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MOMENT EQUATION APPROACH TO
NEOCLASSICAL TRANSPORT THEORY

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BY

S. P. HIRSHMAN

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Moment Equation Approach to Neoclassical
Transport Theory

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ABSTRACT

The neoclassical cross-field fluxes for a toroidally confined, axisymmetric plasma are calculated in terms of the thermodynamic forces from the fluid continuity and momentum balance equations. This macroscopic formulation of neoclassical transport theory unifies the numerous complex expressions for the transport coefficients, previously obtained by solving the Fokker-Planck equation, and elucidates their physical basis. In the large aspect ratio limit, the continuous transition in the scaling of the diffusion coefficient throughout various collisionality regimes is shown to depend on the ratio of parallel viscosity coefficients of the plasma species. Comparison of the present results with the kinetic theory expressions for the neoclassical fluxes determines the parallel viscosity coefficients for a multispecies plasma in the long-mean-free-path regime.

I. INTRODUCTION

It is well known¹ that in an axisymmetric, toroidal plasma, the neoclassical cross-field particle flux $\Gamma_a = \langle \vec{\Gamma}_a \cdot \nabla \psi \rangle$ of species (a) is driven by the flux surface averaged parallel friction force R_a :

$$\Gamma_a = - \frac{F(\psi)}{e_a} \left\langle \frac{R_a + e_a n_a E_{||}^{(A)}}{B} \right\rangle \quad (1)$$

where $F(\psi) = RB_T$ is the toroidal flux function, ψ is the poloidal flux, and $E_{||}^{(A)}$ is the parallel component of the induced electric field. Furthermore, steady-state momentum balance along the magnetic field indicates that there is a long-mean-free-path contribution to the particle flux, denoted Γ_a^{nc} , which is proportional to the parallel viscosity^{2,3,4} (i.e., pressure anisotropy):

$$\Gamma_a^{nc} = - \frac{F}{e_a} \frac{\langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle}{\langle B^2 \rangle} \quad (2)$$

Here, $\vec{\pi}_a$ is the stress tensor. The remaining Pfirsch-Schlüter flux $\Gamma_a^{PS} = \Gamma_a - \Gamma_a^{nc}$ is driven by the variation of the friction force on a flux surface:

$$\Gamma_a^{PS} = - \frac{F}{e_a} \left\langle \frac{R_a + e_a n_a E_{||}^{(A)}}{B} \left(1 - \frac{B^2}{\langle B^2 \rangle} \right) \right\rangle \quad (3)$$

These two components of Γ_a thus arise from distinct physical mechanisms which remain distinguishable in all collision frequency

regimes. Henceforth, only the flux Γ_a^{nc} will be considered. Its contribution to Γ_a is dominant whenever at least one other species is in the low or intermediate collision frequency regime.³ (The flux Γ_a^{PS} has been considered for an ion-impurity plasma in Refs. 3 and 5.)

There have been numerous kinetic theory calculations of the neoclassical particle flux in various collisionality regimes.^{1,3,5-7} A derivation of Γ_a^{nc} , in terms of the thermodynamic forces, directly from the fluid equations has not been heretofore possible, since an accurate expression for the neoclassical parallel viscosity in terms of the species' flows has been lacking. In Sec. III, the averaged parallel viscosity $\langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle$ appearing in Eq. (2) is derived for a neoclassical plasma, thus permitting a completely fluid calculation of the neoclassical particle flux. Comparison of Γ_a^{nc} derived in this way with previous kinetic theory results then allows an accurate determination of the neoclassical parallel viscosity coefficients.

The transition between long-mean-free-path and collision-dominated transport scalings in a multispecies plasma is investigated in Sections IV and VI. For a two ion species plasma, it is shown that the scaling of the mutual diffusion coefficient is determined by the collisionality of the species with the smaller parallel viscosity coefficient, in agreement with the detailed balance principle.^{3,4} However, it is found that interspecies collisional coupling can modify the value of collisionality for which transitions in diffusion coefficient scaling occur.

