CONFORMAL ANOMALIES IN CURVED SPACE-TIME

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ABSTRACT

The general form of the conformal anomaly in a dimensionally regularized theory of massless fermions in a background metric is shown to be determined by the first few terms of weak field perturbation theory.

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There have recently been a number of investigations^{1,2,3,4} concerning the possible existence of conformal anomalies in theories with quantized non-interacting matter fields propagating in curved space-time. These studies depend either on covariant point-separation^{1,2} or dimensional regularization to define a finite stress-energy tensor in the presence of a background metric. There are now conflicting results as to whether the stress-energy tensor in a conformally invariant four-dimensional theory, such as that describing massless free fermions in a background metric (the example to be used throughout this note), necessarily develops a nonvanishing trace after suitable regularization. Various implementations of the point-separation method appear to give both positive¹ and negative² answers to this question, whereas dimensional regularization^{3,4} applied to the weak (gravitational) field perturbation expansion of the expectation $\langle T^{\mu\nu}(x) \rangle_g$ appears to lead always to an anomalous (i.e. non-zero) trace.

It is not proposed in this note to decide which of these two approaches is "correct" (although more on this later); the point is simply to clarify the rather confusing situation which exists with regard to the anomalies arising in dimensional regularization. Namely, the exact form of these anomalies will be derived (without introducing non-local counterterms in the action) and shown to be essentially determined by the low orders of a weak-field perturbation expansion.

Consider first a conformally-invariant two-dimensional field theory - that of massless fermions propagating in curved space-time. It will suffice in this case to work to first order in

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$h_{\mu\nu}$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($\eta_{\mu\nu}$ is the Minkowski metric tensor).

The flat-space energy-momentum tensor is just

$$T^{\mu\nu}(x) = -\frac{1}{4} \bar{\psi}(x) (\gamma^{\mu} \overleftrightarrow{\partial}^{\nu} + \gamma^{\nu} \overleftrightarrow{\partial}^{\mu}) \psi(x)$$

and the first order contribution to the expectation-value $\langle T^{\mu\nu} \rangle_g$ is essentially given (there are no seagulls to this order in dimensional regularization) by the quantity

$$\begin{aligned} T^{\mu\nu;\rho\sigma}(p;\omega) &= \int d^{2\omega} z e^{-ip \cdot z} \langle 0 | T \theta^{\mu\nu}(z) \theta^{\rho\sigma}(0) | 0 \rangle \\ &= \frac{2i\pi^{\omega}}{(2\pi)^{2\omega}} \int_0^1 dx \left\{ \Gamma(-\omega) \left(x(1-x)p^2 \right)^{\omega} \left(\eta^{\mu\nu} \eta^{\rho\sigma} + \frac{1-\omega}{2} (\eta^{\mu\sigma} \eta^{\nu\rho} + \eta^{\mu\rho} \eta^{\nu\sigma}) \right) \right. \\ &\quad + \frac{1}{2} \Gamma(1-\omega) \left(x(1-x)p^2 \right)^{\omega-1} \left[x(1-x) \left(p^2 (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\nu\rho} \eta^{\mu\sigma}) - 4 \eta^{\mu\nu} p^{\rho} p^{\sigma} + \eta^{\rho\sigma} p^{\mu} p^{\nu} \right) \right. \\ &\quad \left. \left. + (\omega-2) (\eta^{\mu\rho} p^{\nu} p^{\sigma} + \eta^{\mu\sigma} p^{\nu} p^{\rho} + \eta^{\nu\rho} p^{\mu} p^{\sigma} + \eta^{\nu\sigma} p^{\mu} p^{\rho}) \right) \right] \\ &\quad + \frac{1}{4} (1-\omega) \left(\eta^{\mu\rho} p^{\nu} p^{\sigma} + \eta^{\mu\sigma} p^{\nu} p^{\rho} + \eta^{\nu\rho} p^{\mu} p^{\sigma} + \eta^{\nu\sigma} p^{\mu} p^{\rho} \right) + \eta^{\mu\nu} p^{\rho} p^{\sigma} + \eta^{\rho\sigma} p^{\mu} p^{\nu} \left. \right\} \\ &\quad + \Gamma(2-\omega) \left(x(1-x)p^2 \right)^{\omega-2} x(1-x) \left(x(1-x) - \frac{1}{4} \right) \\ &\quad \times \left[4 p^{\mu} p^{\nu} p^{\rho} p^{\sigma} - \frac{p^2}{2} \left(\eta^{\mu\rho} p^{\nu} p^{\sigma} + \eta^{\mu\sigma} p^{\nu} p^{\rho} + \eta^{\nu\rho} p^{\mu} p^{\sigma} + \eta^{\nu\sigma} p^{\mu} p^{\rho} \right) \right] \end{aligned} \quad (1)$$

