FLUX TUBES AND THE MAGNETIC CONFINEMENT OF QUARKS

By

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H. W. Wyld and R. T. Cutler

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FLUX TUBES AND THE MAGNETIC CONFINEMENT OF QUARKS*

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ABSTRACT

A numerical study is made of the gauge field model of magnetic confinement. The nonlinear differential equations describing a flux tube are solved by a relaxation method. Particular attention is paid to the boundary conditions at the center of the flux tube and the effect of these boundary conditions in differentiating between Abelian and non-Abelian models and between flux tubes containing different numbers of units of flux.

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I. INTRODUCTION

There is much current speculation aimed at devising a mechanism for quark confinement. Two general classes of models are studied, "electric" confinement and "magnetic" confinement, both based on gauge field theories of a quark gluon interaction. Although the electric confinement mechanism^{1-3/} is perhaps a priori the more plausible of the two, it is much more difficult to make it work, and at present this possibility is speculative. On the other hand the magnetic confinement mechanism^{4-13/} does work, although one may be reluctant to accept the many assumptions made at the outset.

In the magnetic confinement models the quarks are endowed with a "magnetic" monopole moment, albeit in a color space or other space different from the flavor space which contains the usual electric and magnetic charges. There is also assumed a Higgs field which undergoes a spontaneous symmetry breakdown to provide the gauge bosons with a mass. The Higgs field has the usual type of electric coupling to the gauge field, i.e. there are no magnetic monopoles in the Higgs field. The magnetic flux produced by the quark magnetic monopoles is thus shielded by the Higgs field by a mechanism strictly analogous to the Meissner effect in a superconductor. The magnetic field produced by a quark monopole is channeled into a flux tube; a quark antiquark pair is connected by such a flux tube. For appreciable separations of the quark antiquark pair the energy of this flux tube is proportional to its length, and this is the

desired confinement potential.

In the electric confinement models it is hoped that something similar happens: that the electric flux is confined to a flux tube by the infrared divergences and/or the nonlinear interactions of the gauge field theory. We will have nothing more to say about electric confinement in the present paper.

The flux tube mechanism of magnetic confinement was invented by Nielsen and Olesen^{4/}, Nambu^{5/}, and Parisi^{6/} and has been developed in subsequent papers by 't Hooft^{7/}, Polyakov^{8/}, Mandelstam^{9/}, Eguchi^{10/}, Ezawa and Tze^{11/}, Corrigan, et al.^{12/} and many others. For the most part these papers deal with solutions of the field equations interpreted as classical c-number equations. An analytical approximation scheme has been given by Patkós^{13/}. Much of the work is a direct carryover from developments in the theory of superconductivity^{14/}.

In the present paper we attempt to clarify and illustrate the model by giving some numerical solutions for the classical field equations. The theory is reviewed and numerical results for the Abelian version of the model are given in Sec. 2 for the case of an infinitely long flux tube, i.e. the vortex tube of Nielsen and Olesen^{4/}. In Sec. 3 an extension of the theory and numerical results are provided for one interpretation of a non-Abelian model. It turns out that the equations to be solved are the same as for the Abelian model, but the boundary conditions both at the center of the flux tube and at infinity are different. In Sec. 4 we discuss briefly some attempts to carry out the numerical work for a flux tube of finite length,

i.e. a quark antiquark pair a finite distance apart.

The equations to be solved in this work are nonlinear differential equations with boundary conditions given at both $r = 0$ and $r = \infty$. They are solved by the relaxation method of Henyey, Wilets, Böhm, LeLevier and Levee^{15/}, which is widely employed by astrophysicists for solving the differential equations describing stellar structure^{16/}. This method would appear to be generally useful for obtaining numerically soliton type solutions of classical nonlinear field equations. We devote an appendix to a brief discussion of the method and the experience we have with it for the problems discussed in this paper. The method has been used by Nohl^{17/} to solve a problem similar to those discussed here.