II. POLOIDAL DEPENDENCE OF FIRST-ORDER NEOCLASSICAL FLOWS

The neoclassical flow velocity of species (a) is given to first order in the Larmor radius expansion by

$$\vec{u}_a^{(1)} = \vec{u}_{\perp a}^{(1)} + u_{\parallel a} \vec{n} \quad (4)$$

where $\vec{n} = \vec{B}/B$, $u_{\parallel a} = \vec{n} \cdot \vec{u}_a^{(1)}$ is the parallel flow, and $\vec{u}_{\perp a}^{(1)}$ is the first-order diamagnetic velocity, determined by the perpendicular component of the momentum balance of species (a):

$$n_a \vec{u}_{\perp a}^{(1)} = \frac{\vec{n} \times \nabla \psi}{m_a \Omega_a} p_a(\psi) \left(\frac{p'_a}{p_a} + \frac{e_a \Phi'_0}{T_a} \right) . \quad (5)$$

Here, $p_a(\psi)$ and $T_a(\psi)$ are the lowest order pressure and temperature of species (a), respectively (which are uniform on a flux surface), $\Phi'_0(\psi)$ is the zero-order electrostatic potential gradient, and prime denotes $\partial/\partial\psi$. Note that the first-order flow is entirely within a magnetic surface, $\vec{u}_a^{(1)} \cdot \nabla \psi = 0$.

From the continuity equation $\nabla \cdot [n_a \vec{u}_a^{(1)}] = 0$, it follows that

$$u_a(\psi) \equiv \vec{u}_a^{(1)} \cdot \vec{B}_p / B_p^2 \quad (6)$$

is a surface function of ψ alone (in an axisymmetric system). Here, $\vec{B}_p = \nabla \zeta \times \nabla \psi$ is the poloidal magnetic field and ζ is the toroidal angle. Taking the scalar product of Eq. (4) with $\vec{B}_p$ yields:

$$u_{||a} = V_{1a} + u_a(\psi)B \quad (7a)$$

where

$$V_{1a} \equiv - \left(\frac{\vec{B}_p \cdot \vec{u}_a^{(1)}}{B_p^2} \right) B = - \frac{F T_a}{m_a \Omega_a} \left(\frac{p'_a}{p_a} + \frac{e_a \phi'_0}{T_a} \right) \quad (7b)$$

Here, V_{1a} is the flow of species (a) along the magnetic field which would be required to exactly cancel the poloidal component of the diamagnetic drift, thus making $u_a(\psi) = 0$. Combining Eqs. (4) and (7) yields:

$$\vec{u}_a^{(1)} = u_a(\psi) \vec{B} + \langle V_{1a} B \rangle \vec{B}_T / B_T^2 \quad (8)$$

Since $\vec{B}_T = F \nabla \zeta$ ($|\nabla \zeta|^{-1} = R$, the distance from the symmetry axis), the last term in Eq. (8) represents rigid body rotation on each flux surface in the direction of toroidal symmetry.

A similar computation for the first-order heat flux vector $\vec{q}_a^{(1)}$ yields:

$$q_{||a} = \frac{5}{2} p_a V_{2a} + q_a(\psi) B \quad (9a)$$

$$\vec{q}_a^{(1)} = q_a(\psi) \vec{B} + \frac{5}{2} p_a \langle V_{2a} B \rangle \vec{B}_T / B_T^2 \quad (9b)$$

where

$$q_a(\psi) = \vec{q}_a^{(1)} \cdot \vec{B}_p / B_p^2 \quad (10a)$$

$$V_{2a} = - \frac{F T'_a}{m_a \Omega_a} \quad (10b)$$

III. DERIVATION OF AVERAGED PARALLEL VISCOSITY TENSOR

First consider the parallel viscosity tensor for the case when there is no heat flux, $\vec{q}_a^{(1)} = 0$. Then, whatever the structure of the stress tensor, it must be (to first order in Larmor radius) linearly dependent on spatial gradients of the first-order flow $\vec{u}_a^{(1)}$, Eq. (8). Furthermore, there is no first-order parallel viscous force arising from nonuniformities of the flow field in the $\nabla\psi$ direction. (This is physically clear from random walk arguments.⁸) Hence, the parallel viscous stress must involve only components of the gradient (operating on $\vec{u}_a$) which lie within a magnetic flux surface. An immediate consequence of this property is that rigid body rotation about the symmetry axis, which may have a different magnitude on each flux surface is nevertheless undamped by the parallel viscous stress. Thus, the last term in Eq. (8) does not contribute to $\langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle$ (nor to the local value $\vec{B} \cdot \nabla \cdot \vec{\pi}_a$), and we obtain:

