Kinetic Equations for the Collisional Plasma Model

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MARCH 1976

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KINETIC EQUATIONS FOR THE COLLISIONAL PLASMA MODEL*

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ABSTRACT

Using the Collisional Plasma Model (CPM) representation, expressions are derived for the Vlasov operator, both in its general form and in the drift-kinetic approximation following the recursive derivation by Hazeltine. The expressions for the operators give easily calculated couplings between neighboring components of the CPM representation. Expressions for various macroscopic observables in the drift-kinetic approximation are also given.

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1. INTRODUCTION

The Collisional Plasma Model (CPM) is a velocity-space orthogonal-function representation for the distribution function of a collisional plasma, which is described in Ref. 1 (hereafter referred to as I). The CPM assumes that the distribution function for a particular species of particles in a collisional plasma can be represented by the series

$$f(\vec{x}, \vec{u}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=0}^{\infty} F_{\ell mn}(\vec{x}, t) \psi_{\ell mn}(\sqrt{\rho(\vec{x}, t)} \vec{u}) \quad , \quad (1.1)$$

where

$$\vec{u} = \vec{v} - V(\vec{x}, t) \quad , \quad (1.2)$$

$\vec{v}$ being the velocity, $\vec{x}$ the position, and t the time. The functions $\psi_{\ell mn}$ are defined in I; ρ and $\vec{V}$ are essentially arbitrary, although in its most useful form, the CPM sets $\rho = M/kT$, with T and $\vec{V}$ equal to the particles' temperature and average velocity. Thus, for predominantly Maxwellian particles, the infinite series (1.1) can be truncated to a practical length.

The choice of the functions $\psi_{\ell mn}$ is determined by several criteria: 1) the functions must provide an accurate representation of the Coulomb collision operator; 2) the representation must be efficient in that physical plasmas be represented by as few terms as possible in order to minimize both computer time and storage requirements; and 3) the collision and Vlasov operators must lead to simple expressions which can be rapidly and easily calculated. These criteria, as they pertain to the Coulomb collision operator, are discussed in I. In the present paper, the CPM representation of the Vlasov operator (that is, the

operator which describes the particle motion in various electric and magnetic fields) is derived for the general kinetic (Boltzmann) equation, and for a drift-kinetic equation of the Hazeltine type,² which is based on the velocity $\vec{u}$ of Eq. (1.2). The various terms of the Vlasov operator give coupling only between neighboring components of the CPM representation, and these couplings are simple in form and easy to calculate.

Also given in this paper are expressions for macroscopic observables in the drift-kinetic approximation. Presented in another paper³ is the application of the CPM in the drift-kinetic approximation to the calculation of the neoclassical equilibrium of a toroidal plasma. The success of the calculation indeed bears out the fact that the CPM does satisfy the criteria stated above.

2. BOLTZMANN EQUATION IN THE CPM REPRESENTATION

The general kinetic equation governing the motion of a collisional plasma is the Boltzmann equation

$$\frac{Df}{Dt} = Cf ,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + (\vec{a} + \vec{v} \times \vec{\Omega}) \cdot \vec{\nabla}_v , \quad (2.1)$$

$$\vec{a} = \frac{Ze}{M} \vec{E} , \quad \vec{\Omega} = \frac{Ze}{Mc} \vec{B} ,$$

and C is the Coulomb collision operator described in I. Two transformations can in general be applied to Eq. (2.1). The first is a transformation to the velocity variable $\vec{u}$ defined by Eq. (1.2). In the second transformation, the action of $\partial/\partial t$ and $\vec{\nabla}$ on the velocity $\vec{u}$ is separated out, these two operators subsequently no longer acting on the $\vec{u}$ dependence of f . The reason for the transformation is that, although position is measured in a fixed frame of reference, the velocity frame of reference is often a dynamic one. A third transformation, based on the introduction of the CPM representation (1.1), can also be applied to Eq. (2.1). The action of $\partial/\partial t$ and $\vec{\nabla}$ on the ρ dependence of the ψ_{lmn} is separated out, these two operators subsequently no longer acting on that ρ dependence. Applying these three transformations to Eq. (2.1), one obtains

$$\begin{aligned} \frac{D}{Dt} = \frac{d}{dt} + \vec{u} \cdot \vec{\nabla} + \vec{A} \cdot \vec{\nabla}_u + (G + \vec{H} \cdot \vec{u})u \frac{\partial}{\partial u} \\ - i(\vec{\Omega}' + \vec{u} \cdot \underline{\underline{u}}) \cdot \vec{L}_u - \vec{u} \cdot \underline{\underline{W}} \cdot \vec{\nabla}_u , \end{aligned} \quad (2.2)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} ,$$

$$\vec{A} = \vec{a} + \vec{V} \times \vec{\Omega} - \frac{d\vec{V}}{dt} ,$$

$$G = -\frac{1}{3} \operatorname{div} \vec{V} + \frac{1}{2\rho} \frac{d\rho}{dt} ,$$

$$\vec{H} = \frac{1}{2\rho} \vec{\nabla} \rho ,$$

$$\vec{\Omega}' = \vec{\Omega} + \frac{1}{2} \operatorname{curl} \vec{V} + \vec{\omega} ,$$

and

$$\vec{L}_u = -i \vec{u} \times \vec{\nabla}_u$$

is the $\vec{u}$ -space angular momentum operator. The rectangular Cartesian components of the symmetric traceless tensor $\underline{W}$ are given by

$$W_{ij} = \frac{1}{2} (w_{ij} + w_{ji}) - \frac{1}{3} \delta_{ij} \operatorname{div} \vec{V} ,$$

$$\underline{w} = \vec{\nabla} \vec{V} .$$

The rectangular Cartesian components of $\vec{\omega}$ in the arbitrary velocity reference frame are given by

$$\omega_i = -\frac{1}{2} \sum_{jk} \epsilon_{ijk} \hat{u}_j \cdot \frac{d\hat{u}_k}{dt} ,$$

where $\epsilon_{i,jk}$ is the permutation symbol and the $\hat{u}_i$ are the velocity-space unit vectors. The components of the tensor $\underline{\kappa}$ are given by

$$\kappa_{ij} = -\frac{1}{2} \sum_{kl} \epsilon_{jkl} \hat{u}_k \cdot \frac{\partial \hat{u}_l}{\partial x_i} .$$

It should be noted that $\vec{A}$, G , $\vec{H}$, $\vec{\Omega}'$, $\underline{\kappa}$, and $\underline{W}$ are functions of $\vec{x}$ and t only.

