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PRONY'S METHOD FOR THE ANGLE DOMAIN

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PRONY'S METHOD FOR THE ANGLE DOMAIN

ABSTRACT

Prony's method, a technique for obtaining the parameters of a series of complex exponentials, is used here to analyze the radiation pattern of a linear array of isotropic radiators. The locations and amplitudes of the radiators are derivable from the pattern. Possible applications include imaging and array synthesis.

INTRODUCTION

Those involved in electromagnetics have recently shown considerable interest in obtaining the complex resonances or poles of conducting objects. Such poles were originally derived from either analytical considerations¹ or numerical searching of the complex frequency plane,² although, of late, attention has centered around their possible extraction from transient data. Prony's method for data processing is of particular interest in transient analysis since it offers the capability for obtaining poles from experimental time-domain measurements. Spectral data, using a frequency-domain analog of Prony's method,³ can also be used for this purpose, thus exploiting the more widely used frequency-domain measurements. While initial work in pole finding was motivated by the Singularity Expansion Method (SEM),¹ its future impetus will come from the various potential applications of Prony's method that continue to be identified.

Our purpose in this brief presentation is to discuss one of these applications that departs radically from the SEM pole-finding applications first used in electromagnetics. This application involves sampling the far field of a given radiator and using the Prony method to determine the source distribution. The following sections discuss some analytical aspects of the procedure, some preliminary numerical results for a line distribution of isotropic point radiators, and some problems needing resolution before the procedure can be generalized.

ANALYTICAL CONSIDERATIONS

Prony's method⁴⁻⁶ provides a way to find the poles, s_α , and residues, R_α , of the expression

$$f(x) = \sum_{\alpha=1}^N R_\alpha \exp(s_\alpha x) \quad (1)$$

from samples $f_i = f(x_i)$, $x_i = i\Delta x$, $i = 0, 1, \dots, M$, where $M - 1 \geq 2N$. It has been used in electromagnetics on spatially sampled data for determining the angle of arrival of two or more plane waves⁷ and on temporally sampled data for determining the object pole. Our concern here is related to the former use, but is significantly different.

Suppose specifically that $f(x) = f(\theta)$ is the radiation pattern of a linear array of isotropic point sources having coordinates ρ_α and amplitudes A_α , $\alpha = 1, \dots, N$. Then the field strength is given by

$$f(\theta) = \sum_{\alpha=1}^N A_\alpha \exp(ik \rho_\alpha \cos\theta), \quad (2)$$

where $k = 2\pi/\lambda = \omega/c$ and θ is the observation angle relative to the array axis. The transformation $x = \cos\theta$ leads to

$$f(\theta) = f(\cos^{-1}x) = \sum_{\alpha=1}^N A_\alpha \exp(ik \rho_\alpha x) \quad (3)$$

so that $s_\alpha = ik\rho_\alpha$.

A comparison of Eq. (3) with the corresponding expression for transient data (typified by Eq. (1)) reveals several important differences between transient and pattern data, suggesting that their processing by Prony's method may also differ.

- The poles of pattern data are purely imaginary;* those of transient data are generally complex. While the poles associated with transient data are those of the object that generated it, they generally do not provide any explicit information concerning its geometry. On the other hand, the poles of pattern data can give the physical locations ρ_α of individual sources.

* A pattern factor could be introduced via a real pole component, but it would be limited to what is describable by an exponential.

- The requirement that data be sampled in equal increments of the exponent results in equally spaced samples of transient data, but unequally spaced samples of pattern data. Sampling of pattern data is, instead, in equal steps of $\cos\theta$.
- The width and location of the transient-data sampling window are somewhat arbitrary (although the window must be of minimum width and must not include the forced part of the waveform) so that sampling can occur over $0 \leq x < \infty$. The pattern data, on the other hand, has $-1 \leq x \leq 1$, which represents the visible range of observation angles available.
- Because transient data are real valued, their poles and residues must occur in complex conjugate pairs. However, there is no such requirement for pattern-data poles, which are generally nonconjugate.