$$- \langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle \Big|_{\vec{q}_a=0} = - 3\mu_{a1} \langle (\vec{n} \cdot \nabla B)^2 \rangle u_a(\psi) . \quad (11)$$

Note that only the magnitude of the poloidal velocity $\propto u_a(\psi)$ is damped by the parallel viscosity.⁹ Here, μ_{a1} is the parallel viscosity coefficient associated with the plasma flow velocity. It is a function of the neoclassical collision frequency regime of species (a) and will be determined in Sec. V. The constant $3\langle (\vec{n} \cdot \nabla B)^2 \rangle$ has been chosen so that μ_{a1} reduces to the classical

viscosity coefficient¹⁰ η_0^a when species (a) is in the collision-dominated regime. It represents the "signature" of the characteristic $\nabla \nabla$ operator appearing in the viscous force. Equation (11) was previously derived¹¹ in the Pfirsch-Schlüter regime using the explicit collisional form¹⁰ for $\vec{\pi}_a$. The present result has been deduced on the basis of the general structure of the neoclassical flow, and is therefore applicable in all neoclassical regimes.

Similar arguments may be given to construct the contribution to $\langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle$ arising from the heat flux. Then, the general result for the parallel viscosity tensor becomes:

$$\begin{aligned} - \langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle = & - \frac{3 \langle (\vec{n} \cdot \nabla B)^2 \rangle}{\langle B^2 \rangle} \left[\mu_{a1} \langle (u_{\parallel a} - v_{1a}) B \rangle \right. \\ & \left. + \frac{2}{5} \mu_{a2} \left\langle \left(\frac{q_{\parallel a}}{p_a} - \frac{5}{2} v_{2a} \right) B \right\rangle \right]. \end{aligned} \quad (12)$$

Equations (7) and (9) have been used to eliminate the poloidal velocity and heat flux in terms of the respective parallel flows. The existence of a viscous stress in response to a nonuniform heat flux had been previously noted for a classical plasma,¹² although the coefficient μ_{a2} has only recently been accurately determined from the Fokker-Planck equation for a collisional plasma.¹³

An important feature of the neoclassical stress tensor, Eq. (12), is that contributions to the stress of species (a) arise only from its own nonuniform flow fields. In principle, contributions to $\vec{\pi}_a$ proportional to $\vec{u}_b$ and $\vec{q}_b$ might be expected

due to the collisional interaction of species (a) with other plasma species (b). (Collisional coupling certainly influences the form of the friction force R_a .) For $\vec{\pi}_a$, however, direct coupling to the flows of other species can be shown to be weak. (This property can be demonstrated rigorously by inspection of the field particle contribution to the $\ell = 2$ spherical harmonic of the linearized Fokker-Planck collision operator.¹⁴) Therefore, Eq. (12) is a valid representation for the parallel viscosity even in a multispecies plasma. Equation (12) implies that the pressure anisotropy of other plasma species (b) can only be transferred indirectly to species (a) through a collisional modification of its own flow fields $u_{||a}$ and $q_{||a}$.

IV. FLUID DERIVATION OF THE NEOCLASSICAL FLUXES

The parallel component of the fluid momentum balance equation averaged over a magnetic flux surface yields:

$$- \langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle + \langle B(R_a + e_a n_a E_{||}^{(A)}) \rangle = 0 . \quad (13)$$

Equation (13), together with the transport relation in Eq. (12), will now be used to compute the neoclassical parallel flows and cross-field particle flux Γ_a^{nc} in terms of the thermodynamic forces $\alpha(V_{1a}, V_{2a})$. Since the net interspecies friction force vanishes, $\sum_a R_a = 0$, Eq. (13) yields a constraint equation on the parallel viscosity:

$$\sum_a \langle \vec{B} \cdot \nabla \cdot \vec{\pi}_a \rangle = 0 \quad . \quad (14)$$

