**Comparison of Model Distribution Functions
with Collisional Plasma Model (CPM)
Results in Various Collisionality
Regimes**

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COMPARISON OF MODEL DISTRIBUTION FUNCTIONS
WITH COLLISIONAL PLASMA MODEL (CPM)
RESULTS IN VARIOUS COLLISIONALITY
REGIMES

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ABSTRACT

Neoclassical equilibrium and transport have been calculated with analytic solutions of a drift-kinetic equation in which the effect of collisions is included. The necessary introduction of approximations precluded the full description of the distribution function in all regimes and over all particle energies. C. O. Beasley, H. K. Meier, W. I. van Rij, and J. E. McCune have used the techniques inherent to the collisional plasma model (CPM) to develop a computer simulation for the dynamics of toroidally confined plasmas in the drift-kinetic, Fokker-Planck description. The CPM permits study of the neoclassical equilibrium without approximation in the drift-kinetic ordering. The neoclassical electron distribution function derived by M. N. Rosenbluth, R. D. Hazeltine, and F. L. Hinton is compared with the CPM for the banana regime. Good general agreement is obtained; discrepancies are attributed to a poor modeling of the collisional (low energy) particles in the model of Rosenbluth et al. The CPM is also compared with calculations by K. T. Tsang and J. D. Callen in the Pfirsch-Schlüter regime. When approximations made in these calculations are taken into consideration, good agreement is observed.

1. INTRODUCTION

Various calculations of neoclassical equilibrium and transport have made use of analytic solutions of a drift-kinetic equation in which the effect of collisions is included. In order to obtain analytic solutions, model collision operators of varying complexity were assumed.¹⁻³ The necessary introduction of approximations precluded the full description of the distribution function in all regimes and over all particle energies. In the interest of achieving more completeness and accuracy, use of numerical simulation has also been introduced.⁴

Recently, Beasley et al.⁵ developed a computer simulation for the dynamics of toroidally confined plasmas in the drift-kinetic, Fokker-Planck description by using techniques inherent to the collisional plasma model (CPM), which offers a distinct alternative to numerical simulation. The CPM permits a study of the neoclassical equilibrium without approximation in the drift-kinetic ordering. Special features of the method used to generate the results given here are described in detail in Refs. 5, 6, and 7. For the sake of completeness, a few of these features are also reviewed in Sect. 2.

In the present work, the newly developed numerical method (CPM) is used to compare the model distribution functions with CPM results in various collisionality regimes. The results presented here remain subject to limitations in applicability, primarily because approximations have been imposed for the sake of convenience in both theoretical^{1,2} and initial CPM⁵ studies.

In Sect. 2, the basic equations of the CPM are briefly discussed and the underlying physical assumptions and their implications are described. In Sect. 3, the neoclassical electron distribution function derived by Rosenbluth et al.¹ is compared with the CPM⁵ in the collisionless or banana regime, and calculations by Tsang and Callen² are compared with the CPM in the Pfirsch-Schlüter regime. The conclusions are given in Sect. 4.

2. THE CPM REPRESENTATION

The time evolution of a plasma discharge in a toroidal confining field and the quasi-equilibrium state to which the plasma evolves are governed by the drift-kinetic, Fokker-Planck equation. The equation governing the (guiding center) distribution functions of the various species in a plasma can be written in a variety of different coordinate systems for both velocity and configuration space. In the CPM, in which sets of orthogonal transcendental functions are used in the description of velocity space, it is convenient to use scalar velocity and pitch-angle variables in the velocity-space description. The rationale for and characteristic features of this approach are described in detail in Ref. 6. The transformation from one coordinate system to the other is straightforward but tedious.

The distribution function for a particular species of particles in a collisional plasma can be expressed as a function of position $\vec{x}$, velocity $\vec{v}$, and time t by the series⁶

$$f(x, u, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \sum_{N=0}^{\infty} F_{\ell mn}(\vec{x}, t) \psi_{\ell mn} \left[\sqrt{\rho(\vec{x}, t)} \vec{u} \right] \quad (2.1)$$

where

$$\vec{u} = \vec{v} - \vec{V}(\vec{x}, t). \quad (2.2)$$

The functions $\psi_{\ell mn}$ are defined in Ref. 6 and ρ and $\vec{V}$ are arbitrary; however, the CPM provides an efficient representation if $\rho = M/kT$ with M , T , and $\vec{V}$ equal to the particles' mass, temperature, and average velocity. The $\ell = m = n = 0$ term in Eq. (2.1) is identifiable as the Maxwellian component. Thus, for predominantly Maxwellian particles, the appropriate choice of ρ and $\vec{V}$ can give a series expansion of f that converges after relatively few terms.

