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DYNAMIC RESPONSE OF A LINEAR ELASTIC/

LINEAR STRAIN-HARDENING THIN RING

by

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DYNAMIC RESPONSE OF A
LINEAR ELASTIC/LINEAR STRAIN HARDENING
THIN RING

1. Introduction

In studying the dynamic response of fluid filled cylindrical shells to large internal pressure pulses, it would be advantageous to have a rapid method of computing the final deformed shape of the shell. Such a computational method would be of particular value as the basis for a subroutine in a comprehensive computer program whose primary objective was calculating the dynamic response of a complex piping system. For example, in treating the dynamic response of liquid metal heat transport systems to pressures generated by large sodium/water reactions, determinations of the structural response of various components is only a fraction of the information required to describe the entire event. The use of large finite-element programs to study structural response would not be appropriate due to extensive core storage requirements and excessive computational time required.

The problem treated in this report is an initial investigation of the value of correlation parameters for removing the effect of pulse shape on predicted plastic deformation. It is an attempt to extend previous studies [1] by inclusion of elastic/plastic response rather than limiting the model to a rigid/plastic description. Since many of the pressure pulses of interest in piping system analysis do not involve pressures that produce large plastic deformation, it is necessary to include the effect of elastic response. A simple bilinear elastic/strain hardening model was thus chosen for this study.

The analysis to follow considers the dynamic response of a bilinear thin ring to various spatially uniform pressure pulses. Solutions have been obtained for rectangular and for linear and exponentially decaying pulses. Numerical procedures have been employed to obtain maximum inelastic displacement as a function of material and pulse parameters. It did not prove

possible to determine correlation parameters that would remove the effect of pulse shape; however, the resulting deformation data should provide the basic information needed to evaluate various correlations that may be proposed.

2. Problem Formulation and Solution

Consider a thin circular ring of thickness H , radius R , and density ρ , subjected to a uniform spacial pressure pulse $P(t)$ as shown in Figure 1.

The equation of motion of the ring is

$$\rho H \ddot{u} = P(t) - \frac{\sigma H}{R} \quad (2.1)$$

where u is the radial displacement of the ring and σ is the circumferential stress. It is assumed that the ring is initially at rest so that

$$u(0) = \dot{u}(0) = 0 \quad (2.2)$$

Circumferential strain in the ring is defined by

$$\epsilon = uR^{-1} \quad (2.3)$$

the bilinear constitutive relationship, $\sigma = \sigma(\epsilon)$, is that shown in Fig.2, where E and E_p are, respectively, the elastic and plastic moduli, and the yield stress and yield strain are related by

$$\sigma_y = E \epsilon_y \quad (2.4)$$

For a given pulse, we are interested in determining the maximum inelastic displacement

$$u_m = R \epsilon_m \quad (2.5)$$

occurring at the end of the initial loading phase, $t = t_m$. The simple pulse shapes considered do not involve unloading and subsequent reloading, and the results to follow hold only for pulses of this type. That is, u_m always represents the maximum inelastic displacement.

Denoting the time at which yield occurs by t_y then we have the elastic response phase

$$\sigma = E\epsilon = E\frac{u}{R} \quad , \quad 0 \leq t \leq t_y \quad , \quad (2.6)$$

followed by the plastic phase

$$\sigma = \sigma_y + E_p (\epsilon - \epsilon_y) \quad , \quad t_y \leq t \leq t_m \quad . \quad (2.7)$$

Thus, the equation of motion for elastic behavior is

$$\ddot{u} + \omega^2 u = P(t) (\rho H)^{-1} \quad , \quad 0 \leq t \leq t_y \quad , \quad (2.8)$$

where

$$\omega^2 = E(\rho R^2)^{-1} \quad ,$$

and that for plastic behavior is

$$\rho H \ddot{u} + E_p H R^{-2} u + \left[\sigma_y H R^{-1} - E_p H u_y R^{-2} \right] = P(t) \quad , \quad t_y \leq t \leq t_m \quad . \quad (2.9)$$

For a pulse of infinite duration, the solutions of (2.8) and (2.9) are*

$$u(t) = \frac{1}{\rho H \omega_p} \int_0^t P(\tau) \sin \omega(t - \tau) d\tau \quad , \quad 0 \leq t \leq t_y \quad , \quad (2.10)$$

and

$$u(t) = \frac{1}{\rho H \omega_p} \int_0^{t-t_y} P(\tau) \sin \omega_p(t - t_y - \tau) d\tau + u_y \cos \omega_p(t - t_y) + \frac{\dot{u}_y}{\omega_p} \sin \omega_p(t - t_y) + u_y \left(\frac{\alpha^2 - 1}{\alpha^2} \right) \left[1 - \cos \omega_p(t - t_y) \right] \quad , \quad t_y \leq t \leq t_m \quad (2.11)$$

where

$$\omega_p^2 \equiv E_p(\rho R^2)^{-1} \quad , \quad (2.12)$$

$$\alpha^2 \equiv E_p E^{-1} \quad .$$

* See Appendix, p. 10

In the case of a pulse of finite duration, equations (2.10) and (2.11) must be modified to allow for the possibility of yield occurring before or after pulse termination.*

Solutions have been obtained for the rectangular, linear, and exponential pulses shown in Fig. 3. Detailed analytical results for these cases is given in the Appendix. It is not possible to express the maximum displacement, u_m , as an explicit function of the material parameters (ω , σ_y , α) and the pulse parameters (P_0, t_1) and numerical solution is thus a necessity.

3. Summary of Results

The data obtained for each of the three pulse shapes is presented in Figures 4 through 24. The principal results are those giving the locus of pulse and material parameters producing a specified inelastic displacement. For example, Fig. 5 gives the locus of (non-dimensional) pressure and time parameters for a rectangular pulse which produce maximum inelastic displacements of u_y , $10u_y$ and $20u_y$.

A test was made to see whether the correlation parameters appropriate to rigid - perfectly plastic response** could be utilized to remove the dependence of pulse shape in the elastic-strain hardening case. The results are given in Figs. 8-10 for the rectangular pulse, Figs. 15-17 for a linear pulse, and Figs. 22-24 for an exponential pulse. These figures clearly show that the previously developed correlation parameters are not useful in the case under consideration.

* See Appendix , p. 11

**Slightly modified as shown on p. 26

Appendix: Summary of Analysis

A.1 Solution for a Rigid/Perfectly Plastic Material:

Rectangular Pulse:

Let P_y denote the static yield pressure

$$P_y = \sigma_y HR^{-1} \quad (A.1.1)$$

and t_f the time at which motion ceases. Clearly $P_0 \geq P_y$ and $t_f \geq t_1$.