For simplicity, $E_{||}^{(A)} = 0$ will be assumed in what follows, and only the response to p'_a and T'_a is considered. (The Ware fluxes in response to $E_{||}^{(A)}$ are considered in Ref. 15.) Using the explicit forms for μ_{aj} ($j = 1, 2$) given in Sec. V, it is readily verified that the normalized viscosity coefficient

$$\xi_a \equiv \frac{\langle (\vec{n} \cdot \nabla B)^2 \rangle}{\langle B^2 \rangle} \frac{\mu_{aj}}{m_a n_a \nu_a} \quad , \quad (15)$$

is a small parameter¹⁶ in the large aspect ratio limit, i.e., $\xi_a < 1$. Here, ν_a is the total collision frequency of species (a) with all other plasma species. (Although for simplicity only the large aspect ratio limit is considered here, the present procedure is extendable to arbitrary aspect ratios.¹⁵) For example, in the banana regime, $\xi_a \sim (r/R)^{1/2}$ is the fraction of trapped particles.⁶ Note, further, that the friction force R_a can always be written as a linear combination of the relative parallel flows $u_{||a} - u_{||b}$ and parallel heat flows, $q_{||b}/p_b$, e.g., $R_a \sim m_a n_a \nu_{ab} [(u_{||a} - u_{||b}) + q_{||b}/p_b + \dots]$. Thus, we conclude from Eqs. (12) and (13), that to lowest order in ξ_a , the parallel velocities of all the plasma species are equal to a common flow V :

$$\langle u_{||a} B \rangle = \langle V B \rangle [1 + O(\xi_a)] \quad . \quad (16a)$$

Also, the parallel heat fluxes must be $O(\xi_a)$:

$$\frac{\langle q_{\parallel a}^B \rangle}{p_a} = \langle V B \rangle O(\xi_a) \quad . \quad (16b)$$

To rigorously derive Eqs. (16a,b) from fluid equations requires an additional equation for the heat flux balance,³ analogous to Eq. (13). For simplicity, this equation has been suppressed in the present calculation, although its retention becomes necessary to justify the present results. Moreover, by including the heat flux balance equation in the calculation, the fluid approach developed here can be extended to compute the neoclassical cross-field heat fluxes, as indicated at the end of this section.

The lowest order parallel speed V may be determined by substituting the expansions, Eqs. (16a,b), into Eq. (12) and using the ambipolar constraint Eq. (14), with the result:

$$\langle V B \rangle = \sum_a (\mu_{a1} \langle V_{1a}^B \rangle + \mu_{a2} \langle V_{2a}^B \rangle) / \sum_a \mu_{a1} \quad . \quad (17)$$

The common flow is a collisionally modified average of the individual species responses V_{ja} . Thus, by influencing $u_{\parallel a}$, inter-particle collisions (indirectly) affect the viscous stress of species (a). [See the discussion following Eq. (19c).]

The parallel flow to lowest order in ξ_a is determined by the species with the largest parallel viscosity coefficient (s). Physically, this is a consequence of the parallel momentum balance Eq. (14). For example, in a simple electron-ion plasma where

$\mu_{ij}/\mu_{ej} \approx (m_i/m_e)^{1/2}$ for $j = 1, 2$ in all collisionality regimes, Eq. (17) reduces to:

$$\langle u_{\parallel i} B \rangle \approx \langle V B \rangle = \langle V_{1i} B \rangle + \frac{\mu_{i2}}{\mu_{i1}} \langle V_{2i} B \rangle. \quad (18)$$

Thus, there is a residual ion poloidal rotation $u_i(\psi) = (\mu_{i2}/\mu_{i1}) \langle V_{2i} B \rangle / \langle B^2 \rangle$ driven by the ion temperature gradient,^{17,18} which is required to balance the heat flux driven contribution to the ion viscous stress. When there is an additional impurity species (z) present, the common flow V will still be determined by the ion gradients whenever $\mu_{i1} > \mu_{z1}$. This occurs, for example, when the ions are in the long-mean-free-path regime and the impurities are well into the collisional regime¹⁹ (see Sec. VI).