The numerical technique we have been using must be modified to handle complex data, and the residues must be corrected to account for the series beginning at $x = -1$, rather than 0 as for the time domain. Other aspects of transient- and pattern-data processing are discussed below.

NUMERICAL RESULTS

At first, we tried to process pattern data the same way as transient data. We specified a source distribution along a straight line. Then, selecting a convenient starting point in the pattern, we successively expanded the width of the data window by increasing Δx while maintaining $M = 2N$.

This approach does not produce consistently good results, as demonstrated by Fig. 1. The figure illustrates the results of a numerical experiment using Prony's method to extract from the far-field radiation pattern the locations of four isotopic sources in a linear array excited with nonconjugate-pair residues. One pair of poles, located at ± 50 m, was excited by residues of $1 + j1$ and $2 - j3$, respectively. The other pair of poles, located at ± 15 m, was excited by residues of $0.5 + j1.5$ and $-1 - j2$, respectively. The values of the extracted poles and residues are listed in Table 1, where poles have been normalized to distance from the origin.

Figure 1(a) is a plot of the far-field radiation pattern, which has been normalized to a peak value of unity. Data points indicate the samples used for extracting the poles. For a step size between samples of $\Delta \cos\theta = 5.59 \times 10^{-2}$, the algorithm was unsuccessful at extracting the poles, as shown

(a) Polar plot of the far-field radiation pattern normalized to a peak value of unity (samples used for extracting the poles are indicated by data points) and the s-plane plot of the extracted poles.

(b) Polar plot of the far-field radiation pattern (data points indicate the samples used) and the s-plane plot of the extracted poles.

Fig. 1. Results of a numerical experiment using Prony's method to extract the locations of four isotropic sources in a linear array excited with nonconjugate-pair residues from the far-field radiation pattern. Wavelength is 100 m.

in the s-plane plot (with the z-axis proportional to the magnitude of the residues on a 3-decade log' scale).

In figure 1(b), another view of the pattern plot, samples were taken using a spacing of $\Delta\cos\theta = 0.117$. The algorithm successfully extracted the poles and their residues, as shown in the s-plane plot.

From these and similar results for other source spacings (changed from that shown by a multiplicative scale factor), we found that the relative accuracy of pole extraction evidently depends on $\Delta\cos\theta$ (or the sample sequence used) and the source distribution. (See Table 1.)

The above approach seemed to work better for wider window widths, so we decided as an alternative to use the fixed angular range $0 \leq \theta \leq \pi$; because of the source geometry, the other half of the pattern is identical. We performed successive calculations whereby the number of data points, M , was increased by decreasing Δx . In order to maintain a square system, we also increased the number of poles, N_p , requested in the Prony process to $M = 2N_p$. As a result, of course, $N_p > N$, so curve-fitting poles, which are not present in the data, always occur. In the pattern-data application, such poles are usually, but not always, indentifiable by their small residues and their large real components. The former property derives from the fact that curve-fitting

Table 1. Values of extracted poles and residues for Fig. 1. Pole values were normalized to distance from the origin. True poles have a real part (R) of zero and an imaginary part (I). The sampling spacing was $\Delta\cos\theta = 5.59 \times 10^{-2}$ for (a) and $\Delta\cos\theta = 0.117$ for (b).

Residue		Pole (m)		True pole (m)
R	I	R	I	I
(a)				
-6.52910E-01	-1.86135E+00	2.27024E+00	-2.58092E+01	1.50000E+01
7.29415E-01	9.12199E-01	8.54338E+00	2.99471E+01	-1.50000E+01
1.93970E+00	-2.55120E+00	-4.82711E-01	-5.10532E+01	5.00000E+01
4.83795E-01	1.00035E+00	2.84362E+00	5.20662E+01	-5.00000E+01
(b)				
-1.00117E+00	-1.99808E+00	2.16052E-02	-1.50000E+01	1.50000E+01
5.01791E-01	1.49864E+00	-1.93886E-02	1.50236E+01	-1.50000E+01
1.99951E+00	-3.00008E+00	1.25769E-03	-5.00002E+01	5.00000E+01
9.99869E-01	9.99520E-01	-9.27328E-04	5.00031E+01	-5.00000E+01

poles associated with computed data, like that considered here, should have residues of computer round-off magnitude. The latter property is equally applicable since actual physical sources have purely imaginary poles (except as previously noted). Consequently, any pole with a significant real component would not be due to an actual source.

