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A Review of Analysis Methods for Prestressed Concrete Reactor Vessels

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A REVIEW OF ANALYSIS METHODS FOR PRESTRESSED
CONCRETE REACTOR VESSELS

W. G. Dodge Z. P. Bazant R. H. Gallagher

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PREFACE

This report was prepared as part of the Oak Ridge National Laboratory Prestressed Concrete Nuclear Pressure Vessel Development Program (189a No. OH003). The goal of the program is to develop and further improve prestressed concrete reactor vessels (PCRVs) for high-temperature gas-cooled reactors in particular and gas-cooled reactors in general.

In addition to contributing to the establishment of the overall state of the art of PCRV analysis capabilities, this report is intended to assist in identifying and interpreting specific needs in this particular area. It will therefore provide the basis for future work undertaken to develop improved methods of analysis for complex concrete structures such as the PCRV.

SUMMARY

The prestressed concrete reactor vessels (PCRVs) of the type currently used in large high-temperature gas-cooled reactors are massive structures with complex geometric characteristics that require three-dimensional analytical models. These structures exhibit time-dependent quasi-static response under normal operating conditions. Their response to accidental overpressurization requires a nonlinear constitutive model to describe material behavior, and the brittle mode of failure exhibited by concrete introduces localized violation of the compatibility requirements of continuum mechanics.

The finite-element method is the only numerical procedure currently available that is capable of providing a realistic model for a structure having the geometric complexity of a modern PCRV. Even with this method, difficulties arise in the structure of the system of algebraic equations that define the unknown parameters, and judicious utilization of the most advanced discretization techniques and solution algorithms is required if computation costs are to remain reasonable. This is particularly true for incremental time-dependent and nonlinear analyses that require repeated resolution of the system of equations.

The time-dependent response of the vessel material under isothermal operating conditions and normal levels of stress seems to be adequately represented by the linear viscoelastic constitutive model. It is not clear that completely satisfactory analytical models have been identified for the cases of thermal transients and changes in moisture, although some satisfactory results have been achieved in the former case utilizing the time-temperature shift principle of thermo-rheologically simple materials. The implementation of the linear viscoelastic model within the framework of the finite-element method is understood; however, the required computational effort is strongly affected by the analytical representation of the creep or relaxation functions. When an exponential series representation corresponding to Maxwell or Kelvin rheological chain models is used, it is possible to develop recursive relations for computing the stress and strain increments occurring during a time increment and avoid the normal requirement of utilizing the total stress-strain history in

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this computation. The resulting saving in computational effort is somewhat offset by increased difficulty in obtaining an accurate model for the actual material response using these representations.

The nonlinear response of a PCRV to moderate overpressurization is the result of two separate phenomena: nonlinearity in the concrete stress-strain relations and the propagation of cracks within the concrete as the failure limits of concrete are exceeded. As the ultimate pressure is approached, the response is primarily determined by the plastic response of the steel reinforcing and prestressing members.

Two material models are commonly used to simulate concrete nonlinear stress-strain response: plasticity and hyperelasticity. There are two fundamental difficulties associated with the use of the plasticity model. The first is the apparent difficulty in achieving an adequate representation for the yield function and hardening law to reproduce the proper stress-strain response under multiaxial loadings. The second, and possibly related, difficulty is that the use of this model implies isotropic material damage, while the microcracking that accompanies the nonlinear response of concrete is known to exhibit directional preference depending on the state of stress.

Two fundamental difficulties are also associated with the use of the hyperelastic model. The first is the inability of the model to account for irreversibility upon unloading, and the second is the apparent difficulty in achieving functional relations for the coefficients that properly account for the stress-induced anisotropy accompanying the nonlinear stress-strain response. At this time, there does not appear to be a completely satisfactory constitutive model for representing the nonlinear response of concrete, although recent work on utilizing the "endochronic theory" developed for work-hardening metals shows significant promise.

Because the trial functions used in finite-element-method analyses are continuous functions, the treatment of the displacement discontinuities associated with cracking poses serious difficulties. The modeling of concrete cracking has proceeded along two distinct lines. Some practitioners model the cracks as actual discontinuities in the displacement field by utilizing techniques of mesh redefinition, while others retain continuity in the displacement field but modify the material constitutive

relations in an attempt to achieve proper force-displacement relations for the postcrack condition. Neither of the techniques is completely satisfactory. In the former case, the algorithms must be developed for each specific case and, consequently, are applicable only for academic studies, while the algorithms used in the latter case do not, in general, yield the proper force-displacement relations. At this time, a completely satisfactory technique for modeling concrete cracking in a PCRV does not exist.

A REVIEW OF ANALYSIS METHODS FOR PRESTRESSED CONCRETE REACTOR VESSELS

W. G. Dodge Z. P. Bazant R. H. Gallagher

ABSTRACT

Theoretical and practical aspects of analytical models and numerical procedures for detailed analysis of prestressed concrete reactor vessels are reviewed. Constitutive models and numerical algorithms for time-dependent and nonlinear response of concrete and various methods for modeling crack propagation are discussed. Published comparisons between experimental and theoretical results are used to assess the accuracy of these analytical methods.

1. INTRODUCTION

The prestressed concrete reactor vessel (PCRVR) is extensively used in reactor designs having a gas as the heat-transfer medium for the primary loop. Because of the characteristics of this coolant, a large volume is required for the reactor core. A current 1160-MW(e) commercial reactor vessel¹ has a core cavity height and diameter of 14.4 m (47.3 ft) and 9.97 m (32.7 ft), respectively, and an external height and diameter of 27.8 m (91.2 ft) and 29.3 m (96.0 ft). Because it is fabricated by on-site casting techniques utilizing readily available materials, the PCRVR offers distinct advantages in the construction of these large vessels.

The nature of the PCRVR design also offers unique safety advantages. Due to the low tensile strength of concrete, steel prestressing tendons are used to develop compressive stresses in the concrete during normal operating conditions. These tendons represent the primary load-bearing elements in the vessel and are installed in such a manner that they may be inspected and/or replaced at any time during the life of the vessel. Insulation and cooling coils are attached to the steel liner at the inside vessel wall to protect the concrete from exposure to excessive temperatures, and, consequently, the steel load-bearing portions of the vessel are not subjected to the plastic thermal cycling or creep that occurs in high-temperature steel reactor pressure vessels. Reinforcing steel is used to control any localized cracking of the concrete. Also, since the

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concrete provides radiation shielding, the steel liner on the inside vessel wall is the only structural component of the vessel that is exposed to significant radiation. The high degree of redundancy in the prestressing and reinforcing elements and the protection afforded them by the concrete results in an extremely low probability of undetected extensive failure leading to vessel collapse.

From the standpoint of public safety, the most significant aspect of the PCRV is that the complete primary loop (consisting of reactor core, coolant circulators, and steam generators) is contained within a vessel that can exhibit a gradual mode of failure as its ultimate pressure is approached. This response has been demonstrated in numerous model tests of various vessel configurations.^{2,3} In the early reactor designs (mid-1960s), the complete primary loop was contained within a single central cavity of the vessel. The predominant vessel shape was cylindrical with two distinct internal configurations depending upon the location of the steam generators.

In the British Oldbury⁴ and Dungeness B⁵ reactors, the steam generators are located around the periphery of the core (see Fig. 1). Ultimate-load tests of scale models^{6,7} indicated cracking and associated nonlinear behavior at 1.5 to 2 times the operating pressure and an ultimate-load capability in excess of 3 times the operating pressure. The steam generators in the French St. Laurent II and Bugey I reactors⁸ are located directly below and in line with the reactor core (see Fig. 2). An ultimate-load test of a one-fifth-scale model of the Bugey I vessel⁹ indicated cracking at 2.5 to 3 times the operating pressure and an ultimate-load capability in excess of 4 times operating pressure. As in the case of the British designs, the failure was gradual and was the result of excessive leakage due to liner failure. The configuration of the American Fort St. Vrain vessel¹⁰⁻¹² is similar to the French designs discussed above (see Fig. 3). The performance of the vessel was verified by a sequence of tests using a one-fourth-scale model¹³ that simulated various operating conditions including overpressurization to twice the operating pressure.

Fig. 1. Transverse elevation of one of the reactors for Oldbury power station showing the prestressed concrete pressure vessel (Ref. 4).
(1 ft = 0.305 m)

Fig. 2. Comparison of St. Laurent and Bugey I power stations (Ref. 8).

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Fig. 3. General arrangement of the primary reactor components (Ref. 11).

Although the performance of the early PCRV designs discussed above was satisfactory, there are certain difficulties associated with these configurations. The large unsupported span across the head regions in the British design is not desirable, while the French configuration leads to a more complex path for coolant flow. In both designs, the steam-generator systems must be assembled within the vessel cavity, and thus a closely integrated schedule of delivery and installation is required. Access for repair work on these systems after installation is by manhole only. These

problems are avoided in the modern high-temperature gas-cooled reactor (HTGR) vessel¹⁴ by placing the steam generators and circulators in vertical cylindrical cavities within the vessel wall (see Fig. 4). By using removable closure plugs, it is possible to design the circulator-steam generator system so that it can be shop fabricated and tested before installation and removed for repair if necessary. The British Hartlepool

Fig. 4. The major components, system, and fuel enclosed in the HTGR PCRV for the Summit plant (Ref. 1).

reactor¹⁵ was the first reactor to utilize this multicavity vessel concept. A test of one-tenth-scale model of this vessel¹⁶ demonstrated a gradual mode of failure for this configuration similar to that of the previous single-cavity designs.

The model tests have demonstrated the gradual and predictable response of the PCRV to increasing pressure. At the lower pressure levels, the response is linearly elastic with only highly localized nonlinearity or cracking. As pressure is increased, significant nonlinear response develops. This nonlinearity is due to two sources: nonlinearity in the stress-strain relation of concrete and the development of significant zones of concrete cracking. The latter effect becomes more pronounced as the pressure is increased, and as the ultimate pressure is approached, the vessel is reduced to a state that can be characterized as discrete blocks of concrete held together by steel reinforcing and prestressing members. The deformations in the latter phase are due primarily to plastic deformation of the steel members. Failure is generally the result of excessive leakage due to rupture of the liner. Because of the distinctly different mechanisms involved, it is convenient to group the discussion of PCRV analysis into linear elastic analysis, nonlinear analysis, and time-dependent effects of shrinkage and creep in the concrete.

Although previous experience has demonstrated that safe and reliable prestressed concrete vessels can be designed and constructed, it should be recognized that (1) the PCRV is a more complex structure than a corresponding steel vessel, (2) the behavior of concrete is complex and subject to many factors such as changes in temperature and moisture content, and (3) the totality of design experience for prestressed concrete vessels extends for only two decades. The performance of the vessels discussed above was the result of careful design based upon extensive use of analytical and experimental results. The purpose of this report is to review the analytical methods available to the designer of modern PCRV configurations and to assess their capability.

2. ANALYTICAL REPRESENTATION OF A PCRV

2.1 The Continuum Model

At the macroscopic level, a PCRV is a quasi-homogeneous structure composed of mortar, aggregate, and steel. At a more refined level of gradation, the mortar may be viewed as a porous solid framework with voids containing gas and liquid. The liquid is held within the pores of the solid by intermolecular forces. Obviously, this level of modeling is impractical, and the first step in simplification is to view the PCRV as a three-dimensional continuum composed of two components — concrete and steel. The equilibrium configuration of a continuum is uniquely defined by three conditions: (1) the deformations must be continuous and single valued (compatibility conditions), (2) the state of stress must satisfy conditions of equilibrium, and (3) the stresses and strains must satisfy appropriate constitutive relations for the material.¹⁷ The compatibility of deformations and equilibrium conditions are physical relations, while material behavior is described by the constitutive relations.

In a modern finite-element code, the three requirements of compatibility, equilibrium, and constitutive relations are considered in functionally independent portions of the program. This modular structure, combined with the ability to handle problems having complex geometry, has led to the wide acceptance of the method and has stimulated a rapid evolution in engineering analysis capabilities. It is anticipated that the method will play the dominant role in future PCRV analysis; consequently, it is pertinent to discuss such analyses in this context.

2.2 The Finite-Element Formulation

The finite-element method is a procedure within the broad class of numerical techniques called methods of trial functions.¹⁸ A functional form of the solution is assumed that contains unknown parameters — usually constants — which are determined so as to achieve an approximate solution to the governing equations. The unique feature of the finite-element method is that the trial functions are defined only for finite subregions,

or elements, of the domain. The principal restrictions placed on the selection of trial functions are associated with the continuity requirements of the minimization principle. There are a number of general texts on the finite-element method, two of which are Refs. 19 and 20.

