Classical solutions with both instantons
and anti-instantons*

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Abstract

We formulate the $SO(4)$ Yang-Mills theory in terms of matrices and find solutions with both instantons and anti-instantons. The "left-handed" and "right-handed" projections of the gauge field yield the instanton and anti-instanton sectors, respectively. There are no interactions among instantons and anti-instantons in this model. The matrix formalism developed here will facilitate the analysis of various $SO(4)$ models in which the gauge field is coupled with other fields.

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More than a year ago Belavin, Polyakov, Schwartz and Tyupkin¹ obtained a one instanton² solution to the Euclidean SU(2) Yang-Mills Lagrangian. Solutions with n instantons or n anti-instantons were recently discovered by Witten³ and 't Hooft⁴. The number of parameters required to specify the multi-instanton solutions was discussed by Jackiw, et al.⁵.

The SO(4) Euclidean Yang-Mills model has been studied by several authors⁶ and they found a classical solution with one instanton and one anti-instanton. The purpose of this paper is to construct solutions with n_+ instantons and n_- anti-instantons in the SO(4) model. We employ a matrix representation which makes transparent the relationship between instanton solutions in SU(2) and SO(4).

The SO(4) Yang-Mills model is the simplest generalization of the SU(2) model that has solutions which simultaneously include both instantons and anti-instantons in four space-time dimensions. In SO(4) two quantum numbers are required to specify a solution: the topological quantum number q which is found to be $n_+ - n_-$ and the number k associated with the action which is equal to $n_+ + n_-$, the total number of instantons and anti-instantons. Our matrix notation is constructed so that "left-handed" and "right-handed" projections split the solutions into instanton and anti-instanton sectors, respectively.

The SO(4) Lagrangian in matrix notation is

$$\mathcal{L} = \frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} , \quad (1)$$

where the field strength $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] . \quad (2)$$

Here, the matrices A_μ and $F_{\mu\nu}$ are defined as $A_\mu = \Sigma_{\alpha\beta} A_\mu^{\alpha\beta} / 8i$ and $F_{\mu\nu} = \Sigma_{\alpha\beta} F_{\mu\nu}^{\alpha\beta} / 8i$. The Greek indices range from 1 to 4 and the matrices $\Sigma_{\alpha\beta} / 2 \equiv [\gamma_\alpha, \gamma_\beta] / 4i$ are the generators of $SO(4)$. The A_μ and $F_{\mu\nu}$ are conveniently decomposed into left- and right-handed parts as

$$A_\mu = A_\mu(+)\cdot\left(\frac{I+\gamma_5}{2}\right) + A_\mu(-)\cdot\left(\frac{I-\gamma_5}{2}\right), \quad (3)$$

$$F_{\mu\nu} = F_{\mu\nu}(+)\cdot\left(\frac{I+\gamma_5}{2}\right) + F_{\mu\nu}(-)\cdot\left(\frac{I-\gamma_5}{2}\right), \quad (4)$$

where

$$A_\mu(\pm) \equiv A_\mu^k(\pm) \frac{\sigma_k}{2i} = \pm \frac{1}{2} \left(A_\mu^{4k} \pm \frac{1}{2} \epsilon_{k\ell m} A_\mu^{\ell m} \right) \frac{\sigma_k}{2i}, \quad (5)$$

$$F_{\mu\nu}(\pm) \equiv F_{\mu\nu}^k(\pm) \frac{\sigma_k}{2i} = \pm \frac{1}{2} \left(F_{\mu\nu}^{4k} \pm \frac{1}{2} \epsilon_{k\ell m} F_{\mu\nu}^{\ell m} \right) \frac{\sigma_k}{2i}. \quad (6)$$

Here $A_\mu(\pm)$ and $F_{\mu\nu}(\pm)$ are 2×2 matrices and Latin indices run over 1, 2, 3.

Either the upper sign or lower sign is always to be taken. We obtain from (2)-(4),

$$F_{\mu\nu}(\pm) = \partial_\mu A_\nu(\pm) - \partial_\nu A_\mu(\pm) + g [A_\mu(\pm), A_\nu(\pm)]. \quad (7)$$

Note that $A_\mu(\pm)$ and $F_{\mu\nu}(\pm)$ transform as adjoint representations of $SU(2)$.

