing the shear stress at the droplet surface in terms of the symmetric and asymmetric parts [Eq. (18)], one concludes

$$\int_{S'_p} \tau_{\alpha\beta} (w_\alpha + u_\alpha) n_\beta dS = u_\alpha \int_{S'_p} \tau_{s,\alpha\beta} n_\beta dS + \int_{\delta S_p} u_\alpha \tau_{a,\alpha\beta} n_\beta dS . \quad (96)$$

Finally, the sum over all the droplet surfaces of the pressure-velocity and shear stress-velocity integrals can be rewritten as

$$\begin{aligned} \int_{S'_p} [p(w_\alpha + u_\alpha) - \tau_{\alpha\beta} (w_\beta + u_\beta)] n_\alpha dS \\ = u_\alpha \int_{S'_p} \delta p n_\alpha dS - u_\alpha \int_{S'_p} \tau_{s,\alpha\beta} n_\beta dS \\ + \int_{S'_p} \left(\frac{\bar{p}_d}{\rho_v} \right) \rho_v w_\alpha n_\alpha dS + \int_S (1 - \theta) \bar{p}_d u_\alpha n_\alpha dS \\ - \int_S (1 - \theta) u_\alpha \tau_{a,\alpha\beta} n_\beta dS + \int_{\delta S_p} \frac{\delta p}{\rho_v} \rho_v w_\alpha n_\alpha dS . \end{aligned} \quad (97)$$

The first two integrals on the right-hand side can be reduced to

$$u_\alpha \left(\int_{S'_p} \delta p n_\alpha dS - \int_{S'_p} \tau_{s,\alpha\beta} n_\beta dS \right) = u_\alpha F_{d,\alpha} \quad (98)$$

because n_i is the unit normal vector toward the droplet. The third and last integrals combine with the droplet energy efflux term in Eq. (87) to yield the enthalpy and mechanical energy efflux at the droplet surface:

$$\begin{aligned} & \int_{S'_p} \rho_v e w_\alpha n_\alpha dS + \int_{S'_p} \frac{\bar{p}_d}{\rho_v} \rho_v w_\alpha n_\alpha dS + \int_{\delta S_p} \frac{\delta p}{\rho_v} \rho_v w_\alpha n_\alpha dS \\ &= -S_m \left(\bar{h}_s + \frac{u^2}{2} + \frac{w^2}{2} \right) + \int_{\delta S_p} \rho_v w_\alpha \delta h_s n_\alpha dS + \int_{\delta S_p} (\rho_v w^2)_s u_\alpha n_\alpha dS . \end{aligned} \quad (99)$$

The work due to the body force on the vapor is simply

$$\dot{W}_b = - \int_V v_\alpha \rho'_v f_{b,\alpha} dV , \quad (100)$$

where $f_{b,i}$ is the body force on the vapor per unit mass of vapor.

The heat transfer term must also be decomposed into integrals over the vapor and droplet surfaces:

$$\dot{Q} = - \int_{S'} \dot{q}_\alpha n_\alpha dS - \int_{S'_p} \dot{q}_\alpha n_\alpha dS . \quad (101)$$

The heat transfer across the droplet surface can be rewritten as

$$\begin{aligned} \int_{S'_p} \dot{q}_\alpha n_\alpha dS &= \int_{S'_p} \dot{q}_{d,\alpha} n_\alpha dS \\ &= \int_{S'_p} \dot{q}_{k,\alpha} n_\alpha dS + \int_{\delta S_p} \delta q_\alpha n_\alpha dS . \end{aligned} \quad (102)$$

The first integral on the right-hand side is the heat transfer rate to the droplets resulting from a bulk temperature difference between the vapor and droplets:

$$\int_{S'_p} \dot{q}_{k,\alpha} n_\alpha dS = \dot{q}_d V . \quad (103)$$

Finally, substituting the expressions for heat transfer [Eq. (101)], work due to body forces [Eq. (100)], and work due to surface forces [Eq. (97)]

into the energy equation [Eq. (87)] and using Eqs. (98), (99), and (103) to identify certain terms gives the following integral form:

$$\begin{aligned}
& \frac{d}{dt} \left(\int_V \rho'_v (i_v + v^2/2) dV \right) + \int_S \rho'_v \left(h_v + \frac{v^2}{2} \right) v_\alpha n_\alpha dS \\
& = - \int_S \theta \dot{q}_\alpha n_\alpha dS - v \dot{q}_d - \int_{\delta S_p} \delta \dot{q}_\alpha n_\alpha dS + S_m \left(\bar{h}_s + \frac{u^2}{2} + \frac{w^2}{2} \right) \\
& - \int_{\delta S_p} (\rho_v w_\alpha) \delta h_s n_\alpha dS - \int_S (1 - \theta) (\rho_v w^2)_s u_\alpha n_\alpha dS \\
& - u_\alpha F_{d,\alpha} - \int_S (1 - \theta) \bar{p}_d u_\alpha n_\alpha dS + \int_S (1 - \theta) u_\alpha \tau_{a,\alpha\beta} n_\beta dS \\
& + \int_S \theta \tau_{\alpha\beta} v_\alpha n_\beta dS + \int_V v_\alpha \rho'_v f_{b,\alpha} dV . \quad (104)
\end{aligned}$$

Referring to Eq. (63), one sees that the second and fourth terms on the right-hand side of Eq. (104) can be combined to yield the heat transfer on the separated surface:

$$\int_{\delta S_p} \delta \dot{q}_\alpha n_\alpha dS + (\rho_v w)_s \int_{\delta S_p} \delta h_s dS = \int_S (1 - \theta) \dot{q}_{s,\alpha} n_\alpha dS . \quad (105)$$

Also, the sixth and eighth terms can be combined with the use of Eq. (82). Incorporating these combinations of terms, the integral form of the energy equation reduces to

$$\begin{aligned}
& \frac{d}{dt} \left[\int_V \rho'_v (i_v + v^2/2) dV \right] + \int_S \rho'_v \left(h_v + \frac{v^2}{2} \right) v_\alpha n_\alpha dS \\
& = - \int_S \theta \dot{q}_\alpha n_\alpha dS - v \dot{q}_d - \int_S (1 - \theta) \dot{q}_{s,\alpha} n_\alpha dS + S_m \left(\bar{h}_s + \frac{u^2}{2} + \frac{w^2}{2} \right) \\
& - \int_S (1 - \theta) p u_\alpha n_\alpha dS - u_\alpha F_{d,\alpha} + \int_S (1 - \theta) u_\alpha \tau_{a,\alpha\beta} n_\beta dS' \\
& + \int_S \theta \tau_{\alpha\beta} v_\alpha n_\beta dS' + \int_V v_\alpha \rho'_v f_{b,\alpha} dV . \quad (106)
\end{aligned}$$

The corresponding differential equation is

$$\begin{aligned}
 & \frac{\partial [\rho'_v (i_v + v^2/2)]}{\partial t} + \frac{\partial [\rho'_v (h_v + v^2/2) v_\alpha]}{\partial x_\alpha} \\
 &= - \frac{\partial (\theta \dot{q}_\alpha)}{\partial x_\alpha} - \frac{\partial [(1 - \theta) \dot{q}_{s,\alpha}]}{\partial x_\alpha} - \dot{q}_d + s_m \left(\bar{h}_s + \frac{u^2}{2} + \frac{w^2}{2} \right) - \frac{\partial [(1 - \theta) p u_\alpha]}{\partial x_\alpha} \\
 & \quad + \frac{\partial [(1 - \theta) u_\alpha \tau_{a,\alpha\beta}]}{\partial x_\beta} + \frac{\partial (\theta \tau_{\alpha\beta} v_\alpha)}{\partial x_\beta} - u_\alpha f_{d,\alpha} + \rho'_v f_{b,\alpha} v_\alpha. \quad (107)
 \end{aligned}$$

It is instructive at this point to identify the terms contributing to the vapor-phase energy equation. The first group of terms on the right-hand side, the two heat transfer terms, represent the composite heat transfer to the vapor due to heat transfer through the vapor and heat transfer over the droplet-vapor interface of the boundary droplets due to the local temperature gradient in the vapor. The third term $-\dot{q}_d$ represents the heat transfer to the vapor from the droplets because of a bulk temperature difference between the droplets and vapor. The fourth term is the energy supplied to the vapor because of mass transfer at the droplet surface. The fifth term is the flow work rate due to the motion of the droplets through the vapor. The sixth and seventh terms represent the composite work rate due to shear stresses in the gas and the shear stress on the vapor-droplet interface of the boundary droplets. The eighth term is the work rate due to aerodynamic forces on the droplets. The ninth and last term is the work rate associated with body forces on the vapor. No terms appear relating to the work associated with the change in droplet volume because the regression rate of the droplet surface has been neglected.