The domain is subdivided into discrete elements by specifying spatial points called nodes that define the boundary surface of each element (see Fig. 5). In the more common implementation, the trial functions are continuous displacement fields within each element. Thus, if the functional form is such that compatibility of deformations is not violated along adjacent element boundaries, the compatibility relations of the continuum are automatically satisfied. Connectivity is accomplished by specifying that adjacent elements share common nodes. It is convenient, and the usual practice, to use the value of the components of the displacement vector at the nodal points as the unknown parameters in the trial functions associated with each element. These values are referred to as the degrees of freedom of the structural model. Thus, the trial functions have the form

$$\delta(x, y, z) = \sum_{i=1}^n \{N_i(x, y, z)\}^T \{\delta_i\}, \quad (1)$$

where n is the number of nodal points associated with the element, $\delta(x,$

Fig. 5. Subdivision of the region into finite elements.

y, z) is the displacement vector, δ_i is the value of displacement field at the i th node point, and $N_i(x, y, z)$ is a function that is unity at the i th node and zero at all other nodes (see Fig. 6). The functions N_i are the interpolating functions for the displacement field and are known collectively as the shape functions for the element. Note that $\{ \}^T$ indicates the matrix transpose and $\vec{\delta}$ is a vector whose components are the values of the displacements in the coordinate directions (i.e., δ_x , δ_y , and δ_z).

Fig. 6. Shape functions for quadratic elements of "Serendipity" and "Lagrange" functions (Ref. 19).

Families of element formulations have been developed for specific problem classes (see Fig. 7). Three-dimensional tetrahedron and hexahedron elements are used for solid bodies with generally three degrees of freedom per node. In planar situations (plane stress, plane strain, or axisymmetric problems), triangular and quadrilateral elements are used with two degrees of freedom per node. Shell and plate-bending problems utilize special forms for the shape functions and rotational degrees of freedom to better approximate the deformations associated with these structures. The important point is that the element formulations provide displacement trial functions that satisfy the compatibility requirements for the continuum and contain the nodal displacements as unknown parameters. The solution to the problem is obtained by determining the unknown parameters so as to satisfy an appropriate minimization criteria.

Fig. 7. Types of finite elements (Ref. 20).

2.3 The Elastic Analysis

Most structural materials behave elastically within a certain stress range, and for the majority of these materials, the relation between the stresses and strains can be approximated with sufficient accuracy by a linear relationship, that is,

$$\{\sigma\} - \{\sigma_0\} = [D](\{\epsilon\} - \{\epsilon_0\}) , \quad (2)$$

where $\{\sigma\}$ is a vector containing the components of stress and $\{\epsilon\}$ is the vector of the components of strain. The matrix $[D]$ is called the material matrix and depends, in the case of homogeneous isotropic materials, on the modulus of elasticity E and Poisson's ratio μ of the material. The vectors $\{\sigma_0\}$ and $\{\epsilon_0\}$ are the initial components of stress and strain, respectively.

The high-strength concrete typically used in PCRV construction responds elastically to applied mechanical loads at stress levels up to approximately 60% of the ultimate strength.²¹ Within the elastic range, the initial strains are recoverable on unloading, and the structural response remains unchanged under repeated application of the load. Hence, it is standard practice to design the PCRV to remain elastic under credible combinations of operating, accident, and environmental loads.²² Some highly localized regions of inelastic stresses may be tolerated if adequate reinforcement is used to control cracking. The effect of these inelastic stresses on an elastic analysis is negligible, except in the immediate neighborhood of the region involved. Thus, the elastic analysis serves as the primary analysis of the PCRV for credible design-base conditions.

The principle of stationary potential energy²³ provides a variational statement for determining the unknown parameters of the trial functions in the analysis of a linearly elastic structure. This principle states that equilibrium corresponds to the position at which the total potential energy has a stationary value. Consider an element having n degrees of freedom that is in equilibrium under the action of forces $\{F_i\}$ applied at the nodes. Let $\{\delta_i\}$ be the equilibrium displacements of the nodes and $\{\sigma\}$ be the corresponding stresses. The necessary condition for stationary

potential energy is

$$\int \{\Delta \epsilon\}^T \{\sigma\} dv - \sum_{i=1}^n \{\Delta \delta_i\}^T \{F_i\} = 0, \quad (3)$$

where $\{\Delta \delta_i\}$ is an arbitrary infinitesimal displacement from the equilibrium position, $\{\Delta \epsilon\}$ is the corresponding increment in the strains, and the integration is over the element volume. The strains are related to the displacements by

$$\{\epsilon\} = \sum_{i=1}^n [B_i] \{\delta_i\}, \quad (4)$$

where the matrix $[B_i]$ is obtained from differential operations on the element shape functions N_i . Utilizing the stress-strain relations given by Eq. (2) and the strain-displacement relation above, Eq. (3) becomes

$$\{F_i\} = \sum_{j=1}^n [k_{ij}] \{\delta_j\} + \int [B_i]^T \{\sigma_0\} dv - \int [B_i]^T [D] \{\epsilon_0\} dv, \quad (5)$$

where

$$[k_{ij}] = \int [B_i]^T [D] [B_j] dv \quad (6)$$

is called the element stiffness matrix and the integration is over the volume of the element. This equation states the relationship between the equilibrium values of the degrees of freedom and the corresponding nodal forces for the element.

Requiring that the external forces at each node equal the resultant of the forces exerted by the node on adjacent elements yields a system of N linear algebraic equations

$$[K]\{\delta\} = \{R\}, \quad (7)$$

where N is the total number of degrees of freedom in the complete structural model, $[K]$ is an $N \times N$ matrix called the global stiffness matrix, and $\{R\}$ is the nodal force vector. The nodal force vector $\{R\}$ is the sum of the external forces $\{F_i\}_{ex}$ and the pseudonodal forces due to the initial strains $\{F_i\}_{\epsilon_0}$ and the initial stresses $\{F_i\}_{\sigma_0}$, where

$$\{F_i\}_{\epsilon_0} = \int [B_i]^T [D] \{\epsilon_0\} dv \quad (8)$$

and

$$\{F_i\}_{\sigma_0} = - \int [B_i]^T \{\sigma_0\} dv. \quad (9)$$

The integration in the above equations extends over all elements containing the i th degree of freedom.

2.4 Sparsity of the Stiffness Matrix and Solution Algorithms

Suppose the equilibrium values of the degrees of freedom of the structure are

$$\delta_l = \begin{cases} 0 & \text{if } l \neq j \\ 1 & \text{if } l = j \end{cases}; \quad (10)$$

then

$$R_i = \sum_{l=1}^n K_{il} \delta_l \quad (11)$$

is the value of the i th generalized nodal force required to maintain equilibrium for a unity value of the j th degree of freedom, all other degrees of freedom being zero. Only those elements that contain the j th degree of freedom will be deformed by this displacement, and thus the force R_i will be zero if the i th and j th degrees of freedom do not share a common element. Consequently, all K_{ij} terms for which the i th and j th degrees of freedom do

not share a common element will be zero. It follows that the global stiffness matrix contains relatively few nonzero terms, and by a suitable numbering of the degrees of freedom, these terms can be restricted to a narrow band containing the main diagonal, as shown in Fig. 8.

The fact that the matrix is sparse and banded is the key to developing efficient algorithms that make it possible to obtain a solution to the large number of algebraic equations involved in a typical finite-element analysis. The maximum bandwidth in a particular problem is determined by the maximum difference of the subscripts assigned to the degrees of freedom within any of the elements of the structure.

The early single-cavity PCRV configurations could be modeled to a reasonable degree of accuracy as axisymmetric structures and hence required only a two-dimensional analysis. For such an analysis, the bandwidth of the global stiffness matrix is related to twice the number of nodes along a line through the structure. The multicavity PCRV is a truly three-dimensional structure of relatively complex geometry. For this type of problem, the bandwidth is related to three times the number of nodes on a plane

Fig. 8. Banded global stiffness matrix achieved by suitable numbering of degrees of freedom.

through the structure. Thus, in addition to the difficulty of handling the larger number of equations involved in a three-dimensional analysis, there is the problem that the bandwidth is considerably larger. The solution algorithms in virtually all existing finite-element programs utilize a direct method for which the number of operations required is proportional to the product of the number of equations and the square of the bandwidth. Thus, analyses of three-dimensional structures with the geometric characteristics of the multicavity PCRV are several orders of magnitude more expensive than analyses of plane or axisymmetric structures of equivalent detail.

The bandwidth problem can be avoided completely by utilizing an iterative-solution procedure based on the Gauss-Seidel method.^{24,25} Since the procedure requires only successive matrix multiplications and does not alter the original stiffness matrix, it is possible to design the algorithm such that only the nonzero elements of the stiffness matrix need be stored or manipulated. Unfortunately, slow convergence is often characteristic of the iterative-solution method, and known procedures for accelerating the convergence are not uniformly effective and may produce divergence. Until better acceleration techniques are developed for the iterative-solution algorithms, the most efficient solution procedures will be based on direct methods.²⁶

The method for the direct solution of banded equations commonly implemented in early codes is based upon the "block diagonalization" method developed by Clough and Wilson.²⁷ In this method, the diagonal band is partitioned into N square submatrices lying along the diagonal and triangular off-diagonal submatrices. By a procedure involving inversion of the diagonal submatrices and multiplication by the off-diagonal matrices, N groups of equations are formed for which the unknowns in the last group are obtained directly and the remaining unknowns are obtained by back-substitution. If m is the bandwidth, the process requires N inversions of $(m \times m)$ matrices and in-core storage of the equivalent of two $(m \times m)$ matrices.

More efficient algorithms based on triangular decomposition²⁸ have been implemented in recent codes.²⁹ In the later algorithms, the matrix

is decomposed into an upper and lower triangular form by a process equivalent to Gaussian elimination, and the solution is obtained by back-substitution, thus avoiding the matrix inversions required in the block diagonalization method. A technique called skylining is used to take partial advantage of sparsity within the bandwidth and can be implemented in such a manner as to provide a better balance between input and output. The latter comment is particularly true for the frontal-solution technique developed by Irons.³⁰ A more detailed survey and discussion of the various solution techniques is given in Refs. 31 and 32.

2.5 Special Element Formulations

Because of the strongly adverse effect that the matrix bandwidth has on solution cost, it becomes critical to utilize an efficient discretization for modeling three-dimensional problems. To achieve an adequate model for the structure with a minimum number of degrees of freedom and bandwidth, two characteristics are necessary: (1) the elements must have sufficient flexibility in their geometric configurations so that the geometry of the structure can be modeled with relatively few nodal points, and (2) the element shape functions must be of sufficiently high order to model the existing strain gradients. The isoparametric element formulation provides a relatively simple means of achieving these requirements by utilizing a normalized-basis coordinate system to express the functional representation of the shape functions, that is,

$$\vec{\delta}(\xi, \eta, \zeta) = \sum_{i=1}^m \{N_i(\xi, \eta, \zeta)\}^T \{\vec{\delta}_i\}, \quad (12)$$

where ξ, η, ζ are variables with a range of -1 to $+1$ (see Fig. 9); $\vec{\delta}_i$ is the value of the displacement vector $\vec{\delta}$ at node i ; and m is the number of nodes associated with the element. These shape functions are also used to describe the element geometry in the global coordinate system in terms

Fig. 9. The isoparametric mapping.

of the coordinates of the node points of the element, that is,

$$\vec{P}(\xi, \eta, \zeta) = \sum_{i=1}^n \{N_i(\xi, \eta, \zeta)\}^T \{\vec{P}_i\}, \quad (13)$$

where $\vec{P}$ is the position vector of a point in the global system and $\vec{P}_i$ is the position vector of the i th node of the element. If bilinear interpolation is used in the definition of the shape functions, curved contours of second degree are produced in describing the element geometry. Thus, the isoparametric representation provides a systematic procedure for developing appropriate high-order displacement trial functions for elements having complex geometry. Most modern codes contain elements that are based on this formulation.

The use of the high-order isoparametric element provides a means of representing the geometrical outline of the PCRV and simultaneously modeling the strain gradients with relatively few elements; however, because of the high cost of formulating the element stiffnesses and the large number of degrees of freedom associated with these elements, it is desirable to utilize them efficiently. Thus, it is necessary to have available the special elements required to make transitions in the element mesh. These

elements are of two types: hierarchical and degenerate. The hierarchical elements are those for which the number of nodes and associated order of the shape functions may be different along different sides of the element (see Fig. 10). With the use of these elements, additional degrees of freedom may be introduced where higher-order strain gradients are expected. The degenerate elements are those for which two adjacent corner nodes are coalesced, as shown in Fig. 10. These elements are required at the center-line apex of three-dimensional cylindrical models and at certain regions of mesh transition. The use of both hierarchical and degenerate elements for a typical multicavity PCRV model is discussed in Ref. 33, and an example is shown in Fig. 11. Some of the special considerations necessary in the formulation of the stiffnesses for such elements are discussed in Ref. 34.

Fig. 10. Typical elements: (a) hierarchical, (b) "degenerate" (Ref. 33).

Fig. 11. Three-dimensional analysis of a pressure vessel indicating advantages in idealization provided by hierarchical and "degenerate" elements (Ref. 33).

Because only relatively few high-order elements are needed in an analysis, the idealization of the reinforcing bars and membrane liners requires special treatment. Limiting these elements to coincide with boundaries of the solid elements places an intolerable restriction on the mesh layout. A less restricted formulation is presented in Ref. 33 in which the bar and membrane elements are embedded within the solid elements but restricted to lines of constant ξ , η , or ζ in the basis-coordinate system (see Fig. 12). By assuming displacement compatibility between the solid elements and the embedded members, one can obtain the combined stiffness contributions. The major problem associated with this formulation is that, in general, a bar or membrane element is located by its position in the global coordinate system, and the corresponding position in the basis coordinate system may not coincide with positions of constant ξ , η , or ζ . This problem could be overcome by a suitable generalization.