Upon substituting from Eq. (1.1) it is possible to obtain the following expansion for Df/Dt , D/Dt being defined by Eq. (2.2):

$$\frac{Df}{Dt} = \sum_{lmn} \left[\frac{dF_{lmn}}{dt} + \sum_{LMN} K \begin{pmatrix} \ell & m & n \\ L & M & N \end{pmatrix} F_{\ell+L \ m+M \ n+N} \right] \psi_{lmn} . \quad (2.3)$$

The reality of f and the relation

$$\psi_{lmn}^* = (-1)^m \psi_{\ell-mn}$$

give

$$F_{lmn}^* = (-1)^m F_{\ell-mn} ,$$

$$K \begin{pmatrix} \ell & m & n \\ L & M & N \end{pmatrix}^* = (-1)^M K \begin{pmatrix} \ell & -m & n \\ L & -M & N \end{pmatrix} . \quad (2.4)$$

The restrictions on L , M , and N are

$$-2 \leq L \leq 2 , \quad -2 \leq M \leq 2 , \quad -2 \leq N \leq 2 , \quad (2.5)$$

that is, the Vlasov operator gives coupling only between neighboring components of the CPM representation. Of course, one must also have

$$\ell + L \geq |m + M| , \quad n + N \geq 0 ,$$

but for a given (lmn) there are, according to Eq. (2.5), at most 125 K matrix elements. Equation (2.4) can be used to calculate the negative m matrix elements. However, many of the 125 vanish; in fact, there are only 65 non-zero elements for a given (lmn) . If one chooses $\vec{V} = 0$, the number is further reduced to 37. The non zero elements for a given (lmn) and $M \geq 0$ are presented in the Appendix, with the notation abbreviated to $K(LMN)$. The ℓ , m , and n dependence of all the matrix elements is simple in form and easy to calculate.

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3. DRIFT-KINETIC EQUATION FOR THE CPM

For problems characterized by time scales that are much larger than the gyroperiod Ω^{-1} , it is possible to average the Boltzmann equation with respect to the gyromotion to obtain a drift-kinetic equation. Recently, Hazeltine² gave a recursive derivation of such a drift-kinetic equation. Here we give a generalization of that drift-kinetic equation. This generalization is based on the velocity transformation Eq. (1.2), in which some unspecified average or drift velocity $\vec{V}$ is subtracted from the total velocity $\vec{v}$. Thus, the picture is one where the velocity $\vec{u}$ gyrates with frequency Ω about the $\vec{B}$ field.

Up until this point, the CPM velocity reference frame has been arbitrary, but we now assume that

$$\hat{u}_3 = \hat{n} = \vec{B}/B ,$$

so that α and β , the spherical polar angles of $\vec{u}$, are the pitch and gyro angles. (The angle $-\beta$ is the counterpart of Hazeltine's ζ .) The β dependence of the $\psi_{\ell mn}$ of Eq. (1.1) is $\exp(im\beta)$ (see I), so that the gyroaveraged distribution function

$$\bar{f} = \frac{1}{2\pi} \int_0^{2\pi} d\beta f \quad (3.1)$$

is simply given by the $m = 0$ part of Eq. (1.1). Because the $\psi_{\ell mn}$ are defined as functions of u , α , and β , it is natural to derive the CPM drift-kinetic equation in (u, α) variables rather than the (energy, magnetic moment) variables used by Hazeltine.²

Our recursive derivation follows closely that given by Hazeltine.² However, in treating the Coulomb collisions, we assume that, for species r and s , the velocity $\vec{v}_r - \vec{v}_s$ is negligible in comparison

to the thermal velocity or parallel to $\vec{B}$. We then obtain the following first-order drift-kinetic distribution function

$$f = \bar{f} + \tilde{f}_1 + \tilde{f}_2 ,$$

where $\bar{f}$ is the solution of the drift-kinetic equation (3.6), and

$$\begin{aligned} \tilde{f}_1 = & -\frac{1}{\Omega} \hat{\beta} \cdot \left[u \sin \alpha (\vec{\nabla} + \vec{H} u \frac{\partial}{\partial u}) + \vec{A} \hat{s} \cdot \vec{\nabla}_u \right. \\ & - \vec{\omega}' \frac{\partial}{\partial \alpha} + \hat{n} \times \text{curl } \hat{n} u \cos \alpha \frac{\partial}{\partial \alpha} \\ & \left. - 2\vec{d} u \sin \alpha \hat{n} \cdot \vec{\nabla}_u \right] \bar{f} , \end{aligned} \quad (3.2)$$

$$\begin{aligned} \tilde{f}_2 = & \frac{1}{4\Omega} u \sin \alpha \left[(\hat{\beta} \cdot \text{curl } \hat{\beta} - \hat{s} \cdot \text{curl } \hat{s}) \frac{\partial}{\partial \alpha} \right. \\ & \left. + 2\hat{s} \cdot \underline{\underline{w}} \cdot \hat{\beta} \hat{s} \cdot \vec{\nabla}_u \right] \bar{f} , \end{aligned} \quad (3.3)$$

with

$$\begin{aligned} \vec{d} &= \hat{n} \cdot \underline{\underline{w}} , \\ \vec{\omega}' &= \frac{d\hat{n}}{dt} + \hat{n} \cdot \underline{\underline{w}} , \\ \hat{\beta} &= -\sin \beta \hat{u}_1 + \cos \beta \hat{u}_2 , \\ \hat{s} &= \cos \beta \hat{u}_1 + \sin \beta \hat{u}_2 . \end{aligned}$$

Three sets of right-handed orthogonal unit vectors are involved here: rectangular Cartesian ($\hat{u}_1, \hat{u}_2, \hat{u}_3 = \hat{n}$), spherical polar ($\hat{u}, \hat{\alpha}, \hat{\beta}$), and the set ($\hat{n}, \hat{s}, \hat{\beta}$) with

$$\begin{aligned} \hat{n} &= \cos \alpha \hat{u} - \sin \alpha \hat{\alpha} , \\ \hat{s} &= \sin \alpha \hat{u} + \cos \alpha \hat{\alpha} . \end{aligned}$$