The result for V in Eq. (17) can be used in Eq. (12) to evaluate the stress tensor of each species to first order in ξ_a . Then, we obtain the particle flux from Eq. (2):

$$e_a \Gamma_a^{nc} = -g(\psi) \mu_{a1} \sum_j \mu_{j1} \langle (V_j^* - V_a^*) B \rangle / \sum_j \mu_{j1} \quad (19a)$$

where

$$V_j^* = V_{1j} + (\mu_{j2}/\mu_{j1}) V_{2j} \quad (19b)$$

and $g(\psi) = 3F \langle (\vec{n} \cdot \nabla B)^2 \rangle / \langle B^2 \rangle^2$ is a coefficient determined by the equilibrium magnetic field configuration. Equations (17) and (19) are new results which extend previous kinetic theory expressions¹⁹ for $\langle V B \rangle$ and Γ_a^{nc} into the collisional regime (once the

μ_{aj} are determined; cf. Sec. V) and provide an interpretation of the physical origin of these quantities. (The ion flow for an electron-ion plasma in all neoclassical regimes has been previously considered in Refs. 17 and 18.)

For the case of two plasma species (a) and (b), Eq. (19a) reduces to:

$$e_a \Gamma_a^{nc} = -g(\psi) (\mu_{al}^{-1} + \mu_{bl}^{-1})^{-1} \langle (V_b^* - V_a^*) B \rangle = -e_b \Gamma_b^{nc} . \quad (19c)$$

Thus, in this case the scaling with collision frequency of the mutual diffusion coefficient is determined by the species with the smaller viscosity coefficient. In addition, when $\mu_{bl} < \mu_{al}$, the magnitude of the flux Γ_a^{nc} is reduced by the factor μ_{bl}/μ_{al} from its uncoupled value $\Gamma_a^{nc} \propto \mu_{al}$. (This effect was previously noted in the banana-plateau regimes in Ref. 7.) Note the similarity of the result Eq. (19c) with the classical ambipolar diffusion coefficient along a magnetic field. The parallel viscosity coefficient enters the neoclassical diffusion coefficient in a manner analogous to the parallel mobility in the classical coefficient.

Equation (19c) may now be applied to the situation when there is an impurity ion (z) in the Pfirsch-Schlüter regime, and an ion (i) in one of the long-mean-free-path regimes. In the collisional regime,¹⁰ $\mu_{z1} \sim \nu_z^{-1}$. Then, provided the impurity is sufficiently collisional so that $\mu_{z1} < \mu_{i1}$, we conclude Γ_i^{nc} decays as ν_z^{-1} . This is consistent with the vanishing of

the neoclassical flux in this mixed collisionality regime, which was previously predicted by a detailed balance argument.^{3,4} The present result provides additional insight inaccessible from the general detailed balance arguments alone. Besides yielding a formula for the flux Γ_a^{nc} , Eq. (19c) predicts that the transition in the scaling of the diffusion coefficient will occur for values of the collisionality where $\mu_{z1} = \mu_{i1}$. As shown in Sec. VI, these may not correspond with the uncoupled values of collisionality which demark the usual transitions between the three neoclassical regimes.

In order not to obscure the fundamental physical effects contributing to the neoclassical flux, the heat flux balance equation,³

$$- \langle \vec{B} \cdot \nabla \cdot \vec{\Theta}_a \rangle + \langle B H_a \rangle = 0, \quad (20)$$

has been thus far neglected [except when it was required to justify Eqs. (16a,b)]. Here, $\vec{\Theta}_a = \int m_a [(v/\alpha_a)^2 - 5/2] [\vec{v} \vec{v} - (v^2/3) \vec{I}] f_a d\vec{v}$ is the heat flux tensor (analogous to $\vec{\pi}_a$) and $H_a = \int m_a (x_a^2 - 5/2) v_{||} C_a d\vec{v}$ is the "heat friction" (analogous to R_a). Note that like the collisional friction, H_a is generally a linear combination of the relative parallel particle flows and parallel heat flows. For completeness, we now consider the role of Eq. (20) in determining the neoclassical cross-field heat flux. The parallel component of the heat flux tensor may be expressed in a form analogous to Eq. (12) for the stress tensor:

$$- \langle \vec{B} \cdot \nabla \cdot \vec{\theta}_a \rangle = - \frac{3 \langle (\vec{n} \cdot \nabla B)^2 \rangle}{\langle B^2 \rangle} [\mu_{a2} \langle (u_{||a} - v_{1a}) B \rangle \quad (21)$$

$$+ \frac{2}{5} \mu_{a3} \langle (q_{||a}/p_a - \frac{5}{2} v_{2a}) B \rangle] .$$

Note the relation³ $Q_a^{nc}/T_a = - (F/e_a) \langle \vec{B} \cdot \nabla \cdot \vec{\theta}_a \rangle / \langle B^2 \rangle$, where $Q_a^{nc} = T_a \langle \int \vec{v}_{Da} \cdot \nabla \psi (x_a^2 - 5/2) f_a d\vec{v} \rangle$ is the neoclassical cross-field heat conduction. It is then possible to compute Q_a^{nc} using $\langle u_{||a} B \rangle = \langle VB \rangle$, where $\langle VB \rangle$ is the lowest order common flow velocity given by Eq. (17), in the transport relation Eq. (21).

The results of this section may be summarized in terms of the neoclassical transport coefficients L_{ij}^{ab} , which relate the cross-field fluxes $I_1^a = \Gamma_a^{nc}$ and $I_2^a = Q_a^{nc}/T_a$ to the thermodynamic forces $A_1^a = p'_a/p_a$ and $A_2^a = T'_a/T_a$ by the linear equations $I_i^a = \sum_{b,j=1,2} L_{ij}^{ab} A_j^b$ (neglecting the contribution from $\langle \vec{E} \cdot \vec{B} \rangle$). From Eq. (19a) and an equation for Q_a^{nc} derivable from Eq. (21), we obtain (for $i, j \neq 3$):

$$T_a L_{ij}^{ab} = G(\psi) \left(\frac{T_a T_b}{e_a e_b} \right) \left(\frac{\mu_{ai} \mu_{bj}}{\sum_k \mu_{kl}} - \mu_{a(i+j-1)} \delta_{ab} \right) \quad (22)$$

where $G(\psi) = 3F^2 \langle (\vec{n} \cdot \nabla B)^2 \rangle / \langle B^2 \rangle^2$ is a positive constant. It is obvious from this expression that the transport coefficients satisfy the Onsager symmetry relations $T_a L_{ij}^{ab} = T_b L_{ji}^{ba}$.

V. DETERMINATION OF PARALLEL VISCOSITY COEFFICIENTS

By comparing Eq. (22) with Eq. (29a) of Ref. 19, the parallel viscosity coefficients in the long-mean-free-path regimes (banana/plateau) are readily identified:

$$\mu_{aj} = f_t n_a m_a \{ (x_a^2 - 5/2)^{j-1} \hat{v}_d^a \} \frac{\langle B^2 \rangle}{3 \langle (\vec{n} \cdot \nabla B)^2 \rangle} \quad j=1,2,3 \quad (23)$$

The notation used here is defined in Ref. 19.

In the collisional regime, the viscosity coefficients have the form¹⁰ $\mu_{aj} = p_a \tau_{aa} l_{aj}$, where l_{aj} are dimensionless constants given by the approximate expressions:¹³

$$l_{aj} = \frac{2}{5} \{ x_a^2 (x_a^2 - \frac{5}{2})^{j-1} / (v_T^a \tau_{aa}) \} \quad (24)$$