Fig. 12. Typical bar and membrane elements in curvilinear directions within a solid element (Ref. 33).

2.6 Demonstrated Results

The theoretical results for one of the earliest published comparisons of analytical and experimental strain distributions in a complex three-dimensional body³⁵ were obtained with a computer code³⁶ that utilizes an iterative-solution method and the constant-strain tetrahedral solid element. A 30° symmetry sector of the top half of a concrete pressure vessel was modeled using 12,231 tetrahedral elements, as shown in Fig. 13. The solution of the 8034 algebraic equations by iteration required several hours of IBM 360/75 computer time; however, the use of a direct method for this system of equations, whose bandwidth was approximately 500, would have been virtually impossible. The large number of elements were required in the model because the constant-strain tetrahedron is rather inefficient in modeling strain distributions. Of the 12 degrees of freedom available in this element, 6 must be used to account for rigid-body motion

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Fig. 13. Finite-element analysis of prestressed concrete reactor vessel (Ref. 35).

of the element; thus, only half are available for approximating the strain distribution. The excessive computing time was the result of the slow convergence of the solution for successive iterations. Although this study demonstrated that relatively accurate results can be obtained using the constant-strain tetrahedral element and an iterative-solution algorithm, the effort involved was considerable.

With the high-order isoparametric elements and more efficient direct-solution algorithms, it became possible to develop reasonably realistic models of complex three-dimensional bodies that are simultaneously within the capability of these algorithms. This was demonstrated by a comparison between analytical and experimental results for a one-tenth-scale model of the Hartlepool reactor.³⁷ Although the geometry of this vessel was more complex than that of the vessel previously discussed, satisfactory accuracy was obtained using a model composed of 41 elements and 978 degrees of freedom (see Fig. 14). The frontal-solution technique was used, and

Fig. 14. Finite element mesh (Ref. 37).

the solution was obtained with only 10 min computing time on the IBM 360/75. A more complex model having 177 elements and 3519 degrees of freedom was also used and required 55 min computing time. The differences between the two solutions were not significant, and both agreed relatively well with the experimental values obtained from the test. A histogram indicates that the major portion of the variations between experimental and analytical results were within -30 and $+40$ $\mu\text{m}/\text{m}$ ($\mu\text{in.}/\text{in.}$) of strain. The standard deviation of the variation was 17.5 $\mu\text{m}/\text{m}$ ($\mu\text{in.}/\text{in.}$). This agrees favorably with the 25 - $\mu\text{m}/\text{m}$ ($\mu\text{in.}/\text{in.}$) standard deviation for similar results obtained from a test of the Wylfa prestressed concrete pressure vessel³⁸ and with the results given by Bremer.³⁹ Thus, it appears that agreement within this order of magnitude is as good as can be expected for currently available experimental instrumentation. A rather detailed discussion of the problems associated with the assessment of experimental results for PCRV models given in Ref. 40 tends to confirm this conclusion.

3. TIME-DEPENDENT BEHAVIOR OF CONCRETE

3.1 Effects of Time-Dependent Deformations on a PCRV

The time-dependent deformations (creep) of the concrete in a PCRV under normal operating conditions are significant from a design consideration. The amount of deformation may be several times the instantaneous elastic deformations and must be considered to ensure proper equipment alignment and the maintenance of minimum tolerances and prestressing forces. They also produce increases in the compressive stress in the vessel liner that may result in liner buckling. Moreover, differences in the creep properties across the vessel wall, caused by differences in temperature, hydration, moisture migration, etc., may lead to significant redistribution of the stresses. If a vessel is subjected to extended periods of prestress without internal pressure, the stress relaxation may result in tensile cracking of concrete along the inside vessel wall upon subsequent pressurization, thus jeopardizing the integrity of the liner anchor studs. For these reasons, it is pertinent to consider the

nature of the time-dependent response of concrete and the analytical methods available for evaluating this effect.

3.2 Basic Characteristics of Concrete Creep

Hardened portland cement paste is the binder for the concrete aggregate, and the properties of the paste and its bond with the aggregate determine: (1) the manner in which the concrete responds to changes in temperature and moisture, (2) the characteristics of its nonlinear response, and (3) the nature of its failure. The interior structure of the aggregate has little effect on these properties, provided sound aggregates are used. The paste is an ultrafine gel-type structure consisting principally of calcium silicate hydrate formed by hydration of the cement. This hydration process is rapid during the first few days and continues at a decreasing rate thereafter, but remains significant throughout the vessel life. The gradual change in the properties of concrete that occurs during hydration is called aging, and aging introduces one of the major complications into the analysis of concrete structures.

The hardened cement paste is a strongly hydrophilic, porous material with a typical porosity of 0.40 to 0.55 and normally contains a substantial amount of evaporable water.^{41,42} Much of the evaporable water is held within pores several molecules thick by strong surface forces (see Fig. 15). The internal surface of the pores is enormous — roughly 500 m²/cm³ (12.7×10^6 in.²/in.³) of cement paste. Concrete has a relatively low degree of chemical stability, and its microstructure reacts to any changes in temperature, water content, or stress. The principal mechanism responsible for the time-dependent deformation (creep) of concrete is thought to be the migration of molecules or ions of the solid particles of cement paste (perhaps Ca ions) from the loaded areas to the unloaded areas, as shown in Fig. 16. The presence of moisture facilitates this migration, especially when the water within the pores of the microstructure is moving as a result of differences in relative humidity or temperature. The loss of evaporable water within the pores also produces substantial capillary tensions and surface tensions on the walls of the pores, resulting in volumetric shrinkage.

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Fig. 15. Idealized schematic of microstructure of hardened portland cement paste; a = free adsorbed water molecules, Ca = calcium ions (crosses), c = remaining anhydrous cement, ch = calcium hydroxide, d = hindered adsorbed water molecules, f = fracture surface, g = cement gel (lines represent sheets of calcium silicate hydrates), i = interlayer water, m = capillary miniscus, o = original surface of anhydrous cement grains, p = capillary pore, v = water vapor and air, w = capillary water (liquid). Water molecules and calcium ions are not shown everywhere they exist. Width of picture $\approx 30 \mu\text{m}$ (1180 $\mu\text{in.}$). Spacing of sheets ($\sim 30 \text{ \AA}$) relative to length of sheets is magnified about 100 times.

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Fig. 16. Idealized drawing of the creep mechanism in cement paste.

3.3 The Superposition Principle

Attempts to relate creep to processes at the microstructural level are plagued by the many and complex interactions; consequently, the earliest and most widely used methods for creep analysis are based upon a phenomenological approach. In this approach, the creep response is obtained by performing tests on concrete samples under representative environmental conditions. To limit the number of tests, it is important to seek relations between the various variables and other simplifications.

An important simplifying assumption commonly employed is that the creep response due to an increment of stress is independent of any previous stress increments and may be superimposed linearly on the response due to the previous increments. This assumption is called the principle of superposition, and without such an assumption, it would be necessary to conduct tests for each stress history of every element in a finite-element analysis. If the validity of this principle is accepted, the creep law takes the form

$$\epsilon(t) = J(t, t')\sigma, \quad (14)$$

where ϵ is the strain caused by the stress σ and $J(t, t')$ is the function (called the creep compliance function) that yields the strain at time t caused by a constant unit stress applied at time t' (see Fig. 17), both t and t' being measured from the time of casting. The fact that concrete undergoes a significant aging effect requires that J depend on two separate variables (t and t') rather than on the one variable $(t - t')$, as is normal in classical linear viscoelasticity. Note that $1/J(t, t)$ equals the instantaneous elastic modulus $E(t)$. The generalization of Eq. (14) to continuously varying stress, according to the principle of superposition, yields the relation

$$\epsilon(t) = \int_0^t J(t, t') d\sigma(t'), \quad (15)$$

where $d\sigma(t')$ is an infinitesimal increment in stress occurring at time t' .

Fig. 17. Creep test and creep compliance function.

Test results obtained by the U.S. Bureau of Reclamation⁴³ for mass concrete show fairly close agreement with those predicted by the principle of superposition. These and later results indicate that under constant temperature and moisture the principle of superposition yields reasonably accurate results within a stress range approximately the same as that for which linear elasticity is valid.⁴⁴

With the assumption that Poisson's ratio is independent of time, the generalization of the uniaxial creep law to the case of constant multi-axial stress follows directly and takes the form of the corresponding law for linear elasticity,⁴⁵ that is,

$$\begin{aligned}\epsilon_x &= J(t, t')\sigma_x - w(t, t') [\sigma_y + \sigma_z] , \\ \epsilon_y &= J(t, t')\sigma_y - w(t, t') [\sigma_x + \sigma_z] , \\ \epsilon_z &= J(t, t')\sigma_z - w(t, t') [\sigma_x + \sigma_y] ,\end{aligned}\tag{16}$$

where

$$w(t, t') = \mu J(t', t') + \mu_c [J(t, t') - J(t', t')]\tag{17}$$

and μ and μ_c are the instantaneous and creep Poisson's ratios respectively. The experimental results given in Refs. 45 and 46 indicate that

the assumption of a constant-creep Poisson's ratio is reasonable, and the value of this constant is approximately 0.15 to 0.18.

3.4 Integral-Type Formulations

Structural analysis based on the stress-strain law given by Eq. (15) may be performed by the following method. The total time interval is subdivided at discrete times t_r ($r = 1, 2, \dots$) into time steps Δt , and the integral is approximated by a finite sum, that is,

$$\epsilon_r = \sum J(t_r, t_{s-1/2}) \Delta \sigma_s \quad (18)$$

$$\Delta \sigma_s = \sigma(t_s) - \sigma(t_{s-1}) ,$$

where subscripts r and s refer to times t_r and t_s , and $t_{s-1/2}$ is an average time within the interval (t_{s-1}, t_s) . Using this approximation, it is possible to formulate the incremental stress-strain law as follows:^{4,2}

$$\Delta \epsilon_r = \frac{\Delta \sigma_r}{E_r''} + \Delta \epsilon_r'' \quad (19)$$

with

$$\frac{1}{E_r''} = J(t_r, t_{r-1/2})$$

$$\Delta \epsilon_r'' = \sum_{s=1}^{r-1} [J(t_r, t_{s-1/2}) - J(t_{r-1}, t_{s-1/2})] \Delta \sigma_s , \quad (20)$$

where E_r'' is a pseudoelastic modulus and $\Delta \epsilon_r''$ is the inelastic strain increment due to creep. Obviously, both E_r'' and $\Delta \epsilon_r''$ can be calculated on the basis of previous stress increments prior to calculating $\Delta \epsilon_r$ and $\Delta \sigma_r$. Therefore, the analysis may be performed as a sequence of elastic analyses in individual time steps Δt in which the modulus of elasticity is E_r'' and the accumulated creep strains $\Delta \epsilon_r''$ are incorporated as initial strains using Eq. (9). This procedure is generally applicable to any form of the creep function $J(t, t')$.

The form of the creep function that enjoys widest international acceptance is

$$J(t, t') = \frac{1}{E_0} + \phi(t') F(t - t') , \quad (21)$$

where E_0 is a constant representing a typical value of the elastic modulus and ϕ and F are continuous, positive, monotonic functions of one variable. This form is recommended by the Comité Européen de Béton-Fédération Internationale de la Précontrainte (CEB-FIP)⁴⁷ and the American Concrete Institute⁴⁸. Excellent fits were obtained by Bazant⁴² using a double-power law for the functions ϕ and F , that is,

$$J(t, t') = \frac{1}{E_0} + \frac{\phi_1}{E_0} (t')^{-m} (t - t')^n , \quad (22)$$

where ϕ_1 , m , and n are nondimensional constants. Typically m and n are approximately 1/3 and 1/8, respectively.

Simple, but relatively accurate, approximations are also obtained⁴⁹ by taking $J(t, t')$ to be the sum of an age-dependent instantaneous elastic part plus a logarithmic function of time after loading multiplied by a function that accounts for the effect of aging. This form has the advantage that the coefficients in the approximation can be obtained directly from semilog plots of creep strain vs time.

3.5 Rate-Type Formulations

A difficulty associated with the analysis procedure just discussed is that at each time step Eq. (19) requires the evaluations of sums over all previous steps for each stress component of every element in the model. This results in extensive storage and computational requirements. The problem can be avoided by approximating the creep function by a series of real exponentials (Dirichlet series),⁵⁰ that is,

$$J(t, t') = \frac{1}{E(t')} + \sum_{i=1}^n \frac{1}{\hat{E}_i(t')} \left[1 - e^{-(t-t')/\tau_i} \right] , \quad (23)$$

in which τ_i are constants called retardation times and $\hat{E}_i$ are coefficients. When this function is introduced into the superposition integral [Eq. (15)], the integrand degenerates into the product of a function of t' and a function of t . The latter function does not involve the variable of integration and can be extracted from the integral, leaving only an integration of functions that are independent of t . Thus, at each new time step, it is only necessary to compute the change in value of the integral from the last time step rather than from the time of initial loading, as is required in the general case.