Using the momentum and energy equations for individual droplets [Eqs. (11) and (32)] to replace the third, fourth, and eighth terms on the right-hand side of the vapor-phase energy equation [Eq. (107)] yields

$$\begin{aligned}
 & \frac{\partial [\rho'_v (i_v + v^2/2)]}{\partial t} + \frac{\partial [\rho'_v (h_v + v^2/2) v_\alpha]}{\partial x_\alpha} \\
 &= - \frac{\partial [\theta \dot{q}_\alpha + (1 - \theta) \dot{q}_{s,\alpha}]}{\partial x_\alpha} - \frac{\partial [(1 - \theta) p u_\alpha]}{\partial x_\alpha} + \frac{\partial [(1 - \theta) u_\alpha \tau_{a,\alpha\beta} + \theta \tau_{\alpha\beta} v_\alpha]}{\partial x_\beta} \\
 & \quad + \rho'_v f_{b,\alpha} v_\alpha + f_{c,\alpha} u_\alpha - \frac{d[(Nm/V)(i_d + u^2/2)]}{dt}, \quad (108)
 \end{aligned}$$

which illustrates how energy lost by the droplet phase appears as a source of energy to the vapor phase. The second to last term on the right-hand side is the work rate resulting from nonaerodynamic forces on the droplets such as a Coulomb force or a body force.

It is also interesting to study the thermal form of the energy equation obtained by subtracting the mechanical energy equation and continuity equation from the total energy equation [Eq. (107)]. The mechanical energy equation is obtained by multiplying the momentum equation [Eq. (85)] by v_i and summing over all i 's to obtain the kinetic energy. The continuity equation is given by Eq. (74). The resulting thermal energy equation for the vapor is

$$\begin{aligned} \rho'_v \frac{\partial i_v}{\partial t} + \rho'_v v_\alpha \frac{\partial i_v}{\partial x_\alpha} \\ = -p \frac{\partial v_\alpha}{\partial x_\alpha} + \left[\theta \tau_{\alpha\beta} + (1 - \theta) \tau_{a,\alpha\beta} \right] \frac{\partial v_\alpha}{\partial x_\beta} - \dot{q}_d - \frac{\partial [\theta \dot{q}_\alpha + (1 - \theta) \dot{q}_{s,\alpha}]}{\partial x_\alpha} \\ + \frac{\partial [(1 - \theta) p (v_\alpha - u_\alpha)]}{\partial x_\alpha} + (v_\alpha - u_\alpha) f_{d,\alpha} \\ + s_m \left(\bar{h}_s - i_v + \frac{|u_i - v_i|^2}{2} + \frac{w^2}{2} \right) + \frac{\partial [(1 - \theta) (u_\alpha - v_\alpha) \tau_{a,\alpha\beta}]}{\partial x_\beta} . \quad (109) \end{aligned}$$

One recognizes the many source terms that arise because of velocity nonequilibrium, such as that due to aerodynamic drag and asymmetric shear stress on the particle. In the special case of steady flow, vapor-droplet velocity equilibrium (no mass transfer, heat transfer, or shear stress), Eq. (109) reduces to

$$\rho'_v v_\alpha \frac{\partial i_v}{\partial x_\alpha} = -p \frac{\partial v_\alpha}{\partial x_\alpha} , \quad (110)$$

which corresponds to the energy equation for single-phase, compressible, inviscid, adiabatic flows, the only difference being the bulk density in lieu of the material density for the vapor. This difference occurs because of the volume occupied by the droplets. As the flow is compressed ($\partial v_\alpha / \partial x_\alpha < 0$), the vapor will experience a larger compression rate because of volume occupied by the droplets.

Mixture Equations

The addition of the droplet-cloud and vapor-phase equations yields the mixture equations. The mixture equations can be easily derived from first

principles, so the equations resulting from adding the droplet and vapor equations provide a check as to the validity of each.

