EFFECT OF CIRCULATING ION
RESONANCES AND IMPURITIES ON
THE DISSIPATIVE TRAPPED-ION
INSTABILITY

BY

R. MARCHAND

MASTER

**PLASMA PHYSICS
LABORATORY**

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Effect of Circulating Ion Resonances and Impurities
on the Dissipative Trapped-Ion Instability

R. Marchand
Princeton University Plasma Physics Laboratory
Princeton, New Jersey 08540

ABSTRACT

Trapped-ion instabilities associated with both electron and ion diamagnetic branches are studied. In particular, the influence of circulating ion Landau resonances and impurities are determined. These effects are studied in the local limit and have been implemented in a computer code used to calculate the relevant mode frequencies.

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I. INTRODUCTION

It has recently been pointed out by Tang et al⁽¹⁾ that in addition to the familiar trapped-ion modes associated with the electron diamagnetic drift, i.e. with phase velocity in the electron diamagnetic direction, ion diamagnetic drift instabilities of this type can also be important. In fact, such modes are found to be the most dangerous ones if the temperature gradients are sufficiently strong, i.e. $\eta_H \equiv d \ln T_H / d \ln n_H > 2/3$. Also, for flat or reversed density profiles, the ion diamagnetic modes can persist as "residual" instabilities, after the electron branch modes are eliminated.

In the present analysis, attention is focused on the influence of circulating ion Landau resonances and impurities, for both the electron and ion diamagnetic branches of the trapped-ion mode. These effects have already been discussed in the literature in connection with the usual electron branch. For example, circulating ion resonances were studied by Rosenbluth, Ross and Kostomarov², who used a perturbative formalism to calculate their contribution to the growth rate. This was also investigated by Gladd and Ross³ who carried out a radially non-local analysis. Their calculations showed that because of the fine structure associated with mode-particle resonance the eigenfrequency of the mode is nearly independent of the boundary condition imposed at the edge of the plasma. The influence of impurities on the usual diamagnetic branch has been investigated, for example, by Dobrowolny and Ross⁴ who used the perturbative approach of Rosenbluth et al¹ to calculate the growth rates.

The particle resonance terms used in the present paper are similar in form to those used by Gladd and Ross³. Collisions and the average toroidal drift are not included in these terms. This neglect is a good approximation for circulating particles since the collision frequency and the average drift frequency are generally smaller than the corresponding quantities for trapped particles by a factor $\epsilon (\equiv r/R)$. As noted by Gladd and Ross³, circulating electron resonances can be ignored since they are smaller than the ion terms by a factor $(m_e/m_H)^{1/2}$.

Corrections due to the impurities are obtained by including their charge density response to the quasi-neutrality equation:

$$\sum_j \rho_j = 0 \quad . \quad (1)$$

The impurity charge density is calculated from the drift-kinetic equation in which collisions are modeled by an energy independent, number conserving Krook collision operator. Such a simple form for the collision operator is valid for the low collisionality regime considered here. In that limit, the energy dependence of the collision frequency (ν) is not particularly important and number conservation alone is adequate.

With these new terms included in the dispersion equation, it is found that circulating ion resonances have a significant effect on both the magnitude and the phase of the mode frequency. In particular, some modes which were nearly purely growing, $|R_e \{\omega\}| \ll |I_m \{\omega\}|$, are found to have the real part of their frequency increased, so that $|R_e \{\tilde{\omega}\}| \sim |I_m \{\tilde{\omega}\}|$, when Landau resonances are

taken into account. In the range of parameters studied here, the impurities are usually found to be stabilizing. When the impurity density gradient is in the opposite direction to that of the electrons, the basic effect remains stabilizing but tends to be weakened.

The plan of the report is as follows: In section II, we present a dispersion equation for the trapped-ion mode, which includes the effects of circulating ion resonances. Corrections to this dispersion equation, due to the presence of impurities, are given in section III. Section IV deals with the asymptotic limits of the dissipative trapped ion mode when the effect of mode-particle resonance is taken into account. An approximate analytic dispersion relation is then found and compared with numerical results obtained in the same limit. Computed mode frequencies corresponding to typical PLT parameters are given in section V. Finally, we give our conclusions in section VI.

II. INCLUSION OF CIRCULATING ION RESONANCES

Following Gladd and Ross², we find the approximate dispersion equation for trapped ion modes in the local approximation has the form

$$\begin{aligned}
 & 1 + \tau_H - (2\epsilon)^{1/2} \left\langle \frac{\omega - \omega_{*e} [1 + \eta_e (E/T_e - 3/2)]}{\omega - \omega_{De} + i v_{f,e}} \right\rangle \\
 & - (2\epsilon)^{1/2} \tau_H \left\langle \frac{\omega - \omega_{*H} [1 + \eta_H (E/T_H - 3/2)]}{\omega - \omega_{DH} + i v_{f,H}} \right\rangle \\
 & - \tau_H \frac{\omega - \omega_{*H}}{k_{||} v_H} \left[Z \left(\frac{\omega}{\epsilon^{1/2} k_{||} v_H} \right) - Z \left(\frac{\omega}{k_{||} v_H} \right) \right] \\
 & + \tau_H \frac{\omega_{*H}}{k_{||} v_H} \eta_H \left\{ \frac{\omega}{k_{||} v_H} (\epsilon^{-1/2} - 1) + \left[\left(\frac{\omega}{\epsilon^{1/2} k_{||} v_H} \right)^2 - \frac{1}{2} \right] Z \left(\frac{\omega}{\epsilon^{1/2} k_{||} v_H} \right) \right. \\
 & \left. - \left[\left(\frac{\omega}{k_{||} v_H} \right)^2 - \frac{1}{2} \right] Z \left(\frac{\omega}{k_{||} v_H} \right) \right\} = 0 \quad , \quad (2)
 \end{aligned}$$