II. ABELIAN MODEL

We treat the magnetic monopole quarks as external sources and account for them through boundary conditions on the gauge fields and Higgs field. For the Abelian model the Lagrangian for the gauge field A_μ coupled to the scalar Higgs field ψ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \left(\frac{\partial}{\partial x_\mu} - ieA_\mu \right) \psi \left(\frac{\partial}{\partial x_\mu} + ieA_\mu \right) \psi^* + \mu^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2 \quad (2.1)$$

with

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} . \quad (2.2)$$

This leads to field equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = -ie \left(\psi^* \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \psi^*}{\partial x_\mu} \psi \right) - 2e^2 A_\mu \psi^* \psi \quad (2.3)$$

and

$$\left(\frac{\partial}{\partial x_\mu} - ieA_\mu \right)^2 \psi = -\mu^2 \psi + \lambda (\psi^* \psi) \psi. \quad (2.4)$$

For a time independent magnetic solution, take $A_4 = 0$ and A_i ($i = 1, 2, 3$) and ψ independent of time. With

$$\vec{B} = \text{curl } \vec{A} \quad (2.5)$$

the field equations (2.3), (2.4) become

$$\text{curl } \vec{B} = -ie \left[\psi^* \nabla \psi - (\nabla \psi^*) \psi \right] - 2e^2 \vec{A} \psi^* \psi \quad (2.6)$$

and

$$(\nabla^2 - 2ie\vec{A} \cdot \nabla - e^2 \vec{A}^2) \psi = -\mu^2 \psi + \lambda (\psi^* \psi) \psi, \quad (2.7)$$

provided the gauge is chosen so that

$$\text{div } \vec{A} = 0. \quad (2.8)$$

For one or more magnetic monopoles along the z axis, take

$$\vec{A} = \hat{\phi} A(\rho, z), \quad (2.9)$$

$$\psi = e^{im\phi} \psi(\rho, z), \quad m = \text{integer}, \quad (2.10)$$

where $\hat{\phi}$ is a unit vector in the ϕ direction, ρ, ϕ, z are the usual cylindrical coordinate, and $\psi(\rho, z)$ is a real function.

With these assumptions (2.8) is satisfied and the field

equations (2.6), (2.7) reduce to

$$\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A}{\partial \rho} - \frac{1}{\rho^2} A = 2e^2 \left(A - \frac{m}{e\rho} \right) \psi^2, \quad (2.11)$$

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = e^2 \psi \left(A - \frac{m}{e\rho} \right)^2 - \mu^2 \psi + \lambda \psi^3. \quad (2.12)$$

If the only sources are magnetic monopoles of opposite sign at $z = \pm \infty$, A and ψ are independent of z , and we have the vortex tube of Nielson and Olesen. For this case the coupled differential equations are ordinary differential equations:

$$\frac{d^2 A}{d\rho^2} + \frac{1}{\rho} \frac{dA}{d\rho} - \frac{1}{\rho^2} A = 2e^2 \left(A - \frac{m}{e\rho} \right) \psi^2, \quad (2.13)$$

$$\frac{d^2 \psi}{d\rho^2} + \frac{1}{\rho} \frac{d\psi}{d\rho} = e^2 \left(A - \frac{m}{e\rho} \right)^2 \psi - \mu^2 \psi + \lambda \psi^3. \quad (2.14)$$

In addition to (2.13), (2.14) we need boundary conditions. For the vortex tube the appropriate conditions are

$$\rho \rightarrow \infty : \quad \psi \rightarrow \sqrt{\mu^2/\lambda}, \quad (2.15)$$

$$A \rightarrow m/e\rho, \quad (2.16)$$

$$\rho \rightarrow 0 : \quad \psi \rightarrow a\rho^m, \quad (2.17)$$

$$A \rightarrow b\rho, \quad (2.18)$$

with a and b unknown constants. In (2.15) we assume for $\rho \rightarrow \infty$

the usual symmetry breakdown value for the Higgs field. If this value is substituted on the right of Eq. (2.13) we obtain a form of Bessel's equation for $A_1 = A - m/ep$. [Note that m/ep is a pure gauge field and reduces the left side of (2.13) to zero]. One solution of the Bessel equation grows exponentially with ρ and is rejected. The other decays exponentially, leading to the boundary condition (2.16).