Some representative results obtained from the use of this approach for a fixed N_p value are shown in Fig. 2. Using Prony's method with $N_p = 16$, we extracted the locations of 11 isotropic sources in a linear array from the far-field radiation pattern for various array lengths. The outermost sources were excited with a residue of $1 + j1$, the next at $2 + j2$, etc., and the innermost at $6 + j6$.

We found that accurate pole and residue values resulted from certain combinations of parameters. Although the parameter combinations that worked best are not obvious, two points are worth noting.

- Our observation regarding the influence of Δx is reinforced.
- As the sources move closer together, those sources nearer the origin than $\lambda/2$ coalesce into one or more composite, nonphysical poles, while those farther from the origin remain extractable.

More study is clearly required to resolve the interrelationships of the various parameters. However, the latter result would appear analogous to the super-gain phenomenon associated with antenna arrays.

We have concentrated thus far on the results obtained by applying the Prony method to pattern data. It is also instructive to examine the eigenvalue characteristics of the data as they appear in the process. A matrix of data values, $\overline{\overline{F}}$, is first obtained for $M = 2N$.

$$\overline{\overline{F}} = \begin{bmatrix} f_0 & f_1 & f_2 & \dots & f_{N-1} \\ f_1 & f_2 & f_3 & \dots & f_N \\ . & . & . & . & . \\ f_{N-1} & f_N & f_{N+1} & \dots & f_{2N-2} \end{bmatrix} \quad (4)$$

Its solution yields the coefficients of a polynomial whose roots give the poles. We might then expect the matrix of data values, $\overline{\overline{F}}$, to strongly

(a) Using a 100-m aperture.
Algorithm unable to determine source locations.

(b) Using a 300-m aperture.
Locations of outermost sources were found.

(c) Using a 500-m aperture.
Outermost pair of sources was found.

Fig. 2. Polar-radiation patterns and s-plane plots for poles extracted from various aperture widths. The vertical axes are proportional to the order of magnitude of residue. Values are listed in Table 2. Wavelength is 100 m.

(d) Using a 700-m aperture.
All sources and residues
extracted with accuracy of
four significant figures.

(e) Using a 900-m aperture.
All sources and residues
extracted with accuracy of
11 significant figures.

Fig. 2. (cont.). Polar-radiation patterns and s-plane plots for poles extracted from various aperture widths.

Table 2. Values of extracted poles and residues for Fig. 2. Pole values were normalized to distance from the origin. True poles have a real part (R) of zero and an imaginary part (I), corresponding to the distance from the center of each of the 11 sources for aperture widths of (a) 100 m, (b) 300 m, (c) 500 m, (d) 700 m, and (e) 900 m.

