APPLICATIONS OF GROUP-IN Variant ANALYTIC SOLUTIONS TO INERTIAL CONFINEMENT FUSION

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Applications of Group-Invariant Analytic Solutions to Inertial Confinement Fusion

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Abstract

Analytic solutions to 1-, 2-, and 3-D hydrodynamics are presented which can be applied to specific aspects of Inertial Confinement Fusion target evolutions. These solutions include drive source time profiles for 1-D isentropic implosions, 2-D \( P_2 \) asymmetric compressions, 1- and 2-D solutions for the stagnation phase of implosions, and a 3-D spinning (mixing) solution.

The quest for controlled thermonuclear fusion requires the study and solution of the equations of magnetohydrodynamics and/or radiation hydrodynamics. These systems of nonlinear partial differential equations describe the dynamics of plasma evolution in both magnetic and inertial fusion devices. In practice, the solutions are explored numerically through the use of computer simulations of specific plasma configurations. A parallel effort searches for relevant analytic solutions to these equations. Such analytic solutions can provide (i) numerical benchmark problems, (ii) the basis for analytic models, and (iii) insight into more general solutions.

The evolution of inertial fusion targets is described by the equations of radiation hydrodynamics. Useful analytic solutions to the equations of hydrodynamics have been found through the use of dimensional analysis and other insightful Ansätze. An alternate approach has been the use of invariance of the differential equations under Lie group transformations.\(^1\) Lie group invariance provides a systematic and deterministic technique to search for group-invariant solutions, which include similarity solutions. The equations of hydrodynamics and radiation hydrodynamics have been extensively explored using Lie group analysis and a large number of analytic solutions have been found.\(^2,3\) Recent emphasis has been on multidimensional (2 and 3-D) solutions, which require a more detailed examination of the structure of the allowed Lie groups.\(^4\) The discovered solutions can be used as models for certain aspects of inertial fusion target performance.

The equations considered in this particular study are the 3-D, 1-temperature perfect gas hydrodynamic equations including thermal conduction and an energy source \( S \):

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho + \rho \nabla \cdot u &= 0, \\
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla (\Gamma \rho T) &= 0, \\
T_t + \mathbf{u} \cdot \nabla T + (\gamma - 1)T \nabla \cdot \mathbf{u} - \frac{\gamma - 1}{\Gamma \rho} \nabla \cdot \kappa \nabla T - \frac{\gamma - 1}{\Gamma} S &= 0.
\end{align*}
\]

The invariance transformations groups are calculated for these equations and used to introduce new sets of independent and dependent variables under which these equations take a simpler form.\(^4\) These special combinations of variables represent a reduction in phase
space to special classes of solutions which evolve for certain initial/boundary conditions. The partial differential equations (PDE's) can be reduced to ordinary differential equations (ODE's) which must then be solved to obtain explicit analytic solutions. The ordinary differential equations can often be solved through judicious choice of functional forms of the solutions.

One specific application of these solutions to inertial fusion is the identification of a variety of drive source time profiles which provide an isentropic implosion of a 1-D spherical capsule. These profiles generally take the form where the pressure $\sim (r^n - t^n)^{-a}$, where $a=2.5$ for $n=2$ and $a=2, 2.5, \text{ or } 5$ for $n=1$. The particular value of the exponents depends on the details of the initial conditions.

A two-dimensional flow example provides an analytic solution of an axisymmetric implosion with a $P_2$ asymmetric drive. This solution is a consequence of three independent scaling groups (space, time and density) allowed by the differential equations. These groups provide the similarity variables

\[ H_7 : \rho = H(\theta) r^a t^b, \quad T = G(\theta) \frac{r^2}{t^2}, \quad u^r = U(\theta) \frac{r}{t}, \quad u^\theta = V(\theta) \frac{r}{t}, \]

which reduce the PDE's to ODE's. A particular analytic solution is found through the ansatz $U = a + b \cos^2 \theta$ and $V = V_0 \sin \theta \cos \theta$ and is given by

\[ \rho(r, \theta, t) = \rho_0 t^{-\beta-3+3(\beta+1)(\gamma-1)/(\gamma+1)} \left( r \cos \theta \right)^\beta, \]
\[ u^r(r, \theta, t) = \frac{r}{t} \left( 1 - \frac{3}{\gamma+1} \cos^2 \theta \right), \]
\[ u^\theta(r, \theta, t) = \frac{3}{\gamma+1} \frac{\gamma-1}{\gamma} \frac{r}{t} \sin \theta \cos \theta, \]
\[ T(r, \theta, t) = \frac{6(\gamma-1)(2-\gamma)}{\Gamma(\gamma+1)^2(\beta+2)} \left( \frac{r}{t} \right)^2 \cos^2 \theta. \]

Another choice of groups, space and density scaling along with time translation and a projective group, form the similarity variables

\[ H_{1+} : \rho = \frac{H(\theta) r^\beta}{(a^2 + t^2)^{\beta/2}} \psi_{+}(t; a, \alpha), \quad T = \frac{G(\theta) r^2}{(a^2 + t^2)^2}, \quad u^r = \frac{U(\theta) + t}{a^2 + t^2} r, \quad u^\theta = \frac{V(\theta)r}{a^2 + t^2}, \]

with \( \psi_{+}(t; a, \alpha) = \exp \left( \frac{\alpha}{a} \tan^{-1} \frac{t}{a} \right) \) and \( \psi_{-}(t; a, \alpha) = \left( \frac{a-t}{a+t} \right)^{\alpha/2a} \),

again reducing the 2-D PDE's to ODE's.