A completely stable incremental analysis can be obtained by numerically approximating the integral at discrete times $t_0, t_1, \dots, t_N$ and using the assumption that $d\sigma(t)/dt$ and $\hat{E}_i(t')$ are constant within each time interval.⁵⁰ The incremental stress-strain law becomes

$$\Delta \epsilon_r = \frac{\Delta \sigma_r}{E_r''} + \Delta \epsilon_r'' , \quad (24)$$

in which

$$\frac{1}{E_r''} = \frac{1}{E_{r-1/2}} + \sum_{i=1}^n \frac{1 - \left[1 - e^{-\Delta t_r / \tau_i} \right] \tau_i / \Delta t_r}{(\hat{E}_i)_{r-1/2}}$$

and

$$\Delta \epsilon_r'' = \sum_{i=1}^n \left[1 - e^{-\Delta t_r / \tau_i} \right] (\epsilon_i^*)_{r-1} . \quad (25)$$

The effect of prior stress history is contained in the variable ϵ_i^* , which is defined by the recurrence formula

$$(\epsilon_i^*)_r = \frac{\Delta \sigma_r \left[1 - e^{-\Delta t_r / \tau_i} \right] \tau_i / \Delta t_r}{(E_i)_{r-1/2}} + (\epsilon_i^*)_{r-1} e^{-\Delta t_r / \tau_i} . \quad (26)$$

Thus, the difficulty of computing sums involving the complete stress history at each time step is avoided.

It can be shown⁵⁰ that the Dirichlet series is the solution to the system of differential equations

$$\begin{aligned}\dot{\epsilon} &= \dot{\sigma}/E + \sum_{i=1}^n \dot{\epsilon}_i, \\ \dot{\epsilon}_i &= \sigma_i/\eta_i, \\ \dot{\sigma} &= \dot{\sigma}_i + \hat{E}_i \dot{\epsilon}_i,\end{aligned}\tag{27}$$

when a unit step stress $\sigma(t)$ is applied at time t' . This system of equations, which corresponds to the physical system shown in Fig. 18, is called the Kelvin chain model. The response of a typical element of the Kelvin chain in series with the spring element to a stress pulse is shown in Fig. 19. Since the formulation defined by Eqs. (27) states the relations between the rates of stress and strain, it is referred to as a rate-type formulation.

Although the rate-type formulation offers computational advantages, the identification of the material parameters from experimental data is difficult. This problem was discussed by Bazant in Ref. 50, and an algorithm was presented that was successfully used to fit various published data. In a later publication,⁵¹ an improved algorithm was presented along with its FORTRAN program listing.

Fig. 18. Kelvin chain model.

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Fig. 19. Typical response of a Kelvin element in series with a spring element.

An alternate rate-type form is obtained if relaxation functions, rather than creep compliance functions, are expanded in a Dirichlet series.⁵² Mathematically, the relaxation function $E(t, t')$ is defined by

$$\sigma(t) = E_r(t, t')\epsilon, \quad (28)$$

where $\sigma(t)$ is the stress as a function of time due to the application of a constant strain ϵ at time t' . The function $E_r(t, t')$ yields the stress at time t caused by a constant unit strain applied at time t' (see Fig. 20). By direct substitution, it can be shown that the Dirichlet series

$$E_r(t, t') = \sum_{i=1}^n E_i(t') e^{-(t-t')/\tau_i} \quad (29)$$

is the solution to the system of differential equations

$$\begin{aligned} \sigma &= \sum_{i=1}^n \sigma_i, \\ \dot{\epsilon} &= \left[\dot{\sigma}_i + \sigma_i/\tau_i \right] / E_i(t'), \end{aligned} \quad (30)$$

Fig. 20. Relaxation test and the relaxation function.

when a unit strain $\epsilon(t)$ is applied at time t' . This system of equations corresponds to the physical system shown in Fig. 21 and is called the Maxwell chain model. The response of a typical element of the Maxwell chain in parallel with the spring element to a stress pulse is shown in Fig. 22. One of the major differences between the Maxwell and Kelvin chain models is the part played by the single spring element. In the Kelvin chain model, all the instantaneous response occurs in this element, while in the Maxwell model, there would be no time-dependent recovery upon unloading without the single spring element.

One difficulty associated with the Maxwell model is that the time-dependent properties are usually characterized by the creep function $J(t, t')$, while the determination of the parameters requires knowledge of the relaxation function. If the superposition principle is assumed to be valid, the relaxation function is uniquely defined in terms of the creep function as the solution to the integral equation

$$h(t - t') = \int_{t'}^t J(t, \lambda) \frac{\partial E_r(\lambda, t')}{\partial \lambda} d\lambda, \quad (31)$$

where $h(t - t')$ is the unit step function. The listing of a computer program to numerically determine the relaxation values corresponding to a general creep function is contained in Ref. 53 along with an algorithm for computing the material parameters associated with the Maxwell chain model.

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Fig. 21. Maxwell chain model.

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Fig. 22. Response of a Maxwell element in parallel with a spring element.

An incremental stress-strain law corresponding to Eq. (24) but based on the Maxwell chain is contained in Ref. 52. As with the Kelvin model, the stress history in the incremental law is defined by a recurrence relation, thus eliminating the need to sum the complete stress history at each time step.

It was found that equivalent accuracy was obtained for the Maxwell chain with simpler expressions than those required for the Kelvin chain.

Additional advantages arise for the Maxwell model when the effects of temperature and humidity changes are included.

3.6 The Effect of Temperature Changes

As a structural material, concrete has a relatively low degree of chemical stability and consequently its properties are strongly affected by changes in temperature. The process of cement hydration that gives rise to aging is a chemical process that is accelerated by increases in temperature. The migration of molecules or ions of solid particles that produces creep deformation is also accelerated by increases in temperature. Thus, temperature directly influences the deformation rate and the rate of aging of concrete.

A type of temperature dependence that is particularly amenable to analytical description is applicable to the class of materials that are referred to as being thermo-rheologically simple.⁵⁴ With materials of this class, changes in temperature cause the creep or relaxation functions to undergo a simple horizontal shift when plotted with $\log(t - t')$ as the abscissa (see Fig. 23). Thus, if

$$C_T(t, t') = J(t, t') - J(t', t') \quad (32)$$

is the specific creep at temperature T and $C_{T_0}(t, t')$ is the creep at temperature T_0 , then

$$C_T[\ln(t - t')] = C_{T_0}[\ln(t - t') + \phi(T)] \quad (33)$$

or

$$C_T(t - t') = C_{T_0} \left[(t - t_0) e^{\phi(T)} \right], \quad (34)$$

where $\phi(T)$ is called the temperature-shift function.

Using published data, Sackman⁵⁵ found that concrete behavior could be modeled with sufficient accuracy with this model. Mukaddam and Bresler⁵⁶ generalized the model to include the effect of temperature on concrete

Fig. 23. Time-temperature shift principle.

aging by introducing a second shift function, $\psi(g)$, where g is an equivalent age as a function of temperature. This concept was applied to the Kelvin chain model to yield a creep law of the form

$$C_T(t - t') = \sum_{i=1}^n c_i e^{-\lambda_i (t-t') \phi(T) \psi(g)}, \quad (35)$$

where

$$\psi(g) = \alpha_0 + \alpha_1 e^{-\alpha_2 g} \quad (36)$$

and

$$g = \tau_0 + \sum_{i=1}^n \Delta \tau_i e^{-\Lambda(T_i - T_0)}. \quad (37)$$

This form of the law was used to fit the Wylfa reactor data,⁵⁷ and a satisfactory fit was obtained.

Incorporating the temperature effect into the creep law corresponding to the Maxwell model is rather simple. Equation (27) is rewritten as

$$\sigma = \sum \sigma_i$$

$$\dot{\epsilon} = \frac{\dot{\sigma}_i}{E_i(t_e)} + \frac{\sigma_i}{\eta_i(t_e)}, \quad (38)$$

where η_i is the age-dependent viscosity associated with the i th Maxwell unit, which equals $\tau_i E_i$ at constant temperature T . Since creep is a thermally activated process, it is known from physics that η_i should depend on temperature according to the Arrhenius equation,

$$\frac{1}{\eta_i} = \frac{1}{\eta_0} \exp \left[\frac{U_i}{R} \left(\frac{1}{T_0} - \frac{1}{T} \right) \right], \quad (39)$$

where T_0 is the reference temperature, η_0 is the value of η at T_0 , R is the universal gas constant, and U_i is the activation energy of the i th Maxwell unit. The effect of temperature on aging is represented by making E_i and η_i dependent on T_e rather than on t , where Δt_e is an equivalent hydration period that yields the same degree of hydration at temperature T as occurs during a period Δt at temperature T_0 . This equivalent time is given by

$$\Delta t_e = \beta_T \Delta t \quad (40)$$

with

$$\beta_T = \exp \left[\frac{U_h}{R} \left(\frac{1}{T_0} - \frac{1}{T} \right) \right], \quad (41)$$

in which U_h equals the activation energy for hydration. The method of identifying U_i and U_h from test data, given in Ref. 58, is related to a time-temperature shift of the relaxation curves. It has been demonstrated that various published data on elevated-temperature creep are properly fitted by the concept.

Although the creep laws discussed above seem to provide adequate fits of the experimental data, most have not been validated for states of stress or environmental conditions simulating those existing under PCRV operating conditions. In particular, even if the theories provide adequate fits of isothermal creep data, it does not necessarily follow that the implied mathematical generalization to time-varying temperature conditions is valid. In particular, it is known that under high rates of change in temperature, a temporary acceleration of creep, called transitional thermal creep, occurs. Although this effect may not be of interest under normal PCRV operations, it may be significant under certain accident situations. Some test data are available in the literature, and a theory based upon thermodynamic principles has been developed.⁴² In the simplest form, this effect can be included by replacing $\eta_i(t_e)$ in Eq. (35) by

$$\eta_i(t_e)[1 + k\dot{T}] , \quad (42)$$

where k is a constant to be found from test data. In general, additional research to experimentally validate the computational procedures under time-varying temperatures would be desirable.

3.7 The Effect of Moisture Changes

The behavior of concrete becomes more complex when the moisture content varies within the pores. This movement of moisture produces an acceleration of the creep in the zones of changing moisture content. The development of constitutive equations for this condition is enhanced by utilizing the principles of irreversible thermodynamics of multiphase systems. Two kinds of pores in the cement paste must be distinguished: the micropores, which are less than 10 molecules thick, and the macropores, which are larger. The micropores probably are predominantly laminar in shape, and the water molecules are held by very strong adsorption forces. The macropores contain capillary water and air of varying amounts, depending on the degree of saturation. Thermodynamic equilibrium between the adsorbed water (referred to by the subscript d) and capillary water (subscript c) exists when

$$G_d = G_c , \quad (43)$$

where G equals the Gibbs' free energy of water per unit mass (also called the chemical potential). The free energy G is defined by its total differential

$$\begin{aligned} G_d &= -s_d dT + \Gamma_w^{-1} d\pi + p_d \Gamma_w dl_d \\ G_c &= -s_c dT + v_c dp_c, \end{aligned} \quad (44)$$

in which s is the entropy per unit mass, Γ_w is the mass of water per unit area of the pore's surface, π is the spreading pressure of the water at the surface of the adsorbed layer, p_d is the disjoining pressure of water across the layer, l_d is the thickness of the water layer, p_c is the pressure within the capillary water, and v_c is the specific volume of the capillary water. The micropores are part of the load-carrying microstructure, and thus changes of the thickness l_d are associated with macroscopic strains.

The conditions of thermodynamic equilibrium require equality of the Gibbs' free energies per unit mass of the water within the micropores and the macropores and similarly for the walls of the solid microstructure. The mass flux of the water and solids between the micropores and the macropores is thus assumed to be proportional to the differences in their respective free energies. Using these basic relations and certain auxiliary ones, it is possible to deduce the form of the constitutive equations for the volumetric deformations:

$$\begin{aligned} \dot{\sigma}_{s_i} + \phi_{s_i} \sigma_{s_i} &= E_{s_i} \left(\dot{\epsilon} - \dot{\epsilon}_s - \alpha_{s_i} \dot{T} \right) \\ \dot{\sigma}_{w_i} + \phi_{w_i} \sigma_{w_i} &= E_{w_i} \left(\dot{\epsilon} - \dot{\epsilon}_s - \alpha_{w_i} \dot{T} \right), \end{aligned} \quad (45)$$

where T is the temperature, α_{s_i} and α_{w_i} are thermal dilation coefficients, $\dot{\epsilon}_s$ is the instantaneous shrinkage strain due to a change in the specific water content, and ϕ_{s_i} and ϕ_{w_i} are coefficients.

The basic difference between the present equation and the previous rate-type formulation for constant moisture, Eq. (28), is that in the present equation it is necessary to distinguish between two types of hidden stresses (those carried by the water in the micropores, σ_{w_i} , and

those carried by the solids, σ_{s_i}) and the two associated elastic moduli E_{w_i} and E_{s_i} . For the case when $\dot{\epsilon}_s = \dot{\epsilon}_w = 0$, the water becomes immobile and one may lump the hidden stresses and elastic moduli as follows:

$$\begin{aligned}\sigma_i &= \sigma_{s_i} + \sigma_{w_i} \\ E_i &= E_{s_i} + E_{w_i},\end{aligned}\tag{46}$$

and Eq. (45) reduces to Eq. (28). The rate coefficients $\dot{\phi}_{s_i}$ and $\dot{\phi}_{w_i}$ are the principal coefficients determining the effect of humidity on creep. These coefficients decrease with decreasing water content and sharply increase when there is a difference in the Gibbs' free energy potential ($G_d - G_c$), thus correctly modeling the creep of predried concrete and the acceleration of creep due to induced moisture movement as in drying.