where

$$\langle A \rangle = \frac{2}{(\pi)^{1/2}} \frac{1}{(T_j)^{3/2}} \int_0^\infty E^{1/2} e^{-E/T_j} A \, dE \quad (3)$$

is the energy average of A ,

$$\tau_j = \frac{T_j}{T_e} \quad , \quad (4)$$

$$\epsilon = r/R, \quad (5)$$

with r and R the minor and major radii respectively,

$$\omega_{*j} = \frac{\ell q}{r} \frac{T_j c}{e_j B} \frac{1}{n_j} \frac{dn_j}{dr}, \quad (6)$$

$$\eta_j = \frac{d \ln T_j}{d \ln n_j}, \quad (7)$$

$$\omega_{Dj} = G \frac{\ell q}{rR} \frac{cE}{e_j B} \quad (8)$$

is the average toroidal drift frequency of a trapped particle,

$$v_j = \left(\frac{2T_j}{M_j} \right)^{1/2}, \quad (9)$$

$$k_{||} = \frac{n}{qR}. \quad (10)$$

ω is the (complex) frequency of the mode, q is the safety factor, G is a constant of order unity, $v_{f,e}$ and $v_{f,H}$ are respectively the trapped electron and hydrogen effective collision frequencies. Their energy dependence will be assumed to be $E^{-3/2}$:

$$v_{f,j} = v_j / \epsilon \left(\frac{E}{T_j} \right)^{-3/2}, \quad (11)$$

where v_j is the Spitzer collision frequency of species j , ℓ is the toroidal wave number of the perturbed quantities and n is given by $(m - \ell q)$, where, corresponding to flute-like

perturbations, m is the closest integer to ℓq . Z is the usual plasma dispersion function.³

In deriving Eq. (2) we assume the usual trapped ion ordering:

$$\left(v_H / \epsilon, \omega \right) \ll (\epsilon)^{1/2} \omega_{tH} \ll \Omega_H, \quad (12)$$

where ω_{tj} is the transit frequency of species j :

$$\omega_{tj} \sim \frac{v_j}{qR}. \quad (13)$$

Also, we assume that all first order perturbed quantities vary like

$$g(\vec{x}, t) = \tilde{g}(r) \exp [i(m\theta - \ell\phi - \omega t)] , \quad (14)$$

and that the typical banana width ρ_{bj} of a trapped particle is much smaller than the radial wavelength of the mode, which in turn, must be much smaller than the minor radius of the plasma:

$$\rho_{bj} \ll \left| \frac{d \ln g}{dr} \right|^{-1} \ll a. \quad (15)$$

For computational purposes, it is convenient to normalize all frequencies with respect to $(-\bar{\omega}_{DH}/G)$, where

$$\bar{\omega}_{DH} = G \frac{\ell q}{rR} \frac{c^T_H}{eB} \quad (16)$$

and

$$\omega_{DH} = \bar{\omega}_{DH} E / T_H. \quad (17)$$

Defining $\tilde{\omega} = -G \omega / \bar{\omega}_{DH}$, $\tilde{v}_j = -G v_j / \epsilon \bar{\omega}_{DH}$, $\tilde{\omega}_{||j} = -G k_{||} v_j / \bar{\omega}_{DH}$ and making use of the above definitions, we can write Eq. (2) as:

$$\begin{aligned}
 & 1 + \tau_H - (2\varepsilon)^{1/2} \left\langle \frac{\tilde{\omega} - \tau_H (R/r_{ne}) [1 + \eta_e (E/T_e - 3/2)]}{\tilde{\omega} - \tau_H G(E/T_e) + i \tilde{\nu}_e (E/T_e)^{-3/2}} \right\rangle \\
 & - (2\varepsilon)^{1/2} \tau_H \left\langle \frac{\tilde{\omega} + R/r_{nH} [1 + \eta_H (E/T_H - 3/2)]}{\tilde{\omega} + G(E/T_H) + i \tilde{\nu}_H (E/T_H)^{-3/2}} \right\rangle \\
 & - \tau_H \frac{R/r_{nH} + \tilde{\omega}}{\tilde{\omega}_{||H}} \left[z\left(\frac{\tilde{\omega}}{\varepsilon^{1/2} \tilde{\omega}_{||H}}\right) - z\left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}}\right) \right] \\
 & - \tau_H \frac{R/r_{TH}}{\tilde{\omega}_{||H}} \left\{ \frac{\tilde{\omega}}{\tilde{\omega}_{||H}} (\varepsilon^{-1/2} - 1) + \left[\left(\frac{\tilde{\omega}}{\varepsilon^{1/2} \tilde{\omega}_{||H}} \right)^2 - \frac{1}{2} \right] z\left(\frac{\tilde{\omega}}{\varepsilon^{1/2} \tilde{\omega}_{||H}}\right) \right. \\
 & \left. - \left[\left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}} \right)^2 - \frac{1}{2} \right] z\left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}}\right) \right\} = 0, \tag{18}
 \end{aligned}$$