From the equation of motion 2.1 one has

$$\rho H \ddot{u} = P(t) - P_y, \quad t \leq t_1$$

and thus

$$\rho H u = (P_0 - P_y) \frac{t^2}{2}, \quad t \leq t_1 \quad (A.1.2)$$

After the pressure has returned to zero, the equation of motion implies

$$\rho H u(t) = \rho H u(t_1) + \rho H \dot{u}(t_1)(t-t_1) - \frac{P_y}{2} (t-t_1)^2, \quad t > t_1, \quad (A.1.3)$$

and, using (A.1.2), one has

$$\frac{\rho H}{P_y} u(t) = \left(\frac{P_0}{P_y} - 1 \right) \frac{t_1^2}{2} + \left(\frac{P_0}{P_y} - 1 \right) t_1(t-t_1) - \frac{1}{2} (t-t_1)^2, \quad t > t_1. \quad (A.1.4)$$

Motion stops at t_f when $\dot{u}(t_f) = 0$ and, hence, from (A.1.4)

$$t_f = \frac{P_0}{P_y} t_1. \quad (A.1.5)$$

The final deformation is thus

$$u_f \equiv u(t_f) = \left(\frac{P_0}{P_y} \right) \frac{t_1^2}{2} \left(\frac{P_0}{P_y} \right) \left(\frac{P_0}{P_y} - 1 \right). \quad (A.1.6)$$

Linear Pulse:

From simple integrations of the equation of motion one finds

$$\rho H u(t) = (P_0 - P_y) \frac{t^2}{2} - \frac{P_0 t^3}{6t_1}, \quad t \leq t_1, \quad (A.1.7)$$

and

$$\rho H u(t) = -\frac{P_0}{6} t_1^2 + \frac{P_0}{2} t t_1 - \frac{P_y}{2} t^2, \quad t > t_1. \quad (A.1.8)$$

From (A.1.7) it follows that motion stops during the pulse if and only if

$$1 \leq \frac{P_0}{P_y} \leq 2 \quad ; \quad (\text{A.1.9})$$

and, in this case,

$$t_f = 2t_1 \left(1 - \frac{P_y}{P_0}\right) \quad , \quad (\text{A.1.10})$$

and

$$\frac{\rho H u_f}{P_y} = \frac{2}{3} t_1^2 \left(\frac{P_0}{P_y} - 1\right) \left(1 - \frac{P_y}{P_0}\right)^2 \quad . \quad (\text{A.1.11})$$

If $P_0 P_y^{-1} > 2$ then motion stops after the pulse and

$$t_f = \frac{1}{2} \frac{P_0}{P_y} t_1 \quad , \quad (\text{A.1.12})$$

which implies

$$\frac{\rho H u_f}{P_y} = \frac{t_1^2}{24} \left(\frac{P_0}{P_y}\right) \left(3 \frac{P_0}{P_y} - 4\right) \quad . \quad (\text{A.1.13})$$

Exponential Pulse:

There is a single equation defining the deformation:

$$\rho H u(t) = P_0 t_1^2 \left[\exp \left(-\frac{t}{t_1}\right) - 1 \right] + P_0 t t_1 - P_y \frac{1}{2} t^2 \quad . \quad (\text{A.1.14})$$

Motion stops at a time, t_f , defined by the first positive zero of

$$\frac{P_0}{P_y} \left[\exp \left(-t_f t_1^{-1}\right) - 1 \right] + \frac{t_f}{t_1} = 0 \quad , \quad (\text{A.1.15})$$

with the final deformation calculated by substitution of t_f into (A.1.14).

A.2 Correlation Parameters for a Rigid/Perfectly Plastic Material

Correlation parameters which eliminate the effect of pulse shape on the final deformation of a rigid/perfectly plastic material (for certain geometries) have been developed by Youngdahl [1]. These parameters involve the effective impulse,

$$I = \int_{t_y}^{t_f} P(t) dt \quad , \quad (A.2.1)$$

where t_y is the time of initial yielding, and the effective load

$$P_e = \frac{I}{\hat{t}} \quad , \quad (A.2.2)$$

where

$$\hat{t} = \frac{1}{I} \int_{t_y}^{t_f} (t - t_y) P(t) dt \quad (A.2.3)$$

is the mean load duration. The use of these parameters relative to the results developed in (A.1) is illustrated below for each of the three pulse shapes being considered.

Rectangular Pulse:

For the rectangular pulse one notes that

$$I = P_0 t_1 \quad , \quad \hat{t} = \frac{P_0 t_1^2}{2} \quad (A.2.4)$$

and, thus, from (A.2.2),

$$\frac{P_e}{P_y} = \frac{P_0}{P_y} \quad . \quad (A.2.5)$$

Define

$$K \equiv \rho H P_y \quad , \quad (A.2.6)$$

and note from (A.1.6) that, in terms of the effective pressure the final plastic deformation is given as

$$\frac{K}{I^2} u_f = \frac{1}{2} \left[1 - \left(\frac{P_e}{P_y} \right)^{-1} \right] \quad . \quad (A.2.7)$$

Linear Pulse:

In this case, one finds

$$I = \begin{cases} P_0 t_f \left(1 - \frac{t_f}{2t_1} \right) & , \quad t_f < t_1 \\ \frac{P_0 t_1}{2} & , \quad t_f > t_1 \end{cases} \quad (A.2.8)$$

and

$$\hat{I}t = \begin{cases} P_0 t_f^2 \left(\frac{1}{2} - \frac{t_f}{3t_1} \right) & , \quad t_f < t_1 \\ \frac{P_0 t_1^2}{6} & , \quad t_f > t_1 \end{cases} \quad (A.2.9)$$

The definition of effective pressure then yields

$$\frac{P_e}{P_y} = \begin{cases} 3 \frac{P_y}{P_0} \left[4 \frac{P_y}{P_0} - 1 \right]^{-1} & , \quad t_f < t_1 \\ \frac{3}{4} \frac{P_0}{P_y} & , \quad t_f > t_1 \end{cases} \quad (A.2.10)$$

and, upon substitution of these results into (A.1.11) and (A.1.13), it follows that

$$\frac{K}{I^2} u_f = \frac{1}{2} \left[1 - \left(\frac{P_e}{P_y} \right)^{-1} \right] \quad . \quad (A.2.11)$$

Thus, when expressed in terms of I and P_e , the linear result (A.2.11) is identical to that determined for the rectangular pulse, (A.2.7).