In Refs. 6 and 7, the drift-kinetic, Fokker-Planck equation in the CPM formulation is given in considerable generality. In Ref. 6, attention is focused on the Fokker-Planck collision operator in the CPM description. In Ref. 7, the CPM representation of the Vlasov operator is derived for the general kinetic (Boltzmann) equation, and the drift-kinetic version of the Vlasov operator is treated in sufficient generality to allow for time-varying (self-consistent) fields and for a quite accurate description of the particle drifts.

In the initial studies, two approximations have been imposed for the sake of convenience. First, the ion kinetics has not been included; the ions are simply treated as a massive neutralizing background. Second, we make the approximation of radial localization. Since the ions are treated as a stationary background, only the kinetic equation for the electrons is required. The confining magnetic field and the induction electric field are given by

$$\vec{B} = B_0 (\hat{e}_\zeta + \hat{\theta} \epsilon/q) / (1 + \epsilon \cos \theta)$$

(2.3)

and

$$\vec{E}_\zeta = \hat{e}_\zeta V / [2\pi R_0 (1 + \epsilon \cos \theta)]$$

where $\epsilon = r/R_0$. In Eq. (2.3) toroidal field ripple effects have been neglected. If a low β approximation is assumed and the Ω^{-1} drift terms (where Ω is the electron gyrofrequency) in B_θ are neglected, the general equation in Ref. 7 [Eq. (3.6), $\vec{V} = \vec{H} = G = 0$ and $\hat{n} \equiv \vec{B}/|\vec{B}|$] simplifies and is reduced to

$$\begin{aligned} \frac{\partial \bar{f}}{\partial t} + \left[v \cos \alpha \hat{n} - \frac{v^2(1 + \cos^2 \alpha)}{2\Omega_{\zeta} B_{\zeta}} \hat{\zeta} \times \vec{\nabla} B_{\zeta} \right] \cdot \vec{\nabla} \bar{f} + \hat{n} \\ \cdot \left[\frac{v^2 \sin \alpha}{2B_{\zeta}} \vec{\nabla} B_{\zeta} \alpha - \frac{e}{M_e} \vec{E}_{\zeta} \hat{n} \right] \cdot \vec{\nabla} \bar{f} = \bar{C} \bar{f} \end{aligned} \quad (2.4)$$

where $\bar{f}$ is the gyroaveraged distribution function [from the $m = 0$ part of Eq. (2.1)]. Equation (2.4) becomes a matrix equation for the time evolution of the various amplitudes making up the electron guiding center distribution $\bar{f}(\vec{x}, v, \alpha, t)$ by expanding Eq. (2.1), as suggested in the CPM description.

A radial localization approximation is further imposed on Eq. (2.4). Specifically, it is assumed that

$$\bar{f}(r, \theta, v, \alpha, t) \simeq N_0(r) \frac{\rho(r)^{3/2}}{2\pi} \tilde{f}(\theta, \sqrt{\rho} v, \alpha, t) \quad (2.5)$$

where $\rho = M_e/kT_0$ and N_0 and T_0 are the electron density and temperature at $t = 0$. The radial gradients $(d \ln N_0 / d \ln r)$ and $(d \ln T_0 / d \ln r)$ entering in Eq. (2.4) are specified as parameters related directly to moments of $\bar{f}$. Neglecting $\partial \tilde{f} / \partial r$, the kinetic equation can then be solved locally as long as $\bar{f}$ remains approximately a local Maxwellian [Eq. (2.4)]. It should be noted that in a finite β analysis, nonlocal phenomena arise (even in the time-independent problem) due to the distortion of flux surfaces by the pressure gradient; these nonlocal phenomena cannot be studied properly with a fixed model field. However, time-asymptotic results are satisfactory and are comparable to earlier work,^{1,2} at least within the applicability of the use of the model $\vec{B}$ field. The results from the comparison of these model distribution functions^{1,2} with the CPM are discussed in Sect. 3. Results are given for cases with a radial density gradient alone and with zero induction voltage. These results are obtained over a wide range of

collisionality: $0.1 \leq \nu_* \leq 100$, where ν_* is defined as $\nu_{\text{eff}}/\omega_{\text{be}}$; ν_{eff} is the reciprocal time for scattering particles out of the trapped region and ω_{be} is the electron bounce frequency.