To obtain the boundary conditions (2.17), (2.18) at $\rho \rightarrow 0$, first note that the flux through a circle of radius ρ centered on the flux tube is given by

$$\Phi(\rho) = \int \text{curl } \vec{A} \cdot d\vec{s} = \oint \vec{A} \cdot d\vec{l} = 2\pi\rho A(\rho). \quad (2.19)$$

If we assume that $\Phi(\rho) \rightarrow 0$ as $\rho \rightarrow 0$, thus for example eliminating a Dirac string at $\rho = 0$, we see that $A(\rho)$ is less singular than ρ^{-1} at $\rho \rightarrow 0$. The leading terms in (2.14) for small ρ are then

$$\frac{d^2\psi}{d\rho^2} + \frac{1}{\rho} \frac{d\psi}{d\rho} - \frac{m^2}{\rho^2}\psi = 0. \quad (2.20)$$

This has solutions $\rho^{\pm m}$. Eliminating the singular solution leads to the boundary condition (2.17). The leading terms in (2.13) at small ρ are then

$$\frac{d^2A}{d\rho^2} + \frac{1}{\rho} \frac{dA}{d\rho} - \frac{1}{\rho^2}A = 0 \quad (2.21)$$

with solutions $\rho^{\pm 1}$. The condition that A be less singular than ρ^{-1} then leads to the boundary condition (2.18).

The integer m is the number of flux quanta in the flux tube. According to (2.19) for $\rho \rightarrow \infty$ and (2.16) we find for the total flux

$$\Phi = \frac{2\pi m}{e} . \quad (2.22)$$

In terms of the magnetic monopole moment g of the monopoles at the end of the flux tube, the total flux is $\Phi = g$ (in rationalized units), so we have the Dirac quantization condition

$$eg = 2\pi m . \quad (2.23)$$

The energy per unit length of the flux tube relative to the Higgs vacuum is given by

$$\begin{aligned} \frac{E}{L} = 2\pi \int_0^\infty \rho d\rho \left[\left(\frac{\partial \psi}{\partial \rho} \right)^2 + e^2 \psi^2 \left(A - \frac{m}{e\rho} \right)^2 + \frac{1}{2} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A) \right\}^2 \right. \\ \left. - \mu^2 \psi^2 + \frac{\lambda}{2} \psi^4 + \frac{\mu^4}{2\lambda} \right] . \end{aligned} \quad (2.24)$$

Eliminating the derivatives in this expression by integrating by parts and using the field equations (2.13), (2.14), we find

$$\frac{E}{L} = \frac{\pi}{e} B(0) + 2\pi \int_0^\infty \rho d\rho \left[-e^2 \psi^2 \left(A - \frac{m}{e\rho} \right)^2 - \frac{\lambda}{2} \left(\psi^4 - \frac{\mu^4}{\lambda} \right) \right] , \quad (2.25)$$

where the magnetic field is given by

$$B(\rho) = \frac{1}{\rho} \frac{d}{d\rho} (\rho A) . \quad (2.26)$$

For numerical purposes it is convenient to introduce dimensionless variables. The masses of the scalar and vector mesons