Exponential Pulse:

In this case, the definition of I yields

$$I = P_0 t_1 \left[1 - \exp(-t_f t_1^{-1}) \right] \quad (A.2.12)$$

and $\hat{t}$ is given by

$$I \hat{t} = P_0 \left\{ -t_f t_1 \exp(-t_f t_1^{-1}) + t_1^2 \left[1 - \exp(-t_f t_1^{-1}) \right] \right\} \quad (A.2.13)$$

Using the transcendental equation (A.1.15) which defines t_f , one finds

$$\frac{P_e}{P_y} = \frac{1}{2} \left[1 + \frac{t_1}{t_f} \left(1 - \frac{P_0}{P_y} \right) \right]^{-1} \quad (A.2.14)$$

Using (A.2.12), (A.1.14), (A.1.15), and (A.2.14) finally yields

$$\frac{K u_f}{I^2} = \frac{1}{2} \left[1 - \left(\frac{P_e}{P_y} \right)^{-1} \right] \quad (A.2.15)$$

which again is the same result obtained for the rectangular pulse.

A.3 Solution for a Linear Elastic/Linear Strain Hardening Material Subjected to an Arbitrary, Spatially Uniform, Pressure Pulse

It is assumed that the stress-strain relation is bilinear as shown in Fig. 2. Prior to obtaining results for the three pulse shapes considered above, it is necessary to derive general expressions for the deformation which account for the various types of possible response.

A.3.1 Infinite Duration Pulse:

If one assumes a pulse of infinite duration*, the problem is relatively

* It is assumed that the pulse eventually shows monotone decay to a pressure below static yield.

simple since it is not necessary to determine whether yielding occurs before or after the end of the pulse. In the case of the infinite pulse, for elastic behavior, the equation of motion and initial conditions are simply

$$\ddot{u} + \omega^2 u = \frac{P(t)}{\rho H} \quad , \quad 0 \leq t \leq t_y \quad , \quad (\text{A.3.1})$$

with

$$u(0) = \dot{u}(0) = 0 \quad . \quad (\text{A.3.2})$$

A direct Laplace transform solution then gives

$$\left. \begin{aligned} u(t) &= \frac{1}{\rho H \omega} \int_0^t P(\tau) \sin \omega(t-\tau) d\tau \\ \dot{u}(t) &= \frac{1}{\rho H} \int_0^t P(\tau) \cos \omega(t-\tau) d\tau \end{aligned} \right\} \quad (\text{A.3.3})$$

Plastic response occurs for $t_y \leq t \leq t_m$ where t_m is the time at which unloading is initiated. The governing equation is

$$\rho H \ddot{u} + \frac{E_p H}{R^2} u + \left(\alpha_y \frac{H}{R} - \frac{E_p H}{R^2} u_y \right) = P(t) \quad , \quad t \geq t_y \quad (\text{A.3.4})$$

with known initial conditions for $u(t_y)$ and $\dot{u}(t_y)$. A Laplace transform solution of (A.3.4) yields

$$\begin{aligned} u(t) &= \frac{1}{\rho H \omega_p} \int_0^{t-t_y} P(\tau) \sin \omega_p(t-t_y-\tau) d\tau + u_y \cos \omega_p(t-t_y) \\ &+ \frac{\dot{u}_y}{\omega_p} \sin \omega_p(t-t_y) + u_y \left(\frac{\alpha^2 - 1}{\alpha^2} \right) \left[1 - \cos \omega_p(t-t_y) \right] \quad . \end{aligned} \quad (\text{A.3.5})$$

A.3.2 Finite Duration Pulse

In the case of a pulse of finite duration, there are two possibilities - each of which involves three response regimes. Yield will occur either during the pulse, $t_y \leq t_1$, or after the pulse, $t_y > t_1$. If yield occurs during the pulse then there is one elastic and two plastic regimes. If yield occurs after the pulse, there are two elastic and one plastic regime.

That is,

a) yield during the pulse, $t_y \leq t_1$:

- i) elastic behavior; eq. (A.3.1); $0 \leq t \leq t_y$; $u(0) = \dot{u}(0) = 0$
- ii) plastic behavior; eq. (A.3.4); $t_y \leq t \leq t_1$; $u(t_y) = u_y$, $\dot{u}(t_y) = \dot{u}_y$
- iii) plastic behavior; eq. (A.3.4); $t_1 \leq t \leq t_m$; $P(t) \equiv 0$;
 $u(t_1) = u^*$, $\dot{u}(t_1) = \dot{u}^*$ where u^* , $\dot{u}^*$ are computed from ii) at $t = t_1$.

b) yield after the pulse, $t_1 \leq t_y$:

- i) elastic behavior; eq. (A.3.1); $0 \leq t \leq t_1$; $u(0) = \dot{u}(0) = 0$
- ii) elastic behavior; eq. (A.3.1); $t_1 \leq t \leq t_y$; $P \equiv 0$; $u(t_1) = u^*$,
 $\dot{u}(t_1) = \dot{u}^*$ with u^* , $\dot{u}^*$ computed from i) at $t = t_1$.
- iii) plastic behavior; eq. (A.3.4); $t_y \leq t \leq t_m$; $P \equiv 0$; $u(t_y) = u_y$,
 $\dot{u}(t_y) = \dot{u}_y$ with u_y , $\dot{u}_y$ computed from ii) at $t = t_y$.

Using the solutions of (A.3.1) and (A.3.4), the various expressions for the displacement corresponding to the above regimes are:

$$a) i): u(t) = \frac{1}{\rho H \omega} \int_0^t P(\tau) \sin \omega(t-\tau) d\tau, \quad 0 \leq t \leq t_y, \quad (A.3.6)$$

$$a) ii): u(t) = \frac{1}{\rho H \omega_p} \int_0^{t-t_y} P(\tau) \sin \omega_p(t-t_y-\tau) d\tau + u_y \cos \omega_p(t-t_y) + \frac{\dot{u}_y}{\omega_p} \sin \omega_p(t-t_y) + u_y \frac{(\alpha^2 - 1)}{\alpha^2} \left[1 - \cos \omega_p(t-t_y) \right], \quad t_y \leq t \leq t_1, \quad (A.3.7)$$

$$\begin{aligned} \text{a) iii): } u(t) = & u^* \cos \omega_p(t-t_1) + \frac{\dot{u}^*}{\omega_p} \sin \omega_p(t-t_1) \\ & + u_y \frac{(\alpha^2-1)}{\alpha^2} \left[1 - \cos \omega_p(t-t_1) \right], \quad t \geq t_1 \end{aligned} \quad (\text{A.3.8})$$

$$\text{b) i) } \equiv \text{ a) i) except } 0 \leq t \leq t_1 \quad (\text{A.3.9})$$

$$\text{b) ii): } u(t) = \frac{\dot{u}^*}{\omega} \sin \omega(t-t_1) + u^* \cos \omega(t-t_1), \quad t_1 \leq t \leq t_y \quad (\text{A.3.10})$$

$$\begin{aligned} \text{b) iii): } u(t) = & u_y \cos \omega_p(t-t_y) + \frac{\dot{u}_y}{\omega_p} \sin \omega_p(t-t_y) \\ & + u_y \frac{(\alpha^2-1)}{\alpha^2} \left[1 - \cos \omega_p(t-t_y) \right], \quad t \geq t_y \end{aligned} \quad (\text{A.3.11})$$

