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A COMPUTER SIMULATION OF THE CREATION OF A TRANSIENT, HIGH-DENSITY PLASMA BY CONVERGENT NEUTRAL BEAMS

G. J. Eggens (Hartman)

June 1, 1976

Prepared for U.S. Energy Research & Development
Administration under contract No. W-7405-Eng-48

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Printed in the United States of America

Available from

National Technical Information Service

U.S. Department of Commerce

5285 Port Royal Road

Springfield, VA 22161

Price: Printed Copy \$: Microfiche \$3.00

Page Range	Domestic Price	Page Range	Domestic Price
001-025	\$ 3.50	326-350	10.00
026-050	4.00	351-375	10.50
051-075	4.50	376-400	10.75
076-100	5.00	401-425	11.00
101-125	5.50	426-450	11.75
126-150	6.00	451-475	12.00
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LAWRENCE LIVERMORE LABORATORY
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Introduction

Neutral-beam injection is emerging as a powerful tool in fusion research. Preliminary calculations showed that, because of recent advances in the technology of high-current, pulsed neutral beams, it might be possible to create a transient, high-density plasma by combining focused, spatial convergence with the temporal bunching achieved by velocity modulation.^{1,2} However, a more rigorous proof was required.

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This report describes a computer code written to establish a formalism through which accurate predictions of the geometrical convergence and the temporal bunching of neutral beams could be made as they proceed.³ Approximate calculations⁴ are extended and verified in detailed calculations. In the latter, an initial distribution function at the source is derived from a solution to the Vlasov equation through the electrode structure of the ion source. Liouville's theorem is then applied to follow the evolution of this distribution function. The code evaluates the distribution function at any position and time within the beam region. Additionally, it calculates the neutral atom density, the relative velocity, the rate parameters $\bar{\sigma}_{NN}$, $\bar{\sigma}_{NI}$, the probabilities of ionization, and the ion density. The results of these calculations confirm that the convergence-bunching technique could have important application in fusion power research.

The report concludes with predictions of the subsequent history of the newly created plasma. Appendices provide supplementary information and derivations.

Neutral Beams

Achievement of intense, highly collimated neutral beams is crucial to the success of the convergence-bunching technique. Only recently have appropriate sources been available, and much research is still needed.^{5,6,7}

Ion sources and neutral beams have had a long and progressive history of varied applications and requirements. Some of the considerations of intense highly collimated neutral beams discussed here were a result of much effort for other applications and at other energies.

Neutral beams are produced from ions and have an intrinsic divergence due to physical and mechanical limitations of the plasma sources. The ions must be formed, extracted, accelerated, and charge neutralized. Each process must be optimized to limit, as much as possible, the beam divergence. Any mechanism that would disperse the intense ion beam before it is converted into neutral particles must be avoided. Appendix A discusses the fundamental processes that limit the important beam parameters of divergence and intensity.

Energetic neutral atomic and molecular beams were first suggested in the 1950's as a means of filling magnetic bottles with fusion-temperature plasmas. The injection of neutral (and ion) beams into mirror machines to create, sustain, and heat plasmas evolved from a complicated and continuing technology.

This application is a major component of the program to develop the magnetic mirror concept for fusion and was a major impetus for the development of high-intensity sources.⁸

Recent technology has led to the extension of beams to higher power and better focusing. Large-volume, highly uniform plasmas⁷ are being used, and the beam-focusing optics are being aided by sophisticated computer simulation.⁵

The Lawrence Berkeley Laboratory (LBL) 50-A source⁶ is one of the most promising for intense, pulsed neutral-beam applications. Development of these beams has progressed to the point where alternate proposals for controlled thermonuclear fusion seem feasible. In particular, it is hoped that these beams could be velocity modulated so that the bunching technique proposed here could be applied.

THE CONVERGENCE-BUNCHING TECHNIQUE

The proposed method for creating transient, high-density plasma² is based on the theory that the coincidence of temporal and spatial convergence of the intense, low-divergence neutral beams would create a maximum density at the target center that would correspond to an enhancement of the initial beam densities by many orders of magnitude. The beams would be radially focused on the surface of a sphere or cylinder and directed toward a center target site by the directed motion imparted to the particle currents by the curved surface. For example, as the center of a spherical containment vessel is approached, particle density would increase proportionally to $1/r^2$. Figure 1 shows how an array of inwardly directed, pulsed neutral-beam sources located on a spherical surface would spatially converge.

In addition to spatial convergence, it is proposed to velocity modulate the beams so that all the neutral particles arrive at the target site simultaneously. This technique, which utilizes the idea of a klystron, is shown schematically in Fig. 2.

The proposed energy (velocity) modulation of the neutral beam is

$$W_0(t) = \frac{W_I}{(1 - t/t_0)^2} \quad ,$$

$$v_0(t) = \frac{v_I}{(1 - t/t_0)} \quad . \quad (1)$$

At time $t = 0$, neutral atoms are launched with an initial energy W_I (and velocity v_I). With the progress of time, the beam energy is programmed up (by changing the acceleration potential in the ion gun portion of the source) so that particles launched late in time would catch up with the earlier ones. Because neutral beams are essentially noninteractive, it may be possible to arrange an array of sources on a sphere or cylinder to create high instantaneous power densities at the coincidence of temporal and spatial convergence. At the target point, either collisional self-ionization or ionization by impact on a target plasma would result in the conversion of the neutral particle cloud to a transient, high-density plasma.

Note that the required configuration would be large (≈ 300 cm); also, the sources now available would be applicable only to a cylindrical containment vessel. Further source development is necessary before the spherical geometries and velocity modulation hoped for in the proposed technique would be possible.

MATHEMATICAL REPRESENTATION OF PHYSICS OF CONVERGENT NEUTRAL BEAMS

Characteristic of the convergent neutral-beam system is the accuracy of the spatial and temporal focusing. The uncertainties involved with beam collimation at the target site are represented as follows in variables pertinent to the system:

- J_0 = current density at source.
- $W_I(v_I)$ = initial energy (velocity) of neutrals at an emission time $t = 0$.
- t_0 = characteristic flight time
- τ_p = pulse duration
- ΔW_I = angular beam divergence (predominately determined by ion thermal energy).
- $\Delta W_{||}$ = uncertainty due to velocity modulation.
- R = radial distance between source and target center.

Spatial Divergence Due to Ion Optics

As discussed earlier, physical and mechanical limitations cause all beams to diverge. The degree of convergence that can be achieved is primarily limited by the angular divergence of the neutral atom beam. This in turn is ultimately determined by the thermal energy ΔW_I of the initial plasma source

from which the ions were extracted. The angular divergence is perpendicular to the beam. The neutrals, then, arrive in the vicinity of the target site within an uncertainty given by

$$\Delta S_{\perp} = R \sqrt{\frac{\Delta W_{\perp}}{W_I}} \quad (2)$$

As shown in Fig. 3(a), this relationship is simply derived by applying the small-angle formula for the spread δ after the beam has traversed a distance R from the source, S . If there were no divergence, the beam would be radially directed inward along the axis ξ to the center of the sphere, 0. For straight-line trajectories, the velocity vectors are in the same direction and are proportional to the position vector, so that

$$\delta = \tan \delta = \frac{\Delta S_{\perp}}{R} = \frac{\Delta v_{\perp}}{v_I} = \sqrt{\frac{\Delta W_{\perp}}{W_I}}.$$

Focusing Uncertainty Along Axis

The ability to temporally bunch the neutral beam is also intrinsic to the proposed technique. If the pulse could be timed to collapse at the exact center of the sphere or cylinder, the target region would receive a uniform energy density input.

It is assumed that the neutral-beam sources provided can be energy (velocity) modulated without further degradation to the ion optics just described. If the angular divergence can be maintained independently of the accelerating potential, a longitudinal uncertainty corresponding to the spread in the distribution function as it emerges from the source is

$$\Delta S_{\parallel} = \frac{R}{2} \frac{\Delta W_{\parallel}}{W_I} \quad (3)$$

This is shown schematically in Fig. 3(b).

The energy of the beam along the axis in this direction is given by the sum of the initial energy of the neutrals W_I plus or minus the increment $\Delta W_{\parallel}$.

Fig. 3. Beam divergence.