where $r_{nj} = - (d \ln n_j / dr)^{-1}$, $r_{Tj} = (d \ln T_j / dr)^{-1}$ are the density and temperature gradient scale lengths respectively. The correction terms which give circulating ion resonance are the last two terms, on the left side of Eq. (18). Notice that there are no corresponding terms for the electrons. The reason is that the correction terms due to circulating electron resonances would be roughly $(M_e/M_H)^{3/2}$ smaller than those of the ions. This is seen by expanding the particle resonant correction terms in $1/\tilde{\omega}_{||}$, and noting that $\tilde{\omega}_{||e} \sim (M_H/M_e)^{1/2} \tilde{\omega}_{||H}$.

III. IMPURITY EFFECTS

Here we use the drift kinetic equation with a number conserving collision operator⁴ to calculate the density response of an impurity species α caused by a perturbed electric potential. Assuming that the impurities are massive enough, so that $\omega > \omega_{t\alpha}$, we can describe their dynamics in slab geometry. With x and z along the radial and equilibrium magnetic field directions respectively, we write the drift kinetic equation as

$$\frac{\partial f}{\partial t} + v_{||} \frac{\partial f}{\partial z} + c \frac{\vec{B} \times \nabla \phi}{B^2} \frac{\partial f}{\partial \vec{x}} - \frac{Ze}{M} v_{||} \phi \frac{\partial f}{\partial v_{||}} = - \nu \left(f - \frac{n}{n_0} f_0 \right), \quad (19)$$

where Ze is the impurity charge.

We have suppressed the species subscript α temporarily. Here the subscript 0 refers to equilibrium quantities. The density n is calculated self-consistently through:

$$n(\vec{x}, t) = 2\pi \int_{-\infty}^{\infty} dv_{||} \int_{-\infty}^{\infty} v_{\perp} dv_{\perp} f(\vec{x}, v_{||}, v_{\perp}). \quad (20)$$

We now linearize Eq. (19). Assuming that first order quantities vary in space and time like $\exp[i(k_{||}z + k_{\perp}y - \omega t)]$, where $k_{\perp} = m/r$, the linearized form of Eq. (19) becomes

$$i(k_{||}v_{||} - \omega)f_1 + \frac{ize\phi_1}{T} \left\{ k_{||}v_{||} - \omega_* \left[1 + \eta \left(\frac{E}{T} - \frac{3}{2} \right) \right] \right\} = - \nu \left(\tilde{f}_1 - \frac{\tilde{n}_1}{n_0} f_0 \right), \quad (21)$$

where ω_* for the impurities is still defined by Eq. (6). This leads to an explicit expression for $\tilde{f}_1$ which can be integrated

over velocity space to give the perturbed density. Since f_0 is a local Maxwellian, the result can be expressed in terms of the plasma dispersion function Z . Introducing the subscript α to label the impurity species; we find

$$\tilde{n}_{\alpha 1} = -n_{\alpha 0} \frac{Z_{\alpha} e \tilde{\phi}_1}{T_{\alpha}} \frac{1 + \zeta_{\alpha} Z(\zeta_{\alpha}) - \omega_{* \alpha} / k_{\parallel} v_{\alpha} \{Z(\zeta_{\alpha}) + \eta_{\alpha} [\zeta_{\alpha} + \zeta_{\alpha}^2 Z(\zeta_{\alpha}) - (1/2) Z(\zeta_{\alpha})]\}}{1 + i(v_{\alpha} / k_{\parallel} v_{\alpha}) [Z(\zeta_{\alpha})]} \quad (22)$$

where

$$\zeta_{\alpha} = \frac{\omega + i\nu_{\alpha}}{k_{\parallel} v_{\alpha}} \quad (23)$$

Adding the perturbed charge density $\tilde{\rho}_{\alpha 1} = Z_{\alpha} e \tilde{n}_{\alpha 1}$ to Eq. (1), we find that the new dispersion equation is the same as Eq. (2) except that the hydrogen terms are now multiplied by the hydrogen charge fraction f_H and the left side contains the extra terms

$$\sum_{\alpha} f_{\alpha} Z_{\alpha} \tau_{\alpha} \frac{1 + \zeta_{\alpha} Z(\zeta_{\alpha}) - \omega_{* \alpha} / k_{\parallel} v_{\alpha} \{Z(\zeta_{\alpha}) + \eta_{\alpha} [\zeta_{\alpha} + \zeta_{\alpha}^2 Z(\zeta_{\alpha}) - (1/2) Z(\zeta_{\alpha})]\}}{1 + i(v_{\alpha} / k_{\parallel} v_{\alpha}) [Z(\zeta_{\alpha})]} \quad (24)$$

where f_j , the charge fraction of species j , is defined by

$$f_j = \frac{n_{j0} Z_j}{n_{e0}} \quad (25)$$

Also the summation over α allows for the possibility of more than one impurity species. Note that bulk neutrality at equilibrium requires that:

$$\sum_j f_j = 1 \quad (26)$$

and

$$\sum_j f_j (R/r_{nj}) = R/r_{ne} , \quad (27)$$

In terms of the $(-G/\tilde{\omega}_{DH})$ normalization introduced earlier, the new dispersion equation is:

$$\begin{aligned} & 1 + f_H \tau_H - (2\epsilon)^{1/2} \left\langle \frac{\tilde{\omega} - \tau_H (R/r_{ne}) [1 + \eta_e (E/T_e - 3/2)]}{\tilde{\omega} - \tau_H G(E/T_e) + i\tilde{v}_e (E/T_e)^{-3/2}} \right\rangle \\ & - f_H \tau_H (2\epsilon)^{1/2} \left\langle \frac{\tilde{\omega} + (R/r_{nH}) [1 + \eta_H (E/T_H - 3/2)]}{\tilde{\omega} + G(E/T_H) + i\tilde{v}_H (E/T_H)^{-3/2}} \right\rangle \\ & - f_H \tau_H \frac{R/r_{nH} + \tilde{\omega}}{\tilde{\omega}_{||H}} \left[Z\left(\frac{\tilde{\omega}}{\epsilon^{1/2} \tilde{\omega}_{||H}}\right) - Z\left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}}\right) \right] \\ & - f_H \tau_H \frac{R/r_{TH}}{\tilde{\omega}_{||H}} \left\{ \frac{\tilde{\omega}}{\tilde{\omega}_{||H}} (\epsilon^{-1/2} - 1) + \left[\left(\frac{\tilde{\omega}}{\epsilon^{1/2} \tilde{\omega}_{||H}} \right)^2 - \frac{1}{2} \right] Z\left(\frac{\tilde{\omega}}{\epsilon^{1/2} \tilde{\omega}_{||H}}\right) \right. \\ & \left. - \left[\left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}} \right)^2 - \frac{1}{2} \right] Z\left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}}\right) \right\} \\ & + \sum_{\alpha} f_{\alpha} Z_{\alpha} \left[\frac{1 + [(R/r_{T\alpha}) \tau_H / Z_{\alpha} \tau_{\alpha} \tilde{\omega}_{||\alpha}] \zeta_{\alpha}}{1 + (i\tilde{v}_{\alpha} / \tilde{\omega}_{||\alpha}) Z(\zeta_{\alpha})} \right. \\ & \left. + \frac{(\zeta_{\alpha} + [(R/r_{n\alpha}) \tau_H / Z_{\alpha} \tau_{\alpha} \tilde{\omega}_{||\alpha}] \{1 + \eta_{\alpha} [\zeta_{\alpha}^2 - (1/2)]\}) Z(\zeta_{\alpha})}{1 + (i\tilde{v}_{\alpha} / \tilde{\omega}_{||\alpha}) Z(\zeta_{\alpha})} \right] = 0 , \quad (28) \end{aligned}$$

where ζ_{α} can also be written as:

$$\zeta_{\alpha} = \frac{\tilde{\omega} + i\tilde{v}_{\alpha}}{\tilde{\omega}_{||\alpha}} . \quad (29)$$

We assume for simplicity that the impurities mainly collide with the hydrogen background and take:

$$\tilde{\nu}_\alpha = \frac{4\tau_\alpha f_\alpha z_\alpha^2}{(\sum_\beta f_\beta z_\beta)} \frac{(\tau_H M_e M_H)^{1/2}}{M_\alpha} \frac{r}{R} \tilde{\nu}_e \quad (30)$$

IV. ASYMPTOTIC LIMIT

In this section we consider the limit $G = 0$, $v_e/\epsilon \rightarrow \infty$, $v_H/\epsilon \rightarrow 0$ and $\omega/\epsilon^{1/2} \tilde{\omega}_{||H} \rightarrow 0$. For simplicity we restrict our attention to the case of a pure hydrogenic plasma. In this limit each term of Eq. (18) can be evaluated analytically. We find the approximate dispersion equation, to leading order in v_e^{-1} , v_i and $\tilde{\omega}/\tilde{\omega}_{||H}$:

$$\begin{aligned}
 1 + \tau_H - (2\epsilon)^{1/2} \frac{4}{(\pi)^{1/2}} \frac{\tilde{\omega} - \tau_H (R/r_{ne}) [1 + (3/2)\eta_e]}{i\tilde{v}_e} \\
 - \tau_H (2\epsilon)^{1/2} \left(\frac{\tilde{\omega} + (R/r_{nH})}{\tilde{\omega}} + \frac{4i}{3(\pi)^{1/2}} \frac{\{\tilde{\omega} + (R/r_{nH}) [1 - (3/2)\eta_H]\}}{\tilde{\omega}^2} \right. \\
 \left. \times \tilde{v}_i \ln(|\tilde{v}_i|/\tilde{\omega}) \right) - \tau_H \frac{\tilde{\omega}}{\tilde{\omega}_{||H}} \frac{R/r_{nH}}{\tilde{\omega}_{||H}} \\
 \times \left(2(\eta_H - 1)(\epsilon^{-1/2} - 1) + \frac{4}{3} \left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}} \right)^2 (\epsilon^{-3/2} - 1)(1 - 2\eta_H) + i(\pi)^{1/2} \right. \\
 \left. \times \frac{\tilde{\omega}}{\tilde{\omega}_{||H}} [(3/2)\eta_H - 1](\epsilon^{-1} - 1) \right) + 2\tau_H \left(\frac{\tilde{\omega}}{\tilde{\omega}_{||H}} \right)^2 (\epsilon^{-1/2} - 1) = 0 \quad (31)
 \end{aligned}$$