A.4 Solution for a Rectangular Pulse

A.4.1 Analytical Solution

A rectangular pulse is described by

$$p = \begin{cases} p_0, & t \leq t_1 \\ 0, & t > t_1 \end{cases}$$

Assuming that yield occurs during the pulse, eq. (A.3.6) gives

$$u(t) = \frac{p_0}{\rho H \omega^2} \left[1 - \cos \omega t \right], \quad t \leq t_y$$

Since $\omega^2 = \frac{E}{\rho R^2}$ and $p_y = \frac{\sigma_y H}{R} = \frac{u_y E H}{R^2}$, this may be expressed as

$$\frac{u(t)}{u_y} = \frac{p_0}{p_y} \left[1 - \cos \omega t \right], \quad t \leq t_y \quad (\text{A.4.1})$$

and, since at $t = t_y$, $u = u_y$, the condition determining t_y is, from (A.4.1),

$$t_y = \omega^{-1} \cos^{-1} \left(1 - \frac{p_y}{p_0} \right) \quad (\text{A.4.2})$$

From (A.4.1) and (A.4.2) it follows that the velocity at $t = t_y$ is

$$\frac{\dot{u}_y}{\omega u_y} = \left(2 \frac{p_0}{p_y} - 1 \right)^{\frac{1}{2}} \quad (A.4.3)$$

It is necessary to develop the plastic response during the two intervals $t_y \leq t \leq t_1$ and $t > t_1$. For $t_y \leq t \leq t_1$, eq. (A.3.7) yields for the displacement and velocity

$$\left. \begin{aligned} \frac{u(t)}{u_y} &= \left[1 + \frac{1}{\alpha^2} \left(\frac{p_0}{p_y} - 1 \right) \right] \left[1 - \cos \omega_p(t-t_y) \right] + \cos \omega_p(t-t_y) \\ &\quad + \frac{1}{\alpha} \left(2 \frac{p_0}{p_y} - 1 \right)^{\frac{1}{2}} \sin \omega_p(t-t_y), \\ \frac{\dot{u}(t)}{\dot{u}_y} &= \omega_p \left[1 + \frac{1}{\alpha^2} \left(\frac{p_0}{p_y} - 1 \right) \right] \sin \omega_p(t-t_y) - \omega_p \sin \omega_p(t-t_y) \\ &\quad + \frac{\omega_p}{\alpha} \cos \omega_p(t-t_y) \left(2 \frac{p_0}{p_y} - 1 \right)^{\frac{1}{2}}. \end{aligned} \right\} (A.4.4)$$

The displacement and velocity values u^* , $\dot{u}^*$ are computed from (A.4.4) at $t = t_1$. Using these and eq. (A.3.8) then gives the following result for $t \geq t_1$:

$$\frac{u(t)}{u_y} = \frac{u^*}{u_y} \cos \omega_p(t-t_1) + \frac{\dot{u}^*}{\omega_p u_y} \sin \omega_p(t-t_1) + \frac{\alpha^2 - 1}{\alpha^2} \left[1 - \cos \omega_p(t-t_1) \right],$$

$$\frac{\dot{u}(t)}{\dot{u}_y} = - \frac{u^*}{u_y} \omega_p \sin \omega_p(t-t_1) + \frac{\dot{u}^*}{\dot{u}_y} \cos \omega_p(t-t_1) + \frac{(\alpha^2 - 1)}{\alpha^2} \omega_p \sin \omega_p(t-t_1). \quad (A.4.5)$$

If yield occurs after the pulse, the solution is given by equations (A.3.9), (A.3.10), and (A.3.11). The initial elastic response is:

$$\begin{aligned}\frac{u(t)}{u_y} &= \frac{P_0}{P_y}(1 - \cos \omega t) \quad , \\ \frac{\dot{u}(t)}{u_y} &= \frac{P_0}{P_y} \omega \sin \omega t \quad , \quad 0 \leq t \leq t_1 \quad .\end{aligned}\tag{A.4.6}$$

The elastic response for $t_1 \leq t \leq t_y$ is, from (A.3.10) ,

$$\begin{aligned}\frac{u(t)}{u_y} &= \frac{P_0}{P_y} \left[\sin \omega t_1 \sin \omega(t-t_1) + (1 - \cos \omega t_1) \cos \omega(t-t_1) \right] \quad , \\ \frac{\dot{u}(t)}{u_y} &= \frac{P_0}{P_y} \omega \left[\sin \omega t_1 \cos \omega(t-t_1) - (1 - \cos \omega t_1) \sin \omega(t-t_1) \right] \quad .\end{aligned}\tag{A.4.7}$$

The yield time, t_y , is determined from the first of (A.4.7) by setting

$$\begin{aligned}\frac{u(t_y)}{u_y} &= 1. \quad \text{From (A.3.11), the plastic response is } (t > t_y): \\ \frac{u(t)}{u_y} &= \cos \omega_p(t-t_y) + \frac{\dot{u}_y}{\omega_p u_y} \sin \omega_p(t-t_y) + \frac{\alpha^2 - 1}{\alpha^2} \left[1 - \cos \omega_p(t-t_y) \right] \quad , \\ \frac{\dot{u}(t)}{u_y} &= -\frac{\omega_p}{\alpha^2} \sin \omega_p(t-t_y) + \frac{\dot{u}_y}{u_y} \cos \omega_p(t-t_y) \quad ,\end{aligned}\tag{A.4.8}$$

where $\dot{u}_y$ is calculated from the second of equations (A.4.7).

A.4.2 Numerical Solution - Rectangular Pulse

It is convenient to consider two different classes of pulse length. A "long" pulse is defined as $\omega t_1 \geq \pi$. If yield occurs at all, it must occur before or at the end of the pulse since the time of maximum elastic displacement is $\omega t_m = \pi$ and the displacement is given by

$$\frac{u_m}{u_y} = \frac{u(t_m)}{u_y} = 2 \frac{p_o}{p_y}.$$

Thus, the critical pressure to just produce yield is

$$\left(\frac{p_o}{p_y}\right)_c = \frac{1}{2}. \quad (\text{A.4.9})$$

For all pressure ratios greater than this critical value (for long pulses), the time of yield, $\omega t_y < \omega t_1$, is established from eq. (A.4.2) and the velocity at yield is given by (A.4.3). For $\frac{p_o}{p_y} > \left(\frac{p_o}{p_y}\right)_c$ it remains to determine the time of maximum inelastic displacement and there are two possibilities: the time of maximum displacement occurring before pulse termination or after.