The numerical values of the parameters involved in Eq. (45) were identified, and good agreement with previous test data was demonstrated in Ref. 59. The stable time-step algorithm developed for the Maxwell chain representation, given in Ref. 52, has been extended for Eq. (45); however, the use of independent hidden stresses in the water and the solid will require twice as much computer machine time and storage as required for Eq. (28).

For mass concrete structures, the effect of moisture changes has been generally ignored, because the loss of moisture to the atmosphere occurs only to a depth of several feet from the surface. However, this does not preclude the existence of significant moisture movement that might be induced by the sustained temperature gradient across the wall that exists in the PCRV during normal operations. The significance of the effect of moisture movements on the creep occurring in a PCRV under normal operating conditions has not been evaluated at this time. A coordinated experimental and analytical program to ascertain the validity of this constitutive model and then proceed to an analytical investigation of the effect of moisture migration on a PCRV under various operating conditions would be desirable. Such a study would provide greater insight into the significance of this effect and a basis for either rejecting or accepting the hypothesis that the effect can be ignored.

3.8 Effective Modulus Method

When the changes of σ from time t' to time t are very small, an acceptable approximation of the resulting strain is

$$\epsilon(t) = \frac{\sigma(t)}{E_{\text{eff}}(t, t')} \quad (47)$$

with

$$E_{\text{eff}} = \frac{1}{J(t, t')}, \quad (48)$$

where E_{eff} is called the effective modulus. This equation has the form of a linearly elastic law, and linearly elastic finite-element codes may be used to analyze structures for these conditions. The primary deficiency in applying this method to the analysis of a PCRV is that differences in creep strains within the structure (due to differences in temperature and age of the concrete and the presence of steel) produce redistributions of stress with time. As the approximation is only valid for small changes in stress, this method is only applicable for relatively short time intervals.

3.9 Demonstrated Results

It is difficult to obtain reliable experimental data on the time-dependent response of a PCRV for environmental conditions and loadings representative of actual operating conditions. As previously discussed, concrete is sensitive to both temperature and moisture conditions, and each of these must be carefully controlled. One is also faced with the problem that the experimental instrumentation (in particular, embedded strain gages) is neither very reliable nor very accurate.⁶⁰ Because of these problems and the high cost associated with tests of this type, there are only a limited number of experimental validations of the analytical methods for time-dependent conditions approximating those existing in a PCRV under operating conditions. In addition, it is not possible to establish a consistent set of factors for model scaling that meets the requirements of dimensional similitude.⁶⁰ Thus, the trend has been either

to use a full-size vessel as a direct verification of the vessel response or to use a scale model to validate an analytical method and then use the analytical method to extrapolate the model results to the full-size vessel.

Embedded strain gages were used in the Oldbury⁶¹ and Wylfa⁶² vessels to experimentally determine their strain response during commissioning and early life. The scatter in the experimental results from the Oldbury vessel was so excessive that general conclusions are not justified (see Fig. 24). For the Wylfa vessel, analytical results were obtained using linear

Fig. 24. Measured temperature and strain and calculated hoop stress and strain at gage 41 in Oldbury vessel (Ref. 61). (1 psi = 6895 Pa)

superposition for changes in load and temperature and a power-law type of creep law. The agreement between these results and the experimental results was very good (see Fig. 25).

For the Fort St. Vrain reactor, a scale model of the vessel was used to validate the analytical method. Two separate analyses were published. In the first,¹³ a logarithmic form of the creep law was used with the superposition principle to account for changes in load and temperature. A non-linear modification was introduced into the superposition principle in an attempt to correct the tendency of the linear theory to overestimate the strain recovery during unloading. Generally good agreement was obtained between the analytical and experimental results (see Fig. 26). In a later analysis,⁵⁷ linear superposition was combined with an analysis that included a cracking model for the concrete. The time-temperature shift algorithm due to Sackman⁵⁵ was used to account for the effect of temperature on creep. (The effect of temperature on aging was not included in the analysis.) The results of both analyses were essentially the same.

The latter analysis method was also used to obtain results for the Oak Ridge thermal cylinder experiment.⁶³ A sealed, thick-wall, cylindrical vessel, without restraining heads, was subjected to internal pressure, axial and circumferential prestressing, and a temperature crossfall. Good agreement was obtained between the experimental and analytical response throughout the loading history; however, the analysis significantly underestimated the creep strains during the heatup phase of the experiment (see Fig. 27), and the resulting error remained essentially constant throughout the remainder of the test. The source of this error is not fully understood but may be related to transitional thermal creep previously discussed in Sect. 3.6.

These test results seem to indicate that, with the possible exception of cases involving large thermal transients at high rates, the time-dependent response of a PCRV can be determined with sufficient accuracy by utilizing a constitutive model based on the linear superposition principle, an appropriate creep law, and constant Poisson's ratios for both elastic and creep strains. However, the results of the Oak Ridge thermal cylinder test seem to indicate that the creep response during thermal transients may not be accurately accounted for if the temperature change is large.

Fig. 25. Measured and predicted strain changes with time related to operating conditions for the Wylfa reactor vessel. (Gages located at four positions at 90° intervals around the equator and 300 mm (0.98 ft) from the inside face of the vessel wall (Ref. 62). (1 N/mm² = 1 MPa = 145 psi)

Fig. 26. Circumferential concrete strain at inner midplane (Ref. 13). See Ref. 13 for explanation of test activities. ($1 \mu\text{in./in.} = 1 \mu\text{m/m}$)

4. TIME-INDEPENDENT NONLINEAR BEHAVIOR OF CONCRETE

4.1 Basic Characteristics

Perhaps the most important consideration in the design of a prestressed concrete reactor vessel is the assurance of an adequate margin of safety against failure by internal pressure. If adequate margins are obtained for normal operating conditions by the use of elastic and time-dependent analysis methods, it remains only to validate the vessel performance for overpressures due to potential accident conditions. Because concrete is relatively weak in tension, the basic philosophy in the design of concrete pressure vessels is to maintain compressive stresses under the action of internal pressure through the use of prestressing tendons. At a certain level of overpressure, the effect of prestressing is overcome and tensile cracking develops in the concrete. In addition, due to the restraining

Fig. 27. Theoretical vs experimental strains for average of three circumferential type A vibrating-wire embedment gages in position closest to inner liner (Ref. 63). ($1 \mu\text{in./in.} = 1 \mu\text{m/m}$)

effect of the head on the wall deformations of the vessel, large values of shearing and compressive stresses develop in the head and head-to-vessel intersection. To predict the response of the vessel to accidental overpressures, it is necessary to conduct analyses that include tensile cracking and the nonlinear stress-strain response of concrete at high stress levels. Since it is inconceivable that an accident causing high internal pressure would be of long duration, only short-time nonlinear behavior is of interest.

It is useful to regard concrete as a structure consisting of large aggregates (gravel) bonded by mortar, which is a structure of fine aggregates (sand) bonded by cement paste. The elastic moduli, Poisson's ratios, thermal expansion coefficients, and long-time deformation properties of the gravel, mortar, and cement paste usually differ, and therefore, any loading or temperature change produces self-equilibrated internal stresses in the microstructure. These stresses are further intensified by any drying shrinkage in the cement paste.

Under low triaxial confinement pressure, the weakest links in the microstructure are the paste-aggregate interfaces; therefore, the internal stresses first produce microcracking at the interface bonds. This microcracking at the bonds between the mortar and the large aggregates, shown in Fig. 28, is the main source of the nonlinear stress-strain response at high stresses and subsequent concrete failure. The inelastic strain due to this microcracking is accompanied by volume dilatation, a circumstance that differs significantly from the inelastic response of ductile metals. Failure occurs due to coalescence of bond microcracks by various modes into interconnecting cracks within the mortar.

The behavior of concrete under high triaxial confining pressure is completely different. Formation of bond microcracks is prevented and very large shear strains can be achieved without significant damage to the material. Particles of cement paste, subjected to shear, are able to slide relative to each other while continuously breaking and reforming bonds. The same phenomena exist within the bond zone between the mortar and aggregate. Under very high confining pressure, concrete exhibits as much ductility as metals do, and, as with metals, no volume change accompanies this type of inelastic strain.

Fig. 28. Schematic drawing of microcracking in concrete.

There have been a number of efforts directed toward understanding the complex behavior of concrete through the use of idealized models. An interesting example of this work is described in Ref. 64. Nine circular disks representing aggregates were embedded in a mortar matrix to produce a square array (see Fig. 29). This model was tested under biaxial stress, and the results were compared with those obtained from a finite-element analytical idealization. In the analysis, the material components were assumed to be elastic, and the Mohr-Coulomb shear theory of failure⁶⁵ was used to predict the loss of cohesion at the aggregate interfaces. If the tensile stress in the mortar exceeded its measured tensile strength, a "crack" was introduced by appropriate modification of the material properties. Even with this highly idealized model, the agreement between the experimental and analytical stresses at failure was very good, as seen in Fig. 30, and both results exhibited the characteristic features of the biaxial failure surface of conventional concrete. The stress-strain response of the two models, however, did not agree very well, and neither reproduced the features that characterize conventional concrete (see Figs. 31 and 32).

Fig. 29. The concrete model (Ref. 64).

Although idealized models provide qualitative descriptions of concrete behavior and confirm that the primary mechanism is initiation and propagation of microcracks, they do not provide a quantitative description of sufficient accuracy for detailed analysis of concrete structures. Thus, as in the case of time-dependent response, the description of the time-independent nonlinear response of concrete must be based on an empirical description obtained from detailed testing.

One of the major difficulties encountered in the testing of concrete subjected to biaxial and triaxial stresses is the development of loading fixtures capable of producing a well-defined and uniform stress state in the specimen. This is thought to be the primary reason for the considerable deviation commonly found in biaxial test data reported by various investigators. For example, reported strength values for equal biaxial compression are found to vary from 80 to 350% of the uniaxial compressive strength, with the higher values obtained when rigid platens are used and the lower values obtained when well-lubricated platens are used. Although Poppel⁶⁶ had shown in 1900 that a prismatic specimen may be confined along

To overcome the problem of end restraint, Hilsdorf⁶⁷ developed a brush bearing platen consisting of closely spaced small steel bars that are flexible enough to follow lateral deformations without significant restraint but yet have sufficient buckling stiffness to transmit the required compressive forces (see Fig. 33). Using the brush platen system, Kupfer, Hilsdorf, and Rusch⁶⁸ conducted a series of biaxial tests of concrete specimens of various strengths. The failure stresses for the various concretes, normalized by dividing by the magnitude of compressive strength, are shown in Fig. 34. It can be seen in the figure that the normalized

Fig. 31. Experimental stress-strain curves for models with $d/r = 0.5$ (Ref. 64). ($1 \mu\text{in./in.} = 1 \mu\text{m/m}$)

strengths for various biaxial stress ratios are only slightly affected by the concrete mix. Also, the strengths under biaxial tension are essentially the same as those under uniaxial tension.

The stress-strain relations obtained in these tests for biaxial compression, combined tension and compression, and biaxial tension loading are shown in Figs. 35, 36, and 37, respectively. As seen in these figures, the stress-strain relations for compressive loading are quite nonlinear,

Fig. 32. Stress-strain curves obtained from finite-element analysis (Ref. 64). (1 μ in./in. = 1 μ m/m)

while under tensile loading the relations are essentially linear. Plots of volumetric strains under biaxial compression are shown in Fig. 38. It can be seen in the figure that slight nonlinearity begins at about 40% of the failure stress, while at approximately 95% of the failure stress, the minimum volume is obtained and a major inflection exists in the curve. It is generally agreed that this inflection point coincides with the stress

Fig. 33. Brush bearing platens; dimensions of filaments: 5×3 mm (0.195×0.118 in.); spacing of filaments: 0.2 mm (0.0008 in.) (Ref. 68).

Fig. 34. Biaxial strength of concrete; results of experimental investigation (Ref. 68).

at which major microcracking occurs. An important aspect of these test results is that the ratios between the various significant stress levels appear to be essentially independent of the biaxial stress ratio (see Fig. 39). This implies that the same functional form can be used to define these various stress levels in stress space, a fact that is often used in modeling concrete response.

Recently, there have been a number of experimental investigations into the behavior of concrete under biaxial and triaxial loadings, with the major emphasis placed on establishing appropriate failure criteria.⁶⁹⁻⁷¹

Fig. 35. Stress-strain relationships of concrete under biaxial compression (Ref. 68).

Fig. 36. Stress-strain relationships of concrete under combined tension and compression (Ref. 68).

Fig. 37. Stress-strain relationships of concrete under biaxial tension (Ref. 68).

Fig. 38. Volumetric strain of concrete under biaxial compression (Ref. 68).

Fig. 39. Stresses at the elastic limit, minimum volume, and failure of concrete subjected to biaxial stress states (Ref. 68).