- a) The spatial divergence due to ion optics. b) The focusing uncertainty in the longitudinal velocity direction. c) Disc formed at target site (from one source) due to both uncertainties.

$$W_{\parallel} = W_I \pm \Delta W_{\parallel} = W_I \left(1 + \frac{\Delta W_{\parallel}}{W_I} \right).$$

The velocity then is approximately

$$v_{\parallel} = v_I \sqrt{1 + \frac{\Delta W_{\parallel}}{W_I}} = v_I \left(1 + \frac{\Delta W_{\parallel}}{2W_I} \right),$$

so that the longitudinal velocity uncertainty may be defined as

$$\Delta v_{\parallel} \triangleq v_{\parallel} - v_I = \frac{v_I}{2} \frac{\Delta W_{\parallel}}{W_I}.$$

A spatial uncertainty $\Delta S_{\parallel}$ may now be defined in terms of the longitudinal velocity uncertainty $\Delta v_{\parallel}$ and the characteristic flight time t_0 as

$$\Delta S_{\parallel} \triangleq \Delta v_{\parallel} t_0$$

$$\Delta S_{\parallel} = \frac{R}{2} \frac{\Delta W_{\parallel}}{W_I}.$$

FEASIBILITY OF METHOD

From the preceding discussion, it is seen that the degree of density enhancement achievable will be determined by the uncertainties, in angle and in energy, associated with the nature of the neutral-beam sources, i.e., with the design and mechanical tolerances of the sources and with the initial thermal energy components of the accelerated ions. The dominant positional uncertainty is $\Delta S_{\perp}$. Thus, if the sources are located on a spherical surface of radius R , the heating energy will be concentrated within a spherical shell region whose radius is of order $\Delta S_{\perp}$ and whose thickness is of order $\Delta S_{\parallel}$. Each elementary converged pulse bunch takes on the shape of a thin disc, as may be seen in Fig. 3(c).

An estimate of the mean density of the converged particles may now be made by recognizing that the bunching process has the effect of gathering up within a smaller spherical region of volume $4/3\pi(\Delta S_{\perp})^3$ an incoming pulse of particles that initially occupied a larger spherical volume. Thus, if the average current density of the neutral sources is J_0 , the mean density at convergence may be estimated by dividing the total number of particles emitted during a pulse time, t_p , by the converged volume.

$$\bar{n} = \frac{\text{total particles emitted}}{\text{converged volume}} \approx \frac{4\pi R^2 J_0 t_p}{4/3\pi(\Delta S_I)^3} \quad (4)$$

For a 4:1 energy modulation, the pulse time is taken to be half the characteristic flight time t_0 . The mean density at convergence may then be rewritten as

$$\bar{n} \approx \frac{3}{2\sqrt{2}} J_0 \left(\frac{M}{W_I} \right)^{1/2} \left(\frac{W_I}{\Delta W_I} \right)^{3/2} \quad (5)$$

By taking current values for density and thermal energies from the existing 50-A LBL source⁶ and assuming that similar values could be achieved in an energy-modulated spherical source array, the density of the converged particles may be calculated from the source parameters listed in Table 1.

Table 1. Source parameters assumed for system.

Parameter	Value
Current density, J_0	0.25 A/cm^2
Initial energy (4:1 energy modulation), W_I	20 keV
Initial velocity (2:1 velocity modulation), v_I	$1.4 \times 10^8 \text{ cm/s}$
Characteristic flight time, t_0	$R/v_I = 2.14 \text{ } \mu\text{s}$
Pulse length, t_p	$t_0/2 = 1.07 \text{ } \mu\text{s}$
Ion thermal energy, T_I	$3/2\Delta W_I = 3/2kT_I = 1 \text{ eV}$
Radius of sphere, R	300 cm

The feasibility of using this unique method of convergent neutral beams to create a high-density plasma at a target site may thereby be confirmed.

For a deuterium beam (using Eq. 4), $\bar{n} = 0.9 \times 10^{17} \text{ cm}^{-3}$, a high density in the fusion context ($\beta = 1$ at 30.0 T). Furthermore, it should be possible to achieve substantially higher values by further source development. Owing to the short duration of the beam pulse, marked increase in both the emission current density and the beam energy may be possible. Under pulsed conditions, neutral-beam fluxes of several amperes equivalent per cm^2 at beam energies of order 100 keV seem attainable.⁹ Experiments with pulsed ion sources that suggest this possibility have been reported. Under these latter assumptions, the predicted densities are of order 10^{19} cm^{-3} .

In the next chapter, exact numerical solutions for this scheme will be derived so that subsequent investigation and analysis can be made.

Additional pertinent characteristics and parameters of the proposed system may also be estimated from basic physical equations. Using the parameter values in Table 1, the streaming density at the source is

$$n_0 \approx \frac{J_0}{v_I} \approx 10^{10} \text{ neutrals/cm}^3.$$

A target site may be defined as a spherical region of radius $\Delta S_{\perp}$. Because of beam divergence, all the energy of the neutral beams will be deposited within this spherical volume rather than on a point target. The mean volume of this region is

$$\Delta V \approx 4/3\pi \Delta S_{\perp}^3 \approx 20 \text{ cm}^3.$$

The duration of this energy pulse Δt may be simply estimated from the uncertainty parameter $\Delta S_{\perp}$.

$$\Delta t = \frac{\Delta S_{\perp}}{v_I} = \frac{t_0 \Delta S_{\perp}}{R} = t_0 \sqrt{\frac{\Delta W_{\perp}}{W_I}} \approx 10 \text{ ns}.$$

The power associated with the deposition of energy into this target site is

$$P = \frac{\bar{n} W_I}{\Delta t} = \frac{10^{17} \cdot 20 \text{ keV}}{10 \text{ ns}} = 10^{10} \text{ W}.$$

Numerical Solution of Model

GENERAL APPROACH

To establish a formalism through which accurate predictions of the convergence-bunching processes can be made as they proceed, a computer code was written that predicts the evolution in time of a distribution function

$f(r,v,t)$. The initial source distribution function is taken to be a solution of the Vlasov equation (derived in Appendix B) for a half-Maxwellian accelerated through the source electrode structure.

The code then follows the evolution of the distribution function in time by using Liouville's theorem (Appendix C). It can use the distribution function to get exact numerical solutions (within the uncertainties discussed in the previous chapter) for the convergence phenomena. In particular, the distribution function is necessary for calculating number densities, rate parameters, and ionization probabilities.

A major assumption made for this technique of using convergent neutral beams to create a transient, high-density plasma was that the beams are essentially noninteractive as they proceed toward the target site. The numerical solutions of the ionization probabilities for these beams show that this assumption of weak interaction is well verified.

Subsequent sections show the equations and calculations made. The general approach to the numerical solution may be described as follows:

- Number density: The number density is found by numerical integration of the distribution function over appropriate velocity-space limits.
- Rate parameter: The rate parameters $\overline{\sigma v}_{NN}$ and $\overline{\sigma v}_{NI}$ are calculated from a five-dimensional integration scheme over the velocity-space limits of the two reacting particles. The cross sections are tabulated in the code from curve fits to existing experimental data.
- Ionization probabilities: These probabilities are calculated from the rate parameters through numerical integration up to a time of interest (corresponding to when the particles are said to be within the target site ($p \leq AS_1$)).

DISTRIBUTION FUNCTION

The variation of the distribution during the acceleration process was derived as an exact solution to the Boltzman equation without collisions or forces.

Figure B-1 shows the source distribution of the ions that result from acceleration through a potential $V = Mv_0^2/2e$. After acceleration, the ions are charge neutralized and emerge at the source with a directed velocity in the longitudinal direction much greater than the velocity components in the transverse directions. It is important to note that the source distribution that emerges is Gaussian in the velocity components perpendicular to the normal and approximately exponential in the longitudinal components. This characteristic of the distribution function, which is especially true for observation points greater than the uncertainty ΔS_1 , will allow intuitive analytical approximations¹⁰ for the system of convergent neutral beams.

The distribution function at the source (Appendix B) is

$$f_s(0) = \begin{cases} \frac{2\alpha^2}{\pi} J_0 \exp \left[-\alpha (v^2 - v_0^2) \right] & \text{for } v_\xi > v_0 \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

The neutrals will be emitted radially from the sources directed toward the center of the sphere or cylinder. In terms of spherical coordinates, the radial component of the velocity is $v \cos \theta$. The spherical coordinate system used in this analysis is defined as follows (see Fig. 1):

R is the radius of the sphere, the distance from the source S to the center of the sphere O .

r is the distance from the source S to an observation point P .

ρ is the distance from the center of the sphere O to an observation point P .

θ is the angle that the velocity vector makes with the normal.

η is the spherical polar angle in velocity space.

ξ is the longitudinal coordinate (usual spherical coordinate z).

The modulating velocity will now be expressed in terms of an emission time, i.e., the time when a particle would have to be emitted so that it could reach the observation point in a time t . The modulating velocity at the source had the dependence expressed in Eq. (1). The new dependence, defined in terms of the emission time $t_e = t - r/v$, is

$$v_0(t - r/v) = \frac{v_1}{1 - \frac{(t - r/v)}{t_0}} = \frac{R}{t_0 - t + \frac{r}{v}} \quad (7)$$

The distribution function (using Liouville's theorem, Appendix C) translated in terms of these new coordinates has the form

$$f(r, v, t) = \begin{cases} \frac{M^2 J_0}{2\pi \Delta W_1^2} \exp \left\{ -\frac{M}{2\Delta W_1} \left[v^2 - v_0^2 (t - r/v) \right] \right\} & \text{for } v \cos \theta > v_0 (t - r/v) \\ & \text{and } 0 \leq (t - r/v) \leq t_p \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

This expression is now the distribution function for a spherical geometry at the observation point P and corresponds to Eq. (6) with

$$t \longrightarrow t - r/v$$

$$v_\xi \longrightarrow v \cos \theta.$$

In doing the cylindrical problem, it will be shown that it is possible to express these quantities in terms of variables appropriate to a cylindrical coordinate system so that a similar equation holds.