Residue		Pole (m)		True pole (m)
R	I	R	I	I
(a)				
1.43518E+01	1.43625E+01	-8.10871E-01	-4.19336E-02	-2.27374E-13
8.88638E+00	8.22785E+00	-7.23929E-01	2.30313E+01	9.09091E+00
8.19518E+00	8.85024E+00	-7.11355E-01	-2.31042E+01	-9.09091E+00
2.40587E+00	2.16857E+00	-2.65795E-01	4.25299E+01	1.81818E+01
2.16078E+00	2.39084E+00	-2.57176E-01	-4.25552E+01	-1.81818E+01
Negligible residues				2.72727E+01
1.47495E-07	-8.97786E-08	-1.37586E+01	-2.87862E+02	-2.72727E+01
-5.63697E-08	1.47697E-07	-1.08669E+01	2.92698E+02	3.63636E+01
3.56054E-08	-1.24561E-07	-1.11097E+01	-5.07569E+02	-3.63636E+01
-1.08547E-08	-2.30152E-08	-7.14880E+01	-6.27036E+02	4.54545E+01
-1.38572E-08	-5.33183E-09	-9.71569E+01	4.40779E+02	-4.54545E+01
-1.15256E-08	-8.00030E-08	1.45860E+00	-7.05082E+02	
-1.03706E-07	-1.12278E-08	2.22335E+00	6.99772E+02	
1.65890E-08	-1.17210E-08	-8.56585E+01	-3.98637E+02	
-1.44618E-07	5.75388E-08	-1.32033E+01	5.10744E+02	
-5.54458E-09	-1.53969E-08	-8.49054E+01	6.55391E+02	
-8.60352E-16	-9.32450E-16	-4.00465E+02	1.97349E+02	
(b)				
8.39206E+00	8.47465E+00	5.61226E-01	2.11168E+01	0.
8.47280E+00	8.38844E+00	5.58650E-01	-2.11544E+01	2.72727E+01
5.53580E+00	5.63290E+00	4.52717E-01	6.44232E+01	-2.72727E+01
5.62959E+00	5.53271E+00	4.46069E-01	-6.44540E+01	5.45455E+01
2.86120E+00	2.92268E+00	2.34791E-01	1.03516E+02	-5.45455E+01
2.91999E+00	2.85967E+00	2.28958E-01	-1.03532E+02	8.18182E+01
1.08870E+00	1.10048E+00	4.24253E-02	1.35749E+02	-8.18182E+01
1.05987E+00	1.08847E+00	4.09376E-02	-1.35752E+02	1.09091E+02
Negligible residues				-1.09091E+02
4.78635E-08	-1.42400E-07	1.08951E+01	3.72786E+02	1.36364E+02
-1.17980E-07	1.04491E-07	6.03404E+00	-3.78256E+02	-1.36364E+02
-7.97270E-08	8.60029E-08	2.73233E+01	-4.91566E+02	
9.61255E-08	-9.07929E-08	2.37879E+01	4.92484E+02	
8.41070E-08	-2.08465E-08	1.24365E+01	5.92589E+02	
-2.42829E-08	8.53198E-08	9.94295E+00	-5.90302E+02	
-3.63244E-08	-3.11993E-08	6.74989E-01	-7.23179E+02	
-2.00232E-08	-2.83302E-08	-2.48106E-01	7.26278E+02	

Table 2. (continued).