If we further assume that ϵ is small, we can solve (31) iteratively. Keeping electron and ion collision frequencies and circulating ion resonance to lowest order in ϵ , we find:

$$\begin{aligned}
 \tilde{\omega} \approx & \frac{\tau_H (2\epsilon)^{1/2}}{1 + \tau_H} R/r_{nH} + \frac{i 8 \tau_H^2 \epsilon}{(\pi)^{1/2}} \frac{[1 + (3/2) \eta_e]}{(1 + \tau_H)^2} \frac{(R/r_{nH})^2}{\tilde{v}_e} \\
 & - i \frac{4}{3(\pi)^{1/2}} [1 - (3/2) \eta_H] \tilde{v}_H \ln \{ | [\tau_H (2\epsilon)^{1/2} R/r_{nH}] / (1 + \tau_H) \tilde{v}_H | \} \\
 & + i (2\pi\epsilon)^{1/2} \frac{2\tau_H^4}{(1 + \tau_H)^4} \frac{(R/r_{nH})^4}{(\tilde{\omega}_{||H})^3} [(3/2) \eta_H - 1] . \quad (32)
 \end{aligned}$$

We note here that both $\tilde{v}_e$ and $\tilde{v}_H$ can be either stabilizing or destabilizing: $\tilde{v}_e$ is destabilizing for $\eta_e > -2/3$ and stabilizing otherwise. $\tilde{v}_H$ is stabilizing for $\eta_H < 2/3$ and destabilizing otherwise. Comparing this with the dispersion relation obtained by Kadomtsev and Pogutse,⁵ who used an energy independent collision operator, and the one obtained by Rosenbluth, et al.,⁶ who solved the Fokker-Planck equation with a Lorentz collision operator, we see that our result is considerably better than that of Ref. 5 since, even though Eq. (32) still underestimates the ion-collision terms, it does give the correct qualitative effect of η on collisional (de)stabilization.

The analytic approximation given in Eq. (32) is compared with some numerical results in Table I. To facilitate comparison, convenient but unrealistic parameters are chosen such that the small quantities $[(2\epsilon)^{1/2}, \tilde{\omega}/\tilde{\omega}_{||H}, \dots]$ be at most 1/10. From Table I we see that Eq. (32) agrees well with the numerical results. In particular, the real part of $\tilde{\omega}$ predicted by Eq. (32) is better than 10% accurate. The imaginary parts however are up to 30% inaccurate. In particular, we note that $\tilde{\gamma}_L \neq 0$

for $\eta_H = 2/3$. This shift of the point where $\tilde{\gamma}_L = 0$ is due to the fact that the arguments of the Z-function in Eq. (2) are small but finite $(\tilde{\omega}/(\epsilon^{1/2}\omega_{||H}) \approx 0.054)$. Finally we observe that the analytical approximation in Eq. (32) does give the correct qualitative effect of particle resonances on the mode, namely that it is stabilizing for $\eta_H < 2/3$ and destabilizing for $\eta_H > 2/3$.

V. NUMERICAL RESULTS

Here we present some numerical results for cases that are believed to be relevant to a real machine such as PLT, and consistent with the regimes assumed thus far which, as we recall, correspond to the ordering:

$$(|\omega|, v_e/\epsilon) < (\epsilon)^{1/2} \omega_{te} \ll |\Omega_e|, \quad (33)$$

$$(|\omega|, v_H/\epsilon) < (\epsilon)^{1/2} \omega_{tH} \ll \Omega_H, \quad (34)$$

$$\omega_{t\alpha} < \omega \ll \Omega_\alpha, \quad (35)$$

where α refers to an impurity species. The Appendix gives a list of standard parameters which will be used in all calculations below, unless otherwise specified. Also, except for the impurity charge fractions, they are approximately those of PLT.

The numerical results of this section are presented in four steps. First, in order to give an idea of the effect and the importance of the new terms in the dispersion equations, we give, in Table II, the real and imaginary parts of the frequency corresponding to several values of the scale lengths. These were obtained from Eq.'s (18) and (28) and from the original dispersion equation, which ignored particle resonances and impurity effects altogether. The first point to note about