3. COMPARISON WITH CPM RESULTS

In Sect. 3.1 the neoclassical distribution function derived by Rosenbluth et al.¹ is compared with the CPM in the banana regime, in which $v_* = 0.1$. The value of $v_* = 0.1$ indicates that a typical trapped particle can successfully traverse its bounce (or banana) orbit roughly ten times before being scattered out of that orbit by particle-particle interaction. In Sect 3.2 calculations by Tsang and Callen² are compared with the CPM in the Pfirsch-Schlüter regime, in which $v_* = 100$. A value of $v_* = 100$ indicates that a particle that would be trapped in the absence of collisions has a very small chance of traversing its assigned bounce orbits before being scattered away by its neighbors.

3.1 NEOCLASSICAL ELECTRON DISTRIBUTION FUNCTION (ROSENBLUTH ET AL.)

For the purpose of the work, the neoclassical electron distribution function in the banana regime is rederived. The equilibrium distribution function is the solution of the drift-kinetic equation,¹

$$V_{dr} \frac{\partial f_o}{\partial r} + v_{\parallel} \frac{B_{\theta}}{B} \frac{f_o}{r} \frac{\partial \hat{f}_e}{\partial \theta} = C(f) . \quad (3.1)$$

The radial drift velocity V_{dr} is given by¹

$$V_{dr} \simeq \frac{M_e}{eB_0} \frac{v_{\parallel}}{r} \frac{\partial}{\partial \theta} (v_{\parallel} h) \hat{e}_r , \quad (3.2)$$

$C(f)$ is the linearized collision operator, $h = 1 + \epsilon \cos \theta$, and $v_{\parallel}$ is the parallel velocity

$$v_{\parallel} = \sigma v (1 - \frac{\lambda}{h})^{1/2} \quad (3.3)$$

where

$$\sigma = v_{\parallel} |v_{\parallel}| = \pm 1,$$

$$\lambda = \mu B_0 / (E - e\phi) = h \sin^2 \alpha,$$

$$\mu = M_e v_{\perp}^2 / 2B,$$

$$E = M_e v^2 / 2 + e\phi = M_e W + e\phi,$$

α is the pitch angle, and ϕ is the electrostatic potential. In Eq. (3.1) the distribution function $\bar{f}$ is assumed to be $\bar{f}_e = f_0(1 + \hat{f}_e)$, $\hat{f}_e \ll 1$, with $f_0 = f_{\text{Maxwellian}} = f_M$. From Eq. (3.3), the trapped particle region is $(1 - \epsilon) < \lambda < (1 + \epsilon)$.

In the low collision frequency limit, $v_{\text{eff}} \ll \omega_{be}$ or $v_* \ll 1$, the perturbed electron distribution function [the solution of Eq. (3.1)] is given by¹

$$\hat{f}_e = \frac{M_e}{eb} v_{\parallel} h \left[A_{1e} + A_{2e} M_e W \right] + g_c(\mu, \sigma, \epsilon) \quad (3.4)$$

where

$$b = hB_0;$$

$$A_{1\alpha} = \frac{N'}{N} - \frac{3}{2} \frac{T_{\alpha}'}{T_{\alpha}} + \frac{e_{\alpha}}{T_{\alpha}} \phi';$$

$$A_{2\alpha} = \frac{1}{2} \frac{T_{\alpha}'}{T_{\alpha}} \quad \text{with } \alpha = i, e;$$

$$N = N_e = N_i.$$

For untrapped particles,

$$\frac{\partial g_e}{\partial \lambda} = \sigma \left(\frac{W}{2}\right)^{\frac{1}{2}} \frac{G_e(W)}{\langle (1 - \frac{\lambda}{h})^{\frac{1}{2}} \rangle}; \quad (3.5)$$

for trapped particles,

$$\frac{\partial g_e}{\partial \lambda} = 0. \quad (3.6)$$

In Eq. (3.5),

$$G_e(W) = \frac{M_e}{eb} \left[A_{1e} + A_{2e} M_e W - eb \frac{\tilde{D}}{T_e} \right], \quad (3.7)$$

with

$$-eb\tilde{D} = T_i (A_{1i} + 1.33 T_i A_{2i}) \quad (3.8)$$

where the sharp brackets, $\langle \rangle$, represent the gyroaverage.