It is instructive to consider the time dependence characteristics of $\bar{f}$, $\tilde{f}_1$, and $\tilde{f}_2$. For the drift-kinetic derivation, it is assumed that

the dominant term in the Boltzmann equation is given by the term $-i\vec{\Omega} \cdot \vec{L}_u$ (see Eq. (2.2)). Thus, if one neglects all other terms of the Boltzmann equation, it follows that the distribution function has a β and t dependence which is given by the expansion

$$f = \sum_m f_m \exp [i m(\beta + \Omega t)] ,$$

so that m exactly identifies the gyroharmonics of f . Re-introduction of the neglected terms leads to an expansion of the form

$$f = \sum_{mM} f_{mM}(t) \exp [i(m\beta + M\Omega t)] , \quad (3.4)$$

with the diagonal amplitude f_{mm} dominating. Consequently, m now only approximately identifies the gyroharmonics of f . The amplitudes f_{mM} vary on a time scale much larger than Ω^{-1} ; therefore, performing the gyromotion average of the drift-kinetic derivation, one obtains

$$f(\text{drift-kinetic}) \approx \sum_m f_{m0}(t) \exp (im\beta) . \quad (3.5)$$

Discarded in this approximation are the possibly large, but rapidly varying $M \neq 0$ terms. In our first-order drift-kinetic approximation we only have $m = 0$ ($\bar{f}$), $m = \pm 1$ ($\tilde{f}_1$), and $m = \pm 2$ ($\tilde{f}_2$) terms, all varying on the larger time scale.

The first-order drift-kinetic equation is given by

$$\left[\frac{\partial}{\partial t} + \vec{V}_{\text{tot}} \cdot \vec{\nabla} + \vec{A}_{\text{tot}} \cdot \vec{\nabla}_u \right] \bar{f} = \bar{C} \bar{f} , \quad (3.6)$$

where the collision operator $\bar{C}$ is to be calculated with the

$\sqrt{\rho} (\vec{V}_r - \vec{V}_s) = 0$ or $\hat{n} \cdot (\vec{V}_r - \vec{V}_s) = 0$ assumption referred to previously.

The velocity $\vec{V}_{\text{tot}}$ is given by

$$\begin{aligned}
\vec{V}_{\text{tot}} &= \vec{V} + \vec{V}_D + (\hat{n} + \frac{1}{\Omega} \hat{n} \times \vec{\omega}') u \cos \alpha \\
&+ \frac{1}{2\Omega} [\hat{n} \times \vec{b} + \hat{n} (\hat{n} \cdot \text{curl } \hat{n})] u^2 \\
&- \frac{1}{2\Omega} [\hat{n} \times \vec{b} + 3\hat{n} (\hat{n} \cdot \text{curl } \hat{n}) - 2 \text{curl } \hat{n}] u^2 \cos^2 \alpha, \quad (3.7)
\end{aligned}$$

with

$$\vec{V}_D = \frac{1}{\Omega} \vec{A} \times \hat{n} = \vec{V}_E - \vec{V}_\perp + \frac{1}{\Omega} \hat{n} \times \frac{d\vec{V}}{dt}, \quad (3.8)$$

$$\vec{V}_E = c \frac{\vec{E} \times \vec{B}}{B^2},$$

$$\vec{V}_\perp = \vec{V} - \hat{n} (\hat{n} \cdot \vec{V}),$$

$$\vec{b} = \frac{1}{B} \vec{\nabla} B.$$

The velocity $\vec{V}_E$ is just the electric drift velocity, and $\vec{V}_\perp$ is the component of $\vec{V}$ perpendicular to $\vec{B}$. Thus, $\vec{V}_D$ is the transverse drift velocity that remains after subtraction of the drift $\vec{V}$ in Eq. (1.1). However, if one chooses $\vec{V} = \vec{V}_E$, it does not follow that $\vec{V}_D = 0$, because of the first-order term in Eq. (3.8).

The acceleration $\vec{A}_{\text{tot}}$ is given by

$$\begin{aligned}
\vec{A}_{\text{tot}} &= \vec{u} \left\{ \vec{G} + \frac{1}{2} \vec{d} \cdot (\hat{n} + \frac{2}{\Omega} \hat{n} \times \vec{\omega}') - \frac{1}{2} \text{div } \vec{V}_D \right. \\
&+ \vec{V}_D \cdot (\vec{H} + \frac{1}{2} \hat{n} \times \text{curl } \hat{n}) - [\frac{1}{2} \text{div } (\hat{n} + \frac{1}{\Omega} \hat{n} \times \vec{\omega}')] \\
&- (\hat{n} + \frac{1}{\Omega} \hat{n} \times \vec{\omega}') \cdot \vec{H} - \frac{2}{\Omega} (\hat{n} \times \vec{d}) \cdot (\hat{n} \times \text{curl } \hat{n}) \left. \right\} u \cos \alpha \\
&+ \frac{1}{2\Omega} [\hat{n} \times \vec{b} + \hat{n} (\hat{n} \cdot \text{curl } \hat{n})] \cdot \vec{H} u^2 - [\frac{1}{2} \hat{n} \cdot \text{curl } (\frac{1}{\Omega} \hat{n} \times \text{curl } \hat{n}) \\
&+ \frac{1}{2\Omega} (\hat{n} \times \vec{b} + 3 \hat{n} (\hat{n} \cdot \text{curl } \hat{n}) - 2 \text{curl } \hat{n}) \cdot \vec{H}] u^2 \cos^2 \alpha \\
&+ \hat{n} \{ \hat{n} \cdot \vec{A} + (\vec{\omega}' - 2 \vec{d}) \cdot \vec{V}_D + [\frac{1}{2} \text{div } \vec{V}_D -
\end{aligned}$$