If $\omega t_m < \omega t_1$ then ωt_m is determined from the condition $\frac{\dot{u}(t_m)}{u_y} = 0$ using the second of eqs. (A.4.4). Thus ωt_m is the first positive zero of

$$f(t_m) = \left(\frac{p_o}{p_y} - 1\right) \sin \omega_p(t_m - t_y) + \alpha \left(2 \frac{p_o}{p_y} - 1\right)^{\frac{1}{2}} \cos \omega_p(t_m - t_y) = 0. \quad (\text{A.4.10})$$

The numerical solution of (A.4.10) was started with $\omega t_m = \omega t_y$, for

which $f(\omega t_y) = \alpha \left(2 \frac{p_o}{p_y} - 1\right)^{\frac{1}{2}} > 0$. The argument was incremented until the first sign change of f . A simple iteration then determines the root to the required accuracy. If $\omega t_m < \omega t_1$ the maximum

displacement was then calculated from the first of eqs. (A.4.4).

If no zero is found in the range $t_y \leq t \leq t_1$, then the maximum inelastic displacement occurs after the pulse and t_m is determined from the second of eq. (A.4.5). Thus ωt_m is the first positive zero of

$$f(t_m) = \frac{\alpha^2 - 1}{\alpha^2} \sin \omega_p(t_m - t_1) - \frac{u^*}{u_y} \sin \omega_p(t_m - t_1) + \frac{\dot{u}^*}{\dot{u}_y} \cos \omega_p(t_m - t_1) = 0, \quad (\text{A.4.11})$$

The numerical solution is started with $t_m = t_1$ for which $f(t_1) > 0$.

Using the same simple search routine, the time t_m is computed and used in the first of eq. (A.4.5) to determine the maximum inelastic displacement.

For "short" pulses where $\omega t_1 \leq \pi$, yield may occur before or after the pulse. Assuming yield after the pulse, the time of maximum elastic displacement and the critical pressure needed to just reach yield can be determined from (A.3.10) as follows. For $t_1 \leq t \leq t_y$ we have

$$\frac{u(t)}{u_y} = \frac{p_0}{p_y} \left[\sin \omega t_1 \sin \omega(t - t_1) + (1 - \cos \omega t_1) \cos \omega(t - t_1) \right].$$

Differentiating and imposing the condition $\dot{u}(t_m) = 0$ (where $\omega t_m > \omega t_1$) gives as the equation determining the time of maximum elastic displacement

$$\tan \omega t_m = \frac{\sin \omega t_1}{\cos \omega t_1 - 1}. \quad (\text{A.4.12})$$

substitution of (A.4.12) into the expression for displacement and use of the condition that yield will just occur when $u(t_m) = u_y$ gives as the critical pressure

$$\left(\frac{p_0}{p_y} \right)_c = \left[2 \cos \omega(t_m - t_1) \right]^{-1}. \quad (\text{A.4.13})$$

From (A.4.1), the pressure required to cause yield at the end of the pulse (i.e. $\omega t_y = \omega t_1$) is given by

$$\frac{P_o}{P_y} = (1 - \cos \omega t_1)^{-1} \quad (A.4.14)$$

Increasing the pressure above that given by (A.4.14) implies yield will occur before the pulse is over. (The numerical analysis for this case is identical to that for a "long" pulse). It remains to consider values of pressure in the range

$$\left[2 \cos \omega(t_m - t_1) \right]^{-1} \leq \frac{P_o}{P_y} \leq (1 - \cos \omega t_1)^{-1} \quad (A.4.15)$$

From the first of (A.4.7), the dimensionless yield time, ωt_y , is given by first positive zero of

$$f(\omega t_y) = \frac{P_o}{P_y} \left[\sin \omega t_1 \sin \omega(t_y - t_1) + (1 - \cos \omega t_1) \cos \omega(t_y - t_1) \right] - 1 \quad (A.4.16)$$

The numerical solution of (A.4.16) is started with $\omega t_y = \omega t_1$ (for which $f < 0$) and stepped in time until f becomes positive. At the time of maximum inelastic displacement $\dot{u}(t_m) = 0$. From the second of (A.4.7) and (A.4.16) the equation determining t_m is

$$\tan \omega_p(t_m - t_y) = \frac{P_o}{\alpha P_y} \left[(1 - \cos \omega t_1) \sin \omega t_y + \sin \omega t_1 \cos \omega t_y \right] \quad (A.4.17)$$

Using (A.4.17), the maximum inelastic displacement is

$$\frac{u_m}{u_y} = 1 + \frac{1}{\alpha^2} \left[\frac{1}{\cos \omega_p(t_m - t_y)} - 1 \right] \quad (A.4.18)$$

The various computations indicated above were performed using a small program, RPULSE, which determines the maximum inelastic displacement as a function of the pulse parameters (P_o, t_1) and the material property

parameter $\alpha^2 = E_p E^{-1}$.

A.5 Solution for a Linear Pulse

A.5.1 Analytical Solution

The linear pulse is given by

$$P(t) = \begin{cases} P_0(1 - t/t_1) & , \quad 0 \leq t \leq t_1 \\ 0 & , \quad t \geq t_1 \end{cases} \quad (A.5.1)$$

Assuming that yield occurs during the pulse, (A.3.6) gives as the elastic response ($0 \leq t \leq t_y$) :

$$\begin{aligned} \frac{u(t)}{u_y} &= \frac{P_0}{P_y} \left[1 - t/t_1 - \cos \omega t + (\omega t_1)^{-1} \sin \omega t \right] , \\ \frac{\dot{u}(t)}{u_y} &= \omega \frac{P_0}{P_y} \left[-(\omega t_1)^{-1} + \sin \omega t + (\omega t_1)^{-1} \cos \omega t \right] . \end{aligned} \quad (A.5.2)$$

The dimensionless time for maximum elastic displacement as determined by the condition $\dot{u}(\bar{t}_m) = 0$ and the second of (A.5.2), is given by the root of

$$-1 + \omega t_1 \sin \omega \bar{t}_m + \cos \omega \bar{t}_m = 0 \quad (A.5.3)$$

Thus, from (A.5.3) and the first of (A.5.2):

$$\frac{u_m}{u_y} = \frac{P_0}{P_y} \left[-\bar{t}_m/t_1 + \omega t_1 \sin \omega \bar{t}_m + (\omega t_1)^{-1} \sin \omega \bar{t}_m \right] \quad (A.5.4)$$

This gives as the critical pressure to just produce yield

$$\left(\frac{P_0}{P_y} \right)_c = \omega t_1 \left\{ \left[1 + (\omega t_1)^2 \right] \sin \omega \bar{t}_m - \omega \bar{t}_m \right\}^{-1} \quad (A.5.5)$$

For pressures greater than this critical value, the dimensionless yield time ωt_y is, from (A.5.2), the solution of

$$1 = \frac{P_0}{P_y} \left[1 - (\omega t_y)(\omega t_1)^{-1} - \cos \omega t_y + (\omega t_1)^{-1} \sin \omega t_y \right], \quad (\text{A.5.6})$$

with the velocity at yield calculated from the second of (A.5.2) .