Relatively less effort^{72,73} has been devoted to establishing the nonlinear relationship between the stress and strain that exists prior to failure. From an analytical viewpoint, this sparsity of effort seems rather short-sighted in that there is little advantage to having accurately defined failure criteria available if one is unable to accurately calculate the stresses in the neighborhood of the failure limit. Also, since failure of the concrete in a reinforced concrete structure does not necessarily

imply failure of the structure but only a transfer of load from the concrete to the reinforcing steel, an accurate estimation of maximum loads requires appropriate models for the postfailure response.

4.2 Failure Surfaces for Plain Concrete

Any state of stress is completely defined by its three principal values and their corresponding three orthogonal directions. If the strength of the material is independent of direction (an isotropic characteristic), the state of stress at failure can be represented as a point in the three-dimensional space whose coordinate axes are the three principal stresses. The plot of all such points defines a surface in the stress space called the failure surface for the material.

In recent years a number of investigators have undertaken the development of analytical descriptions of experimentally determined failure surfaces for concrete. Basically, the failure envelope is a conical-like surface with curved meridians and noncircular cross sections, augmented by a cutoff in the tensile regime (see Fig. 40).

One of the simplest descriptions of the surface is given by Hansson and Schimmelpfennig.⁷⁴ The surface is described by a vector ξ on the line of hydrostatic compression,

$$\xi = \frac{1}{\beta_c \sqrt{3}} (\sigma_1 + \sigma_2 + \sigma_3) , \quad (49)$$

and a perpendicular vector,

$$\zeta = -\frac{1}{\beta_c} \sqrt{\frac{1}{3} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} . \quad (50)$$

The relationship between ξ and ζ , $\zeta_0(\xi)$, on the meridian $\sigma_1 = \sigma_2 > \sigma_3$ is approximated with a quadratic function of the form

$$\xi = c_0 + c_1 \zeta_0 + c_2 \zeta_0^2 , \quad (51)$$

and the shape of the other meridians are defined by

$$\zeta = H(\theta) \zeta_0(\xi) , \quad (52)$$

Fig. 40. Failure envelope of plain concrete (Ref. 78).

where

$$H(\theta) = \frac{\lambda}{1 - (1 - \lambda)(1 - 3/\pi \arccos \theta)^{1.9}} \quad (53)$$

and θ is the angle perpendicular to the axis of hydrostatic compression, with $\theta = 0$ for $\sigma_1 = \sigma_2 > \sigma_3$ and $\theta = 60$ for $\sigma_1 > \sigma_2 = \sigma_3$.

The four parameters c_0 , c_1 , c_2 , and λ are determined by using the uniaxial tensile and compressive strength, the biaxial compressive strength, and the requirement that the intersection of $\zeta_0(\xi)$ with the ξ -axis be determined by a straight line through the uniaxial and the biaxial compressive strengths. The comparison of the fits with experimental data along the meridians $\theta = 0$ and $\theta = 60$ is shown in Fig. 41. The shape of

Fig. 41. Comparison of fits with experimental data along meridians $\nu = 0$ and $\nu = 60$ (Ref. 74).

the surface is shown in Fig. 42, and the trace of the surface in the bi-axial plane is shown in Fig. 43.

William and Warnke⁷⁵ also developed an analytical expression for the failure surface based upon the uniaxial tensile and compressive strengths and the biaxial compressive strength. Straight meridians were used in this model, and, hence, it was not possible to obtain an accurate fit at high levels of hydrostatic stress; however, the failure surface is continuous and has continuous derivatives, a property not exhibited by the previous model. This latter property ensures that a unique normal vector exists at each point on the surface for defining the direction of inelastic deformations according to the normality principle of the theory of plasticity. The functional form of the expression for the failure surface is rather complex and will not be restated herein; however, the shape of the surface is shown in Fig. 44. An even more complicated five-parameter model was developed to obtain a more accurate fit of the experimental data at high levels of hydrostatic stress. As can be seen in Fig. 45, this

Fig. 42. Shape of the failure surface approximation (Ref. 74).

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Fig. 43. Biaxial trace of the failure surface approximation (Ref. 74).

model has curved meridians and produces a better fit with the experimental results.

The primary advantage of the expressions presented in this work is their applicability to an elastic-plastic computational scheme. With this in mind, the authors presented in detail all the required additional relations required for such a computation. Another interesting feature of the proposed failure surface is that it includes, as degenerate cases, the Drucker-Prager criterion, which is a three-dimensional approximation of the Mohr-Coulomb criterion, and the von Mises criterion, commonly used to represent metal behavior.

Fig. 44. Fitting of triaxial test data — low compression regime (Ref. 75).

Other efforts⁷⁶⁻⁸⁰ to develop criteria for triaxial failure surfaces have generally either led to more complicated representations and/or did not provide fits of the experimental data that were as accurate. Some of these^{78,79} include attempts to use the Mohr-Coulomb criterion that defines the relation between shear stress τ and normal stress σ at a plane of failure according to

$$|\tau| = c - \sigma \tan \phi, \quad (54)$$

where c is the material cohesion and ϕ is the angle of internal friction. One of the basic deficiencies of this model is that, contrary to experimental observations, it predicts failure independently of the intermediate principal stress.

Fig. 45. Five-parameter model - fitting of triaxial test data (Ref. 75).

4.3 Postfailure Models for Tensile Failures

One of the basic difficulties that arises in attempting to establish appropriate models for failed concrete is that, for plain concrete, the test is terminated when the specimen fails. One might attempt to avoid this problem by testing reinforced concrete specimens, but then complex interactions are introduced between the concrete and the reinforcement. The common approach is to simply hypothesize the postfailure model and then justify it by comparing the results obtained using the model with results from structural tests. The problem with this approach is that other assumptions are generally introduced into the analysis so that the comparison does not necessarily provide a true test of the hypothetical model. Also, a model which is valid for certain situations may not be valid for others.

Under tensile loading, the failure of plain concrete is rather conspicuous and takes the form of cracks in planes transverse to the direction of maximum tensile stress. After failure, the material has no resistance to tensile stress in the direction normal to the crack plane, but it retains its ability to transmit compressive stresses in this direction. Also, depending on the amount of crack opening, it may retain some ability to transmit shearing stresses across the crack due to aggregate interlock. Except for aggregate interlock, for which there is no satisfactory model at this time, the hypothetical model for a tensile crack is clear, and the main question remaining is how to achieve the model within the context of the analytical procedure.

Even for this rather simple mode of failure, a fundamental difficulty arises in its implementation in a finite-element analysis. In the finite-element method, continuous trial functions are assumed for the displacement fields within each element (see Sect. 2.2), while a crack is characterized by the existence of displacement discontinuities. Thus interelement cracking cannot be readily accommodated.

Displacement discontinuities can, however, be accommodated along element boundaries by "disconnecting" the adjacent elements at the nodal points (see Fig. 46). This approach was used by several analysts^{60, 61, 62} to account for the displacement discontinuities, but it gives rise to problems of another nature. When adjacent elements are "disconnected," additional degrees of freedom are introduced into the problem that, in general, abolish the banded nature of the stiffness matrix and give rise to difficulties in obtaining a solution to the algebraic stiffness equations (see Sect. 2.4). This problem can be avoided if the solution algorithm is based on an iterative technique. An alternate approach would be to terminate the analysis when tensile cracking is indicated, redefine the structure with due regard paid to the associated bandwidth such that the faces of the crack become boundaries of the body, input the revised information, and continue the calculations.

While the techniques discussed above are useful in theoretical studies of simple structures, it would be extremely difficult to develop the required algorithms to automate the procedures for use in an analysis of a

Fig. 46. Accounting for displacement discontinuities at the crack by "disconnection" of nodes.

complex structure for which the crack planes cannot be predetermined. Because of this, the more common approach is to ignore the displacement discontinuities and attempt only to achieve force-displacement relations for the cracked element that are consistent with the existence of the crack. When the tensile stress in an element exceeds the failure criteria, the material properties of the element are modified to produce an orthotropic material (see Fig. 47) with zero elastic modulus in the direction normal to the crack and zero shear modulus for the appropriate orthogonal directions.

For the constant-strain elements to which it was first applied,^{83,84} this procedure can be implemented in a simple and consistent manner. Incorporating the new material properties into the material matrix, $[D]$, and computing the integral (see Sect. 2.3),

$$[k_{ij}] = \int [B_i]^T [D] [B_j] dv, \quad (55)$$

yields an element stiffness matrix that produces consistent force-displacement relations for the cracked element.

Fig. 47. Cracked constant-strain triangular element (Ref. 83).

Certain theoretical difficulties arise when one attempts to use the material-modification technique for modeling cracks in the high-order isoparametric elements commonly used for analysis of multicavity PCRVs.^{78,85,86} For these types of elements, the element stiffness is determined by numerical integration of Eq. (55). The usual procedure is to calculate the stresses at the integration points and compare them to the failure criteria. If the failure criteria is exceeded, the material properties at that point are modified in the manner discussed above (see Fig. 48), and the revised element stiffness is determined by summing the contributions associated with each integration point. The number and position of the integration points depend upon the type and order of integration used, but, typically,

Fig. 48. Cracking in the isoparametric element.

second- or third-order Gaussian integration is used. One difficulty with this procedure is that the amount of material modified is affected by the integration order used. If a single-point integration is used, the material properties are modified for the entire element, but as the integration order is increased, the volume of material affected diminishes until the volume becomes zero and the existence of the crack has no effect on the calculated element stiffness. Another difficulty is that when the material properties are set to zero at a point, the polynomial interpolation used in the numerical integration implies, in general, negative values for the material properties over a finite adjoining region. It appears that neither of these undesirable characteristics of the method have been recognized by the authors who have utilized this technique.

A second fundamental difficulty that arises in the analysis of post-crack behavior is associated with the fact that failure of concrete is accompanied by a declining branch in the stress-strain curve (see Figs. 35, 36, and 37). When failure of this type occurs, the material becomes locally unstable, with the result that the deformations tend to localize into a narrow band. In such a situation, the propagation of cracks through the element system depends on the size of element chosen. If the element

just ahead of the crack front is replaced by several smaller elements, the tensile strain will concentrate in them (see Fig. 49). By choosing sufficiently small frontal elements, the stresses in them can be made arbitrarily large. Thus, the failure load depends on the chosen size of the elements, which is a theoretically incorrect feature. A possible solution is to enhance the criterion of cracking by imposing an additional condition on the relative displacements across the band of cracked elements similar to the condition of maximum crack-opening displacement of fracture mechanics.

Fig. 49. The spurious effect of finite-element size on propagation of a crack band.

4.4 Postfailure Models in the Compressive Regime

The failure of concrete in the compressive regime is more complex than that occurring in the tensile regime. In discussing this problem, Newman and Newman⁸⁷ suggest that under uniaxial compressive states of stress, the first microcracks develop in the mortar parallel to the direction of the

compressive stress. As the load increases, the crack path extends, dividing and traveling around the hard aggregates until, after final disruption, isolated particles of aggregates are produced to which are adhering small cones of mortar aligned in the direction of the principal compressive stress (see Fig. 50). Under biaxial compression, the fracture paths extend in the directions of both principal compressive stresses, dividing around the aggregates to produce, at final disruption, isolated particles surrounded by a "halo" of adhering mortar (see Fig. 51).

The particles that have been isolated by the dividing crack path exert a wedging influence on the crack that gives rise to the development of cleavage planes. The fracture patterns commonly observed at failure for specimens tested under biaxial stress conditions are shown in Fig. 52. At low biaxial compression-tension stress ratios (zone 1), the failure is

FIG. 50. Particles of coarse aggregate removed from concrete specimens after loading to ultimate failure under uniaxial compression showing typical cone formation (Ref. 87). (1 in. = 25.4 mm)

Fig. 51. Particles of coarse aggregate removed from concrete specimens after loading to ultimate failure under biaxial compression showing typical "halo" formation (Ref. 87). (1 in. = 25.4 mm)

characterized by a single cleavage crack as discussed in the previous section. The failure is abrupt with an essentially linear stress-strain relation up to failure. For stress states close to uniaxial compression (zone 2), the crack pattern undergoes a transition to multicracking fractures for which the propagation is more gradual, giving rise to nonlinear stress-strain relations. In the major part of the compression-compression zone (zone 3), the pattern becomes one of several parallel cleavage planes whose propagation is still more gradual, producing a greater nonlinearity in the stress-strain relations prior to failure.

It is clear from the previous discussion that the single-cleavage model used as the postfailure model in the tension regime will not suffice in the compressive regime. The commonly used model for the compressive

Fig. 52. Idealization of the macroscopic fracture patterns observed at ultimate failure for specimens loaded under biaxial stress states (Ref. 87).

postfailure condition is to assume that the concrete yields plastically when the compressive failure criteria is satisfied by the stress state.