after the spontaneous symmetry breakdown are

$$m_s = \sqrt{2\mu^2} \quad , \quad (2.27)$$

$$m_v = \sqrt{\frac{2e^2\mu^2}{\lambda}} \quad . \quad (2.28)$$

Introduce a dimensionless length ρ' ,

$$\rho = \frac{1}{m_v} \rho' \quad , \quad (2.29)$$

and dimensionless scalar and vector fields ψ', A' ,

$$\psi = \sqrt{\frac{\mu}{\lambda}} \psi' \quad , \quad (2.30)$$

$$A = \frac{m_v}{e} A' \quad . \quad (2.31)$$

In terms of these variables there is a dimensionless coupling constant

$$G = \frac{\lambda}{e^2} = \frac{m_s^2}{m_v^2} \quad . \quad (2.32)$$

The field equations (2.13), (2.14) in terms of dimensionless variables become (dropping primes)

$$\frac{d^2 A}{d\rho^2} + \frac{1}{\rho} \frac{dA}{d\rho} - \frac{1}{\rho^2} A = \psi^2 \left(A - \frac{m}{\rho} \right) \quad , \quad (2.33)$$

$$\frac{d^2 \psi}{d\rho^2} + \frac{1}{\rho} \frac{d\psi}{d\rho} = \psi \left(A - \frac{m}{\rho} \right)^2 - \frac{1}{2} G (\psi - \psi^3) \quad , \quad (2.34)$$

with boundary conditions

$$\rho \rightarrow \infty: \psi \rightarrow 1, \quad (2.35)$$

$$A \rightarrow m/\rho,$$

$$\rho \rightarrow 0: \psi \rightarrow a\rho^m, \quad (2.36)$$

$$A \rightarrow b\rho.$$

and the energy per unit length (2.25) becomes

$$\frac{E}{L} = \pi \frac{m_V^2}{e^2} f(G), \quad (2.37)$$

$$f(G) = B(0) + C, \quad C = \int_0^\infty \rho d\rho \left\{ -\psi^2 \left(A - \frac{m}{\rho} \right)^2 + \frac{1}{4} G \left(1 - \psi^4 \right) \right\}, \quad (2.38)$$

with

$$B(\rho) = \frac{1}{\rho} \frac{d}{d\rho} (\rho A). \quad (2.39)$$

Some numerical results for the equations (2.33)-(2.39) for the case $m = 1$ (one unit of flux) are presented in Figs. 1,2,3. These results were obtained with the relaxation method discussed in the appendix. Fig. 1 presents $B(\rho)$ as a function of ρ for various values of G , ranging from weak coupling ($G=0.1$) to strong coupling ($G=100.0$). Corresponding results for the Higgs field $\psi(\rho)$ are presented in Fig. 2. For strong coupling ψ maintains its asymptotic value outside of a narrow core. This is the basis of an approximation technique in which ψ is replaced by its asymptotic value in Eq.(2.13)-- see for example Ref.[3]. In Fig. 3 we give $B(0)$, the magnetic field at the center of the flux tube, as a function of the coupling parameter G . Also plotted in Fig. 3 is the quantity $f(G) = B(0) + C$, which is directly proportional to the energy per unit length

of the flux tube. We see that these quantities are very slowly varying function of G .

We can attempt to attach some physical significance to these numerical values by comparison with phenomenological potentials used in recent studies of charmonium spectroscopy. Eichten et al.^{18/} use a potential

$$V(r) = - \frac{\alpha_s}{r} \left[1 - \left(\frac{r}{a} \right)^2 \right] \quad (2.40)$$

and find that the values $\alpha_s = 0.2$, $a = 0.2\text{fm.}$, fit the data. The Coulomb potential part of (2.40) determines the monopole strength g and hence the gauge coupling constant by (2.23):

$$\alpha_s = 0.2 = \frac{g^2}{4\pi} = \frac{\pi}{e^2}, \quad \frac{e^2}{4\pi} = 1.25. \quad (2.41)$$

The linear term in the potential (2.40) is the energy per unit length we have calculated, (2.37), (2.38):

$$\pi \frac{m_v^2}{e^2} f(G) = \frac{\alpha_s}{a^2} = \frac{0.2}{(0.2\text{fm})^2} = 10 m_\pi^2, \quad m_v^2 = \frac{10 e^2}{\pi} m_\pi^2 \frac{1}{f(G)}. \quad (2.42)$$

With e given by (2.41), the vector meson mass is given by (2.42). The parameter G is not fixed by these considerations, but $f(G)$ is a slowly varying function. Using the value (2.41) for e and the curve in Fig. 3, we calculated the scalar Higgs meson and vector meson masses for typical values of G , as given in Table 1.