Eq. (1) is valid for arbitrary space-time dimension 2ω . The pole part of $T^{\mu\nu;\rho\sigma}$ as $\omega \rightarrow 1$ is readily seen to be proportional to

$$\begin{aligned} A^{\mu\nu;\rho\sigma}(p,\omega) &= \frac{1}{\omega-1} \left(-p^2 \eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\mu\nu} p^{\rho} p^{\sigma} + \eta^{\rho\sigma} p^{\mu} p^{\nu} \right. \\ &\quad \left. - \frac{1}{2} (\eta^{\mu\rho} p^{\nu} p^{\sigma} + \eta^{\mu\sigma} p^{\nu} p^{\rho} + \eta^{\nu\rho} p^{\mu} p^{\sigma} + \eta^{\nu\sigma} p^{\mu} p^{\rho}) \right. \\ &\quad \left. + \frac{1}{2} p^2 (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho}) \right) \end{aligned} \quad (2)$$

The tensor in brackets actually vanishes identically in two dimensions. However, the rules of dimensional regularization forbid us to use identities valid only at $\omega = 1$, before taking the limit $\omega \rightarrow 1$. In fact, it must be possible to cancel this term by adding to the action a generally covariant local term quadratic in derivatives and constructed solely from the metric tensor (this argument is essentially that of DeWitt and Utiyama⁵). There is one such term, and indeed we find that the action

$$S_{\text{total}} = S_{\text{matter}}[\psi, \bar{\psi}, g] + \alpha(\omega) \int d^{2\omega} x \sqrt{-g} R(x) \quad (3)$$

$$\text{with } \alpha(\omega) \xrightarrow{\omega \rightarrow 1} \frac{1}{24\pi} \frac{1}{\omega-1} + 0(1)$$

leads to a finite stress-energy tensor to first order in $h_{\mu\nu}$, and in virtue of the uniqueness of the counterterm, to all orders.*

The vanishing of the tensor in (2) at $\omega = 1$ is due to the fact that $\sqrt{-g} R$ is a total derivative in 2 space-time dimensions:

* In fact, this action defines a finite vacuum persistence amplitude $\langle 0 \text{ out} | 0 \text{ in} \rangle_g$ in the presence of an arbitrary asymptotically flat metric, to all orders in h .

nevertheless the counterterm in (3) must be added, as it is not otherwise possible to assign a sensible limit to $\Lambda^{\mu\nu;\rho\sigma}(p, \omega)$ as $\omega \rightarrow 1$.

The action in (3) is immediately seen to give rise to a non-vanishing and unique trace-anomaly in two dimensions. Note that since the stress-tensor is now finite for arbitrary ω and smooth in the limit $\omega \rightarrow 1$, we may compute its trace in two dimensions by computing the trace in 2ω dimensions and then taking $\omega \rightarrow 1$. This procedure yields the unique result

$$g_{\mu\nu} \langle T^{\mu\nu}(x) \rangle_g = \lim_{\omega \rightarrow 1} \left\{ g_{\mu\nu}(x) \cdot \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \int d^{2\omega}z \sqrt{-g} \alpha(\omega) R(z) \right\}$$

$$= \frac{1}{12\pi} R(x) \quad (4)$$

We emphasize that the anomaly here is local, and both its form and magnitude are fixed by the requirement that the poles up to $O(h)$ be removed by local counterterms which, in virtue of their coordinate invariance, automatically preserve conservation of $T^{\mu\nu}$.

The situation is somewhat different in four dimensions. The relevant counterterms here are quartic in derivatives⁵. There are only three, and we begin from the action

$$S_{\text{total}} = S_{\text{matter}}(\phi, \psi, g) + \int d^{2\omega}x \sqrt{-g} \left\{ \beta_1(\omega) R^2(x) + \beta_2(\omega) R_{\kappa\lambda} R^{\kappa\lambda}(x) \right. \\ \left. + \beta_3(\omega) R_{\kappa\lambda\rho\sigma} R^{\kappa\lambda\rho\sigma}(x) \right\} \quad (5)$$