$$\begin{aligned}
& - \frac{3}{2} \vec{V}_D \cdot (\hat{n} \times \text{curl } \hat{n}) - \frac{3}{2} \vec{d} \cdot (\hat{n} + \frac{2}{\Omega} \hat{n} \times \vec{\omega}')] u \cos \alpha \\
& + [\frac{1}{2} \text{div} (\hat{n} + \frac{1}{\Omega} \hat{n} \times (\vec{\omega}' - 2 \vec{d})) - \frac{1}{4\Omega} \text{Re} (P_2 R_2^*)] u^2 \\
& + [\text{div} (\frac{1}{\Omega} \hat{n} \times \vec{d}) - \frac{4}{\Omega} (\hat{n} \times \vec{d}) \cdot (\hat{n} \times \text{curl } \hat{n}) \\
& + \frac{1}{4\Omega} \text{Re} (P_2 R_2^*)] u^2 \cos^2 \alpha \\
& + \frac{1}{2} \hat{n} \cdot \text{curl} (\frac{1}{\Omega} \hat{n} \times \text{curl } \hat{n}) u^3 \cos \alpha \} , \quad (3.9)
\end{aligned}$$

where P_2 and R_2 are defined in the Appendix. This is a rather complicated looking result, but one sees that the velocity dependence is simple, consisting of terms of the type u ($1, u \cos \alpha, u^2, u^2 \cos^2 \alpha$), and $\hat{n}$ ($1, u \cos \alpha, u^2 \cos^2 \alpha, u^3 \cos \alpha$).

The relationship

$$\text{div } \vec{V}_{\text{tot}} = - \text{div}_u \vec{A}_{\text{tot}} \left(\frac{\partial \rho}{\partial t} = \vec{V} \rho = 0 \right)$$

can readily be derived from Eqs. (3.7) and (3.9). This result is useful for calculating moments of the drift-kinetic equation (3.6). An interesting example is the first moment equation,

$$\frac{\partial}{\partial t} N(\vec{x}, t) = - \text{div } \vec{\Gamma}(\vec{x}, t) , \quad (3.10)$$

where

$$N(\vec{x}, t) = \int d^3v \bar{f}$$

is the number density, and

$$\vec{\Gamma}(\vec{x}, t) = \int d^3v \vec{V}_{\text{tot}} \bar{f}$$

is a particle flux vector. Furthermore, one can show that

$$\vec{\Gamma} = \vec{J} + \text{curl} \left(\frac{\hat{n}}{2\Omega} \int d^3u u^2 \sin^2 \alpha \bar{f} \right) , \quad (3.11)$$

where

$$\vec{J}(\vec{x}, t) = \int d^3v \vec{v} (\bar{f} + \tilde{f}_1 + \tilde{f}_2)$$

is the particle current density. It follows from Eq. (3.11) that Eq. (3.10) can be written in the more familiar form

$$\frac{\partial}{\partial t} N(\vec{x}, t) = - \operatorname{div} \vec{J}(\vec{x}, t) ,$$

and it should be emphasized that $\vec{J}$ is the physical quantity, not $\vec{\Gamma}$.

The differing of $\vec{\Gamma}$ and $\vec{J}$ reflects the fact that the drift-kinetic approximation gives a guiding-center solution (Eq. (3.5)), and not an exact solution (Eq. (3.4)).

4. DRIFT-KINETIC EQUATION IN THE CPM REPRESENTATION

Using Eqs. (3.7) and (3.9), the drift-kinetic equation (3.6) can be written in the form

$$\frac{\partial \bar{f}}{\partial t} = \bar{C} \bar{f} - \sum_{j=1}^{13} K_j \left(\sqrt{\rho} \vec{u}, \frac{1}{\sqrt{\rho}} \vec{\nabla}_u \right) S_j(\vec{x}, \vec{\nabla}, t) \bar{f} . \quad (4.1)$$

The K_j are dimensionless velocity-space operators which are defined as follows:

$$K_1 = 1 , \quad (4.2.1)$$

$$K_2 = \sqrt{\rho} u \cos \alpha , \quad (4.2.2)$$

$$K_3 = \rho u^2 \quad (4.2.3)$$

$$K_4 = \rho u^2 \cos^2 \alpha , \quad (4.2.4)$$

$$K_5 = u \frac{\partial}{\partial u} , \quad (4.2.5)$$

$$K_6 = \sqrt{\rho} u^2 \cos \alpha \frac{\partial}{\partial u} , \quad (4.2.6)$$

$$K_7 = \rho u^3 \frac{\partial}{\partial u} , \quad (4.2.7)$$

$$K_8 = \rho u^3 \cos^2 \alpha \frac{\partial}{\partial u} , \quad (4.2.8)$$

$$K_9 = \frac{1}{\sqrt{\rho}} \hat{n} \cdot \vec{\nabla}_u , \quad (4.2.9)$$

$$K_{10} = u \cos \alpha \hat{n} \cdot \vec{\nabla}_u , \quad (4.2.10)$$

$$K_{11} = \sqrt{\rho} u^2 \hat{n} \cdot \vec{\nabla}_u , \quad (4.2.11)$$

$$K_{12} = \sqrt{\rho} u^2 \cos^2 \alpha \hat{n} \cdot \vec{\nabla}_u , \quad (4.2.12)$$

$$K_{13} = \rho u^3 \cos \alpha \hat{n} \cdot \vec{\nabla}_u . \quad (4.2.13)$$

The definitions of the spatial operators S_j follow directly from a comparison of Eqs. (4.1) and (4.2) with Eqs. (3.6), (3.7), and (3.9). The

advantage of the factorization in Eq. (4.1) lies in the fact that only the S_j depend on the nature of the physical problem considered, so that a large class of problems could be studied by the same general computer code.

The gyroaveraged distribution function has a CPM representation given by Eqs. (1.1) and (3.1):

$$\bar{F}(\vec{x}, \vec{u}, t) = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} F_{\ell n}(\vec{x}, t) \psi_{\ell n}(\sqrt{\rho(\vec{x}, t)} \vec{u}), \quad (4.3)$$

where

$$F_{\ell n} = F_{\ell 0 n}, \quad \psi_{\ell n} = \psi_{\ell 0 n}.$$

As in Eq. (2.3), one can make expansions

$$K_j \bar{F} = \sum_{\ell n} \left[\sum_{LN} K_j \begin{pmatrix} \ell & n \\ L & N \end{pmatrix} F_{\ell+L, n+N} \right] \psi_{\ell n}.$$