Residue		Pole (m)		True pole (m)
R	I	R	I	I
(c)				
6.46957E+00	6.52524E+00	-6.79744E-01	-2.60637E+01	0.
6.52410E+00	6.46825E+00	-6.79400E-01	2.60936E+01	4.54545E+01
4.95143E+00	5.01665E+00	-5.36906E-01	-8.02825E+01	-4.54545E+01
5.01522E+00	4.94998E+00	-5.35858E-01	8.03060E+01	9.09091E+01
3.39952E+00	3.44824E+00	-3.00416E-01	-1.32351E+02	-9.09091E+01
3.44712E+00	3.39850E+00	-2.99308E-01	1.32364E+02	1.36364E+02
2.08260E+00	2.09992E+00	-8.91833E-02	-1.81007E+02	-1.36364E+02
2.09951E+00	2.08227E+00	-8.86674E-02	1.81010E+02	1.81818E+02
1.00470E+00	1.00627E+00	-8.40706E-03	-2.27218E+02	-1.81818E+02
1.00623E+00	1.00467E+00	-8.34011E-03	2.27218E+02	2.27273E+02
				-2.27273E+02
Negligible residues				
-5.85591E-08	-2.47763E-08	9.32120E+00	-4.72363E+02	
-2.26756E-08	-6.16229E-08	8.32345E+00	4.72255E+02	
-6.26157E-08	3.84545E-08	-6.01347E+00	6.14614E+02	
3.78569E-08	-6.09668E-08	-7.49517E+00	-6.14979E+02	
-6.33537E-08	5.85894E-08	-2.21030E+01	7.74010E+02	
-2.64950E-09	1.58088E-09	-1.20240E+02	-7.69701E+02	
(d)				
6.00000E+00	6.00001E+00	-4.00807E-04	-2.61832E-05	0.
5.00002E+00	4.99999E+00	-3.96292E-04	6.36364E+01	6.36364E+01
4.99999E+00	5.00001E+00	-3.87125E-04	-6.36364E+01	-6.36364E+01
4.00001E+00	3.99999E+00	-2.65313E-04	1.27273E+02	1.27273E+02
3.99998E+00	4.00001E+00	-2.52871E-04	-1.27273E+02	-1.27273E+02
3.00001E+00	2.99999E+00	-1.14629E-04	1.90909E+02	1.90909E+02
2.99999E+00	3.00000E+00	-1.06329E-04	-1.90909E+02	-1.90909E+02
2.00000E+00	2.00000E+00	-2.84183E-05	2.54545E+02	2.54545E+02
2.00000E+00	2.00000E+00	-2.55567E-05	-2.54545E+02	-2.54545E+02
1.00000E+00	1.00000E+00	-2.99029E-06	3.18182E+02	3.18182E+02
1.00000E+00	1.00000E+00	-2.59500E-06	-3.18182E+02	-3.18182E+02
Negligible residues				
-1.35399E-10	2.70848E-11	1.31309E+01	-6.01102E+02	
4.13438E-11	-1.41282E-10	1.23862E+01	6.02513E+02	
-2.91079E-14	3.30342E-14	-2.41680E+02	5.62621E+02	
5.85618E-11	-7.19752E-11	3.23427E+01	-7.71746E+02	
4.47699E-15	-8.06384E-15	-2.69616E+02	-5.72671E+02	
(e)				
6.00000E+00	6.00000E+00	6.30592E-11	1.45103E-11	1.81899E-12
5.00000E+00	5.00000E+00	6.13076E-11	-8.18182E+01	8.18182E+01
5.00000E+00	5.00000E+00	7.05037E-11	8.18182E+01	-8.18182E+01
4.00000E+00	4.00000E+00	8.36410E-11	-1.63636E+02	1.63636E+02
4.00000E+00	4.00000E+00	6.96279E-11	1.63636E+02	-1.63636E+02
3.00000E+00	3.00000E+00	5.73664E-11	-2.45455E+02	2.45455E+02
3.00000E+00	3.00000E+00	5.73664E-11	2.45455E+02	-2.45455E+02
2.00000E+00	2.00000E+00	3.76604E-11	-3.27273E+02	3.27273E+02
2.00000E+00	2.00000E+00	3.85362E-11	3.27273E+02	-3.27273E+02
1.00000E+00	1.00000E+00	5.69285E-12	-4.09091E+02	4.09091E+02
1.00000E+00	1.00000E+00	7.88240E-12	4.09091E+02	-4.09091E+02
Negligible residues				
-1.39829E-13	4.73484E-13	-6.61616E+00	-5.54152E+02	
3.38545E-13	-7.75098E-14	-7.05581E+00	5.54808E+02	
-1.15496E-13	2.43473E-13	7.49914E+00	-6.49756E+02	
1.28966E-13	-2.33858E-15	6.71977E+00	6.50191E+02	
5.42236E-14	-1.12813E-13	-1.61476E+01	-7.73975E+02	

influence the success of the overall process. In essence, the samples, f_i , constitute a basis set that represents all the waveform, $f(x)$, including that part not sampled (i.e., the sample space must span the waveform space). Therefore, we should be able to test the data matrix, F , for its suitability; the eigenvalue procedure is applicable in this regard.

The eigenvalues (λ) and eigenvectors ($\bar{m}$) of a matrix ($\bar{M}$) are defined by the relation

$$\bar{M} \cdot \bar{m} = \lambda_i \bar{m}; i = 1, \dots, N, \quad (5a)$$

where the λ_i are solutions of the determinantal equation

$$|\bar{M} - \lambda \bar{I}| = 0. \quad (5b)$$

For the given matrix, $\bar{M}$, the eigenvectors, $\bar{m}_i$, represent a basis set for expanding the rows of $\bar{M}$, and the eigenvalues represent their corresponding contribution to that expansion. For example, dominance of the rows of $\bar{M}$ by one function, an exponential in our case, would result in only one eigenvalue of significant magnitude. Thus, an eigenvalue analysis of the matrix $\bar{F}$ can provide the test mentioned above.