Here, $v_T^a(v)$ is the total collision frequency governing the rate of relaxation of the pressure anisotropy of species (a). In the large aspect ratio limit, where $\langle (\vec{n} \cdot \nabla B)^2 \rangle = 1/2 (B_p/R_0)^2$, the viscosity coefficient in Eq. (23) becomes equal to the classical one at the plateau-Pfirsch-Schlüter transition $v_{*a} = \epsilon^{-3/2}$ where $\epsilon = r/R_0$, $v_{*a} = (v_a/\omega_{ta}) \epsilon^{-3/2}$ is the collisionality parameter, and $\omega_{ta} = \alpha_a/(R_0 q)$ is the (thermal particle) transit speed. Comparing Eqs. (23) and (24), note that a continuously valid expression for the viscosity coefficient throughout all neoclassical regimes (in the large aspect ratio limit) is given by

$$\mu_{aj} = f_t n_a m_a \left\{ \left(x_a^2 - \frac{5}{2} \right)^{j-1} \hat{v}_d^a \left[1 + (5\pi/8) v_{Ta}^a / \omega_{Ta}(x_a) \right]^{-1} \right\} \frac{\langle B^2 \rangle}{3 \langle (\vec{n} \cdot \nabla B)^2 \rangle} \quad (25)$$

where $\omega_{Ta}(x_a) = \omega_{Ta} x_a$. (Here, the numerical factor arises from the explicit form for $\hat{v}_d^a$ given in Ref. 19.) A graph of the parallel viscosity coefficient μ_{a1} versus v_{*a} is given in Fig. 1.

It should be noted that the parallel viscosity coefficient of any species (a) is only a function of the collisionality regime of that particular species, i.e., $\mu_{a1} = \mu_{a1}(v_{*a})$. (This follows from the lack of direct coupling of the viscosity tensor π_a to the flows of other species, as discussed previously.) Thus, the formula for μ_{aj} given in Eq. (25) is universally valid (for all values of v_{*a}) independent of the collision frequency regimes of the other plasma species. In contrast, the particle diffusion coefficients derived in Eq. (19a) exhibit a very complex dependence on the collisionalities of all plasma species. In this respect, the parallel viscosity coefficient is a fundamental feature of the collisional transport of each species (i.e., μ_{a1} always varies with v_{*a} as indicated in Fig. 1), whereas the diffusion coefficient is a hybrid quantity resulting from the interspecies collisional interactions in a neoclassical plasma. It is precisely this strong collisional interaction between plasma species which sometimes results in the nonintuitive scaling^{3,4,7} (i.e., in contrast to that predicted on the basis of single particle motion) of the diffusion coefficient.

VI. APPLICATIONS

Finally, let us consider the qualitative scaling as a function of v_* of the diffusion coefficient in a two ion species plasma, given by Eq. (19c). From the quantitative expression for μ_{aj} in Eq. (25), note that the viscosity coefficient scales as:

$$\mu_{al} \approx (R_0 q) \frac{n_a m_a \alpha_a v_{*a}}{1 + v_{*a}} \frac{1}{1 + \epsilon^{3/2} v_{*a}}. \quad (26)$$

There are two cases of interest, which are characterized by the parameter

$$v_{crit} \equiv \frac{\mu_{bl}(v_{*b}=1)}{\mu_{al}(v_{*a}=1)} = \frac{n_b}{n_a} \left(\frac{m_b T_b}{m_a T_a} \right)^{1/2}. \quad (27)$$

Physically, v_{crit} is the ratio of plateau values (maximum values) of the two viscosity coefficients. Note that in an electron (a)-ion (b) plasma, $v_{crit} \gg 1$. However, for an ion impurity plasma, v_{crit} can take on a range of values.

A: $v_{crit} \equiv < 1$

With $(m_b/m_a)^{1/2} > 1$, $(e_a/e_b)^2 < 1$ and $n_b e_b^2/n_a e_a^2 \approx 1$, we find $v_{*b}/v_{*a} \sim (e_b/e_a)^2 > 1$. Thus, the viscosity coefficient of species (b) undergoes transitions to the plateau and Pfirsch-Schlüter regimes at lower collisionalities (measured in terms of v_{*a}) than species (a). This situation, originally