Table 1

Some Numerical Values given by the Formulas (2.41), (2.42)

G	f(G)	m_v	$m_s = \sqrt{G} m_v$
0.1	.610	1.28 GeV	0.40 GeV
1.0	.975	1.01	1.01
10.0	1.626	.78	2.47
100.0	2.53	.63	6.30

III. NON-ABELIAN MODEL

We consider in this section one possible non-Abelian generalization of the calculation presented in section 2. Our model is an SU(3) version of an SU(2) model due to 't Hooft^{19/}. There are now 8 vector meson fields A_μ^a ($a = 1, \dots, 8$) and 8 field tensors $F_{\mu\nu}^a$. Using Gell-Mann's 8 traceless 3x3 Hermitian matrices λ_a ^{20/} we can write the field quantities as 3x3 matrices:

$$A_\mu = A_\mu^a \frac{\lambda_a}{2}, \quad (3.1)$$

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\lambda_a}{2}$$

$$= \frac{\partial}{\partial x_\mu} A_\nu - \frac{\partial}{\partial x_\nu} A_\mu - ie[A_\mu, A_\nu]. \quad (3.2)$$

The Higgs field is now a spin scalar and 3 component color spinor Ψ . The covariant derivative of Ψ is

$$D_\mu \Psi = \left(\frac{\partial}{\partial x_\mu} - ieA_\mu \right) \Psi. \quad (3.3)$$

In terms of these quantities the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 - (D_\mu \Psi)^* (D_\mu \Psi) + \mu^2 \Psi^* \Psi - \frac{\lambda}{2} (\Psi^* \Psi)^2. \quad (3.4)$$

The field equations are

$$\frac{\partial}{\partial x_\nu} F_{\mu\nu} - ie[A_\nu, F_{\mu\nu}] = ie \sum_a \frac{\lambda_a}{2} \left[(D_\mu \Psi)^* \frac{\lambda_a}{2} \Psi - \Psi^* \frac{\lambda_a}{2} D_\mu \Psi \right],$$

$$\left(\frac{\partial}{\partial x_\mu} - ieA_\mu \right)^2 \Psi = -\mu^2 \Psi + \lambda (\Psi^* \Psi) \Psi. \quad (3.5)$$

One can demonstrate the spontaneous symmetry breakdown by taking

$$\Psi = \sqrt{\frac{\mu}{\lambda}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left(Z + i \frac{\lambda_a}{2} \chi_a \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (3.6)$$

and working out the linearized field equations (treating Z , χ_a , A_μ^a as small and real). One finds a scalar meson Z of mass

$$m_s = 2\mu^2 \quad (3.7)$$

and 8 vector mesons

$$V_\mu^a = A_\mu^a - \frac{\sqrt{\lambda}}{e\mu} \frac{\partial \chi_a}{\partial x_\mu}. \quad (3.8)$$

The triplet ($a = 1, 2, 3$) has mass zero, the two doublets ($a = 4, 5, 6, 7$) have mass $= (e^2 \mu^2 / 2\lambda)^{\frac{1}{2}}$, and the singlet ($a = 8$) has mass

$$m_V = \sqrt{\frac{2}{3}} \frac{e^2 \mu^2}{\lambda} . \quad (3.9)$$

For a time independent "magnetic" solution we set $A_4 = 0$ and take all time derivatives to vanish. Then in terms of