Let us consider the condition¹ which yields an absolute minimum of the action,⁸

$$\epsilon_{\alpha\beta\gamma\delta} F_{\mu\nu}^{\gamma\delta} + \epsilon_{\mu\nu\sigma\tau} F_{\sigma\tau}^{\alpha\beta} = 0, \quad (8)$$

where $\varepsilon_{1234} = 1$. Equations (2), (4) and (8) lead to

$$F_{\mu\nu}(\pm) = \pm \frac{1}{2} \varepsilon_{\mu\nu\varrho\sigma} F_{\varrho\sigma}(\pm), \quad (9)$$

i.e. $F_{\mu\nu}(+)$ and $F_{\mu\nu}(-)$ are self-dual and anti-self-dual, respectively.

The solution $A_{\mu}(+)[A_{\mu}(-)]$ of Eq. (9) will be found to correspond to instantons (anti-instantons).

Let us consider the ansatz

$$A_{\mu} = i \bar{\sigma}_{\mu\nu} \partial_{\nu} \psi_{+} \cdot \left(\frac{I + \gamma_5}{2} \right) + i \sigma_{\mu\nu} \partial_{\nu} \psi_{-} \cdot \left(\frac{I - \gamma_5}{2} \right), \quad (10)$$

i.e.

$$A_{\mu}(+) = i \bar{\sigma}_{\mu\nu} \partial_{\nu} \psi_{+}, \quad A_{\mu}(-) = i \sigma_{\mu\nu} \partial_{\nu} \psi_{-}, \quad (11)$$

where $\sigma_{ij} = \bar{\sigma}_{ij} = \varepsilon_{ijk} \sigma_k / 2$ and $\sigma_{i4} = -\bar{\sigma}_{i4} = \sigma_i / 2$. The A_{μ} as defined in (10) automatically satisfies $\partial_{\mu} A_{\mu} = 0$. From Eqs. (8) and (10) or equivalently (9) and (11) we obtain

$$\square \psi_{\pm} + g (\partial_{\mu} \psi_{\pm})^2 = 0. \quad (12)$$

The solutions⁴ of Eq. (12) are

$$\psi_{\pm} = \frac{1}{g} \ln S_{\pm}, \quad (13)$$

where

$$S_{+} = 1 + \sum_{i=1}^{n_{+}} \lambda_{+i}^2 / (\chi - y_{+i})^2, \quad (14)$$

$$S_- = 1 + \sum_{i=1}^{n_-} \lambda_{-i}^2 / (x - y_{-i})^2, \quad (15)$$

where $\lambda_{\pm i}$ and $y_{\pm i}$ are arbitrary parameters.

The solutions (14) and (15) are non-singular except at the locations y_{+1} of the instantons and y_{-1} of the anti-instantons. We indicate below that the singularities can be removed in a manner similar to that of SU(2) so that the action is regular everywhere.

Let us consider the gauge transformation generated by the matrix

$$\Omega = \Omega_+ + \Omega_-, \quad \Omega^{-1} = \tilde{\Omega}_+ + \tilde{\Omega}_-, \quad (16)$$

where

$$\Omega_{\pm} = \omega_{\pm} \cdot \left(\frac{I \pm \gamma_5}{2} \right), \quad \tilde{\Omega}_{\pm} = \omega_{\pm}^{-1} \cdot \left(\frac{I \pm \gamma_5}{2} \right). \quad (17)$$

In Eq. (17) $\omega_{\pm}$ is a 2×2 matrix. Under the gauge transformation generated by (16)

$$\begin{aligned} A'_{\mu} &= \Omega \left(A_{\mu} - \frac{1}{g} \Omega^{-1} \partial_{\mu} \Omega \right) \Omega^{-1} \\ &= \Omega \left\{ \left[A_{\mu}(+) - \frac{1}{g} \omega_+^{-1} \partial_{\mu} \omega_+ \right] \cdot \left(\frac{I + \gamma_5}{2} \right) \right. \\ &\quad \left. + \left[A_{\mu}(-) - \frac{1}{g} \omega_-^{-1} \partial_{\mu} \omega_- \right] \cdot \left(\frac{I - \gamma_5}{2} \right) \right\}, \end{aligned} \quad (18)$$

so that $A_{\mu}(+)$ and $A_{\mu}(-)$ independently transform as SU(2) triplets under the gauge transformation. For SU(2), 't Hooft⁴ showed that there exists a gauge transformation that removes the singularities.⁹ Therefore, we observe from Eq. (18) that the singularities in $A_{\mu}(+)$ can likewise be removed.