We performed an eigenvalue analysis of the data used to derive the results of Figs. 1 and 2; this analysis is summarized in Figs. 3 and 4. Although these results are not yet conclusive, they do demonstrate that the number of poles having residues of significant magnitude is generally equal to the number of significant-size ($>10^{-4}$) eigenvalues. In other words, the dimensionality or rank of the data can be estimated from the eigenvalue set. Thus, eigenvalue analysis provides a potentially useful way to establish the effective number of independent exponential terms in the data.⁸ This topic deserves further study in the context of pole finding even though it has received considerable attention in data processing in general.^{9,10} Noise in the data, which we have not considered here, can also significantly affect the data rank and utility of the eigenvalue test.

Fig. 3. Eigenvalues of the data matrix (Eq. (4)) for (a) and (b) of Fig. 1. The smallest eigenvalue for (a) was well below the 10^{-4} criterion, and extracted poles had significant error. The smallest eigenvalue for (b) was only slightly below the 10^{-4} criterion, and poles were extracted correctly.

Fig. 4. Eigenvalues of the data matrix (Eq. (4)) for sections (a) through (e) of Fig. 2. The most successful pole extractions occurred for (d) and (e). These curves had the smallest slope and the smallest ratio of the first to the 11th (since the system had 11 poles) eigenvalue.

FURTHER DEVELOPMENTS

Our results represent a rather simple extension of the use of Prony's method for transient data, yet its potential applications for pattern data are intriguing. It offers several interesting possibilities, including the interpolation and storage of measured pattern data and, more significantly, the electromagnetic imaging of radiating objects. Some additional work is required, however, before these kinds of applications become practicable. Two items in particular deserve mention.

Two-Dimensional Source Distributions

To this point, we have considered only a one-dimensional problem — a linear array of point sources. In general, a two-dimensional distribution of sources should be permitted over the surface of the plane in which the pattern is observed. Note that this arrangement can also suffice to represent a three-dimensional distribution since the projection of source locations onto the observation plane alone determines the field pattern. Thus, for scalar sources, the two-dimensional form of Eq. (2) is

$$f(\theta) = \sum_{\alpha=1}^N A \exp[ik\rho_{\alpha}\cos(\theta - \theta_{\alpha})], \quad (6)$$

where ρ_{α} and θ_{α} are the source coordinates of the α th pole in the observation plane. But Eq. (6) can be transformed to

$$f(\theta) = \sum_{\alpha=1}^N A_{\alpha} \exp[ik\rho_{\alpha}(\cos\theta \cos\theta_{\alpha} + \sin\theta \sin\theta_{\alpha})]. \quad (7)$$

This expression is not compatible with the Prony method because the exponential depends upon two independent functions of θ . If we were to transform $\cos\theta \rightarrow x$ as before, the exponent would contain x and $\sqrt{1-x^2}$. Consequently, it is impossible to vary x in such a way that the exponent will increase in the uniform steps required by Prony's method.

Vector Point Sources

Acoustic point sources are normally considered isotropic or scalar in nature, while sources in electromagnetics are generally vectoral. Vector sources could readily be treated if they all had the same pattern factor (e.g., $\cos\theta$). However, the pattern factor could be $\cos(\theta-\theta_{\alpha})$, introducing an

additional θ dependence into the exponential to further complicate the problem.

Determining the distribution and direction of vector sources may thus present formidable obstacles, at least in terms of directly using Prony's method for practical problems. The benefits of overcoming these obstacles seem worthwhile enough, however, to warrant some additional study of Prony's method or some alternative to it. One possible application might be the processing of scale-model pattern data obtained on antenna ranges such as that at Naval Oceans System Center. If the pattern data could be reduced to an equivalent source distribution for various antennas on a given ship, then important interactions might be more easily identified. Predicting the effects of structure modification might also be possible if further analysis could show the dependence of the derived source distribution on the physical structure. For these and other reasons, further work on the problem is desirable.

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