For an observation point far enough from the source, it is sufficiently accurate to assume an effective uniform current density J_0 at the wall. The source is turned on for some pulse time t_p and then turned off. Therefore, the particles are primarily located between some radii, $\rho_{\min}$ and $\rho_{\max}$, that correspond roughly to the positions of those particles emitted at time $t = 0$ at velocity v_I and those emitted at time $t = t_p$ at velocity v_F .

For $\cos \theta = 1$, $\eta = \pi$,

$$\rho_{\min} = R - v_I t$$

$$v_I = v_0 (t_e = 0)$$

$$\rho_{\max} = R - v_F t$$

$$v_F = v_0 (t_e = t_p).$$

NUMBER DENSITY

The number density for the system of convergent neutral beams may be calculated by numerically integrating over all velocity space; the density will simply be the first moment of the distribution function.

$$N(r,t) = \int_{\text{all velocity space}} f(r,v,t) d^3v. \quad (9)$$

The integration schemes are given next along with an explanation of the numerical limits used for each integration.

Spherical Geometry

The velocity volume element in spherical coordinates (where azimuthal symmetry has been assumed) is $2\pi v^2 \sin\eta d\eta dv$. The expression to be integrated numerically is

$$N(r,t) = 2\pi \int_0^\pi \sin\eta \, d\eta \int_0^\infty f(|v|, \eta, \rho, t) v^2 dv. \quad (10)$$

The interdependence of the variables in real and velocity space (Fig. 1), can be derived from the law of cosines:

$$\mu = \cos\eta = (\rho^2 - R^2 + r^2)/(2\rho r), \quad (11)$$

$$r = \rho\mu + [R^2 - (1 - \mu^2)\rho^2]^{1/2}, \quad (12)$$

$$\cos\theta = (R^2 + r^2 - \rho^2)/(2rR). \quad (13)$$

Several things must be taken into account when calculating the limits on speed $|v|$ and angle η in velocity space. The primary motivation is to ascertain where most of the particles are in the distribution for any chosen position and time. Mathematical approximations based on the physics involved will show the dependence of various parameters of the system. How these parameters vary in the distribution function will allow definition of appropriate limits of integration.

From Eq. (8), note that the distribution function is zero unless

$$\frac{rcos\theta}{t} > v_0(t_e). \quad (14)$$

There is a minimum velocity associated with the emission time $t_e = t - r/v$. The emission time is determined from what this time must be for particles to reach an observation point in time t . If ρ and μ are chosen, Eqs. (12) and (13) give r and θ .

If Eq. (14) is satisfied,

$$v_{\min} = r/t. \quad (15)$$

Otherwise, the minimum velocity is calculated by solving Eq. (7) for $v = v_{\min}$, with the result

$$v_{\min} = \frac{R - r \cos \theta}{(t_0 - t) \cos \theta}. \quad (16)$$

The maximum allowable velocity would be

$$v_{\max} = r/(t - t_p).$$

Particles with greater velocities would have passed the observation point. These limits are shown roughly in Fig. 4. In the actual numerical integration over the velocity variable $|v|$, $v_{\max}$ is determined by where the integrand drops off significantly so as to include only the pertinent area under the curve.

The integration over speed $|v|$ is

$$F'(\rho, \eta, t) = 2\pi \int_{v_{\min}}^{v_{\max}} f(\rho, v, \eta, t) v^2 dv.$$

The value of the integrand may be defined as

$$F(\rho, \eta, v, t) = f(\rho, v, \eta, t) v^2$$

and evaluated at various points within the velocity limits. A trapezoidal numerical integration for this integral is

Fig. 4. The limits on velocity variable v for $t > t_p$. When $\eta \approx \pi$, $\cos\theta = 1$, $r = R - \rho$.

$$F'(\rho, n, t) \approx 2\pi dv \left[\sum_{N=1}^{N_{\max}} F_N - \frac{F_1}{2} - \frac{F_{N_{\max}}}{2} \right]$$

where $N_{\max}$ is chosen to be a sufficient number of integration steps (See Appendix E), and F_N are values of the integrand for grid steps along $|v|$. A sample integration grid is shown schematically in Fig. 5.

In calculating the limits on the polar velocity angle in spherical coordinates, those regions that correspond to the physical location of the particles as convergence is approached are considered. If appropriate approximations are made,¹⁰ analytical solutions may be found for these regions of interest. The approximate solutions and their dependence on the pertinent physical parameters of the system allow an accurate calculation of the limits needed for numerical integration.

An approximate form for the distribution function is given below:

$$f \approx \begin{cases} \frac{2\alpha^2 J_0}{\pi} \exp \left\{ -\alpha \left[\frac{\mu^2 (1 - \mu^2) \rho^4}{(t_0 - t)^2 R^2} + \frac{2\mu^2 \rho^2 (v - v_{\min})}{(t_0 - t) R} \right] \right\} & \text{for } \rho \geq \rho_{\min}; v_{\min} \leq v \leq v_{\max}; \mu_{\min} \leq \mu \leq \mu_{\max} \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

This approximation is readily integrated (See Appendix D) over the $|v|$ limits $v_{\min}$ and $v_{\max}$ with the result being

$$F'(\rho, \mu, t) \approx \frac{2\alpha J_0 R}{t_0 - t} \exp \left\{ -\alpha \frac{[\rho^4 \mu^2 (1 - \mu^2)]}{(t_0 - t)^2 R^2} \right\} \text{ for } \mu_{\min} \leq \mu \leq \mu_{\max} \quad (19)$$

$$F'(\rho, \mu, t) \approx 0 \quad \text{otherwise.}$$

Figure 6 shows how this function $F'(\rho, \mu, t)$ varies with μ , the cosine of the polar angle in spherical velocity space. At early times, the main part of the distribution is approximately a sharply decaying exponential function of μ . To calculate $\mu_{\max}$ here, the criterion is stipulated that the significant contribution to the integral is that portion where the integrand decreases by two orders of magnitude. This truncation value is defined in terms of the parameters of the system:

$$\mu_L = -\sqrt{0.5 + \sqrt{0.25 - L}},$$

where

$$L \triangleq \frac{4.614 (t_0 - t)^2 R^2}{\alpha \rho^4}.$$

Fig. 5. Sample integration grid for numerically calculating the integral over the velocity variable v , where $F'(\rho, \eta, v, t) = K \exp[-\alpha(v^2 - v_0^2)] v^2$.

Fig. 6. Variation of $F'(\rho, \mu, t)$ with μ .

a) Early time variation ($t \ll t_0$; $\rho \gg \Delta S_L$; $\mu^2(1-\mu^2) \approx 2\delta$;

b) Variation for times very close to convergence ($t \rightarrow t_0$, $\Delta S_{||} \ll \rho < \Delta S_L$, $F' \approx \text{constant} = K \exp[-\alpha(v^2 - v_0^2)] v^2$).

As time progresses, i.e., as the collection of particles proceeds toward convergence, this exponential broadens. When the particles get inside the convergence region, the distribution is mainly located between two values of μ : μ_{co} and μ_{trans} , which are determined by the time t and the position ρ . For these times near convergence, the distribution at any velocity v is roughly constant between μ_{co} and μ_{trans} . These cutoff points are determined by considering those angles at which all particles must have either past or not yet arrived. Thus, μ_{co} represents that value of $\mu = \cos\eta$ for which at some chosen ρ and t all the particles emitted from the sources have gone by, i.e., when the velocity limits are equal. Equating Eqs. (16) and (17) and using Eqs. (12) and (13) to solve for this critical value of μ will give the following result:

$$\mu_{co} = \frac{\frac{-\rho_{max}}{\rho} \left[1 - \frac{\rho^2}{R\rho_{max}} \right]}{\sqrt{1 + \frac{\rho^2}{R^2} - \frac{2\rho_{max}}{R}}} \quad \text{for } \rho > \rho_{max},$$

(20)

$$\mu_{co} = -1 \quad \text{for } \rho < \rho_{max},$$

where μ_{trans} is a transitional point where the values of $F'(\rho, \mu, t)$ start to drop off significantly. Here, the formula for v_{min} goes to r/t instead of Eq. (16) and can be seen schematically in Fig. 6. The value of μ at this point is

$$\mu_{trans} = \frac{\frac{-\rho_{min}}{\rho} \left[1 - \frac{\rho^2}{R\rho_{min}} \right]}{\sqrt{1 + \frac{\rho^2}{R^2} - \frac{2\rho_{min}}{R}}} \quad \text{for } \rho > \rho_{min},$$

(21)

$$\mu_{trans} = -1 \quad \text{for } \rho < \rho_{min}.$$

These two values, μ_{co} and μ_{trans} , effectively limit the contribution to the distribution function. For purposes of accuracy, a small additional term $\Delta\mu_e$ is also necessary to extend the integration domain beyond μ_{trans} . The dependence of this term on the parameters of the system is

$$\Delta\mu_e = \frac{4.614}{2\alpha\rho v_I} \frac{R}{I}. \quad (22)$$

The limits for the spherical polar angle ($\mu = \cos\eta$) in velocity space are listed below:

<ul style="list-style-type: none"> • For $L \leq 0.25$ and $\mu_{trans} + \Delta\mu_e \geq \mu_L$ • For either $L > 0.25$ or $L \leq 0.25$ and $\mu_{trans} + \Delta\mu_e < \mu_L$ 	{	$\begin{aligned} \mu_{min} &= \mu_{co} \\ \mu_{max} &= \mu_L \\ \mu_{min} &= \mu_{co} \\ \mu_{max} &= \mu_{trans} + \Delta\mu_e \end{aligned}$
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Once the limits on μ have been ascertained, the integration itself is accomplished by using a 40-point Gaussian integration scheme. The number density is then

$$N(\rho, t) = \int_0^{2\pi} F'(\rho, \eta, t) \sin\eta d\eta = \int_{-1}^1 F'(\rho, \mu, t) d\mu$$

$$\approx \int_{\mu_{min}}^{\mu_{max}} F'(\rho, \mu, t) d\mu \approx \sum_{k=1}^{N_{max}=40} a_k F'(\mu_k). \quad (23)$$

The function F' is evaluated at the grid point μ_k and weighted by a factor a_k . The Gaussian integration constants are all scaled appropriately for the newly defined limits.

Approximate formulae for the number densities were also derived⁴ for regions of time pertinent to the system. By integrating Eq. (19) (assuming η is sharply peaked about π), an approximate expression for the number density at early times is

$$N(\rho, t) \approx \frac{J_0 R^3 (t_0 - t)}{\rho^4} \text{ for } \rho_{\min} > \rho > \rho_{\max}. \quad (24)$$

To further reinforce the validity of the approximate and exact numerical calculations for early times, the system of convergent neutral beams was analyzed in macroscopic terms. The results derived¹⁰ for the number densities at early times are independent of the distribution assumed for the statistical formulation in this report. This derivation is shown in Appendix G for both the spherical and cylindrical geometries. It was found that a whole family of acceptable solutions is possible.

At late times (times within the convergence region), the number densities vary as $1/\rho$ between $\rho_{\max}$ and $\Delta S_{\perp}$. This may be seen by assuming that Eq. (19) is nearly constant between μ_{trans} and μ_{co} . The number density for this time region is

$$N(\rho, t) \approx \frac{2\alpha J_0 R (\mu_{\text{trans}} - \mu_{\text{co}})}{(t_0 - t)}. \quad (25)$$

Fig. 7. Variation of number density with distance from target site.

Fig. 8. Variation of number density with distance from target site for last few nanoseconds.

To illustrate exact numerical results obtained from the code. Fig. 7 shows on a log-log scale snapshots of the number density at successive time intervals during the convergence process. Figure 8 shows the last stages on an expanded linear time scale. The source parameters are the same as those given in Table 1.

In Fig. 7, a momentary sharp spike in density appears to occur at the exact center. Actually, this spike is a result of the smallness of the longitudinal positional uncertainty $\Delta S_{||}$ of the target cloud. It would be expected that in an actual system, geometrical inaccuracies in the source system would substantially weaken the spiking tendency. The dashed curve shown in Fig. 7 is a rough estimate of this effect for the case where the inaccuracies in source positioning at radius R are about ± 0.5 cm.

Cylindrical Geometry

In doing the cylindrical problem, it was found that the parameters used in the spherical case could be expressed in terms of variables appropriate for a cylindrical coordinate system. A judicious redefinition of variables¹⁰ allows the use of the same equations and coding for the distribution function as in the spherical case.

Figure 9 shows a source position S and an observation point P defined in a typical cylindrical coordinate system. The source position is defined by r , θ , and z . The projection of r on the $\xi = 0$ axis is defined as s . The velocity vector is defined by v , η and v_z . Similarly, the projection of v on the $\xi = 0$ axis is defined as v_s . Then by similar triangles, $r/s = v/v_s$, and $r/z = v/v_z$.

Given the coordinates of an observation point ρ, η , and t , the following relationships can be derived using these redefinitions:

- ρ is now the distance along the x-axis from the center of the cylindrical coordinates system;
- r is the distance from the source to this observation point making an angle θ normal to the ξ axis; and

$$\mu = \cos\eta = \frac{\rho^2 + s^2 - R^2}{2\rho s}$$

$$s = \rho\mu + \sqrt{R^2 - (1 - \mu^2)\rho^2}.$$

The code uses the same equations to calculate s and ϕ as were used for the coordinates in the spherical case (r and θ). From the right triangle made up of the vectors r and R , a relationship for $\cos\theta$ in terms of v_s is

$$v\cos\theta = v_s \cos\phi$$

The distribution function can now be defined for the cylindrical coordinate system. Liouville's theorem is again applicable for translating the definition of the distribution function to the observation point.

The velocity limit $v\cos\theta$ and the term r/v , which enters into the emission time $t-r/v$ in the spherical case, are now expressed in the cylindrical case in terms of the projections onto a plane normal to the axis of the cylinder.

$$r/v \rightarrow s/v_s$$

$$v\cos\theta \rightarrow v_s \cos\phi.$$

The distribution function for the cylindrical geometry is

$$f(\rho, v_s, \eta, v_z, t) = \begin{cases} \frac{2\alpha^2 J_0}{\pi} \exp \left[-\alpha (v_s^2 + v_z^2 - v_0^2) \right] & \text{for } v_s \cos\phi \geq v_0 (t - s/v_s) \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

where the current density J_0 and the modulating velocity v_0 are evaluated at the cylindrical emission time, $t-s/v_s$. The subsequent integrations for the number density are done the same way as for the spherical case.

Figure 10 shows the number density versus ρ for different times and may be compared to a similar plot for spherical geometry (Fig. 7). Note that the densities are smaller than for the spherical case. The distributions look the same as they leave the source. In the spherical case, however, particles come in from both perpendicular directions, and the distribution gets broader in both perpendicular velocity-space coordinates.

IONIZATION PROBABILITIES

In addition to calculating the number densities anywhere inside the containment sphere, other physical parameters of the system may also be calculated. Using the distribution functions and number densities already derived, the ion densities, rate parameters, and ionization probabilities may be calculated for the converging beams.¹¹

In calculating ionization probabilities, several assumptions were made:

- That neutral beams are nearly noninteractive as they proceed toward the target site.
- That the form of the distribution function is not appreciably changed due to subsequent collisions when the beam is within the target site. This is a reasonable assumption because the energy of the neutral atoms is great compared to the energy that they exchange during collisions. The energy exchanged in such a reaction is typically of the order of 10 eV, and the energy of the incoming neutral is about 20 keV. It is therefore assumed that the ion distribution function has the same form as the unperturbed neutral distribution function, but is reduced in magnitude by the ratio of ion density to neutral density.

$$f_i \approx \frac{n_i}{n_0} f_0.$$

Similarly, the new neutral density will be reduced by the number of ions formed.

$$f_n \approx \frac{(n_0 - n_i)}{n_0} f_0.$$

- That certain terms in the rate equation could be neglected because their contribution is small. The equation used for the build up of ions is

$$\frac{\partial n_I}{\partial t} = n_N^2 \overline{\sigma v_{NN}} + n_N n_I \overline{\sigma v_{NI}} \quad (27)$$

The first term represents ionization by impact of two neutrals, and the second term is due to the interaction between neutrals and ions.

The following sections will show methods and results achieved for the numerical solutions of the ionization probabilities for these beams and will verify the major assumption of noninteraction.

The general form for the rate parameter is a six-dimensional integral given by^{12,13}

$$\overline{\sigma v} = \frac{1}{2} \int d^3 v \int d^3 v' \sigma(|v - v'|) |v - v'| f(v) f(v'). \quad (28)$$

The cross sections, evaluated from a table of published experimental data¹⁴ are shown in Fig. 11.

The probability that an atom will be ionized in a time dt is

$$\begin{aligned} P(t)dt &= \left(n_N \overline{\sigma v_{NN}} + n_I \overline{\sigma v_{NI}} \right) dt \\ &= \left[\left(n_0 - n_I \right) \overline{\sigma v_{NN}} + n_I \overline{\sigma v_{NI}} \right] dt. \end{aligned}$$

If this equation is integrated over time, an ionization probability may be calculated.

$$P_{ion}(t) = \int_0^t P(t') dt'. \quad (29)$$

Fig. 10. Number density for a cylindrical coordinate system as a function of distance from the center of the target site.

Fig. 11. Atom-atom and proton-atom ionization cross sections shown as a function of the relative velocity between the participating particles.

The number of ions formed at any time is given by

$$n_I(t) \approx n_0(t) P_{ion}(t). \quad (30)$$

This approximation will remain true as long as the number of ions is much less than the number of neutrals and is [Eq. (27)] over-estimated by a fractional amount n_I/n_0 (see Appendix F).