Plastic behavior in the interval $t_y \leq t \leq t_1$ is obtained from (A.3.7).

Thus it follows that

$$\begin{aligned} \frac{u(t)}{u_y} &= \frac{1}{\alpha^2} \frac{P_0}{P_y} \left[1 - \frac{t-t_y}{t_1} - \cos \omega_p(t-t_y) + \frac{1}{\omega_p t_1} \sin \omega_p(t-t_y) \right] \\ &+ \cos \omega_p(t-t_y) + \frac{1}{\alpha} \frac{P_0}{P_y} \left[-(\omega t_1)^{-1} + \sin \omega t_y + (\omega t_1)^{-1} \cos \omega t_y \right] \sin \omega_p(t-t_y) \\ &+ \frac{\alpha^2 - 1}{\alpha^2} \left[1 - \cos \omega_p(t-t_y) \right] , \\ \dot{u}(t) &= \frac{\omega_p}{\alpha^2} \frac{P_0}{P_y} \left[-(\omega_p t_1)^{-1} + \sin \omega_p(t-t_y) + (\omega_p t_1)^{-1} \cos \omega_p(t-t_y) \right] \\ &+ \frac{\omega_p}{\alpha} \frac{P_0}{P_y} \left[-(\omega t_1)^{-1} + \sin \omega t_y + (\omega t_1)^{-1} \cos \omega t_y \right] \cos \omega_p(t-t_y) \\ &- \omega_p \alpha^{-2} \sin \omega_p(t-t_y) . \end{aligned} \quad (\text{A.5.7})$$

The plastic behavior for $t \geq t_1$ is calculated from (A.3.8) using the values of (A.5.7) at t_1 for u^* and $\dot{u}^*$.

If yield occurs after the pulse there are two elastic regimes and the displacements are evaluated using (A.3.9), (A.3.10), and (A.3.11) . Equations (A.5.2) now hold for $0 \leq t \leq t_1$. For $t_1 \leq t \leq t_y$, (A.3.10) gives

$$\frac{u(t)}{u_y} = \frac{p_0}{p_y} \left\{ \left[-(\omega t_1)^{-1} + \sin \omega t_1 + (\omega t_1)^{-1} \cos \omega t_1 \right] \sin \omega(t-t_1) + \left[-\cos \omega t_1 + (\omega t_1)^{-1} \sin \omega t_1 \right] \cos \omega(t-t_1) \right\}, \quad (A.5.8)$$

$$\frac{\dot{u}(t)}{\dot{u}_y} = \frac{p_0}{p_y} \left\{ \left[-(\omega t_1)^{-1} + \sin \omega t_1 + (\omega t_1)^{-1} \cos \omega t_1 \right] \cos \omega(t-t_1) - \left[-\cos \omega t_1 + (\omega t_1)^{-1} \sin \omega t_1 \right] \sin \omega(t-t_1) \right\}$$

The plastic behavior, $t \geq t_y$, is given by (A.3.11) with $u(t_y)$ and

$\dot{u}(t_y)$ computed from (A.5.8).

A.5.2 Numerical Solution

For a given pulse parameter ωt_1 it is first necessary to determine whether the time for maximum elastic displacement $\omega \bar{t}_m$ is less than or greater than ωt_1 . Assume $\omega \bar{t}_m < \omega t_1$ where $\omega \bar{t}_m$ is the first positive zero of (A.5.3) - computed by a simple sign-change search routine. The critical pressure is then calculated from (A.5.5). For any pressure

$$\left(\frac{p_0}{p_y}\right) > \left(\frac{p_0}{p_y}\right)_c, \quad (A.5.2) \text{ is used to compute the time of yield } \omega t_y;$$

again by a simple search routine. At this point there are two possibilities. The time of maximum inelastic response will occur either before or after the end of the pulse. Assuming that $\omega t_m < \omega t_1$, the time of maximum inelastic displacement is obtained from $\dot{u}(t_m) = 0$ using the second of equations (A.5.7). That is, ωt_m is determined from

$$\begin{aligned}
f(\omega t_m) &\equiv \frac{p_0}{p_y} \left[-(\omega_p t_1)^{-1} + \sin \omega_p (t - t_y) + (\omega_p t_1)^{-1} \cos \omega_p (t_m - t_y) \right] \\
&+ \alpha \frac{p_0}{p_y} \left[-(\omega t_1)^{-1} + \sin \omega t_y + (\omega t_1)^{-1} \cos \omega t_y \right] \cos \omega_p (t_m - t_y) \\
&- \sin \omega_p (t_m - t_y) = 0 \quad , \quad (A.5.9)
\end{aligned}$$

where the search is started with $\omega t_m = \omega t_y$ for which $f(\omega t_y) > 0$.

If this search reveals a root $\omega t_m < \omega t_1$ then the maximum inelastic displacement, $u_m = u(t_m)$, is computed from the first of (A.5.7).

If there is no positive root less than ωt_1 , the maximum inelastic displacement occurs after the pulse is over and $\omega t_m > \omega t_1$ is determined from the equation for $\dot{u}$ which is derived using (A.3.8). That is

$$f(\omega t_m) = -\frac{u^*}{\dot{u}_y} \sin \omega_p (t_m - t_1) + \frac{\dot{u}^*}{\omega_p \dot{u}_y} \cos \omega_p (t_m - t_1) + \frac{\alpha^2 - 1}{\alpha^2} \sin \omega_p (t_m - t_1) = 0 \quad , \quad (A.5.10)$$

where u^* and $\dot{u}^*$ are calculated from (A.5.7) at ωt_1 . The search is started at ωt_1 where $f(\omega t_1) > 0$. The maximum inelastic response is then calculated using (A.3.8). That is,

$$\begin{aligned}
\frac{u_m}{\dot{u}_y} &= \frac{u^*}{\dot{u}_y} \cos \omega_p (t_m - t_1) + \frac{\dot{u}^*}{\omega_p \dot{u}_y} \sin \omega_p (t_m - t_1) \\
&+ \frac{\alpha^2 - 1}{\alpha^2} \left[1 - \cos \omega_p (t_m - t_1) \right] \quad . \quad (A.5.11)
\end{aligned}$$