In a plasticity analysis, yielding is assumed to occur if the stress state $\{\sigma\}$ satisfies the yield criterion

$$F(\{\sigma\}, \kappa) = 0, \quad (56)$$

where κ is a "hardening" parameter. This condition can be visualized as a surface in the n -dimensional stress space whose position depends on the parameter κ (see Fig. 53). If the state of stress satisfies the yield criterion, then an increment of strain, $\{d\epsilon\}$, is divisible into an elastic

Fig. 53. Yield surface and normality criterion in two-dimensional stress space.

part, $\{d\epsilon_e\}$, and a plastic part, $\{d\epsilon^P\}$. Thus

$$\{d\epsilon\} = \{d\epsilon^e\} + \{d\epsilon^P\}, \quad (57)$$

where $\{d\epsilon\}$ is the total strain increment. The component of plastic strain is assumed to be normal to the yield surface; hence

$$\{d\epsilon^P\} = \lambda \frac{\partial F}{\partial \{\sigma\}}, \quad (58)$$

and the elastic strains are related to the stress by the elastic material matrix, $[D]$, where

$$\{d\epsilon\} = [D]^{-1} \{d\sigma\} + \lambda \frac{\partial F}{\partial \{\sigma\}}. \quad (59)$$

During plastic deformation the stress state must remain on the yield surface; thus

$$\left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T d\{\sigma\} + \frac{\partial F}{\partial \kappa} d\kappa = 0. \quad (60)$$

Note that for strain-hardening materials, $d\kappa \neq 0$ and the yield surface is displaced during plastic deformation, while for perfectly plastic materials, $d\kappa = 0$ and the surface remains fixed in stress space. After some manipulation, the incremental stress-strain relation for plastic deformations becomes^{88,89}

$$\{d\sigma\} = [D]_{ep} \{d\epsilon\} , \quad (61)$$

with

$$[D]_{ep} = [D] \left\{ 1.0 - \frac{\left\{ \frac{\partial F}{\partial \{\sigma\}} \right\} \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D]}{A + \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}^T [D] \left\{ \frac{\partial F}{\partial \{\sigma\}} \right\}} \right\} \quad (62)$$

and

$$A = \frac{\partial F}{\partial \kappa} \{\sigma\}^T \frac{\partial F}{\partial \{\sigma\}} . \quad (63)$$

Several analysts^{78,90,91} have utilized the elastic-perfectly plastic model for the postfailure condition in the compressive regime. In these analyses, the failure surface is taken as a yield surface.

An objection to the use of this model is that the normality condition of Eq. (58) gives rise to a volumetric dilatancy during plastic straining that exceeds that which occurs in actual concrete. A possible correction to the latter problem is the use of the "constant volume criterion" suggested in Ref. 92. A more fundamental objection to the use of the plasticity theory for concrete is that the existence of a unique yield surface in a principal stress space implies that the material properties are independent of direction, while the damage due to microcracking is known to be directional in nature.

Although there are theoretical objections to the use of the plasticity model for concrete, the effect of these theoretical inconsistencies on the accuracy of the analysis is not known, and reasonable results have been obtained for practical problems. Some of these results will be discussed in Sect. 4.9.

4.5 Nonlinear Stress-Strain Relations

One of the difficulties with the elastic-perfectly plastic model discussed in the previous section is that it ignores the nonlinearity of the stress-strain relations for compressive loading. This deficiency can be overcome by using the discontinuity surface as the initial yield surface and introducing nonlinear hardening into the plasticity model. Care must be exercised so that the subsequent yield surfaces are bounded by the initial discontinuity and failure surfaces.

Probably the best example of such a model is the one proposed by Chen and Chen.⁸⁰ In this work the yield surfaces are defined by the relation

$$F(\{\sigma\}, \kappa) = \frac{\kappa^2}{3} J_2 - \frac{\kappa^2}{36} I_1 \pm \frac{1}{12} I_1^2 + \frac{1}{3} A_u I_1 - \tau_u^2, \quad (64)$$

where the positive sign on the third term applies in the compressive region and the negative sign applies in the compression-tension region. The term I_1 is the first invariant of the stress tensor, and J_2 is the second invariant of the deviatoric stress tensor. The shape of the surfaces in principal stress space is shown in Fig. 54. It should be noted that the surface has circular cross sections with curved meridians.

The kinematics of the yield surface involves a translation along the hydrostatic axis and a simultaneous isotropic expansion. The form of the functions describing the strain hardening were obtained by generalizing uniaxial stress-strain data utilizing

$$\sigma_e = \tau + \sqrt{f} \quad (65)$$

as the definition of the effective stress and

$$\epsilon^P = \int \sqrt{\{d\epsilon^P\}^T \{d\epsilon^P\}} \quad (66)$$

as the definition of the effective plastic strain. Idealized equivalent stress-strain relations obtained from the experimental data given in Ref. 72 are shown in Fig. 55. It was necessary to define separate strain-hardening functions for the tension-compression and compression regimes.

Fig. 54. Failure and initial discontinuous surfaces in principal stress space (Ref. 80).

A comparison between the experimental and idealized stress-strain relations is shown in Figs. 56 and 57. It can be seen in the figures that there is good agreement, except for some deviations in the stress-volumetric strain curves at high stress levels.

An alternative to the use of the strain-hardening plasticity model is to utilize a model based upon the theory of hyperelasticity to introduce the nonlinearity in the stress-strain relations. In this theory the requirement of reversible elastic deformations is dropped, and it is stipulated instead that the stress increments be linearly dependent on the strain increments, that is,

$$\{d\sigma\} = [C(\{\sigma\}, \{\epsilon\})] \{d\epsilon\}, \quad (67)$$

where the coefficients of the matrix $[C]$ depend on the components of the stress and strain tensors $\{\sigma\}$ and $\{\epsilon\}$. This formulation is complicated, and identification of the material parameters from given test data is difficult. The matrix $[C]$ is a fourth-rank tensor which, in general, is a function of two second-rank tensors and must be considered to be anisotropic as a result of the dependence of $\{\sigma\}$ and $\{\epsilon\}$ (stress-induced anisotropy), even though the material is isotropic with regard to the initial

Fig. 55. Equivalent stress-strain relations: (a) compression zone; (b) tension-compression zone (Ref. 80).

state. Because of these difficulties, it has not been possible, thus far, to consider more sophisticated functions for $[C(\{\sigma\}, \{\epsilon\})]$ than a linear tensor polynomial in terms of $\{\sigma\}$. This has been done by Coon and Evans,⁹³ who demonstrated acceptable agreement with uniaxial, biaxial, and triaxial stress-strain diagrams up to about 85% of the ultimate strength. However, their formulation implies infinite strain at peak stress, which is incorrect, and it does not properly represent the inelastic volume dilatancy due to shear.

Fig. 56. Stress-strain relationships of concrete: (a) under biaxial compression; (b) under combined tension and compression (Ref. 80).

Fig. 57. Stress-strain relations in compression zone: (a) stress-strain; (b) stress-volumetric (Ref. 80).

Others have considered the matrix $[C]$ in simpler forms that either neglect the stress-induced anisotropy, making the components of $[C]$ dependent only upon the invariants of $\{\sigma\}$, or introduce incorrect forms of stress anisotropy in which $[C]$ depends upon certain components of $\{\sigma\}$ but is not an invariant tensorial function.^{72,94} Such formulations cannot be correct, in general, because the material properties are not independent of the choice of direction of coordinate axes. These formulations have served a useful purpose in describing the biaxial stress-strain diagrams of concrete, but they fail for other types of loading.

4.6 Endochronic Theory

A promising new type of constitutive relation that may offer a better description of material behavior is the endochronic theory. This theory, whose first practical application was to work-hardening metals, seems to be particularly effective for materials in which the inelastic strain develops gradually. A preliminary development of the theory for concrete has been made by Bazant and Bhat.⁹⁵ The basic concept in the endochronic theory is that of an intrinsic time z which represents a monotonically increasing variable whose increment dz depends on the actual time increment dt as well as the deformation increments $d\epsilon_{ij}$. Using the condition that the intrinsic time may never decrease, it can be shown that its increment dz must be defined as

$$dz = \sqrt{d\zeta^2 + \left(\frac{dt}{\tau_1}\right)^2}, \quad d\zeta = \sqrt{p_{ijkl} d\epsilon_{ij} d\epsilon_{kl}}, \quad (68)$$

where p_{ijkl} and τ_1 are coefficients depending on the state of the material variables σ_{ij} , ϵ_{ij} , and z . Further simplification of the expression is possible using the conditions of isotropy and the assumption of an absence of inelastic strain in purely volumetric deformation.

The presence of dt in Eq. (68) allows simultaneous consideration of strain-rate effects and creep. Coefficients p_{ijkl} may be selected in a form that reflects the strain hardening as well as the strain softening of concrete and the restraining effect of hydrostatic pressure on the development of inelastic strain. The variable ζ may be useful as a measure

of the accumulation of damage to the microstructure of concrete caused by the deviatoric strain. The damage suffered by concrete as a result of the microcracking that accompanies shear strain has long been regarded as the crucial phenomenon in the inelastic behavior of concrete.

For time-independent deformations the stress-strain relations in the endochronic theory are written in the form

$$d\epsilon'_{ij} = \frac{ds_{ij}}{2G} + d\epsilon''_{ij}, \quad d\epsilon''_{ij} = \frac{S_{ij}}{2G} dz$$

$$de = \frac{d\sigma}{3K} + d\lambda,$$
(69)

where e is the volumetric strain, σ is the volumetric stress, λ is the inelastic dilatancy whose increment $d\lambda$ depends on dz . The deviatoric stress-strain relation is analogous to a similar relation in the theory of plasticity, and it can be shown that equations of the form of the Prandtl-Reuss equations of plasticity can be obtained as a special case of the endochronic theory. However, in contrast to plasticity, the stress-strain relations of endochronic theory cannot, in general, be reduced to a linear relation between $d\sigma$ and $d\epsilon$, as in Eq. (61).

The method by which the endochronic theory models the inelastic strain may be clarified by considering the special case of short-time uniaxial deformation, for which, according to Eq. (69) for $\tau_1 = \infty$, one has

$$dz = k \sqrt{(d\epsilon)^2} = k |d\epsilon|$$
(70)

and so

$$d\sigma = E(de - d\epsilon'') = Ede - \sigma k |de|,$$
(71)

where E is Young's modulus and k is a constant. The stress increment is composed of an elastic increment Ede and an inelastic increment $-\sigma k |de|$ (see Fig. 58). When loading is suddenly reversed to unloading, de is changed from positive to negative, while the inelastic strain does not change its sign. This causes the unloading branch of the stress-strain diagram to have substantially higher slope than the loading branch. The

Fig. 58. (a) Elastic-plastic strain-hardening material; (b) strain increment in endochronic theory.

irreversibility of loading, which is the salient feature of all inelasticity, is thus embodied in the theory without using any inequality conditions.

Relative to implementation in a finite-element analysis, the endochronic theory has an additional advantage. Due to the absence of inequalities in the constitutive equation, no surfaces of discontinuity in material properties exist within the structure, in contrast to elastic-plastic models for which surfaces of discontinuity separate plasticized regions and elastic regions. The absence of a discontinuity surface is consistent with the smooth interpolation polynomials in the finite elements.

4.7 The Effect of Reinforcement

The proper treatment of cracking in reinforced concrete structures should take due account of the reinforcement. Many different paths have been followed in this regard, and they can be grouped under three general headings: (1) discrete idealization, (2) smeared, and (3) embedded.

Discrete idealization. This approach is the most obvious scheme for idealization of reinforced concrete structures and was therefore the first to be employed. In the discrete idealization, different elements are used

for the steel and concrete. Additional "special" elements may be incorporated to represent bond slippage and other phenomena. Although the method has been used widely in research studies and has found application in practical analysis, it is generally not economical for large-scale applications.

Perhaps the earliest report of discrete idealization of concrete structures is that of Ngo and Scordelis,⁸¹ who employed rectangular elements for the concrete, axial members for the steel, and a zero-length spring element to represent bond slippage. Subsequent studies were conducted by Nilson,⁸² Mirza and Mufti,⁸⁶ Valliappan and Doolan,⁸⁷ Nam and Salmon,⁸⁸ and Muto et al.⁸⁹ All of these are planar analyses and, with the exception of the work by Muto, addressed research examinations.

The elements employed in the planar analysis of concrete have been of varied form, depending on the purposes of the investigation. The rectangular elements, based on bilinear displacement fields, found early use in all types of problems and remain a desirable mode of representation when there is no bias. Some analysts have used the 8-node planar element, which is the two-dimensional counterpart of the most popular three-dimensional element (the 20-node brick element). It should be observed that the quadratic displacement fields on which these elements are based implies the definition of bar elements that are three-jointed axial members. References 85 and 100, among others, employ this type of axial member formulations.

Smeared representation. In the smeared element approach, the effects of steel and concrete are assumed to be uniformly distributed over the element, with due account being taken of the directional nature of the steel reinforcement. This approach assumes perfect bonding between steel and concrete. In general, the basic constitutive relationships are expressible in the following form. For concrete alone, which is designated by the subscript "c,"

$$\{\sigma_c\} = [E_c] \{\epsilon_c\} \quad (72)$$

and, for a system of n steel bars,

$$\{\sigma_s\} = [E_s] \{\epsilon_s\} . \quad (73)$$

Now, if there is full strain compatibility,

$$\{\epsilon\} = \{\epsilon_s\} = \{\epsilon_c\} , \quad (74)$$

and, with the total stress given by the sum of steel and concrete,

$$\begin{aligned} \{\sigma\} &= \{\sigma_c\} + \{\sigma_s\} \\ &= \{[E_c] + [E_s]\} \{\epsilon\} . \end{aligned} \quad (75)$$

As in the case of discrete idealization, most of the published efforts have been concerned with planar situations. One of the most notable of these, due to its implementation of certain schemes for cracking and inelastic behavior, is that of Scanlon and Murray,¹⁰¹ but it is concerned exclusively with plate flexural elements. Indeed, many of the developments associated with smeared representations involve plate flexure, since this class of problem is not easily treated by discrete idealizations.