The K_j matrices are sparse, with

$$-3 \leq L \leq 3, \quad -3 \leq N \leq 3, \quad (4.4)$$

subject to the constraints

$$\ell + L \geq 0, \quad n + N \geq 0.$$

Comparing Eq. (4.4) with Eq. (2.5), one sees that a wider L, N range is spanned by the drift-kinetic matrix elements. This is because, in the drift-kinetic approximation, some of the $M \neq 0$ coupling of Eq. (2.3) is converted to L, N coupling. The K_j matrix elements are pure numbers, and the non-zero elements for a given (ℓn) are presented below with the notation abbreviated to $K_j(LN)$. All 98 of them are simple in form and easy to calculate. If one chooses $\vec{V} = 0$, the spatial operator S_{12} vanishes, effectively reducing the number of K_j matrix elements to 82

and the L coupling restriction to $-2 \leq L \leq 2$.

$$K_1(ON) = \delta_{NO} \quad .$$

$$K_2(-1N) = (\delta_{NO} - \delta_{N1})D_\ell \quad ,$$

$$K_2(1N) = -\delta_{-N1}G_{\ell n} - \delta_{NO}H_{\ell n} \quad ,$$

where

$$D_\ell = \frac{\ell}{\sqrt{(2\ell-1)(2\ell+1)}} \quad ,$$

$$G_{\ell n} = n D_{\ell+1} \quad , \quad H_{\ell n} = \left(\ell + n + \frac{3}{2}\right) D_{\ell+1} \quad .$$

$$K_3(ON) = -\delta_{-N1} + \delta_{NO}\left(\ell + 2n + \frac{3}{2}\right) - \delta_{N1}\left(\ell + n + \frac{3}{2}\right) \quad .$$

$$K_4(-2N) = (\delta_{NO} - 2\delta_{N1} + \delta_{N2})J_\ell \quad ,$$

$$K_4(ON) = K_3(ON)I_\ell \quad ,$$

$$K_4(2N) = \delta_{-N2}G_{\ell n}G_{\ell+1, n-1} - 2\delta_{-N1}G_{\ell+1, n}H_{\ell n}$$

$$+ \delta_{NO}H_{\ell n}H_{\ell+1, n} \quad ,$$

where

$$J_\ell = D_{\ell-1}D_\ell \quad , \quad I_\ell = D_\ell^2 + D_{\ell+1}^2 \quad .$$

$$K_5(ON) = \delta_{-N1}n - \frac{3}{2}\delta_{NO} - \delta_{N1}\left(\ell + n + \frac{3}{2}\right) \quad .$$

$$K_6(-1N) = [\delta_{-N1}n - \delta_{NO}\left(n + \frac{5}{2}\right) - \delta_{N1}(\ell + n - 1) \\ + \delta_{N2}\left(\ell + n + \frac{3}{2}\right)]D_\ell \quad ,$$

$$K_6(1N) = -[\delta_{-N2}(n - 1) - \delta_{-N1}(\ell + n + 3)]G_{\ell n}$$

$$+ [\delta_{NO}\left(n - \frac{3}{2}\right) - \delta_{N1}\left(\ell + n + \frac{5}{2}\right)]H_{\ell n} \quad .$$

$$\begin{aligned}
K_7(ON) = & -\delta_{-N2} n (n - 1) + \delta_{-N1} n (\ell + 2n + 3) \\
& - \delta_{NO} \frac{5}{2} (\ell + 2n + \frac{3}{2}) - \delta_{N1} (\ell + 2n)(\ell + n + \frac{3}{2}) \\
& + \delta_{N2} (\ell + n + \frac{3}{2})(\ell + n + \frac{5}{2}) .
\end{aligned}$$

$$\begin{aligned}
K_8(-2N) = & [\delta_{-N1} n - \delta_{NO} (2n + \frac{7}{2}) - \delta_{N1} (\ell - \frac{11}{2}) \\
& + \delta_{N2} (2\ell + 2n - \frac{1}{2}) - \delta_{N3} (\ell + n + \frac{3}{2})] J_\ell ,
\end{aligned}$$

$$K_8(ON) = K_7(ON) I_\ell ,$$

$$\begin{aligned}
K_8(2N) = & [\delta_{-N3} (n - 2) - \delta_{-N2} (2\ell + 2n + \frac{9}{2})] G_{\ell n} G_{\ell+1 n-1} \\
& + \delta_{-N1} (\ell + \frac{13}{2}) G_{\ell+1 n} H_{\ell n} + [\delta_{NO} (2n - \frac{3}{2}) - \delta_{N1} (\ell + n + \frac{7}{2})] H_{\ell n} H_{\ell+1 n} .
\end{aligned}$$

$$K_9(-1N) = -(\delta_{NO} + \delta_{N1}) D_\ell ,$$

$$K_9(1N) = \delta_{-N1} G_{\ell n} + \delta_{NO} H_{\ell n} .$$

$$K_{10}(-2N) = -(\delta_{NO} - \delta_{N2}) J_\ell ,$$

$$K_{10}(ON) = \delta_{-N1} n I_\ell - \frac{1}{2} \delta_{NO} - \delta_{N1} (\ell + n + \frac{3}{2}) I_\ell ,$$

$$K_{10}(2N) = -\delta_{-N2} G_{\ell n} G_{\ell+1 n-1} + \delta_{NO} H_{\ell n} H_{\ell+1 n} .$$

$$K_{11}(-1N) = [(\delta_{-N1} - \delta_{N1}) n - (\delta_{NO} - \delta_{N2})(\ell + n + \frac{3}{2})] D_\ell ,$$

$$K_{11}(1N) = -[\delta_{-N2}(n - 1) - \delta_{-N1}(n + 1)] G_{\ell n}$$

$$+ [\delta_{NO} (\ell + n + \frac{1}{2}) - \delta_{N1} (\ell + n + \frac{5}{2})] H_{\ell n} .$$

$$K_{12}(-3N) = - [\delta_{NO} - \delta_{N1} - \delta_{N2} + \delta_{N3}] D_{\ell} J_{\ell-1} ,$$

$$K_{12}(-1N) = \delta_{-N1} [I_{\ell} + D_{\ell-1}^2] G_{\ell-1n}$$

$$- \delta_{NO} [D_{\ell+1} H_{\ell n} + \frac{1}{2} + (n+1) I_{\ell-1}] D_{\ell}$$

$$- \delta_{N1} [D_{\ell+1} G_{\ell n} - \frac{1}{2} + (\ell + n + \frac{1}{2}) I_{\ell-1}] D_{\ell}$$