this table is the large effect of particle resonances on the eigenfrequency $\tilde{\omega}$. It is noted here that $\tilde{\omega}_r$ can be increased by more than two orders of magnitude. In the absence of this effect $\tilde{\omega}_r \ll \tilde{\gamma}$, corresponding to an almost purely growing mode. However, inclusion of particle resonances makes $\tilde{\omega}_r$ of the same order as $\tilde{\gamma}$. The growth rates and the absolute value of the frequencies $|\tilde{\omega}|$, however are generally only slightly modified (by less than an order of magnitude in most cases). It is interesting to note that even though we are far from the asymptotic limit assumed in Section IV, the qualitative effect of circulating ion resonances is still predicted correctly by Eq. (32) when $R/r_{nH} = 4$, since in that case, the electron branch is stabilized for $\eta_H < 2/3$ and destabilized for $\eta_H > 2/3$. When we consider impurities, we assume a relative concentration of iron to oxygen as suggested by transport calculations made by D. Post.⁷ The exact relative concentration is not very important, since both oxygen and iron have the same qualitative effects on the mode. The total impurity charge fraction $f_O + f_{Fe}$, however, was taken to be smaller than suggested by Ref. 7 in order to keep the effect of impurities small. The impurities are usually found to be stabilizing. It is interesting to note that the electron branch tends to be less stabilized, or in one case more destabilized, when the impurities' density gradient is in the opposite direction from that of the electrons. We recall that the electron and ion branches correspond to waves

propagating in the electron and ion diamagnetic directions respectively. This terminology was introduced in Ref. 1 where, to our knowledge, the ion branch was studied for the first time.

Secondly we look at the effect of collisions on the frequency of the mode. Typical plots of $\tilde{\omega}_r$ and $\tilde{\gamma}$ as a function of $\log_{10}(\tilde{\nu}_e)$ are shown in Fig. 1 for a pure hydrogenic plasma with circulating ion resonance, and in Fig. 2 for a plasma contaminated by iron. Similar plots for a pure plasma without particle resonances were obtained by Tang, et al.¹ All of these plots have an important feature in common: they show a decrease, often monotonic, of the growth rate as a function of the electron collision frequency, which leads to complete stabilization for large enough $\tilde{\nu}_e$.⁸

Thirdly we look at the effect of each impurity species separately. Figures 3 and 4 show plots of $\tilde{\omega}_r$ and $\tilde{\gamma}$ as a function of the impurity charge fraction for both oxygen and iron. We see that in this case, the growth rate is a monotonic decreasing function of the impurity charge fraction. We note that the stabilizing effect of oxygen is less than that of iron. This comes from the fact that for comparable $\tilde{\omega}$, the impurity sound term $(\tilde{\omega}_{||\alpha}/\tilde{\omega})^2$ is smaller by (M_O/M_{Fe}) for iron than for oxygen and, indeed, agrees with earlier calculations in which we treated the iron as hydromagnetic ($\tilde{\omega}/\tilde{\omega}_{||Fe} \rightarrow \infty$) and the oxygen as adiabatic ($\tilde{\omega}/\tilde{\omega}_{||O} \rightarrow 0$). Also we observe a saturation of $\tilde{\gamma}$ in each case, when the impurity ion sound term becomes of order unity.

Finally let us look at the effect of varying the poloidal and toroidal mode numbers. For simplicity we consider the case of a pure hydrogenic plasma with, say, an inverted density profile ($R/r_{ne} = R/r_{nH} = -4$). The only term in the dispersion equation (18) with an explicit dependence on m and n is $\tilde{\omega}_{||H}$. Indeed, from the definition of $\tilde{\omega}_{||H}$ and from Eq. (16), we find that for our choice of parameters,

$$\begin{aligned}\tilde{\omega}_{||H} &= 153 \frac{n}{\ell q} \\ &\approx 153 \frac{n}{m},\end{aligned}$$

where m is a positive number since, consistent with our assumption that $\tilde{v}_e$ be positive, we have to take $(-G/\bar{\omega}_{DH})$ as a positive constant. Hence we see that the only explicit dependence on m and n in Eq. (18), comes through the ratio m/n . Therefore, it suffices to consider $\tilde{\gamma}$ as a function of this ratio. Figure 5 gives a plot of $\tilde{\gamma}$ as a function of $\log_{10}(m/n)$. This plot shows that the normalized growth rate $\tilde{\gamma}$ has a maximum for $m/n \approx 18$ which, fortunately, is not too far from the standard value $m/n = 30$, assumed in the previous calculations.

Let us now look at the more interesting question of finding the maximum growth rate γ . From the definition $\tilde{\gamma} = -G\gamma/\bar{\omega}_{DH}$ and Eq. (16), we see that $\gamma \propto m\tilde{\gamma}$. Hence, the largest m -modes will be the most unstable. However, since $\tilde{\gamma}$ only depends

on m and n through the ratio m/n , we see that for a given $\tilde{\gamma}$, the largest n -modes will be the most unstable. It is important to remember though that n is bounded by inequality (34). Indeed, rewriting Eq. (34) as

$$|\tilde{\omega}| < (\epsilon)^{1/2} \tilde{\omega}_{tH}, \quad (37)$$

with

$$\tilde{\omega}_{tH} = - \frac{G}{\tilde{\omega}_{DH}} \frac{V_H}{qR} \quad (38)$$

$$= \frac{\tilde{\omega}_{||H}}{n}, \quad (39)$$

we see that Eq. (34) is equivalent to

$$|\tilde{\omega}| < (\epsilon)^{1/2} \frac{\tilde{\omega}_{||H}}{n}. \quad (40)$$

For order of magnitude estimates of the maximum growth rate, let us take:

$$n_{\max} \sim \frac{1}{2} \frac{\tilde{\omega}_{||H}}{|\tilde{\omega}|}. \quad (41)$$