Eq. (3.4) can be written as

$$\hat{f}_e = \frac{M_e}{eb} v_{\parallel} h (A_{1e} + A_{2e} M_e W) + H(\lambda_c - \lambda) G_e \quad (3.9)$$

where

$$\begin{aligned} H(\lambda_c - \lambda) &= 1 \text{ for } 1 < \lambda < \lambda_c \text{ (untrapped region)} \\ &= 0 \text{ for } \lambda > \lambda_c \text{ (trapped region)}. \end{aligned}$$

Here λ_c is the value of λ separating untrapped and trapped electrons;

$\lambda_c = 1 - \epsilon$. After simple manipulation, Eq. (3.9) becomes

$$\begin{aligned} \hat{f}_e &= \frac{M_e}{eB_{\theta}} \left\{ \left[\frac{N'}{N} - e \frac{\phi'}{T_e} - \left(\frac{3}{2} - \frac{M_e W}{T_e} \right) \frac{T_e'}{T_e} \right] v_{\parallel} + H(\lambda_c - \lambda) \right. \\ &\quad \left. \left[\left(1 + \frac{T_i'}{T_e} \right) \frac{N'}{N} - \left(\frac{3}{2} - \frac{M_e W}{T_e} \right) \frac{T_e'}{T_e} - 0.172 \frac{T_i'}{T_i} \right] \frac{W}{h} \int_{\lambda_c}^{\lambda} \frac{d\lambda}{\langle v_{\parallel} \rangle} \right\}. \quad (3.10) \end{aligned}$$

It should be noted that the factor h^{-1} in Eq. (3.10) has been replaced

by 1 in Eq. (181) in Ref. 1.

The following quantities are needed in Eq. (3.10):

$$\begin{aligned}
 1) \quad \langle v_{\parallel} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \, v_{\parallel} = \frac{\sigma v}{2\pi} \int_0^{2\pi} d\theta \, (1 - \frac{\lambda}{h})^{\frac{1}{2}} \\
 &= \sigma v \left(\frac{2\epsilon\lambda}{k^2} \right)^{\frac{1}{2}} \frac{2E(k)}{\pi} , \tag{3.11}
 \end{aligned}$$

where $k^2 \equiv 2\epsilon\lambda/(1 - \lambda + \epsilon\lambda)$ and $E(k)$ is the complete elliptic integral, and for small ϵ

$$\begin{aligned}
 \lim_{\epsilon \rightarrow 0} \langle v_{\parallel} \rangle &= \sigma v (1 - \lambda)^{\frac{1}{2}} ; \\
 2) \quad I &\equiv \int_{\lambda_c}^{\lambda} \frac{d\lambda}{\langle v_{\parallel} \rangle} = \frac{1}{\sigma v} \int_{\lambda_c}^{\lambda} \frac{d\lambda}{(1 - \lambda + \epsilon\lambda)^{\frac{1}{2}}} \frac{2/\pi}{E(k)} , \tag{3.12}
 \end{aligned}$$

where again for small ϵ ,

$$\lim_{\epsilon \rightarrow 0} I = - \frac{2}{\sigma v} (1 - \lambda)^{\frac{1}{2}}, \quad \lambda(\epsilon=0) = \sin^2 \alpha .$$

Except for $\epsilon \rightarrow 0$, the evaluation of I may be done numerically. $I_0 = vI$ is plotted as a function of pitch angle α for $\epsilon = 0.2$ in Fig. 1.

For simplicity, the case where $T_e' = T_i' = 0$ is examined. Hence Eq. (3.10) reduces to

$$\begin{aligned}
 \hat{f}_e &= \frac{M_e}{eB_0} \left\{ \left[\frac{N'}{N} - e \frac{\phi'}{T_e} \right] v \cos \alpha \right. \\
 &\quad \left. + H(\lambda_c - \lambda) \left[1 + \frac{T_i}{T_e} \right] \frac{N'}{N} \frac{v}{2h} I_0 \right\} . \tag{3.13}
 \end{aligned}$$

Here

$$\sigma = +1 \quad \text{for} \quad 0 \leq \alpha < \pi/2 ;$$

$$\sigma = -1 \quad \text{for} \quad \pi/2 < \alpha \leq \pi .$$

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Fig. 1. I_0 vs α for $\theta = 0^\circ$ and 180° .