$$+ \delta_{N2} [I_{\ell} + D_{\ell-1}^2] H_{\ell-1 n+1} ,$$

$$K_{12}(1N) = - \delta_{-N2} (n-1) [I_{\ell} + D_{\ell+2}^2] G_{\ell n}$$

$$+ \delta_{-N1} [D_{\ell} G_{\ell-1 n+1} + \frac{1}{2} + (\ell + n + \frac{3}{2}) I_{\ell+1}] G_{\ell n}$$

$$+ \delta_{NO} [D_{\ell} H_{\ell-1n} - \frac{1}{2} + n I_{\ell+1}] H_{\ell n}$$

$$- \delta_{N1} (\ell + n + \frac{3}{2}) [I_{\ell} + D_{\ell+2}^2] H_{\ell n+1} ,$$

$$K_{12}(3N) = \delta_{-N3} G_{\ell n} G_{\ell+1 n-1} G_{\ell+2 n-2}$$

$$- \delta_{-N2} H_{\ell n} G_{\ell+1n} G_{\ell+2 n-1} - \delta_{-N1} G_{\ell n} H_{\ell+1 n-1} H_{\ell+2 n-1}$$

$$+ \delta_{NO} H_{\ell n} H_{\ell+1n} H_{\ell+2n} .$$

$$K_{13}(-2N) = [\delta_{-N1} n - \delta_{NO} (\ell + 2n + \frac{3}{2}) + \delta_{N1} (\ell + \frac{3}{2})$$

$$+ \delta_{N2} (\ell + 2n + \frac{3}{2}) - \delta_{N3} (\ell + n + \frac{3}{2})] J_{\ell} ,$$

$$K_{13}(0N) = - \delta_{-N2} n (n-1) I_{\ell} + \delta_{-N1} n [\frac{1}{2} + (\ell + 2n + \frac{3}{2}) I_{\ell}]$$

$$- \delta_{NO} (\ell + 2n + \frac{3}{2}) (\frac{1}{2} + I_{\ell}) + \delta_{N1} (\ell + n + \frac{3}{2}) [\frac{1}{2} - (\ell + 2n + \frac{3}{2}) I_{\ell}]$$

$$+ \delta_{N2} (\ell + n + \frac{3}{2}) (\ell + n + \frac{5}{2}) I_{\ell} ,$$

$$\begin{aligned}
K_{13}(2N) = & [\delta_{-N3} (n - 2) - \delta_{-N2} (\ell + 2n + \frac{3}{2})] G_{\ell n} G_{\ell+1 \ n-1} \\
& - \delta_{-N1} (\ell - \frac{1}{2}) G_{\ell n} H_{\ell+1 \ n-1} + [\delta_{N0} (\ell + 2n + \frac{3}{2}) \\
& - \delta_{N1} (\ell + n + \frac{7}{2})] H_{\ell n} H_{\ell+1 n} .
\end{aligned}$$

5. DRIFT-KINETIC-APPROXIMATION MACROSCOPIC OBSERVABLES

Expressions for macroscopic observables given in I are in terms of the expansion amplitudes of the exact distribution function. It is possible to determine the $m = \pm 1, \pm 2$ amplitudes from Eqs. (3.2) and (3.3), and to use these and the $m = 0$ amplitudes of Eq. (4.3) in the general equations of I to calculate the drift-kinetic macroscopic observables. However, by substituting directly from Eqs. (3.2) and (3.3) in the appropriate integrals, integration by parts can be used to make simplifications before introducing the expansion Eq. (4.3) for $\bar{f}$. This is the approach that we have followed in deriving the expressions that are given below.

All the observables of interest involve integrals of the type

$$\begin{aligned} I_s(l, q) &= \int d^3u u^{l+2q} P_l(\cos \alpha) \bar{f}_s \\ &= \sqrt{\frac{\pi}{l + \frac{1}{2}}} \left(\frac{1}{\sqrt{\rho_s}} \right)^{l+2q+3} 2^{l+q+1} \Gamma(l + q + \frac{3}{2}) \\ &\times \sum_{n=0}^{\infty} F_{ln}^{(s)} (-1)^n \binom{l+n+\frac{1}{2}}{n} {}_2F_1(-n, -q; l + \frac{3}{2}; 2) , \end{aligned}$$

where s denotes a particular species, P_l is a Legendre polynomial, $\binom{a}{b}$ is a binomial coefficient, and ${}_2F_1$ is the Gauss hypergeometric function.

5.1 Number Density

The number density for a species s is given by

$$N_s = \int d^3v f_s = I_s(0, 0) .$$

The plasma number density and average mass are given by

$$N = \sum_s N_s ,$$

$$M = \sum_s M_s N_s / N .$$

5.2 Particle Current Density

The particle current density for a species s is given by

$$\vec{J} = \int d^3v \vec{v} f_s = N_s \vec{V}_s + \vec{j}_s ,$$

where

$$\begin{aligned} \vec{j}_s &= \int d^3u \vec{u} f_s \\ &= N_s \vec{V}_{D_s} + \left(\hat{n} + \frac{1}{\Omega_s} \hat{n} \times \vec{\omega}'_s \right) I_s (1,0) \\ &\quad + \frac{1}{M_s \Omega_s} \left[\hat{n} \times (\vec{\nabla} - 5\vec{H}_s) w_{\perp s} + (\hat{n} \times \text{curl } \hat{n}) \times \hat{n} D_s \right] , \end{aligned}$$

$$D_s = M_s I_s (2,0) .$$

The transverse energy density $w_{\perp s}$ is defined in Sect. 5.3. The species average velocity is given by

$$\vec{U}_s = \vec{J}_s / N_s = \vec{V}_s + \vec{j}_s / N_s ,$$

and the plasma average velocity is given by

$$\vec{U} = \sum_s M_s N_s \vec{U}_s / MN .$$

5.3 Energy Density

The energy density for a species s is given by

$$\begin{aligned} w_s &= \frac{1}{2} M_s \int d^3v v^2 f_s \\ &= - \frac{1}{2} M_s N_s (V_s^2 - 2\vec{V}_s \cdot \vec{U}_s) + w_s , \end{aligned}$$

where

$$w_s = \frac{1}{2} M_s \int d^3u u^2 \bar{f}_s = \frac{1}{2} M_s I_s (0,1) .$$

The tranverse energy density $w_{\perp s}$ is given by

$$\begin{aligned} w_{\perp s} &= \frac{1}{2} M_s \int d^3 u u^2 \sin^2 \alpha \bar{f}_s \\ &= \frac{2}{3} w_s - \frac{1}{3} D_s . \end{aligned}$$