We see then, that the maximum growth rate corresponding to a given ratio m/n , will be

$$\gamma_{\max} \Big|_{n/m} \propto \frac{m}{n} \frac{\tilde{\omega}_{||H}}{|\tilde{\omega}|} \tilde{\gamma}, \quad (42)$$

or, from Eq. (36)

$$\gamma_{\max} \Big|_{m/n} \propto \frac{\tilde{\gamma}}{|\tilde{\omega}_r|} , \quad (43)$$

where we have taken $|\tilde{\omega}| \sim |\tilde{\omega}_r|$. The absolute maximum growth rate is then given by:

$$\gamma_{\max} \propto \max \left(\frac{\tilde{\gamma}}{|\omega_r|} \Big|_{m/n} \right) . \quad (44)$$

Table III gives the values of $\gamma_{\max} \Big|_{m/n}$ for several values of m/n . These are plotted in Fig. 6 where we observe a maximum at $m/n \sim 21$, corresponding to $n \sim 1$ and $m \sim 21$.

The important point to note here is that according to our simple local theory, the maximum growth rate $\gamma_{\max}$ is not found interior to the domain, in parameter space, corresponding to the trapped-ion ordering. Indeed from Eq.'s (41) and (40), $\gamma_{\max}$ is found to lie at the boundary of this domain. From this we conclude that the determination of the fastest growing mode would require at least the study of the intermediate regime between the trapped-ion and trapped-electron regimes. Although the above discussion was made for one particular choice of the parameters, it is clear that this conclusion is general since, in all cases, $\tilde{\gamma}$ only depends on m and n through the ratio m/n .

VI. CONCLUSION

We have modified the local code for trapped-ion modes by implementing circulating ion resonance and impurity effects. Comparison with the results obtained from the original code, which ignored these two effects, shows that for realistic values of the parameters, particle resonances have a large effect on the frequency of the mode. Impurities were found to be stabilizing in most cases and it was observed that the electron branch tended to be less stabilized when the impurity density gradients were in the opposite direction from that of the electrons. Inclusion of particle resonances in the dispersion equation allowed us, for the first time with this local code, to look at the poloidal and toroidal mode number dependence of the growth rates. We found that the most unstable modes tended to have large m and n and lay at the upper boundary of $|\omega|$ set by the trapped-ion ordering. This suggests that in the context of a local theory, the investigation of the fastest growing mode would require the study of the intermediate regime between the trapped-ion and trapped-electron regimes. We believe however that before doing this other aspects of the trapped-ion regime should be investigated, such as nonlocality in the poloidal or radial coordinates.

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APPENDIX

Here we list the standard values of the parameters which are being used throughout, unless otherwise specified. These are:

$$q = 2.5, B_O = 35 \text{ kG}, G = 1.0, R = 100 \text{ cm},$$

$$R/r = R/r_{ne} = R/r_{nH} = R/r_{nO} = R/r_{nFe} = R/r_{Te} = R/r_{TH}$$

$$= R/r_{TO} = R/r_{TFE} = 4.0,$$

$$T_e = T_H = T_O = T_{Fe} = 1 \text{ keV},$$

$$M_e = 1/1840, M_H = 1, M_O = 16, M_{Fe} = 56 \text{ amu},$$

$$Z_O = 8, Z_{Fe} = 10,$$

$$f_H = 1.0, f_O = f_{Fe} = 0,$$

$$n = 0.5, m = 15.$$

These parameters correspond to:

$$\tilde{\omega}_{||H} = 5.11, \tilde{\omega}_{||O} = 1.28, \tilde{\omega}_{||Fe} = 0.683$$

$$\text{and } (-\bar{\omega}_{DH}/G) = 1.71 \times 10^4 \text{ sec}^{-1}.$$

Reference and Remarks

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Table I.

Comparison of the analytic approximation (a) and numerical results (b). We used $\epsilon = 0.005$, $R/r_{ne} = R/r_{nH} = 20$, $R/r_{Te} = 0$, $\tilde{\omega}_{||H} = 261$, $\tilde{\nu}_e = 2257$, $\nu_H = 4.54 \times 10^{-5}$, $G = 0$. Subscripts e and H refer to electron and hydrogen collisional contributions to $\tilde{\gamma}$ respectively. Subscript L refers to the circulating ion resonance contribution.

η_H	$\tilde{\omega}$	$\tilde{\gamma}_e (10^{-3})$	$\tilde{\gamma}_H (10^{-3})$	$\tilde{\gamma}_L (10^{-3})$	$\tilde{\gamma}_e + \tilde{\gamma}_H (10^{-3})$	$\tilde{\gamma}_e + \tilde{\gamma}_H + \tilde{\gamma}_L (10^{-3})$
1/3	1.	1.	-0.170	-0.1	0.830	0.730
2/3	1.	1.	0.	0.	1.	1.00
1	1.	1.	0.170	0.1	1.17	1.27

(a) Calculated from Eq. (32).

η_H	$\tilde{\omega}_{\text{no part.res.}}$	$\tilde{\gamma}_e + \tilde{\gamma}_H (10^{-3})$	$\tilde{\omega}_{\text{with part.res.}}$	$\tilde{\gamma}_e + \tilde{\gamma}_H + \tilde{\gamma}_L (10^{-3})$	$\tilde{\gamma}_L (10^{-3})$
1/3	1.05	0.859	1.05	0.717	-0.142
2/3	1.05	0.999	1.05	0.980	-0.019
1	1.05	1.14	1.05	1.25	0.11

(b) Computed numerically.