CONTINUITY EQUATION

The continuity equations for the droplet and vapor phases are Eqs. (46) and (74), respectively. Adding these two equations, one finds the source terms cancel to give

$$\frac{\partial(\rho'_d + \rho'_v)}{\partial t} + \frac{\partial(\rho'_d u_\alpha + \rho'_v v_\alpha)}{\partial x_\alpha} = 0, \quad (111)$$

which states that the mass of the mixture is conserved.

MOMENTUM EQUATION

The momentum equations for the droplet and vapor phases are Eqs. (56) and (84), respectively. Adding these two equations, one obtains

$$\begin{aligned} \frac{\partial(\rho'_d u_i + \rho'_v v_i)}{\partial t} + \frac{\partial(\rho'_d u_\alpha u_i + \rho'_v v_\alpha v_i)}{\partial x_\alpha} = & -\frac{\partial p}{\partial x_i} \\ & + \frac{\partial[(1-\theta)\tau_{a,\alpha i} + \theta\tau_{\alpha i}]}{\partial x_\alpha} + \rho'_v f_{b,i} + f_{c,i}, \end{aligned} \quad (112)$$

which is the momentum equation for the mixture. One feature of this equation that may not be readily apparent in a direct derivation of the mixture equations is the θ factor in the shear stress terms, which results from the effective area of the vapor being reduced by the presence of the droplets. Also one notes the asymmetric shear stress on the droplets resulting from a vapor velocity gradient and the contribution of nonaerodynamic forces on the droplets.

ENERGY EQUATION

The energy equations for the droplet and vapor phases are Eqs. (66) and (108). Adding these two equations to eliminate the source terms yields

$$\begin{aligned} \frac{\partial[\rho'_d(i_d + u^2/2) + \rho'_v(i_v + v^2/2)]}{\partial t} \\ + \frac{\partial[\rho'_d(i_d + p/\rho_d + u^2/2)u_\alpha + \rho'_v(h_v + v^2/2)v_\alpha]}{\partial x_\alpha} \\ = - \frac{\partial[\theta\dot{q}_\alpha + (1-\theta)\dot{q}_{s,\alpha}]}{\partial x_\alpha} + \frac{\partial[\theta\tau_{\alpha\beta}v_\alpha + (1-\theta)\tau_{a,\alpha\beta}u_\alpha]}{\partial x_\beta} \\ + \rho'_v f_{b,\alpha} v_\alpha + f_{c,\alpha} u_\alpha. \end{aligned} \quad (113)$$

It is interesting to note that the flow work term in the vapor-phase equations sums with the internal energy term in the droplet cloud equations to yield an enthalpy-like term. The p/ρ_d term is based on the pressure in the vapor, not the internal pressure of the droplet, and therefore the $i_v + p/\rho_d$ term is not truly the droplet-liquid enthalpy. However, for most problems the use of the droplet enthalpy is a valid approximation.

Starting directly with the mixture and considering the forces and heat transfer on the surfaces within the droplets would yield the same form of the energy equation.

The thermal form of the mixture energy equation can also be obtained by adding Eqs. (109) and (67). In this formulation one would recognize the several terms relating to the sources of thermal energy due to velocity non-equilibrium.

Discussion

The equations presented here represent a self-consistent set that approach the correct limits for a single-phase flow or highly dispersed flow. The equations agree with those derived in a different fashion by Drew except for terms that arise because of nonuniformity of droplet evaporation, a distribution of particle sizes, and a pseudostress term to account for particle-particle (or droplet-droplet) interaction. Extending the above approach to incorporate these phenomena would yield Drew's equations.

It must be stressed that Drew's derivation and the one presented here do not treat the condensed phase as another gaseous species that can coexist (in a continuum sense) with the vapor at every point in the flow field. Such an assumption leads to equations that fail to yield the correct limiting solutions.

Conclusions

Considerable care and proper application of the Reynolds transport theorem must be exercised in deriving the Eulerian equations for vapor flow with suspended burning, evaporating, or condensing droplets. Including the effects of the boundary droplets intersected by the control surface ensures a self-consistent set of equations that yield the correct limiting expressions for equilibrium, single-phase, and highly dispersed two-phase flows.

The equations presented here represent a starting point for viable numerical or mathematical models of vapor-droplet flows. Undoubtedly, for

certain applications, many of the terms arising due to the boundary droplets can be neglected. However, by starting with the complete equations and simplifying, the researcher has a measure of the validity of his analytical results.

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