Spherical Geometry

Equation (31) shows the calculation necessary to find a solution for the rate parameter $\overline{\sigma v}$ for any process in a spherical configuration where azimuthal symmetry exists.

$$\begin{aligned}
\left(\frac{\sigma}{v_{ab}}\right)_{\text{sph}} = & \frac{2\pi}{n_a n_b} \int_0^\infty v^2 dv \int_{-1}^1 d\mu f_a(v, \mu) \int_0^\infty v'^2 dv' \\
& \times \int_{-1}^1 d\mu' f_b(v', \mu') \int_0^{2\pi} \sigma(u) u d\zeta. \quad (31)
\end{aligned}$$

The usual angles of spherical coordinate velocity space are contained in μ and ζ , where

$$\mu = \cos \eta,$$

$$\zeta = \phi - \phi'.$$

This reduces a six-dimensional integration in velocity space to a five-dimensional one. The form of the relative velocity in these coordinates is

$$u = \left[v^2 + v'^2 - 2vv' (\mu\mu' + \sqrt{1-\mu^2} \sqrt{1-\mu'^2} \cos\zeta) \right]^{1/2} \quad (32)$$

Figure 12 shows a plot of the rate parameters versus time for a spherical geometry where ρ was chosen to be near the front edge of the neutral-beam pulse. This ρ corresponds to a position of the neutral particles at a particular time assuming they left the source first with the slowest velocity.

Figure 13 shows the integration of the rate parameter to get the ionization probability [Eq. (29)]. As convergence is approached, the ionization probability increases rapidly. This increase is due in part to the increase in the relative velocities of the particles as they approach convergence (which leads to an increase in the cross section until it levels off). There is also a rapid increase in the number density as convergence is approached. For a point in the distribution near the front of the pulse, 10 ns before convergence, only 3% ionization occurs.

Figures 14 and 15 show the rate parameter and ionization probabilities for a position corresponding to the back of the neutral-beam pulse. For this point in the distribution, as the pulse collapses toward the target, the probabilities become larger because the velocities are larger (80 keV instead of 20 keV in the test case). Very close to convergence, the probability approaches one; however, the formulas that were used (Appendix F) overestimate this value.

Cylindrical Geometry

The rate parameter and relative velocity for a cylindrical geometry are

$$n_a n_b \overline{\sigma v}_{ab} = \int_0^\infty v_s dv_s \int_0^{2\pi} d\eta \int_0^\infty v'_s dv'_s \int_0^{2\pi} d\eta' \int_{-\infty}^\infty dv_z \\ \times \int_{-\infty}^\infty dv'_z f_a(v_s, \eta, v_z) f_b(v'_s, \eta', v'_z) \sigma_{ab}(u) u, \quad (33)$$

$$u = \left[v_s^2 + v'_s{}^2 - 2v_s v'_s (\cos\eta \cos\eta' + \sin\eta \sin\eta') + (v_z - v'_z)^2 \right]^{1/2}. \quad (34)$$

The rate equation is again a six-dimensional integral. The velocity-volume coordinates considered now have a v_z component. A judicious transformation of variables and redefinitions¹⁰ again reduces the integrations required to five by defining new variables as the sum and difference of the directed velocity component.

$$w = \frac{1}{\sqrt{2}}(v_z + v'_z)$$

$$z = \frac{1}{\sqrt{2}}(v_z - v'_z)$$

The product of the distribution functions $f_a f_b$ is then proportional to $\exp[-\alpha(w^2 + z^2)]$.

Fig. 12. Rate parameter versus time for a spherical geometry ($\rho = 1.05\rho_{\min}$) near the front of the beam pulse.

Fig. 13. Ionization probability versus time for a spherical geometry ($\rho = 1.05\rho_{\min}$) near the front of the (beam) pulse.

Fig. 14. Rate parameter versus time for a spherical geometry ($\rho = 0.975\rho_{\max}$) near the back of the beam pulse.

Fig. 15. Ionization probability versus time for a spherical geometry ($\rho = 0.975\rho_{\max}$) near the back of the beam pulse.

Since w doesn't enter into the relative velocity u and is only contained in the distribution function, the exponential portion $e^{-\alpha w^2}$ can be taken out and integrated separately, giving the factor $(\alpha/\pi)^{1/2}$. The rate parameter in terms of these variables is

$$\begin{aligned} n_a n_b \overline{\sigma v}_{ab} = & \sqrt{\frac{\alpha}{\pi}} \int_0^\infty v_s dv_s \int_0^{2\pi} d\eta f_a''(v_s, \eta) \int_0^\infty v_s' dv_s' \\ & \times \int_0^{2\pi} d\eta' f_b''(v_s', \eta') \cdot 2 \int_0^\infty dz e^{-\alpha z^2} \sigma(u) u. \quad (35) \end{aligned}$$

The formalism of the code for the case of a cylindrical geometry is the same with the exception of a slightly different u and a different weight function in the integrand. The volume element now contains $v_s dv_s$ instead of $2\pi s \sin \eta$.

Figure 16 shows the rate parameter $\overline{\sigma v}$ versus time for cylindrical coordinates. The same values of ρ as in the spherical case were chosen for comparison.

Figure 17 shows the ionization probabilities for a ρ near the front of the neutral-beam pulse. At 10 ns before convergence, the probability is 0.02%. This probability is smaller than for the spherical case. The distribution looks the same as it leaves the source, but the ionization rate is less in the cylindrical case. In the spherical case, particles come in from both perpendicular directions, and the distribution gets broader in both perpendicular velocity-space coordinates. The density is less in the cylindrical case due to this loss of one dimension.

Figures 18 and 19 show the same information for a point in the distribution toward the back of the beam. The same statements hold as for the spherical case. The particles emitted later have a higher velocity, so the degree of ionization 10 ns before convergence (0.2%) is increased.

COMPUTER CODE

The code is written for use on the CDC 7600 computer and is written in the LRLTRAN version of the FORTRAN IV language.

Fig. 16. Rate parameter versus time for a cylindrical geometry ($\rho = 1.05\rho_{\min}$) for the front of the beam pulse.

Fig. 17. Ionization probability versus time for a cylindrical geometry ($\rho = 1.05\rho_{\min}$) at the front of the beam pulse.

Fig. 18. Rate parameter versus time for a cylindrical geometry ($\rho = 0.975\rho_{\max}$) at the back of the beam pulse.

Fig. 19. Ionization probability versus time for a cylindrical geometry ($\rho = 0.975\rho_{\max}$) at the back of the beam pulse.

The code occupies approximately $(40000)_8$ core locations. The code runs fully in core, but could be easily modified to run using disc storage should subsequent versions require it.

Running time on the CDC 7600 for a sample problem was 3 minutes. The timing depends strongly on the number of steps taken in each of the six nested integration loops for the numerical calculation of Eq. (29). In the flow chart of the code, Fig. 20, the optimum number of integration steps is marked for each loop. By raising the number of steps until no differences appeared in the answer, it was possible to guarantee that sufficient integration steps were used for each integral. However, a more accurate analytical method was derived¹⁰ using the error term for trapezoidal integration (see Appendix E).

Discussion of Plasma Creation

Thus far in this report, no predictions have been made about the subsequent history of the newly created plasma. It is hoped that the energy will be deposited at the target site before scattering and that ionization effects could appreciably influence the final convergence process. However, problems may arise from electron heating, the creation of self-electromagnetic fields, and the onset of instabilities. These possibilities will now be briefly discussed.

Calculations of the ionization probabilities ignored various additional mechanisms such as electron heating that could become important as the ion and electron densities increase. Newly created ions have nearly all the energy of the parent neutral; the electron has little energy (≈ 10 eV). However, once ionization has occurred, the energy gained by the electron between collisions can produce additional ionizations via a cascading effect.¹⁵

Analysis of the subsequent history of the newly created electrons is done by examining the electron transport equations. These equations are derived by Braginskii by taking the moments of the Boltzman equation¹⁶ and incorporating the conservation (mass, momentum, energy) laws.

The electron transport equations delineating the important mechanisms pertinent to this problem⁸ are listed below.

Fig. 20. Flowchart of code to calculate various parameters of convergent neutral beams.

$P(t) = \int dt \int d\delta \int d\delta' \int f v^2 dv \int f' v'^2 dv' \int \sigma(u) u d\xi$, where u is a function of $t, v, v', \delta, \delta'$.