As has been pointed out by Rutherford et al.,⁸ the potential ϕ is necessary in equilibrium to balance the radial ion pressure gradient, i.e.,

$$\frac{\partial P_i}{\partial r} = P_i \frac{1}{N_i} \frac{\partial N_i}{\partial r} = T_i N \frac{N'}{N}, \text{ then}$$

$$- e\phi' = T_i \frac{N'}{N}.$$

Thus Eq. (3.13) becomes

$$\hat{f}_e = \frac{M_e}{eB_\theta} \left(\frac{N'}{N} - e \frac{\phi'}{T_e} \right) v \left[\cos \alpha + H(\lambda_c - \lambda) \frac{I_o}{2h} \right]. \quad (3.14)$$

Further assuming $\phi(\epsilon) = \frac{\partial \phi}{\partial r} \Big|_{\epsilon = r/R_o} = 0$,

$$\begin{aligned} \hat{f}_e &= \frac{M_e}{eB_\theta} \frac{N'}{N} v \cos \alpha + H(\lambda_c - \lambda) \frac{I_o}{2h} \\ &= \frac{\rho_\theta}{\sqrt{2}} \frac{N'}{N} x \cos \alpha + H(\lambda_c - \lambda) \frac{I_o}{2h} \end{aligned} \quad (3.15)$$

where

$$x \equiv \sqrt{\rho} \ v \text{ with } \rho \equiv M_e/T_e; \ \rho_\theta \equiv \frac{V_{Te}}{\Omega_\theta}$$

with

$$\Omega_\theta \equiv \frac{|e| B_\theta}{M_e} \text{ and } V_{Te}^2 = 2/\rho, \text{ the thermal velocity of the electron.}$$

The distribution function which will be compared with Eq. (3.15) has been calculated in Ref. 5; its highlights are summarized in Sect. 2. In showing distribution functions $\bar{f}$ which are almost entirely the initial Maxwellian, the Maxwellian component f_M is subtracted from the entire distribution. All figures will therefore show the difference $\hat{F} = \bar{f}_m \hat{f}_e = \bar{f}_e - f_M$. In three-dimensional plots, the $v_\parallel$ axis is in the direction

of the magnetic field $\vec{B}$; the $v_{\perp}$ axis origin is on the side with the largest function value, in the center of the $v_{\parallel}$ axis (see Fig. 2). The functions $\hat{F}(v_{\perp}, v_{\parallel})$ with the Maxwellian subtracted are shown in three-dimensional plots in Figs. 3 and 4 for the CPM² results and the results of Rosenbluth et al.,³ respectively. For the plots, the following typical tokamak-like parameters are used.

Temperature	$T_e \sim T_i \sim 1 \text{ keV}$
Magnetic field	$B_0 = 10 \text{ kG}$
Safety factor	$q = 2$
Minor radius	$r = 20 \text{ cm}$
Major radius	$R_0 = 100 \text{ cm}$
Radial gradients	$\partial \ln N_e / \partial \ln r = -2$ $\partial \ln T_e / \partial \ln r = 0$
Collisionality	$\nu_* = 0.1 \text{ (for CPM plots)}$

It should be noted that this $\hat{F}$ represents a slice through a cylinder in velocity space, and that to understand the figures in terms of the particle density at a given $(v_{\perp}, v_{\parallel})$, one must multiply $\hat{F}$ by $v_{\perp} = \sin \alpha$.

A group of plots is also given for two different values of Θ , on the outside ($\Theta = 0$) and on the inside ($\Theta = \pi$) of the torus for

$$f(x, \alpha, \Theta) = f_m \hat{F}_e x^2 \sin \alpha$$

where $x^2 \sin \alpha$ is the Jacobian, or velocity-space volume element. Figure 5 shows $f(x, \alpha, \Theta)$ as a function of pitch angle for various energies, while Fig. 6 represents $f(x, \alpha, \Theta)$ as a function of energy for various pitch angles. All curves in these two figures have been normalized to their maximum values.