The plasma energy density is given by

$$W = \sum_s W_s .$$

5.4 Scaling Pressure and Temperature

The partial scalar pressure for a species s is given by

$$\begin{aligned} P_s &= \frac{1}{3} M_s \int d^3 v (\vec{v} - 2\vec{U})^2 f_s \\ &= \frac{2}{3} W_s + \frac{1}{3} M_s N_s (U^2 - \vec{U} \cdot \vec{U}_s) . \end{aligned}$$

The plasma scalar pressure is given by

$$P = \sum_s P_s = \frac{2}{3} W - \frac{1}{3} M N U^2 .$$

Species and plasma temperatures are given by

$$kT_s = P_s / N_s , \quad kT = P / N .$$

5.5 Tensor Pressure

The plasma tensor pressure is given by

$$\begin{aligned} \underline{\underline{P}} &= \sum_s M_s \int d^3 v (\vec{v} - \vec{U})(\vec{v} - \vec{U}) f_s - \frac{1}{3} P \\ &= \sum_s \left\{ \underline{\underline{p}}_s - \frac{2}{3} \frac{1}{N_s} w_s - M_s N_s [\vec{V}_s \vec{V}_s + \vec{U} \vec{U} - \vec{V}_s \vec{U}_s \right. \\ &\quad \left. - \vec{U}_s \vec{V}_s - \frac{1}{3} \frac{1}{N_s} (V_s^2 + U^2 - 2\vec{V}_s \cdot \vec{U}_s)] \right\} , \end{aligned}$$

where

$$\underline{\underline{p}}_s = M_s \int d^3 u \vec{u} \vec{u} f_s .$$

The components of the tensor $\underline{p}_s$ are given by

$$\begin{aligned} p_{11}^{(s)} - p_{22}^{(s)} &= \frac{1}{\Omega_s} [\text{Re}(R_2) g_s - \text{Im}(P_2) w_{1s}] , \\ p_{11}^{(s)} + p_{22}^{(s)} &= 2 w_s - p_{33}^{(s)} = 2 w_{1s} , \\ p_{12}^{(s)} &= \frac{1}{2\Omega_s} [\text{Im}(R_2) g_s + \text{Re}(P_2) w_{1s}] , \\ \hat{u}_1 p_{13}^{(s)} + \hat{u}_2 p_{23}^{(s)} &= M_s \vec{V}_{D_s} I_s (1,0) + \frac{1}{\Omega_s} [\hat{n} \times \vec{\omega}'_s D_s \\ &+ \hat{n} \times (\vec{V} - 6\vec{H}_s) g_s + (\hat{n} \times \text{curl } \hat{n}) \times \hat{n} h_s] , \end{aligned}$$

where P_2 and R_2 are defined in the Appendix and

$$\begin{aligned} g_s &= \frac{1}{5} M_s [I_s (1,1) - I_s (3,0)] , \\ h_s &= \frac{1}{5} M_s [I_s (1,1) + 4I_s (3,0)] . \end{aligned}$$

5.6 Heat Flux Vector

The plasma heat flux vector is given by

$$\begin{aligned} \vec{Q} &= \frac{1}{2} \sum_s M_s \int d^3v (\vec{v} - \vec{U})^2 (\vec{v} - \vec{U}) f_s \\ &= - (\underline{P} + \underline{1} W) \cdot \vec{U} + \sum_s [(\underline{p}_s + \underline{1} w_s) \cdot \vec{V}_s \\ &+ \vec{q}_s + \frac{1}{2} M_s N_s V_s^2 (\vec{U}_s - \vec{V}_s)] , \end{aligned}$$

where

$$\begin{aligned} \vec{q}_s &= \frac{1}{2} M_s \int d^3u u^2 \vec{u} f_s \\ &= \frac{1}{2} M_s (\hat{n} + \frac{1}{\Omega_s} \hat{n} \times \vec{\omega}'_s) I_s (1,1) + (w_s + w_{1s}) \vec{V}_{D_s} \\ &+ \frac{2}{\Omega_s} \hat{n} \times \vec{d}_s g_s + \frac{M_s}{\Omega_s} \left[\frac{1}{6} \hat{n} \times (\vec{V} - 7\vec{H}_s) \right. \\ &\times [I_s (0,2) - I_s (2,1)] + \frac{1}{2} (\hat{n} \times \text{curl } \hat{n}) \times \hat{n} I_s (2,1) \left. \right] . \end{aligned}$$

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APPENDIX - BOLTZMANN EQUATION MATRIX ELEMENTS

Below are given the non-zero matrix elements defined by Eq. (2.3).

Some of them involve the previously undefined quantities

$$P_0 = w_{33} - \frac{1}{2} (w_{11} + w_{22}) ,$$

$$P_1 = w_{13} + w_{31} + i(w_{23} + w_{32}) ,$$

$$P_2 = w_{11} - w_{22} + i (w_{12} + w_{21}) ,$$

$$R_0 = s_{33} - \frac{1}{2} (s_{11} + s_{22}) ,$$

$$R_1 = s_{13} + s_{31} + i (s_{23} + s_{32}) ,$$

$$R_2 = s_{11} - s_{22} + i (s_{12} + s_{21}) ,$$

where

$$s_{ij} = \hat{u}_i \cdot \text{curl } \hat{u}_j .$$

$$K(OON) = - \delta_{NO} \left(\frac{3}{2} G + im \Omega'_3 \right) - \left[\left(\ell + n + \frac{3}{2} \right) \delta_{N1} - n \delta_{-N1} \right] \left[G - \frac{2\ell(\ell+1) - 6m^2}{3(2\ell-1)(2\ell+3)} P_0 \right] ,$$

$$K(O1N) = \sqrt{(\ell-m)(\ell+m+1)} \left\{ \frac{1}{2} \delta_{NO} (\Omega'_2 - i\Omega'_1) - \frac{m + 1/2}{(2\ell-1)(2\ell+3)} \left[\left(\ell + n + \frac{3}{2} \right) \delta_{N1} - n \delta_{-N1} \right] P_1 \right\} ,$$