Mass transport:

$$\frac{\partial n_E}{\partial t} + \text{div}(n_E v_E) = S_E \quad (36)$$

Momentum transport:

$$m_E n_E \frac{dv_E}{dt} = - \frac{\partial p_E}{\partial x} + en_E E + R, \quad (37)$$

Electron energy transport:

$$\frac{3}{2} n_E \frac{dT_E}{dt} + p_E \text{div } v_E = Q_E - \frac{3}{2} n_E S_E. \quad (38)$$

The variables in Eqs. (36) through (38) are:¹⁶

- m_E = mass of the electron
- n_E = electron number density
- $\bar{v}_E$ = average macroscopic velocity of fluid
- p_E = isotropic pressure
- T_E = electron temperature at a time t
- e = charge of an electron
- E = electric field

The electron energy transport equation neglects terms representing anisotropic pressures, radiation, thermal conduction, and the effect of a magnetic field. If $\lambda_{\text{Debye}} \ll \rho$, as in this case, the electrostatic and pressure gradient forces on the electrons must be closely in balance. The electrons, being light and mobile, would be accelerated to high energies very quickly if there were a net force on them. If the electrons left a region, a large ion charge would remain, and the plasma tendency toward charge neutrality would be violated. The slight separation of charge gives rise to an electric field that binds the electron cloud about the ions so that the electrons are constrained to move with them.¹⁷

The source term S_E represents the integral over velocity space of the inelastic collision term in the Boltzman equation.

$$S_E = \sum_{\text{all species}} \int C_{\alpha\beta} dv,$$

where $C_{\alpha\beta}$ describes any possible interaction process between species α and species β .

In a partially ionized plasma, the three constituents (electrons, ions, and neutrals) can produce subsequent ionizations via the following processes:

Therefore,

$$S_E = n_N^2 \overline{ov}_{NN} + n_N n_I \overline{ov}_{NI} + n_N n_E \overline{ov}_{NE}. \quad (39)$$

Various other possible mechanisms are neglected because the rate parameter $\overline{ov}$ is small in the energy range considered here.

The quantity R represents the mean change in the momentum of the electrons due to collisions with all other particles.

$$R = \sum_{\substack{\text{all} \\ \text{species}}} \int m v' C_{\alpha\beta} dv.$$

The rate of electron heat generation Q_E , resulting from collisions between the constituent particles represents the following processes:

$$Q_E = \frac{M_E v'^2}{2} C_{\alpha\beta} dv = \frac{3}{2} n_E \frac{dT_E}{dt} \left\{ + \frac{3}{2} T_{\text{source}} S_E - n_E n_N \overline{ov}_{NE} E_{\text{ion}} \right. \\ \left. \text{Spitzer} \right. \quad (40)$$

The first term on the right-hand side is the Spitzer heating term¹⁸ due to interactions between the ions and the electrons. The second is the effect on the temperature rate due to electrons being born at a low energy (T_{source} is assumed to be 6.8 eV). The last term is a cooling effect on the electron temperature rate since the electrons lose energy when they cause an ionization. The energy required to form an ion pair, E_{ion} , is 40 eV, and includes various processes (such as excitation) for possible interactions between neutrals and electrons.

The electron pressure P_E was considered isotropic since the electrons thermalize (to a Maxwellian distribution) quickly and set equal to $n_E k T_E$. The expression used for the heating of electrons due to the compression term ($P_E \operatorname{div} \bar{v}_E$) in the spherical geometry is

$$P_E \operatorname{div} \bar{v}_E = n_E k T_E \left[\frac{1}{2} \frac{\partial}{\partial \rho} (\rho^2 \bar{v}_E) \right] .$$

Using the approximation for early times of

$$\bar{v}_E \sim -\frac{\rho}{\epsilon} ,$$

$$\frac{2 P_E \operatorname{div} \bar{v}_E}{3 n_E} = -\frac{2}{\epsilon} T_E$$

Eq. (38) then becomes,

$$\frac{dT_E}{dt} = \frac{2}{3} \frac{Q_E}{n_E} - \frac{T_E S_E}{n_E} + \frac{2 T_E}{\epsilon} , \quad (41)$$

or

$$\frac{dT_E}{dt} = \frac{dT_E}{dt}_{\text{Spitzer}} + \frac{T_{\text{source}} S_E}{n_E} - \frac{2}{3} n_N \overline{\sigma v}_{NE \text{ ion}} - \frac{T_E S_E}{n_E} + \frac{2 T_E}{\epsilon} . \quad (42)$$

An equilibrium temperature for which the electron temperature rate was zero was calculated from Eq. (41) and was assumed to provide an upper bound for any position and time. The rate parameter $\overline{\sigma v}_{NE}$ was calculated for this temperature, and the ionizations due to ions was compared to the ionizations due to electrons. Within the constraints of the assumptions used (including a perhaps overly conservative estimate of $\overline{\sigma v}_{NE}$), some pertinent parameters of the system of convergent neutral beams were made for interesting times and positions. These are enumerated in Table 2. Note that even where the rate parameter $\overline{\sigma v}_{NE}$ is greater than $\overline{\sigma v}_{NI}$, the ionization rate due to neutral-neutral interactions is always higher than the electron (or ion) rates, at least up to those times defined to be within the target site. It is therefore hoped that the degree of ionization due to electrons would remain small up to that point ΔS_1 where the convergence process has been essentially completed. There would not be enough time left for any subsequent mechanism to appreciably alter either the final density or the convergence of the beam.

Table 2. System parameters as a function of time and position.

	Time (μ s)	ϵ (ns)	ρ (cm)	Neutral density,	Ion density	$\overline{\sigma v}_{NN}$	$\overline{\sigma v}_{NI}$	$\overline{\sigma v}_{NE}$	T_{eo} (eV)	RNN=	RNI=	RNE=
				n_N	$n_I = n_E$					$n_N \overline{\sigma v}_{NN}$	$n_N n_I \overline{\sigma v}_{NI}$	$n_N n_I \overline{\sigma v}_{NE}$
$\rho = 1.05 \rho_{min}$	2.070	72	10.6	2.5E14	3.0E10	6.0E-11	1.3E-10	6.3E-9	8.6	3.8E18	9.1E14	4.4E16
	2.102	40	6.0	1.0E15	2.9E12	2.2E-10	5.2E-10	1.2E-8	13.7	2.2E20	1.5E18	3.4E19
	2.112	30	4.5	1.7E15	9.9E12	2.8E-10	6.7E-10	1.2E-8	13.7	8.1E20	1.1E19	2.0E20
	2.131	10	1.7	5.7E15	1.5E14	3.5E-10	8.6E-10	1.9E-8	22.7	1.1E22	7.4E20	1.6E22
$\rho = 1.95 \rho_{min}$	2.102	40	10.9	1.2E14	3.0E9	5.3E-11	1.2E-10	5.9E-9	8.5	7.6E17	4.3E13	2.1E15
	2.119	23	6.3	6.3E14	1.2E12	7.3E-10	2.1E-9	9.2E-9	10.6	2.9E20	1.6E18	7.0E18
	2.126	16	4.5	1.9E15	2.6E11	2.1E-9	7.0E-9	1.0E-8	11.7	7.6E21	3.5E20	5.2E20
	2.133	9	2.7	1.1E16	4.6E15	7.5E-9	3.1E-8	2.1E-8	27.0	9.1E23	1.6E24	1.1E24

Another consequence that may possibly occur within the target site due to the creation of a plasma is the growth of instabilities. These would affect the ionization rates. The Buneman two-stream and Weibel instabilities⁸ seem to have growth rates greater than the ionization rate. The creation of self fields could give rise to a plethora of plasma instabilities. Accurate calculations of these growth rates could be done in the program using the distribution function. Since the distribution function becomes more isotropic as time goes on, it is hoped that this would help to damp out the instabilities and that the energy of the neutral-beam pulse would be deposited at the target site unimpeded.

Additionally, subsequent analysis of the newly created plasma would examine the effects of external magnetic fields. These would help to confine the plasma as the neutral cloud turns into plasma. The B lines would be frozen in as the nonconductor became a conductor, thereby counteracting the plasma pressure attempting to push the charged particles to the wall.

Summary

On the basis of the analysis and calculations made in this proposal and in view of advances in high-current, pulsed neutral-beam technology, it is believed that the convergence-bunching technique could have important application in fusion power research.

Some possibilities for using the high power densities and high instantaneous momentum fluxes associated with the converging neutral beams, could have important applications in fusion power research. For example:

- High-beta or reversed-field states in intense magnetic fields might be created on a nanosecond time scale. Some such states appear to be intrinsically inaccessible via conventional methods where plasma-buildup times are long compared to particle loss times.
- In high-beta regions, transients may be stabilized against the onset of instabilities or a potential well could be created so that electrons are trapped in the well as the ion density increases.
- In a cylindrical geometry, a momentum-stabilizing pinch could inhibit the kink instability.
- In tokamaks, it may be possible to overcome the limitations of ohmic heating and heat a plasma to ignition.

- For inertial confinement, the method might be used as an alternative to laser, electron, and ion beam schemes. Radially convergent neutral beams in either spherical or cylindrical geometry would be used to compress a deuterium pellet.

Other possibilities for utilizing the high peak power densities and high instantaneous momentum fluxes that would be associated with such converging beams will no doubt be proposed as this technique is brought to practical realization.

Acknowledgments

The assistance provided by the following people was invaluable:

Richard F. Post for the initial conception of this proposal and for continuous consultation and encouragement.

Kenneth Marx for his continual guidance and patience and formulation of the equations and approximations.

Charles Hartman and Dan Shumaker for insight, advice, and significant moral support.