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Fig. 2. The axes for three-dimensional plots.

Fig. 3. Three-dimensional CPM plots of $\hat{f}(v_{\parallel}, v_{\perp})$ at various θ for $\nu_* = 0.1$.

Fig. 4. Three-dimensional Rosenbluth-Hazeltine-Hinton (RHH) plots of $\hat{f}(v_{\parallel}, v_{\perp})$ at various θ (collisionless).

Fig. 5. (a) The perturbed electron distribution function ($= f_e - f_M$) as a function of pitch angle for various energies ($\theta = 0^\circ$, outside of torus). (b) The perturbed electron distribution function as a function of pitch angle for various energies ($\theta = 180^\circ$, inside of torus).

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Fig. 6(a). The perturbed electron distribution function as a function of pitch angle for various energies ($\theta = 0^\circ$).

Fig. 6(b). The perturbed electron distribution function as a function of pitch angle for various energies ($\theta = 180^\circ$).

It has been observed that good agreement was reached, particularly on the outside of the torus ($\theta = 0$), for higher energy particles. Discrepancies which appear, particularly at low energies on the inside of torus, may be attributed to a poor modeling of the low energy (i.e., more collisional) particles.¹ The agreement between the CPM and Ref. 1 is better for $\theta = 0$ than for $\theta = \pi$, because the current of the collisional low energy particles is opposite to that of the poorly modeled collisionless particles of Ref. 1.

3.2 PERTURBED ELECTRON DISTRIBUTION FUNCTION (TSANG AND CALLEN)

The calculations of Tsang and Callen² apply to a high-Z equilibrium in the Pfirsch-Schlüter regime ($v_* \gg \epsilon^{-3/2}$). Using the drift-kinetic equation, the Tsang-Callen (T-C) perturbed electron distribution function $\hat{f}_e (= f_e - f_M)$ is obtained by assuming the Lorentz model for the linearized collision operator and ignoring the electron-electron collisions due to high-Z ions. Under these assumptions, $\hat{f}_e$ is given as²

$$f_e = \hat{f}_e^{v^{-1}} + f_e^{v^0} \quad (3.16)$$

where

$$f_e^{v^{-1}} \equiv f_e^{-1} = \frac{N_1^{-1}}{N_e} f_M + \frac{T_1^{-1}}{T_e} (v^2/v_{Te}^2 - 5/2) f_M, \quad (3.17)$$

$$\begin{aligned} f_e^{v^0} \equiv f_e^0 = & - \frac{v}{v_{ei}(v)} \frac{\Omega}{r} f_M \left[\frac{1}{N_e} \frac{\partial N_1^{-1}}{\partial \theta} \right. \\ & \left. + (v^2/v_{Te}^2 - 5/2) \frac{1}{T_e} \frac{\partial T_1^{-1}}{\partial \theta} \right] \end{aligned} \quad (3.18)$$

with

$$\frac{N_1^{-1}}{N_e} = - \frac{3\pi}{32} \frac{r^2}{\theta^2 \tau_e} \frac{\sin \theta}{R |\Omega_e|} \left[\frac{25}{8} \frac{1}{N_e} \frac{\partial N_e}{\partial r} + \frac{5}{4} \frac{1}{T_e} \frac{\partial T_e}{\partial r} \right] , \quad (3.19)$$

$$\frac{T_1^{-1}}{T_e} = \frac{3\pi}{32} \frac{r^2}{\theta^2 \tau_e} \frac{\sin \theta}{R |\Omega_e|} \left[\frac{3}{4} \frac{1}{N_e} \frac{\partial N_e}{\partial r} - \frac{1}{2} \frac{1}{T_e} \frac{\partial T_e}{\partial r} \right] , \quad (3.20)$$

$$\theta = \frac{B_\theta}{B_0/h} = \frac{\varepsilon}{q} ;$$

and the electron-ion collision time is given by

$$\tau_e = \frac{1}{v_{ei}} = \frac{\sqrt{2}}{\varepsilon^{3/2}} \frac{r}{\theta V_{T_e}} \frac{1}{v_{*e}} . \quad (3.21)$$

Hence

$$\frac{r^2}{\theta^2 \tau_e R} = \frac{\varepsilon^{3/2}}{\sqrt{2}} q V_{T_e} v_{*e} . \quad (3.22)$$