When the first positive zero of (A.5.3) is such that $\omega \tilde{t}_m \geq \omega t_y$, the maximum elastic displacement occurs after the pulse and $\omega \tilde{t}_m$ is determined from the second of (A.5.8) by the condition $\dot{u}(\omega \tilde{t}_m) = 0$. That is

$$\omega \tilde{t}_m = \omega t_1 + \tan^{-1} (ab^{-1}) \quad , \quad (A.5.12)$$

where

$$a = -(\omega t_1)^{-1} + \sin \omega t_1 + (\omega t_1)^{-1} \cos \omega t_1 \quad ,$$

$$b = -\cos \omega t_1 + (\omega t_1)^{-1} \sin \omega t_1 \quad .$$

From the first of (A.5.8), the maximum elastic displacement is thus

$$\frac{u_m}{u_y} = \frac{p_o}{p_y} \left(\frac{a^2 + b^2}{b} \right) \cos \omega(\bar{t}_m - t_1) \quad (\text{A.5.13})$$

and, hence, the critical pressure is

$$\left(\frac{p_o}{p_y} \right)_c = \left[\left(\frac{a^2 + b^2}{b} \right) \cos \omega(\bar{t}_m - t_1) \right]^{-1}. \quad (\text{A.5.14})$$

The pressure required to produce yield at the end of the pulse is given by the first of (A.5.2) with $u(t_1) = u_y$. That is,

$$\left(\frac{p_o}{p_y} \right)_1 = \left[-\cos \omega t_1 + (\omega t_1)^{-1} \sin \omega t_1 \right]^{-1}. \quad (\text{A.5.15})$$

Thus, for pressures in the range

$$\left(\frac{p_o}{p_y} \right)_c \leq \frac{p_o}{p_y} \leq \left(\frac{p_o}{p_y} \right)_1,$$

yield occurs after the pulse at a dimensionless time, ωt_y ,

determined from the condition that $u(t_y) = u_m$. Using the first of (A.5.8)

it is necessary to determine the first positive root of

$$\bar{f}(\omega t_y) = \frac{p_o}{p_y} \left[a \sin \omega(t_y - t_1) + b \cos \omega(t_y - t_1) \right] - 1 = 0. \quad (\text{A.5.16})$$

This again is done by a simple search routine using the starting value $\omega t_y = \omega t_1$ (for which $f < 0$). The time for maximum inelastic displacement follows from (A.3.11) and the condition $\dot{u}(t_m) = 0$.

That is

$$\tan \omega_p (t_m - t_y) = \alpha \left[\frac{\dot{u}^*}{\omega u_y} \cos \omega(t_y - t_1) - \frac{u^*}{u_y} \sin \omega(t_y - t_1) \right], \quad (\text{A.5.17})$$

where u^* and $\dot{u}^*$ are evaluated from (A.5.2) at $t = t_1$. Using

this value of t_m , $u_m = u(t_m)$ is calculated from (A.3.11).

If $\frac{P_0}{P_y} > \left(\frac{P_0}{P_y} \right)_1$ then yield occurs during the pulse and the analysis

is essentially the same as that used when $\omega \bar{t}_m \leq \omega t_1$. The various calculations outlined above are contained in a simple computer program, LPULSE, which was used to generate the parametric results shown in figures 15 - 17.

A.6 Solution for an Exponential Pulse

A.6.1 Analytical Solution

The pulse is given by

$$P(t) = P_0 \exp(-t/t_1) \quad , \quad 0 \leq t \quad . \quad (A.6.1)$$

It is formally somewhat easier to treat this case since there is no finite cutoff time for the pulse and many of the problems related to the various alternate response regimes do not exist. The elastic response, $0 \leq t \leq t_y$, is

given by substitution of (A.6.1) into (A.3.6) with the result

$$\begin{aligned} \frac{u(t)}{u_y} &= \frac{P_0}{P_y} \left[\frac{(\omega t_1)^2}{1 + (\omega t_1)^2} \right] \left[\exp(-t/t_1) + (\omega t_1)^{-1} \sin \omega t - \cos \omega t \right] , \\ \frac{\dot{u}(t)}{u_y} &= \frac{P_0}{P_y} \left[\frac{\omega t_1}{1 + (\omega t_1)^2} \right] \left[\exp(-t/t_1) + \cos \omega t + \omega t_1 \sin \omega t \right] . \end{aligned} \quad (A.6.2)$$

The second of (A.6.2) and the condition $\dot{u}(\bar{t}_m)$ give as time for maximum elastic displacement the first positive root of

$$f(\omega \bar{t}_m) \equiv -\exp(-\bar{t}_m/t_1) + \cos \omega \bar{t}_m + \omega t_1 \sin \omega \bar{t}_m = 0 \quad (A.6.3)$$

Using (A.6.3) and the first of (A.6.2), the maximum elastic displacement is

$$\frac{u_m}{u_y} = \frac{P_0}{P_y} \omega t_1 \sin \omega \bar{t}_m \quad , \quad (A.6.4)$$

and, thus, the critical pressure is

$$\left(\frac{p_0}{p_y} \right)_c = \left[\omega t_1 \sin \omega \bar{t}_m \right]^{-1} . \quad (A.6.5)$$

For pressures above this critical value the yield time is established by the condition $u(t_y) = u_y$. Thus, from the first of (A.6.2), t_y is the first positive zero of

$$\frac{p_0}{p_y} \left[\frac{(\omega t_1)^2}{1 + (\omega t_1)^2} \right] \left[\exp(-t_y/t_1) - \cos \omega t_y + (\omega t_1)^{-1} \sin \omega t_y \right] - 1 = 0 \quad (A.6.6)$$

and the velocity at yield reduces to

$$\frac{\dot{u}_y}{\omega u_y} = \frac{1}{\omega t_1} \left[\frac{p_0}{p_y} \omega t_1 \sin \omega t_y - 1 \right] . \quad (A.6.7)$$