Appendix A. Characteristics of Neutral Beams

SPACE CHARGE

One of the fundamental processes which limit the collimation of ion beams is space charge. The law for space-charge-limited current between parallel plates implies that for a given voltage the current flowing from a source to an accelerating electrode will not increase beyond a limiting current density. This forms a fundamental limitation on the ion current attainable from an ion source. The space-charge-limited current between two parallel plates is proportional to

$$J \propto \frac{V^{3/2}}{z^2}, \quad (A1)$$

where V is the potential difference, and z is the distance between the two plates. The ion flow between two parallel electrodes is shown in Fig. A-1. From proportionality, it can be seen that the attainment of a large ion current, as desired by fusion research, requires a small electrode spacing. This consideration and the desire to obtain a low-divergence ion beam motivated the use of a large plasma surface with multiaperture or gridded electrodes.

PLASMA CHARACTERISTICS

The plasma density at the extraction surface must be uniform in space and time over the period of interest and must be free of oscillations. From the space-charge Eq. (A1), the current density depends on the spacing between the electrodes. A spacing that is too small is impractical to obtain at high energies; therefore, the ion density must be reduced by allowing the plasma to expand to a larger volume. Many systematic studies have been undertaken with the object of controlling plasma diffusion to improve the optical qualities of the extracted ion beam.

PLASMA MENISCUS

The shape and uniformity of the plasma meniscus is important in the net focusing of the beam. A slightly concave meniscus of homogeneous plasma with uniform curvature has achieved ion beams in which the divergence is commensurate with the temperature of the plasma ($\Delta\theta \approx T_1$).

ELECTRODE SHAPE

To extract and accelerate beams of maximum intensity and minimum divergence, electrodes having good ion optics must be used. Focusing is accomplished by having lens action balanced against space-charge forces to achieve parallel flow from an inwardly focused beam.

Figure A-2 shows models for the behavior of shaped and unshaped extraction systems.¹⁹ Note that the shaped electrode minimizes the divergence of the beam.

NEUTRALIZATION OF ION BEAM

Space-charge neutralization may be attained by adding electrons across the entire cross section of the beam. The efficiency of the process is energy dependent; the efficiency in neutralizing charge particles decreases as their energy increases. Accel-decel electrode systems are utilized to extract these beams and simultaneously to act as an electrostatic trap for electrons. Calculations, done by computer, can predict optimum size and spacing of these apertures.

Fig. A1. Space charge between two parallel electrodes.

Fig. A2. Models for behavior of (a) shaped and (b) unshaped extraction systems.

Appendix B. Solution to Vlasov Equation Within an Ion Source

The variation of the distribution function in phase space as it travels through the ion electrode structure is given by the Boltzman equation (or Vlasov equation for no collisions).

The steady-state, one-dimensional Vlasov equation for this problem is

$$v_{\xi} \frac{\partial f}{\partial \xi} + \frac{eE_{\xi}}{M} \frac{\partial f}{\partial v_{\xi}} = 0$$

The assumption is made that the electric field $E(\xi, t)$ is a slowly varying (nearly constant) quantity. An initial distribution is hypothesized at the entrance of the acceleration region to be a half-Maxwellian. This assumes that there are no backwards-emerging ions from the plasma source. If the distribution function can be written in terms of the constants of motion, the distribution function after acceleration (the solution to the Vlasov equation) is defined.

This will now be shown to be true for this problem, where the constant of motion is v_x^2 , v_y^2 , and

$$v_{\xi}^2 = \frac{2e}{M} E(\xi + \xi_a) \left[v_x^2, v_y^2, v_{\xi}^2 - \frac{2eE}{M\alpha} (\xi + \xi_a) \right] = \text{const.}$$

Therefore,

$$\begin{aligned} f &= A \exp \left(-\alpha \left[v_x^2 + v_y^2 + v_{\xi}^2 - \frac{2eE}{M\alpha} (\xi + \xi_a) \right] \right) \\ &= A \exp \left[-\alpha (v^2 - v_0^2) \right] \end{aligned} \quad (B2)$$

The ions emerge after acceleration with an additional energy component in the longitudinal direction (ξ) equal to $2eV/M$. Then upon emergence, the initial variables can be rewritten as

$$\begin{aligned} v_x^2 &\longrightarrow v_x^2 \\ v_y^2 &\longrightarrow v_y^2 \\ v_{\xi}^2 &\longrightarrow v_{\xi}^2 - 2eV/M = v_{\xi}^2 - \frac{2eE}{M} \xi. \end{aligned}$$

$$\text{and, } \exp \left[-\alpha (v_x^2 + v_y^2 + v_\xi^2) \right] \longrightarrow \exp \left[-\alpha (v_x^2 + v_y^2 + v_\xi^2 - 2eV/m) \right] \\ \longrightarrow \exp \left[-\alpha (v^2 - v_0^2) \right] .$$

The term $\sqrt{2eV/M}$ is the modulating velocity v_0 as described earlier 1.2.1 and will be a function of time as given by Eq. (1). Both the acceleration process and the evolution of the distribution function are shown schematically in Fig. B-1. In Fig. B-1, the parameter α is the inverse square of the thermal velocity:

$$\alpha = \frac{M}{2kT_I} = \frac{M}{2\Delta W_I} = \frac{1}{v_{ch}^2} . \quad (B3)$$

That this new distribution function is indeed a solution to the Vlasov equation may be verified by substitution.

It is important to note that the source distribution that emerges is Gaussian in the velocity components perpendicular to the normal and approximately exponential in the longitudinal components.

The normalization constant for this distribution function is written in terms of the current density J_0 at the source ($\xi=0$).

$$J_0 = n_0 \bar{v} = n_0 \left[A \int_{-\infty}^{\infty} \exp(-\alpha v_x^2) dv_x \int_{-\infty}^{\infty} \exp(-\alpha v_y^2) dv_y \int_{v_0}^{\infty} v_\xi \exp(-\alpha(v_\xi^2 - v_0^2)) dv_\xi \right] / n_0 \\ = A \left(\frac{\pi}{\alpha} \right) \exp(\alpha v_0^2) \int_{v_0}^{\infty} v_\xi \exp(-\alpha v_\xi^2) dv_\xi = \frac{A\pi}{2\alpha^2} \quad (B4)$$

$$A = \frac{2\alpha^2 J_0}{\pi} . \quad (B5)$$

Fig. B1. Acceleration process and evolution of the distribution function (in real and velocity space), showing (a) ions before acceleration and (b) neutrals (charge-exchanged ions) after acceleration.

Appendix C. Liouville's Theorem

Liouville's theorem states that, in the neighborhood of a phase point that is following its trajectory, the probability density (or distribution function) remains constant.¹⁸ The equations of motion determine the trajectory; therefore, given the distribution in phase space at some initial time, the distribution at some later time will be given by rewriting the distribution function in terms of the new coordinates.

$$f(r,v,t) \equiv f(r_0,v_0,t_0). \quad (C1)$$

In particular, for straight-line trajectories of neutral particles where the velocity is constant (no interactions or accelerating fields), the equations of motion are

$$r = r_0 + vt,$$

$$v = v_0.$$

In the problem described in this paper, the distribution function written at some observation point in terms of the distribution function derived at another point (the source position).

Appendix D. Approximate Representation of Distribution Function

An approximate analytical formula for the distribution function may be derived¹⁰ by expanding Eqs. (12) and (13) for $\rho \ll R$ such that

$$r \approx R + \rho u - (1 - u^2) \frac{\rho^2}{2R}, \quad (D1)$$

$$\cos \theta \approx 1 - \frac{(1 - u^2) \rho^2}{2 R^2}. \quad (D2)$$

The value of $v_{\min}$ [Eq. (16)] then becomes

$$v_{\min} \approx \frac{\rho u + (1 - u^2) \frac{\rho^2}{R}}{(t_0 - t)}. \quad (D3)$$

Evaluating the modulating velocity [Eq. (7)] at this value of v results in

$$v_0(v_{\min}) \approx \frac{-\rho u + (1 - u^2) \frac{\rho^2}{R} + \frac{\rho^3 u (1 - u^2)}{R^2}}{(t_0 - t)}. \quad (D4)$$

Additionally,

$$\left. \frac{\partial v_0}{\partial v} \right|_{v=v_{\min}} \approx 1 + \rho u/R - \frac{3(1 - u^2) \rho^2}{2R^2}. \quad (D5)$$

It is now possible to express the distribution function in terms of these values. The distribution function is defined as

$$f \propto \exp(-\alpha(v^2 - v_0^2)). \quad (D6)$$

After expanding the exponential in a Taylor series about $v_{\min}$ and substituting in the approximate expressions [Eqs. (D1) to (D5)]

$$v^2 - v_0^2 \approx \frac{\rho^4 \mu^2 (1 - \mu^2)}{(t_0 - t)^2 R^2} + \frac{2\rho^2 \mu^2 (v - v_{\min})}{R(t_0 - t)}; \quad (D7)$$

the approximate distribution function is then

$$f(\rho, \mu, v, t) \approx \frac{2\alpha^2 J_0}{\pi} \exp \left(-\alpha \left[\frac{(\rho^4 \mu^2 (1 - \mu^2))}{(t_0 - t)^2 R^2} + \frac{2\rho^2 \mu^2 (v - v_{\min})}{R(t_0 - t)} \right] \right). \quad (D8)$$

This expression can be integrated over the speed v to give the approximate expression

$$F'(\rho, \mu, t) \approx \frac{2\alpha J_0 R}{(t_0 - t)} \exp -\alpha \left[\frac{\rho^2 \mu^2 (\mu^2 - 1)}{R^2 (t_0 - t)^2} \right]. \quad (D9)$$

Appendix E. Error Determination For Trapezoidal Integration

All numerical integrations have an intrinsic error associated with them because the analytical integration is replaced by a sum of terms. Enough of these terms, or grid points, must be used to ensure the desired accuracy of a numerical integration scheme.