In a Lorentz model,

$$v_{ei}(v) = \frac{3\sqrt{\pi}}{4} (V_{T_e}/v)^3 \frac{1}{\tau_e} ;$$

thus

$$\frac{1}{v_{ei}(v)} \frac{\theta}{r} = \frac{4\sqrt{2}}{3\sqrt{\pi}} \frac{1}{\varepsilon^{3/2}} \frac{1}{v_{*e} V_{T_e}} (v/V_{T_e})^3 . \quad (3.23)$$

Defining x to be $x \equiv v^2/V_{T_e}^2$, and using Eqs. (3.22) and (3.23) in Eq. (3.16), one then finds

$$\begin{aligned} \hat{f}_e &= f_M \frac{V_{T_e}}{\sqrt{2} |\Omega_e|} T_4 \left[T_1 + \frac{1}{2} (x^2 - 5) T_2 \right] \\ &\quad \cdot \left[\sin \theta - x^4 \cos \alpha \left(\cos \theta - \frac{E}{h} \sin^2 \theta \right) T_3 \right] \end{aligned} \quad (3.24)$$

where

$$T_1 = -\frac{25}{8} \frac{\partial \ln N_e}{\partial r} + \frac{5}{4} \frac{\partial \ln T_e}{\partial r}$$

$$T_2 = \frac{3}{4} \frac{\partial \ln N_e}{\partial r} - \frac{1}{2} \frac{\partial \ln T_e}{\partial r}$$

$$T_3 = \sqrt{\frac{2}{\pi}} / (\epsilon^{3/2} v_{*e})$$

$$T_4 = \frac{3\pi}{32} \epsilon^{3/2} v_{*e} \text{ and } \Omega_\theta = \frac{\epsilon}{q} \Omega_e$$

Since the Pfirsch-Schlüter regime is defined to be

$$v_{*e} \gg \epsilon^{-3/2},$$

this leads to a critical energy, E_c , which is given by²

$$E_c = 0.9695 v_{*e}^{1/2} \epsilon^{3/4} \frac{T_e}{M_e} \quad (3.25)$$

Therefore, the electrons are said to be in the Pfirsch-Schlüter regime if their energy $E = \frac{1}{2}v^2$ is less than the critical energy. For $\epsilon = 0.2$ and $v_* = 100$, it is found that x must be less than 2.4 in order to be in the Pfirsch-Schlüter regime. In other words, Eq. (3.24) is accurate for $x < 2.4$.

On the other hand, the CPM model gives the distribution functions for the following cases:

$$1. \quad C = C_{ei} \text{ with } Z_{eff} = 1 \quad (\tau_{equilibrium} \simeq 400 \tau_e),$$

$$2. \quad C = C_{ee} + C_{ei} \text{ with } Z_{eff} = 5 \quad (\tau_{equilibrium} \simeq 100 \tau_e),$$

where C is the collision operator.

From the results it is found that case 2 shows better agreement with the T-C distribution function, and results are presented for this case only. Three-dimensional plots of neoclassical equilibrium distribution functions of T-C for $v_* = 100$ are given in Fig. 7; corresponding CPM plots are shown in Fig. 8. It is observed that there is a good agreement between the T-C and the CPM distribution functions for $\theta = \pi/2$; only a good agreement is obtained for $x < 1.2$.

CPM calculations are also compared with the results of previous calculations by Tsang and Callen² and Hazeltine and Hinton³ by plotting radial diffusion, $\Gamma_r(v_*)$, as a function of collisionality, where the units of Γ_r are those of Ref. 3. The present CPM results are shown to be in good agreement with the earlier analytical results (see Fig. 9).

Fig. 7. Three-dimensional Tsang-Callen plots of $\hat{f}(v_{\parallel}, v_{\perp})$ at various θ for $v_* = 100$.

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Fig. 8. Three-dimensional CPM plots of $\hat{f}(v_{\parallel}, v_{\perp})$ at various θ for $\nu_* = 100$.

Radial Diffusion as a Function of Collisionality

Fig. 9. Radial diffusion as a function of collisionality.

4. CONCLUSIONS

The present numerical procedure (CPM) has been shown to be a viable method for calculating plasma equilibrium. Comparisons of model distribution functions¹⁻³ with CPM results have shown that good agreement is reached in certain regions.

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