For the 1-D case (\( V=0, H, G, U \) constants), the reduced ODE's can be solved for two types of solutions

\[ \rho(r, t) = \rho_0 r^\beta \left( a^2 + t^2 \right)^{-\beta/2}, \quad u(r, t)(= u^r) = \frac{\pm r t}{a^2 + t^2}, \quad T(r, t) = \frac{\pm a^2 r^2}{\Gamma(\beta-1)(a^2 + t^2)^2}. \]
The material trajectories for these two solutions are given by \( r = R_0 \sqrt{a^2 + t^2} \), which are either hyperbolas (\( \pm \)) or ellipses (\(-\)) in the \( r-t \) plane, shown in Figure 1. The positive-time portion of the elliptical branch (\( \mathcal{H}_{1-} \)) can represent an implosion of an ICF target from initially zero velocity. A special form of this solution was found earlier by Kidder (see Reference 3) and others and is allowed only for \( \beta > 1 \). The negative-time portion of this branch represents an expansion phase from a point explosion which then turns around at \( t = 0 \). The hyperbolic branch, \( \mathcal{H}_{1+} \), has material moving in toward the origin that stagnates at \( t = 0 \) and moves out for \( t > 0 \). This solution, requiring \( \beta < 1 \), can be used to represent the stagnation and subsequent explosion phases of an ICF target.

The reduced ODE’s for \( \mathcal{H}_{1\pm} \) can also be solved for two-dimensional flow \( (V \neq 0) \) with the assumption that \( U = 0 \). For \( \beta \neq 1 \), the equations become

\[
G = \frac{V^2 \mp a^2}{\beta - 1}, \quad V^2 (V^2 \mp a^2)^3 = \frac{c}{\sin^2 \theta}, \quad \frac{H'}{H} = \mp \frac{\alpha}{V} - \frac{1}{\tan \theta} \left( \frac{3V^2}{4V^2 \mp a^2} \right) = \frac{(\beta + 1)V^2}{(4V^2 \mp a^2) \tan \theta}.
\]

When \( \alpha = 0 \), the relations, using \( f = V/a \), become

\[
\beta = -4, \quad G = \frac{a^2}{5} (\pm 1 - f^2), \quad f^2 (\pm 1 - f^2)^3 = \frac{c}{\sin^2 \theta}, \quad \frac{H'}{H} = \frac{-3f^2}{(4f^2 \mp 1) \tan \theta},
\]

we find only the hyperbolic (\( \mathcal{H}_{1+} \)) branch is allowed. At this point we must resort to a numerical solution. We solve first the implicit equation to obtain the function \( f(\theta) \), which then provides \( G(\theta) \). The function \( f \) is then used in the remaining ODE for \( H(\theta) \).

The dominant parameter is the constant \( c \) which determines the extent in \( \theta \) of the solution which extends from \( \theta_s < \theta < \pi - \theta_s, \theta_s = \sin^{-1}(\sqrt{256c/27}) \). Figure 2 shows these solutions for \( c = .05 \ (a = 1) \), and we find two independent upper and lower solution branches.

A 3-dimensional solution with axisymmetry but nonzero \( \phi \)-velocity can be found as a consequence of the above-mentioned scaling groups. Here the similarity variables are

\[
\rho = H(\theta)t^d r^b e^{c\phi}, \quad u^r = U(\theta) r^d, \quad u^\theta = V(\theta) r^d, \quad u^\phi = W(\theta) r^d, \quad \Gamma T = G(\theta) \left( \frac{r^2}{t^2} \right).
\]

The resulting ordinary differential equations can be solved with the ansatz \( V = 0 \), \( H = H_0 + H_1 \sin^a \theta \) to obtain the analytic solution

\[
\rho(r, \theta, \phi, t) = t^{-(b+3)/2} r^b (\rho_0 + \rho_1 \sin^a \theta),
\]

\[
u^r(r, \theta, \phi, t) = \frac{r}{2t},
\]

\[
u^\theta(r, \theta, \phi, t) = 0,
\]

\[
u^\phi(r, \theta, \phi, t) = \pm \frac{r}{t} \left[ \frac{\Gamma T_0 (b + 2) (\sin \theta)^{b+2}}{\rho_0 + \rho_1 \sin^a \theta} - \frac{a \rho_0}{4(\rho_0 + \rho_1 \sin^a \theta)(b + 2 - a)} + \frac{a}{4(b + 2 - a)} \right]^{1/2},
\]

\[
T(r, \theta, \phi, t) = \left( \frac{r}{t} \right)^2 \left[ \frac{T_0 (\sin \theta)^{b+2}}{\rho_0 + \rho_1 \sin^a \theta} - \frac{a \rho_0}{4\Gamma(\rho_0 + \rho_1 \sin^a \theta)(b + 2)(b + 2 - a)} + \frac{1}{4\Gamma(b + 2 - a)} \right].
\]
Here we find a solution where the material flows on fixed cones in $\theta$ and spins in $\phi$ around the $z$-axis. This particular solution can relate an initial nonspherical velocity component to an eventual "mixing" state of the imploded target.


Figure captions

Figure 1. Material trajectories for the solutions $\mathcal{H}_{1\pm}$ show either a point explosion with a later recompression or an implosion, stagnation, expansion solution.

Figure 2. Numerical solution for $\mathcal{H}_{1+}$ show two possible solution branches for each choice of the arbitrary constant $c$. 

Figure 1

\[ u = \frac{-rt}{a^2 - t^2} \quad \text{Radius} \]

\[ u = \frac{rt}{a^2 + t^2} \]

Time