The code written for this report uses six nested trapezoidal integrations to calculate the ionization probabilities (Fig. 20). Each integrand must be evaluated at a number of grid points. It is important to optimize this number since the total number of calculations required to approximate six integrations is

$$\prod_{L=1}^6 n_L.$$

There are two ways to determine that a sufficient number of grid points have been used for each trapezoidal integration loop. One way is simply to increase the number of steps until the answers don't change by some predetermined percentage. As an alternate method, an equation was derived for determining the number of grid points necessary to ensure a given percentage accuracy.¹⁰

If the integrand f can be approximated by a quadratic, the percentage error in approximating the integral with a trapezoidal sum is

$$e \approx \frac{2 - \frac{(f_+ + f_-)}{f_0}}{3n^2} \quad (E1)$$

where f_+ , f_0 , and f_- are, respectively, the beginning, middle, and ending values of the integrand

The number of grid points n necessary to ensure a desired percentage accuracy (e.g., for 1%, $e = 0.01$) is

$$n \approx \left| 2 - \frac{f_+ + f_-}{f_0} \right|^{1/2} / 0.03$$

If the integrand can be approximated by an exponential, $\exp(\alpha x)$, then the percentage error incurred by using the trapezoidal integration scheme is

$$e \approx \frac{[\ln(f_+/f_-)]^2}{12n^2} \quad (E3)$$

Therefore,

$$n \approx \frac{\ln(f_+/f_-)}{2 \sqrt{3e}} \quad (E4)$$

Appendix F. Error in the Probability Calculations

The real increase in ion density due to compression and ionization is

$$\frac{dn_I}{dt} = -n_I \bar{v} \cdot v + n_N \overline{\sigma v_{NN}} + n_N n_I \overline{\sigma v_{NI}}. \quad (F1)$$

Since

$$\frac{dn_0}{dt} = -n_0 \bar{v} \cdot v, \quad (F2)$$

Eq. (F1) can be written

$$\frac{d}{dt} \left(\frac{n_I}{n_0} \right) = \left(\frac{1}{n_0} \right) \frac{dn_I}{dt} - \left(\frac{n_I}{n_0^2} \right) \frac{dn_0}{dt} = \frac{n_N}{n_0} (n_N \overline{\sigma v_{NN}} + n_I \overline{\sigma v_{NI}}). \quad (F3)$$

Therefore, the true probability is overestimated by the fraction n_I/n_0 when the approximation $n_I \approx P_{\text{code}} n_0$ is used.

$$\frac{dP_{\text{true}}}{dt} = \left(\frac{n_N}{n_0} \right) \frac{dP_{\text{code}}}{dt} = \left(1 - \frac{n_I}{n_0} \right) \frac{dP_{\text{code}}}{dt}. \quad (F4)$$

Appendix G. Derivation of Number Densities from the Continuity Equation for Times Prior to Convergence

Since the velocity spread of the distribution remains small compared to the streaming velocity for times prior to convergence, the system behaves like a simple cold fluid with no forces or dissipative effects. Hence, the density at early times can be derived as follows from a macroscopic continuity equation:¹⁰

For mass/charge conservation,

$$\frac{\partial n}{\partial t} = \frac{1}{\rho} \frac{\partial (\rho^m n v)}{\partial \rho} = 0, \quad (G1)$$

where $m = 1$ for cylindrical and $m = 2$ for spherical coordinates.

For momentum conservation where the force is zero,

$$\frac{dv}{dt} = 0,$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v \frac{\partial}{\partial \rho}$.

This simply states that the velocity of a fluid element at some chosen observation point (ρ, t) is the same as the value that it had upon emission at the source.

These Eulerian equations can be solved, but the solution can be obtained more directly by transforming to Lagrangian coordinates.

Let the time of emission t_e be the Lagrangian variable which identifies any fluid element.

$$t_e = t_0 - \frac{R}{\rho} (t_0 - t).$$

Inversely,

$$\rho = R(t_0 - t) / (t_0 - t_e).$$

The modulating velocity v_0 is a function to t_e only. From Eq. (7),

$$v_0(t_e) = R / (t_0 - t_e),$$

so that Eq. (G1) can be written as

$$\left(\frac{1}{n}\right) \frac{dn}{dt} = \left(-\frac{1}{\rho^m}\right) \frac{\partial(\rho^m v_0)}{\partial \rho},$$

which transforms to the Lagrangian equation

$$\left(\frac{1}{n}\right) \frac{\partial n}{\partial t} = \frac{m+1}{t_0 - t},$$

with a solution

$$n(t_e, t) = \frac{A(t_e)}{(t_0 - t)^{m+1}}.$$

The quantity $A(t_e)$ is determined by the boundary condition so that the density as it leaves the source is

$$n_0 = \frac{J_0}{v(t_e)};$$

therefore,

$$n(\rho, t) = \frac{J_0 R^{m+1} (t_0 - t)}{\rho^{m+2}}.$$

This expression is the same as Eq. (24) where $M = 2$ for spherical coordinates.

References

1. R. F. Post, T. K. Fowler, J. Killeen, and A. A. Mirin, *Phys. Rev. Lett.* 31, 280 (1973).
2. R. F. Post, Lawrence Livermore Laboratory, private communication.
3. R. F. Post, K. D. Marx, C. J. Eggers, *Nucl. Fusion* 15 (1975).
4. K. D. Marx, C. J. (Eggers) Hartman, and R. F. Post, "Approximate Analytical Formulas for Systems which Employ Convergent Neutral Beams," presented at the Plasma Physics Meeting of the American Physical Society, St. Petersburg, 1975.
5. W. S. Cooper, K. Halbeck, and S. B. Magyary, "Computer Aided Extractor Design," in *Proc. Symp. Ion Sources and Formation of Beams, 2nd, Berkeley, 1974* (Lawrence Berkeley Laboratory, Berkeley, 1974), LBL-3399, Paper II-1.
6. K. A. Berkner, et al., "Performance of LBL 20-kV, 10-A and 50-A Neutral Beam Injectors," in *Proc. Symp. Ion Sources and Formation of Beams, 2nd, Berkeley, 1974* (Lawrence Berkeley Laboratory, Berkeley, 1974), LBL-3399, Paper VI-12.
7. K. W. Ehlers, et al., "Large-Area Plasma Sources," in *Proc. Symp. Ion Sources and Formation of Beams, 2nd, Berkeley, 1974* (Lawrence Berkeley Laboratory, Berkeley, 1974), LBL-3399, Paper I-5.
8. C. W. Hartman, Lawrence Livermore Laboratory, private communication.
9. R. N. Sudan, S. Humphries, and A. Friedman, *Bull. Am. Phys. Soc.* 19, 871 (1974).
10. K. D. Marx, Lawrence Livermore Laboratory, private communication.
11. C. J. (Eggers) Hartman, K. D. Marx, and R. F. Post, "Numerical Calculations of Processes in Convergent Neutral Beams," presented at the Plasma Physics Meeting of the American Physical Society, St. Petersburg, 1975.
12. K. D. Marx, A. A. Mirin, M. G. McCoy, M. E. Rensink, and J. Killeen, *Calculation of $\langle \sigma v \rangle_{DT}$ for Anisotropic Mirror and Toroidal Distributions*, Lawrence Livermore Laboratory Rept. UCRL-77873 (1976); submitted to *Nucl. Fusion*.
13. A. H. Futch, J. P. Holdren, J. Killeen, and A. A. Mirin, *Plasma Phys.* 14, 211 (1972).

14. D. R. Bates and G. W. Griffing, *Proc. Phys. Soc. Lond.* A66, 961 (1953).
15. J. R. Hiskes, Lawrence Livermore Laboratory, private communication.
16. S. I. Braginskii, *Reviews of Plasma Physics* (Consultant's Bureau, New York, 1965).
17. F. F. Chen, *Introduction to Plasma Physics* (Plenum, New York, 1974).
18. L. Spitzer, Jr., *Physics of Fully Ionized Gases*, (Interscience, New York, 1962).
19. J. R. Pierce, *Theory and Design of Electron Beams* (Van Nostrand, New York, 1954), 2nd ed.