Constant-strain triangles have been employed by a number of authors in smeared representations. Colville and Abbasi¹⁰² use them directly, while Yuzugullu and Schnobrich¹⁰³ contrive a quadrilateral from four triangular elements. Reference 102 gives detailed relationships that account for the arbitrary orientation of a bar element within the overall element.

Three-dimensional analysis has been undertaken using the smeared approach by Suidan and Schnobrich,⁹¹ and again the basic element is the 20-node brick. The median ratios of reinforcement in the respective coordinate directions comprise the terms of a diagonal $[E_s]$ matrix, and the $[E_c]$ matrix is the conventional representation from isotropic elasticity. The treatment of cracking and of dowel action in this work should be noted.

One major unresolved question of the definition of material stiffness properties in the scheme discussed above concerns the shear effects on the crack interface. These effects are due to aggregate interlock, dowel action, and shear reinforcement. Although experimental data show this effect to be nonlinear, the conventional approach to its representation¹⁰⁴ is in a linear manner through multiplication of the material shear modulus by a factor β , where β ranges between 0 and 1. The value $\beta = 0.5$ has been used

5. Stresses and accumulated plastic strains are updated consistent with the results of step 4.

6. The process moves to the next load increment, where steps 3 to 5 are repeated.

The process of load incrementation, represented by step 6, is continued until the desired load amplitude is reached or until the structure is at its ultimate load capacity.

The tangent stiffness method is not often employed by itself in the nonlinear analysis of structures. This is true especially in the analysis of concrete structures, where the method to be described in the next section has been favored. A basic difficulty in tangent stiffness analysis is that the solution tends to drift from the correct solution as the load intensity increases. This effect can be reduced by basing the tangent properties in an increment on the estimated conditions in the middle of the increment and by using smaller increments.

Both of the above methods for reducing the errors in a tangent stiffness analysis have disadvantages. It is rather difficult to incorporate in a general-purpose, finite-element program an automatic and reliable scheme for the estimation of mid-increment stiffness properties. The reduction of increment size enhances a more basic difficulty of the tangent stiffness method — the fact that in each increment the global stiffness matrix must be reconstructed and solved again.

Initial stress method. The objective of the initial stress method in nonlinear finite-element analysis is to eliminate the difficulty of constructing and solving the global stiffness matrix in each iterative cycle. This is done by forming a "reference" global stiffness matrix, which is usually the stiffness for linear elastic behavior, and treating departures from linearity as initial forces. The transformation of departures from linearity into initial forces is not unique and gives rise to various algorithms. Attention is limited in the following discussion to the algorithm that has been most widely used in PCRV finite-element analysis — the "initial stress" approach to formation of initial forces.

In Chap. 2 of this report [see Eqs. (7) and (8)], it was observed that the formula for stiffness analysis is of the following general form:

$$\sum_{\text{vol}} \int [B]^T [D] [B] d(\text{vol}) \{\delta_j\} - \{R\} = 0, \quad (76)$$

where the matrix $[B]$ comprises the transformation of the joint displacements $\{\delta_j\}$ of a typical j th element into strains, and $[D]$ is the matrix of elastic constants. Furthermore, since

$$\{\sigma_j\} = [D] \{\epsilon_j\} = [D] [B] \{\delta_j\} \quad (77)$$

as a consequence of the stress-strain and strain-displacement laws, the stiffness equations can be written as

$$\sum_{\text{vol}} \int [B]^T \{\sigma_j\} d(\text{vol}) - \{R\} = 0. \quad (78)$$

Now, if a linear analysis is performed for an increment of load instead of the correct nonlinear inelastic analysis, the substitution of the calculated stress $\{\hat{\sigma}_j\}$ into the first term of Eq. (78) results in a residual on the right side of the equation, rather than zero -

$$\sum_{\text{vol}} \int [B]^T \{\hat{\sigma}_j\} d(\text{vol}) - \{R\} = \{\psi\} \neq 0. \quad (79)$$

Because the loads $\{R\}$ are independent of the displacements, the change in residual $\{d\psi\}$ due to the stresses $\{d\sigma\}$ associated with a change of displacement $\{d\delta\}$ can be calculated from

$$\sum_{\text{vol}} \int [B]^T \{d\sigma\} d(\text{vol}) = \{d\psi\}. \quad (80)$$

Using the residual of Eq. (80) as a load and preserving the initial stiffness $[K^1]$, the correction to the displacements is

$$\{d\delta^1\} = -[K^1]^{-1} \{\psi^1\}. \quad (81)$$

The changes in displacement so calculated permit the calculation of stress changes and the formation of a revised total stress. The revised total stress will, in general, produce changes to the displacements which will, in turn, cause new residuals, and still further corrections are made until the residuals are negligible.

The foregoing procedure has been described as if the initially imposed load were the desired full intensity of load, although it is generally impossible to achieve a convergent solution for such cases. The initial stress procedure can be identified as a form of Newton-Raphson iterative process, which itself is a linear extrapolation scheme. The linear extrapolation might not succeed when the residual is large. It is therefore customary to divide the load history into a number of intervals, as in the tangent stiffness method. If the stiffness properties pertinent to the start of the interval are used, the first step of the initial stress method is identical to the tangent stiffness procedure. Insertion of the calculated stresses into Eq. (79) gives the residual for the increment, and the correction for this can be made via Eq. (81).

The procedure to be followed in steps beyond the first is ambiguous. It is possible, of course, to construct revised stiffnesses after each cycle within a load increment. Alternatively, the stiffness that is calculated at the start of the interval can be used throughout all the cycles. Indeed, the linear stiffness matrix obtained from the first load increment might be used for this purpose.

Sarne,¹⁰⁶ among others, presents the results from numerical studies of various alternatives in the definition of stiffness properties within an interval. This work demonstrates that it is suitable to update the stiffness matrix at the start of the load interval and use it in all computational cycles within the interval.

4.9 Applications in Practice

The prevalence of numerical analyses of PCRVs using the methods previously discussed is remarkable in view of the high cost of performing such analyses. These studies require hundreds of elements and solution unknowns and involve repeated solution or iterations in the nonlinear

regime. Nevertheless, organizations in many countries (France, Japan, Germany, Denmark, United Kingdom, and Italy, to name a few) have sponsored rather substantial efforts in this respect. It should be noted that these efforts pertain mainly to previously designed PCRV structures that are of practical proportions or to scale models of existing reactor vessel designs. Few, if any, of these analyses have preceded the design phase. The following review of the analyses referred to above is limited to the most recent developments, since the intent of this section is to define more realistically the state of the art.

A useful starting point for this review is work done in the United States by Sarne.^{86,106} These efforts involved a 20-node brick isoparametric finite element to represent the concrete and an 8-node planar element to describe the liner. A scheme of 15 numerical integration points was employed to evaluate the stiffness properties of the brick element. The steel reinforcing bars were treated as axial members.

This work accounts for cracking, time-independent inelasticity, and creep. The treatment of cracking is based on an assessment of the cracking criterion at each numerical integration point; principal stresses are determined, and if the maximum principal stress exceeds that which is allowable, the stiffness in the corresponding direction is set equal to zero. The approach taken in dealing with time-independent inelasticity involves assessment of the respective principal stresses in turn, with due account being taken of the influence of stresses existing in the other two directions. For creep, a viscoelastic model consisting of Kelvin elements in series is adopted. The algorithm for creep analysis is a standard initial strain formulation. The PCRV analyzed in Ref. 86 is of realistic proportions [approximately 27.4 m (90 ft) high and 30.5 m (100 ft) in diameter, with a core cavity 11.3 m (37 ft) in diameter and heads 7.6 m (25 ft) and 9.8 m (32 ft) thick]. The analyses included dead weight, prestressing load, working pressure, and thermal gradients. The effect of creep was examined for the intended life of the structure. The finite-element idealization, it should be noted, was limited to a 30° wedge of the complete structure.

Takeda, Yamaguchi, and Imoto⁹⁰ have described the inelastic finite-element analysis of a multicavity PCRV for which 1/20-scale test data were

available.¹⁰⁷ Two independent analyses, one with axisymmetric triangular ring elements and the other with 20-node brick elements, were performed. In the planar case, the analysis employed the tangent stiffness approach to the inelastic calculations with the Prager-Drucker yield criterion,¹⁶³ wherein the yield function f is given by

$$f = \alpha J_1 + \sqrt{J_2} \quad , \quad (82)$$

where J_1 and J_2 are the first and second stress invariants and α is a material-dependent constant. A secant stiffness approach was taken in the three-dimensional analysis.

In these analyses, as in the work by Sarne,¹⁰⁶ the steel liner is modeled by planar elements and the condition of cracking is determined by testing the principal stresses to see if the maximum value equals or exceeds the allowable. The analysis test results of the paper show, in general, a high degree of correspondence.

Zienkiewicz, Phillips, and Owen¹⁰⁴ have described analyses of a previously tested reactor vessel¹⁰⁹ that are noteworthy on account of the use of isoparametric reinforcing elements embedded in 20-node brick isoparametric elements. Cracking is treated by assessing the principal stresses at the numerical integration points in the same manner as described above for other applications, but after cracking, the aggregate-interlock effect on the crack surface is accounted for by using a reduced shear modulus at the integration point. The nonlinear character of the concrete stress-strain relationship, aside from cracking effects, is accounted for by the initial strain algorithm discussed in a previous section using a constitutive law that accounts for both hydrostatic (J_1) and deviatoric (J_2) stress invariants.

The most extensive reports of applications analyses of PCRVs appear to be those by Saugy et al.^{85,110} They use 20-node isoparametric brick elements in a general three-dimensional analysis. The emphasis of this work is upon constitutive relationships for time-independent plasticity and creep, with cracking accounted for as described in the prior developments but without consideration of interface shear transfer. Analyses

have been conducted of a hypothetical reactor structure of realistic proportions, and more recently¹¹⁰ comments have been made regarding comparisons with test data from an actual structure of this type. Results for the latter, however, have not yet been reported.

Another finite-element PCRV analysis conducted along the above general lines was one described by Wade and Henrywood,¹¹¹ who have been concerned with the design of United Kingdom and German reactors. Hexahedronal elements with 20 nodes were employed in the finite-element discretization. Analyses were performed for pressure, temperature, and prestress loads in the presence of creep. It appears that neither cracking nor time-independent inelasticity were taken into account in this analysis.

Studies in progress in Denmark by Ottosen and Anderson¹¹² and in Italy by Ballio, Castellani, and Finzi¹¹³ bear similarities to the work discussed above.

It can be seen, then, that the 20-node isoparametric brick element is virtually a standard in large-scale practical analyses of PCRVs. Eight-node planar elements, which are compatible with the brick elements, are used to model the liners. There are significant differences among the reported analyses in the number of points used for numerical integration of the element stiffness properties. Customarily, the principal stresses are calculated at each numerical integration point for each load level. If a principal stress is identified as being larger than the tensile strength, a crack is defined in the plane perpendicular to the corresponding principal direction. Some treatments approximate the influence of compressive stresses in the perpendicular direction on the tensile strength. The representation of interface shear transfer is conceptually possible, but few groups have included this effect in numerical analyses because of scarcity of information about the shear stiffness properties.

There appears to be no uniformity in adopted yield functions in the analyses reported to date. Generally, these yield functions are chosen to correspond to test data generated in the past by the group involved in the finite-element analysis. Similarly, there is a divergence of views on the algorithms for inelastic analysis, although there is a decided preference for initial-stress algorithms since these avoid the necessity for repeated

solution of large-scale systems of algebraic equations. Initial-strain algorithms continue to be standard in creep analysis.

As in the case of interface shear stiffness, the proper treatment of reinforcement, including dowel and bond action, continues to be an open question. Although many different attempts at finite-element representation of these phenomena have been made, insufficient PCRV experimental data are available to define their adequacy.

5. CONCLUSIONS

Concrete is a complex composite material with a relatively low degree of chemical stability whose microstructure reacts to changes in temperature, water content, or stress. Fortunately, at low stress levels and quasi-constant temperatures, its response to stress can be approximated within a reasonable degree of accuracy as a linear-viscoelastic material. Because prestressed concrete pressure vessels are designed to operate at low stress levels, the only unresolved question pertaining to their analysis for normal operating conditions is that of resolution and/or validation of an appropriate model for thermal transient conditions. However, the geometric complexity of the PCRV introduces numerical difficulties that require special attention if accurate numerical results are to be obtained.

On the other hand, the proper theoretical model for analysis of a PCRV under conditions of accidental overpressurization is not clear. All too often the models commonly used suffer from deficiencies of either a theoretical or a practical nature. The most critical aspect pertains to modeling of tensile cracks in plain and reinforced concrete. Of slightly less importance is the modeling of nonlinear and postfailure modes under compressive loading conditions. Until these problems are resolved, the results of such analyses must be viewed with some skepticism.

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