Using (1), we find that the expectation value $\langle T^{\mu\nu} \rangle_g$ can be made finite to first order in $h_{\mu\nu}$ with the following choice for $\beta_i(\omega)$:

$$\beta_1(\omega) \rightarrow \frac{b}{\omega-2} + B_1 + O(\omega-2)$$

$$\beta_2(\omega) \rightarrow \frac{K-4b}{\omega-2} + B_2 + O(\omega-2) \quad (6)$$

$$\beta_3(\omega) \rightarrow \frac{b-K}{\omega-2} + B_3 + O(\omega-2)$$

with $K = \frac{1}{960\pi^2}$. Note that the divergences in $T^{\mu\nu;\rho\sigma}$ do not in this case completely fix the divergent part of the counterterms, hence the free parameter b . This ambiguity is due to the fact that the expression $\frac{\delta}{\delta g_{\mu\nu}(x)} \int d^{2\omega}z \sqrt{-g} \left(R^2(z) - 4R_{\kappa\lambda} R^{\kappa\lambda}(z) + R_{\kappa\lambda\rho\sigma} R^{\kappa\lambda\rho\sigma}(z) \right)$ is quadratic in curvature (i.e. of order h^2) and cannot therefore contribute to this order. It seems highly likely that b will however be fixed in the next order of weak field perturbation theory. The reason is that although the linear combination $\sqrt{-g} \left(R^2 - 4R_{\kappa\lambda} R^{\kappa\lambda} + R_{\kappa\lambda\rho\sigma} R^{\kappa\lambda\rho\sigma} \right)$ is a total divergence in four dimensions, it can still conceivably give rise to tensor structures for $\omega \neq 2$ which must be removed by an explicit counterterm before a smooth limit $\omega \rightarrow 2$ can be taken, in analogy to the situation discussed above for two dimensions.

We can now proceed as before to derive the trace of the subtracted stress-energy tensor. A short calculation yields

$$\begin{aligned}
g_{\mu\nu} \langle T^{\mu\nu} \rangle_g &= 4 \left(3B_1 + B_2 + B_3 + \frac{1}{2}K \right) R, \lambda^\lambda + 2K \left(R_{\kappa\lambda} R^{\kappa\lambda} - R_{\kappa\lambda\alpha\beta} R^{\kappa\lambda\alpha\beta} \right) \\
&\quad + 2b \left(R^2 - 4R_{\kappa\lambda} R^{\kappa\lambda} + R_{\kappa\lambda\alpha\beta} R^{\kappa\lambda\alpha\beta} \right) \\
&= 4 \left(3B_1 + B_2 + B_3 + \frac{1}{2}K \right) R, \lambda^\lambda + 2(b-K) C_{\kappa\lambda\alpha\beta} C^{\kappa\lambda\alpha\beta} \\
&\quad - 2(K+2b) \left(R_{\kappa\lambda} R^{\kappa\lambda} - \frac{1}{3} R^2 \right) \quad (7)
\end{aligned}$$

where $C_{\kappa\lambda\alpha\beta}$ is the Weyl tensor.[†] Adjustment of the finite part of the counterterms affects only the term involving R, λ^λ ; by choosing $3B_1 + B_2 + B_3 + \frac{K}{2} = 0$, one can eliminate³ a trace anomaly up to order $\hbar^2$. However, regardless of the value b is forced to assume by the divergence structure in higher orders, it is evident from Eq. (7) that a non-zero trace anomaly persists. It would be interesting to see whether the value of b is indeed fixed by the $O(\hbar^2)$ poles in $\langle T^{\mu\nu} \rangle_g$, as conjectured above.

Finally, we note the amusing fact that the anomalies discussed above appear only in even space-time dimensions. In odd space-time dimensions, one-loop divergences disappear in dimensional regularization and a finite traceless stress-tensor can be defined immediately without recourse to counterterms. The reader can satisfy himself as to a special case of this phenomenon by simply setting $\omega = \frac{3}{2}$ in Eq. (1). In this connection, it is not easy to

see how truncation of direction-dependent terms, as in Ref. (1), can fail to yield an anomaly in odd space-time dimensions.

The above arguments confirm that a conformal trace anomaly is present when the stress-energy tensor is dimensionally regularized. On the other hand, covariant point-separation techniques using tangent-vector averaging² do not give a trace anomaly, so the two methods are clearly inequivalent. The precise reason for the discrepancy is not clear at present but is under active investigation.

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[†]As pointed out by Christensen and Fulling (Ref. 1), an a-priori possible term in (7) proportional to R^2 is in fact absent.

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