We define k , a quantity associated with the action, and q , the topological quantum number, in $SO(4)$ analogously to $SU(2)$ as,

$$k = \frac{g^2}{8\pi^2} \int d^4x \ S(x) , \quad (19)$$

$$q = \frac{g^2}{8\pi^2} \int d^4x \ S^*(x) , \quad (20)$$

where with the aid of (4), (7), (9), (11) and (12),

$$\begin{aligned} S &= -\frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} = \frac{1}{4} \{ [F_{\mu\nu}^k(+)]^2 + [F_{\mu\nu}^k(-)]^2 \} \\ &= -\frac{1}{2g^2} \square\square (\ln S_+ + \ln S_-) , \end{aligned} \quad (21)$$

$$\begin{aligned} S^* &= -\frac{1}{4} \epsilon_{\mu\nu\sigma\tau} \text{Tr} F_{\mu\nu} F_{\sigma\tau} = \frac{1}{4} \{ [F_{\mu\nu}^k(+)]^2 - [F_{\mu\nu}^k(-)]^2 \} \\ &= -\frac{1}{2g^2} \square\square (\ln S_+ - \ln S_-) . \end{aligned} \quad (22)$$

Because the singularities of $A_\mu(+)$ can be removed by a gauge transformation, the S and S^* are non-singular everywhere. Thus, the last equalities in (21) and (22) are only valid away from the singularities of $\rho_\pm$. Therefore, the quantities k and q can, in practice, be calculated using the last equality of Eqs. (21) and (22), and neglecting the singularities.

We obtain⁵ k and q from Eqs. (14), (15) and (19)-(22)

$$k = n_+ + n_- \quad (23)$$

and

$$q = n_+ - n_- , \quad n_+, n_- = 0, 1, 2, \dots . \quad (24)$$

One observes from (24) that the solutions consist of n_+ instantons and n_-

anti-instantons which are parametrized by $5(n_+ + n_-)$ constants describing the positions $y_{\pm i}$ and sizes $\lambda_{\pm i}$. Additional constants are required to parametrize a conformally invariant solution.⁵ When we project the present $SO(4)$ solutions onto $SU(2)$ as shown in (3)-(6), we recover 't Hooft's solutions.⁴ If one uses the condition $\epsilon_{\alpha\beta\gamma\delta} F_{\mu\nu}^{\gamma\delta} - \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}^{\alpha\beta} = 0$ instead of (8) one finds the following constraints on $\psi_{\pm}$:

$$\begin{aligned} Z_{11}(\pm) &= Z_{22}(\pm) = Z_{33}(\pm) = Z_{44}(\pm), \\ Z_{\mu\nu}(\pm) &= 0 \quad \text{for } \mu \neq \nu, \end{aligned} \quad (25)$$

where $Z_{\mu\nu}(\pm) = \partial_{\mu} \partial_{\nu} \psi_{\pm} - g(\partial_{\mu} \psi_{\pm} \partial_{\nu} \psi_{\pm})$. The only solutions we have been able to find are the spherically symmetric solutions⁶ with $n_+ = 0, 1$ and $n_- = 0, 1$. When one projects solutions onto $SU(2)$, one recovers the solutions of Belavin et al.¹

It is worth noting that the present model is similar to a chiral $SU(2) \times SU(2)$ symmetric model in that handedness in a chiral model corresponds to instanton and anti-instanton sectors in this model and chirality corresponds to the sign of the topological quantum number.

For a n_+ instanton solution, if any two instantons are located at the same position, we have an $n_+ - 1$ instanton solution. A similar situation holds for anti-instantons. If an instanton and an anti-instanton are located at the same position, the topological quantum number remains the same. In our formula, there is no interaction among instantons and anti-instantons.

The matrix formalism developed here will facilitate the study of various $SO(4)$ models in which the gauge field is coupled with Higgs fields or fermions.

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Note added. After we completed this work, we received a paper by K.M. Biter, FERMILAB-Pub-77/15 in which he discusses instantons in groups larger than $SU(2)$. While there is some overlap with our paper, our treatment of the $SO(4)$ gauge model is more detailed.

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7. We use the representation $\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}$, $\gamma_5 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$,
 $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and the $\Sigma_{\alpha\beta}$ matrices are then $\Sigma_{4k} = -\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = \sigma_k \cdot \gamma_5$ and
 $\Sigma_{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = \epsilon_{ijk} \sigma_k \cdot I$. Here σ_k are the Pauli matrices.
8. Any solution of Eq. (8) automatically satisfies the equation of motion
 $\partial_\nu F_{\mu\nu} + g[A_\nu, F_{\mu\nu}] = 0$.
9. The removal of the singularity for one instanton case was discussed in R. Jackiw and C. Rebbi, Phys. Rev. D14, 517 (1976).