For $t > t_y$, the plastic response is given by evaluation of (A.3.7). That is,

$$\begin{aligned} \frac{u(t)}{u_y} &= \left[\frac{(\omega t_1)^2}{1 + (\omega t_1)^2} \right] \frac{p_0}{p_y} \left\{ \exp \left[- (t - t_y)/t_1 \right] - \cos \omega_p (t - t_y) + (\omega_p t_1)^{-1} \cdot \right. \\ &\quad \left. \sin \omega_p (t - t_y) \right\} + \left[\frac{p_0}{p_y} \omega t_1 \sin \omega t_y - 1 \right] \frac{\sin \omega_p (t - t_y)}{\omega_p t_1} + 1 - \frac{1}{\alpha^2} \left[1 - \cos \omega_p (t - t_y) \right] , \\ \frac{\dot{u}(t)}{\omega u_y} &= \left[\frac{\omega t_1}{1 + (\omega t_1)^2} \right] \frac{p_0}{p_y} \left\{ \exp \left[- (t - t_y)/t_1 \right] + \omega_p t_1 \sin \omega_p (t - t_y) + \cos \omega_p (t - t_y) \right\} \\ &\quad + \left[\frac{p_0}{p_y} \omega t_1 \sin \omega t_y - 1 \right] \cos \omega_p (t - t_y) - \frac{1}{\alpha^2} \omega_p t_1 \sin \omega_p (t - t_y) . \quad (A.6.8) \end{aligned}$$

A.6.2 Numerical Solution

The search routine establishes the maximum elastic response time from (A.6.3) starting with $\omega \bar{t}_m = 0$ (for which $f = 0$, $df > 0$). The critical pressure is then computed from (A.6.5). For pressures greater than this value, the yield time is established from (A.6.6) with the root search started at time zero. The time of maximum inelastic displacement is found by setting the second of equations (A.6.8) equal to zero and using a simple root search starting with $\omega t_m = \omega t_y$. Having established ωt_m , the maximum inelastic displacement is calculated from the second of (A.6.8). A small computer program, EPULSE, was written to perform this sequence of calculations. The parametric results are shown in the attached figures.

A.7 Application of Correlation Parameters

To eliminate the effect of pulse shape on maximum deformation calculations for an elastic-linear strain hardening material, it appears that correlation parameters must be defined in terms of both pulse parameters and material parameters. In particular, any such correlation will certainly depend on the pulse parameters $\frac{P_0}{P_y}$ and t_1 and also on the material parameters ω , the natural frequency of the ring, α^2 , the ratio of plastic to elastic modulus, and σ_y , the initial yield stress.

No satisfactory set of parameters was found during this study; however, the effective pressure and effective time as defined in section A.2 were slightly modified and were used as a starting point in reviewing alternate formulations. These modified parameters are defined as follows:

$$\text{effective impulse} = I \equiv \int_0^{t_m} P(t) dt ,$$

$$\text{effective mean load duration} = \hat{t} \equiv \frac{1}{I} \int_0^{t_m} tP(t) dt , \quad (\text{A.7.1})$$

$$\text{effective load} = P_e \equiv \frac{I}{2\hat{t}} ,$$

where t_m is the time of maximum inelastic displacement. For the three pulse shapes considered above, these parameters are as follows:

Rectangular pulse:

$$I = \begin{cases} P_0 t_m & , \quad t_m \leq t_1 \\ P_0 t_1 & , \quad t_m \geq t_1 \end{cases}$$

$$\hat{t} = \begin{cases} \frac{1}{2} P_0 t_m^2 & , \quad t_m \leq t_1 \\ \frac{1}{2} P_0 t_1^2 & , \quad t_m \geq t_1 \end{cases}$$

$$\frac{P_e}{P_y} = \frac{P_0}{P_y}$$

A convenient quantity for plotting results is $Ku_m I^{-2}$

where $K = \rho H P_y$. For a rectangular pulse this is

$$\frac{Ku_y}{I^2} = \begin{cases} \left(\frac{P_0}{P_y} \right)^{-2} (\omega t_m)^{-2} & , \quad t_m \leq t_1 \\ \left(\frac{P_0}{P_y} \right)^{-2} (\omega t_1)^{-2} & , \quad t_m \geq t_1 \end{cases}$$

Linear Pulse:

$$I = \begin{cases} P_0 t_m \left[1 - t_m (2t_1)^{-1} \right] & , \quad t_m \leq t_1 \\ \frac{1}{2} P_0 t_1 & , \quad t_m \geq t_1 \end{cases}$$

$$\hat{I}t = \frac{P_0 t^{*2}}{2} \left(1 - \frac{3t^*}{2t_1} \right)$$

$$\text{where } t^* = \begin{cases} t_m & , \quad t_m \leq t_1 \\ t_1 & , \quad t_m \geq t_1 \end{cases}$$

$$\frac{P_e}{P_y} = \begin{cases} \frac{P_0}{P_y} \frac{\left[1 - t_m (2t_1)^{-1} \right]^2}{\left[1 - 2t_m (3t_1)^{-1} \right]} & , \quad t_m \leq t_1 \\ \frac{3}{4} \frac{P_0}{P_y} & , \quad t_m \geq t_1 \end{cases}$$

$$\frac{Ku_y}{I^2} = \begin{cases} \left(\frac{P_0}{P_y} \right)^{-2} \left(\omega t_m \right)^{-2} \left(1 - \frac{t_m}{2t_1} \right)^{-2} & , \quad t_m \leq t_1 \\ 4 \left(\frac{P_0}{P_y} \right)^{-2} \left(\omega t_1 \right)^{-2} & , \quad t_m \geq t_1 \end{cases}$$

Exponential Pulse:

$$I = P_0 t_1 \left[1 - \exp(-t_m/t_1) \right]$$

$$\hat{I}t = P_0 t_1 \left\{ -t_m \exp(-t_m/t_1) - t_1 \left[\exp(-t_m/t_1) - 1 \right] \right\}$$

$$\frac{P_e}{P_y} = \frac{1}{2} \frac{P_0}{P_y} \left[1 - \exp(-t_m/t_1) \right]^2 \left\{ \left[1 - \exp(-t_m/t_1) \right] - \frac{t_m}{t_1} \exp(-t_m/t_1) \right\}^{-1}$$

$$\frac{\alpha u_y}{I^2} = \left(\frac{P_0}{P_y} \right)^{-2} \left(\omega t_1 \right)^{-2} \left[1 - \exp(-t_m/t_1) \right]^{-2}$$

REFERENCES

- [1] C. K. Youngdahl, "Correlation Parameters for Eliminating the Effect of Pulse Shape on Dynamic Plastic Deformation", J. Appl. Mech. 37, p 744 (1970).

Fig. 1

Fig. 2

$$P(t) = \begin{cases} P_0, & 0 \leq t \leq t_1 \\ 0, & t > t_1 \end{cases}$$

$$P(t) = \begin{cases} P_0(1 - t/t_1), & 0 \leq t \leq t_1 \\ 0, & t > t_1 \end{cases}$$

$$P(t) = P_0 e^{-t/t_1}$$

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 9

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17

Fig. 18

Fig. 19

Fig. 20

Fig. 21

Fig. 23

Fig. 24