considered by Hinton and Moore⁷ in the banana-plateau regimes, may be realized in practice by a heavy, relatively highly ionized impurity species (b) embedded in the main hydrogenic plasma (a). From Eq. (26), $\mu_{a1} = \mu_{b1}$ for $v_{*a} = v_{crit}$. When $v_{*a} < v_{crit}$, the main ions (a) are in the banana regime, the impurities (b) are in either the banana or plateau regime, and $\mu_{a1} < \mu_{b1}$, so the light ions (a) determine the banana-like transport scaling of the diffusion coefficient. For $v_{*a} > v_{crit}$, $\mu_{a1} > \mu_{b1}$ pertains, and the mutual particle diffusion coefficient varies as μ_{b1} and is therefore determined by the plateau or collisional value of μ_{b1} , even though species (a) may still be in its own banana or plateau regime.⁷ The transition to collisional scaling occurs for $v_{*a} = (e_a/e_b)^2 \epsilon^{-3/2} \ll \epsilon^{-3/2}$. A graph of the diffusion coefficient $(\mu_{a1}^{-1} + \mu_{b1}^{-1})^{-1}$ versus v_{*a} , together with μ_{a1} and μ_{b1} (the uncoupled viscosity coefficients) is given for this case in Fig. 1. Note that the plateau value for the diffusion coefficient is smaller⁷ than the nominal light ion value based on heuristic random walk arguments (i.e., the plateau value of μ_{a1}) by the factor v_{crit} . The changes in the transitional values of v_{*a} and the magnitude of the diffusion coefficient noted in this example are characteristic results of the collisional coupling between species (a) and (b).

B: $1 < v_{crit} < \epsilon^{-3/2}$

Considering a finite mass ratio $m_a/m_b \approx 1$ and $(e_a/e_b)^2 < 1$,

note that $\nu_{*b}/\nu_{*a} \approx (e_b/e_a)^2 > 1$ still pertains. The condition $\nu_{crit} > 1$ may be realized if species (a) and (b) are, for example, two different charge states of the same high Z element. Alternatively, $\nu_{crit} > 1$ may occur in a hydrogen (a) - impurity (b) plasma with a large proton defect $Z_b n_b/n_a > Z_b^{1/2}$. The diffusion is now determined by species (a), $\mu_{a1} < \mu_{b1}$, for $\nu_{*a} < \nu_2 = \nu_{crit} (e_b/e_a)^{-2} \epsilon^{-3/2}$. For $(e_b/e_a)^2 \epsilon^{3/2} \approx O(1)$, $\nu_{*a} = \nu_2$ is in the plateau regime of species (a). Thus, in the Pfirsch-Schlüter range of species (b) corresponding to $\epsilon^{-3/2} < \nu_{*b} < \nu_{crit} \epsilon^{-3/2}$, the diffusion coefficient is independent of collision frequency. Furthermore, in this range, the long-mean-free-path flux Γ_a^{nc} exceeds the collisional contribution Γ_a^{PS} by the large factor $(\nu_{*a} \epsilon^{3/2})^{-1}$. Thus, in this interval of collisionality, the anisotropy of the plateau species (a) is effectively transferred to the collisional species (b) through a modification of $u_{||b}$. For $\nu_{*a} > \nu_2$, $\mu_{b1} < \mu_{a1}$ and the characteristic ν_b^{-1} decay of the long-mean-free-path diffusion coefficient occurs. Note that although the detailed balance principle predicts the ν_b^{-1} behavior of Γ_a^{nc} when species (b) is well into the collisional regime, the present analysis shows that the more stringent condition $\nu_{*b} > \nu_{crit} \epsilon^{-3/2}$ is required before the impurity pressure tensor becomes nearly isotropic. A graph of the diffusion coefficient for this case is given in Fig. 2.

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Fig. 1: Graph of parallel viscosity coefficients μ_{a1} and μ_{b1} and particle diffusion coefficient $(\mu_{a1}^{-1} + \mu_{b1}^{-1})^{-1}$ versus collisionality parameter ν_{*a} of species (a) for $\nu_{crit} = (n_b/n_a) (m_b/m_a)^{1/2} < 1$.

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Fig. 2: Graph of parallel viscosity coefficients μ_{a1} and μ_{b1} and particle diffusion coefficient $(\mu_{a1}^{-1} + \mu_{b1}^{-1})^{-1}$ versus ν_{*a} for $\nu_{crit} > 1$.

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