$$K(O2N) = - \frac{\sqrt{(\ell-m)(\ell-m-1)(\ell+m+1)(\ell+m+2)}}{2(2\ell-1)(2\ell+3)} \times \left[\left(\ell + n + \frac{3}{2} \right) \delta_{N1} - n \delta_{-N1} \right] P_2 .$$

$$K(2ON) = - \frac{1}{2\ell+3} \sqrt{\frac{(\ell+2-m)(\ell+1-m)(\ell+2+m)(\ell+1+m)}{(2\ell+1)(2\ell+5)}} \times \left[\left(\ell + n + \frac{3}{2} \right) \left(\ell + n + \frac{5}{2} \right) \delta_{NO} - n(n-1) \delta_{-N2} \right] P_0 ,$$

$$K(21N) = \frac{1}{2(2\ell+3)} \sqrt{\frac{(\ell+1-m)(\ell+1+m)(\ell+2+m)(\ell+3+m)}{(2\ell+1)(2\ell+5)}} \times [(\ell+n+\frac{3}{2})(\ell+n+\frac{5}{2})\delta_{NO} - n(n-1)\delta_{-N2}]P_1 ,$$

$$K(22N) = -\frac{1}{4(2\ell+3)} \sqrt{\frac{(\ell+1+m)(\ell+2+m)(\ell+3+m)(\ell+4+m)}{(2\ell+1)(2\ell+5)}} \times [(\ell+n+\frac{3}{2})(\ell+n+\frac{5}{2})\delta_{NO} - n(n-1)\delta_{-N2}]P_2 .$$

$$K(-20N) = \frac{1}{2\ell-1} \sqrt{\frac{(\ell-m)(\ell-1-m)(\ell+m)(\ell-1+m)}{(2\ell-3)(2\ell+1)}} \times (\delta_{NO} - \delta_{N2})P_0 ,$$

$$K(-21N) = \frac{1}{2(2\ell-1)} \sqrt{\frac{(\ell-m)(\ell-1-m)(\ell-2-m)(\ell+m)}{(2\ell-3)(2\ell+1)}} \times (\delta_{NO} - \delta_{N2})P_1 ,$$

$$K(-22N) = \frac{1}{4(2\ell-1)} \sqrt{\frac{(\ell-m)(\ell-1-m)(\ell-2-m)(\ell-3-m)}{(2\ell-3)(2\ell+1)}} \times (\delta_{NO} - \delta_{N2})P_2 .$$

$$K(10N) = \sqrt{\frac{(\ell+1-m)(\ell+1+m)}{\rho(2\ell+1)(2\ell+3)}} \{ [(\ell+n+\frac{3}{2})\delta_{NO} - n\delta_{-N1}] \times [\frac{\partial}{\partial x_3} + \frac{1}{2}(\ell+2) \operatorname{div} \hat{u}_3 - i m R_0] + [(\ell+n+\frac{3}{2})\delta_{NO} + n\delta_{-N1}] \rho A_3 - [(\ell+n+\frac{3}{2})(\ell+n+\frac{5}{2})\delta_{N1} - (n-\frac{3}{2}) \times (\ell+n+\frac{3}{2})\delta_{NO} - n(\ell+n+3)\delta_{-N1} + n(n-1)\delta_{-N2}] H_3 \} ,$$

$$K(11N) = -\frac{1}{2} \sqrt{\frac{(\ell+1+m)(\ell+2+m)}{\rho(2\ell+1)(2\ell+3)}} \{ [(\ell+n+\frac{3}{2})\delta_{NO} - n\delta_{-N1}] \times [\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + \frac{1}{2}(\ell+2) \operatorname{div} (\hat{u}_1 + i\hat{u}_2) - i(m - \frac{1}{2}\ell)R_1] + [(\ell+n+\frac{3}{2})\delta_{NO} + n\delta_{-N1}] \rho (A_1 + iA_2) - [(\ell+n+\frac{3}{2}) \times (\ell+n+\frac{5}{2})\delta_{N1} - (n-\frac{3}{2})(\ell+n+\frac{3}{2})\delta_{NO} - n(\ell+n+3)\delta_{-N1} +$$

$$+ n(n-1)\delta_{-N2}](H_1 + iH_2)\} ,$$

$$K(12N) = \frac{1}{4} i \sqrt{\frac{(\ell-m)(\ell+1+m)(\ell+2+m)(\ell+3+m)}{\rho(2\ell+1)(2\ell+3)}} \times [(\ell+n+\frac{3}{2})\delta_{NO} - n\delta_{-N1}]R_2.$$

$$\begin{aligned} K(-10N) = & \sqrt{\frac{(\ell-m)(\ell+m)}{\rho(2\ell-1)(2\ell+1)}} \{(\delta_{NO} - \delta_{N1})[\frac{\partial}{\partial x_3} - \frac{1}{2}(\ell-1) \operatorname{div} \hat{u}_3 - im R_0] \\ & - (\delta_{NO} + \delta_{N1})\rho A_3 + [(\ell+n+\frac{3}{2})\delta_{N2} - (\ell+n-1)\delta_{N1} \\ & - (n+\frac{5}{2})\delta_{NO} + n\delta_{-N1}]H_3\} , \end{aligned}$$

$$\begin{aligned} K(-11N) = & \frac{1}{2} \sqrt{\frac{(\ell-1-m)(\ell-m)}{\rho(2\ell-1)(2\ell+1)}} \{(\delta_{NO} - \delta_{N1})[\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ & - \frac{1}{2}(\ell-1) \operatorname{div} (\hat{u}_1 + i\hat{u}_2) - i(m+\frac{1}{2}\ell+\frac{1}{2})R_1] \\ & - (\delta_{NO} + \delta_{N1})\rho(A_1 + iA_2) + [(\ell+n+\frac{3}{2})\delta_{N2} \\ & - (\ell+n-1)\delta_{N1} - (n+\frac{5}{2})\delta_{NO} + n\delta_{-N1}](H_1 + iH_2)\} , \end{aligned}$$

$$K(-12N) = -\frac{1}{4} i \sqrt{\frac{(\ell-2-m)(\ell-1-m)(\ell-m)(\ell+1+m)}{\rho(2\ell-1)(2\ell+1)}} \times (\delta_{NO} - \delta_{N1})R_2 .$$

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