Table II.

Frequency of the mode for various values of R/r_{ne} , R/r_{Te} ($= R/r_{TH}$) and R/r_{nO} ($= R/r_{nFe}$). Subscripts L and LK refer to results computed from Eqs. (18) and (28) respectively. Null subscripts refer to the original dispersion equation for a pure hydrogenic plasma without particle resonance. Primed frequencies were calculated with $R/r_{nO} = R/r_{nFe} = -4$ and all the others with $R/r_{nO} = R/r_{nFe} = 4$.

R/r_{ne}	R/r_{Te}	$\tilde{\omega}_r$	$\tilde{\gamma}$	$\tilde{\omega}_{r(L)}$	$\tilde{\gamma}_{(L)}$	R/r_{nH}	$\tilde{\omega}_{r(LK)}$	$\tilde{\gamma}_{(LK)}$	R/r_{nH}	$\tilde{\omega}'_{r(LK)}$	$\tilde{\gamma}'_{(LK)}$
4.	-4.	0.0115	1.64	0.474	1.53	4.	0.462	1.39	4.42	0.428	1.49
4.	0.	0.0069	1.98	0.794	1.87	4.	0.776	1.52	4.42	0.731	1.69
4.	1.333	0.0038	2.21	1.06	2.15	4.	1.02	1.63	4.42	0.957	1.83
4.	2.667	0.00012	2.52	1.44	2.64	4.	1.38	1.86	4.42	1.28	2.06
4.	4.	0.0141	0.137	0.0139	0.107	4.	-0.0370	0.181	4.42	0.0618	0.107
4.	4.	-0.0034	2.91	1.90	3.35	4.	1.73	2.29	4.42	1.64	2.44
-4.	4.	4.13	0.450	3.33	0.614	-4.42	3.32	0.567	-4.00	3.27	0.484
-4.	4.	-4.13	0.462	-	-	-4.42	-3.80	8.16	-4.00	-3.76	8.03
0.	4.	2.46	0.864	2.32	0.755	-0.211	2.48	0.675	0.211	2.37	0.582
0.	4.	-2.46	0.875	-9.21	1.27	-0.211	-3.43	4.42	0.211	-3.24	4.21
2.	4.	1.26	1.14	1.36	1.20	1.89	1.83	1.05	2.32	1.62	0.946
2.	4.	-1.26	1.16	-1.75	2.05	1.89	-1.44	1.70	2.32	-1.06	1.87
3.	4.	0.024	0.755	0.0393	0.705	2.15	-0.304	0.885	3.37	0.0750	0.793
3.	4.	-0.0158	1.88	0.945	0.281	2.15	1.49	1.64	3.37	1.17	1.82

Table III.

Maximum growth rates corresponding to various values of m/n . All results were obtained with $R/r_{ne} = R/r_{nH} = -4$.

m/n	$\tilde{\omega}$	$\tilde{\gamma}$	$\tilde{\omega}_{ }$	$n_{\max}$	$m_{\max}$	$\gamma_{\max}$
10	4.13	0.711	15.3	1.9	19	13.2
14	3.97	0.326	10.9	1.4	19	15.9
16	3.88	0.350	9.58	1.2	20	16.8
18	3.78	0.354	8.52	1.1	20	17.3
20	3.70	0.342	7.66	1.0	21	17.4
22	3.61	0.315	6.97	1.0	21	17.3
24	3.53	0.776	6.39	0.9	22	16.9
26	3.46	0.729	5.90	0.9	22	16.2
30	3.33	0.514	5.11	0.8	23	14.1
40	3.07	0.226	3.83	0.6	25	5.6
60	5.24	0.371	2.55	0.2	15	0.5

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Fig. 1. Real and imaginary parts of $\tilde{m}$ as a function of $\log_{10}(\tilde{\nu}_e)$, for a pure hydrogenic plasma with $R/r_{ne} = R/r_{nH} = -4$.

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Fig. 2. Real and imaginary parts of $\tilde{\omega}$ as a function of $\log_{10}(\tilde{\nu}_e)$ for a plasma contaminated by iron: $f_H = 0.95$, $f_{Fe} = 0.05$, $R/r_{ne} = R/r_{nH} = R/r_{nFe} = -4$.

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Fig. 3. Real and imaginary parts of $\tilde{\omega}$ as a function of the hydrogen charge fraction for a plasma contaminated by oxygen. $R/r_{ne} = R/r_{nH} = r/r_{nO} = -4$.

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Fig. 4. Real and imaginary parts of $\tilde{\omega}$ as a function of the hydrogen charge fraction, for a plasma contaminated by iron. $R/r_{ne} = R/r_{nH} = r/r_{nFe} = -4$.

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Fig. 5. Normalized growth rate $\tilde{\gamma}$ as a function of $\log_{10}(m/n)$, for a pure plasma. $R/r_{ne} = R/r_{nH} = -4$.

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Fig. 6. Maximum growth rate, in arbitrary units, as a function of $\log_{10}(m/n)$ for a pure plasma. $R/r_{ne} = R/r_{nH} = -4$.