$$\vec{B} \equiv \text{curl } \vec{A} - ie\vec{A} \times \vec{A} \quad (3.10)$$

the field equations become

$$\begin{aligned} \text{curl } \vec{B} - ie(\vec{A} \times \vec{B} + \vec{B} \times \vec{A}) &= ie \sum_a \frac{\lambda_a}{2} \left[\left\{ (\nabla - ie\vec{A})\psi \right\}^* \frac{\lambda_a}{2} \psi \right. \\ &\quad \left. - \psi^* \frac{\lambda_a}{2} \left\{ (\nabla - ie\vec{A})\psi \right\} \right] , \end{aligned} \quad (3.11)$$

$$(\nabla - ie\vec{A})^2 \psi = -\mu^2 \psi + \lambda (\psi^* \psi) \psi . \quad (3.12)$$

Eqs. (3.10) and (3.11) have interesting nonlinear terms not present in the Abelian equations of sec. 2. However, they will be eliminated by the ansätze below, which are designed to yield a vortex solution analogous to that of sec. 2. First assume that $\vec{A}$ has only a ϕ -component in 3 space:

$$\vec{A} = A \hat{\phi} . \quad (3.13)$$

Next assume that A has only an $a = 8$ component in $SU(3)$ space,

$$A = \frac{\lambda_8}{2} A(\rho, z) , \quad (3.14)$$

and that ψ is of the form

$$\psi = \begin{pmatrix} 0 \\ 0 \\ -i\frac{m}{3}\phi \\ e^{\psi(\rho, z)} \end{pmatrix}, \quad m = \text{integer}, \quad (3.15)$$

with ψ a real function. The question whether m in (3.15) need be an integer will be discussed presently. With these ansätze the current on the right of (3.11) has only an $a = 8$ component, and Eqs. (3.11), (3.12) reduce to

$$\frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A}{\partial \rho} - \frac{1}{\rho^2} A = \frac{2}{3} e^2 \left(\lambda - \frac{\sqrt{3}m}{3e\rho} \right)^2 \psi^2, \quad (3.16)$$

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = \frac{1}{3} e^2 \left(\lambda - \frac{\sqrt{3}m}{3e\rho} \right)^2 \psi - \mu^2 \psi + \lambda \psi^3. \quad (3.17)$$

Also (3.10) gives a field B in the z -direction with only an $a = 8$ component given by

$$B = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A). \quad (3.18)$$

The equations (3.16)-(3.18) are very similar to their non-Abelian counterparts (2.11), (2.12). Setting $e'^2 = e^2/3$ we find for the z -independent case

$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial A}{\partial \rho} - \frac{1}{\rho^2} A = 2e'^2 \left(\lambda - \frac{m}{3e'\rho} \right)^2 \psi^2 \quad (3.19)$$

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = e'^2 \left(\lambda - \frac{m}{3e'\rho} \right)^2 \psi - \mu^2 \psi + \lambda \psi^3 \quad (3.20)$$

and the $a = 8$ color singlet vector meson mass (3.9) reduces to

$$m_v^2 = \frac{\sqrt{2} e'^2 \mu^2}{\lambda} . \quad (3.21)$$

If we identify e' of sec. 3 with e of sec. 2, Eqs. (3.19), (3.20), and (3.21) differ from (2.13), (2.14), and (2.28) only in that m in (2.13), (2.14) is replaced by $m/3$. This is directly related to the ansatz (3.15)

This brings us to the question of boundary conditions. For the Abelian model of sec. 2, m in (2.10) is taken to be an integer in order to ensure that Ψ , the Higgs field, be single valued. For the SU(3) case this is generalized to the requirement that increasing ϕ by 2π change the fields by a gauge transformation from the center of the SU(3) group. The center of a group is those elements which commute with all group members. The requirement above thus ensures that matrix elements of any operator O are invariant under the transformation $\phi \rightarrow \phi + 2\pi$:

$$\langle \Psi | C^{-1} O C | \Psi' \rangle = \langle \Psi | C^{-1} C O | \Psi' \rangle = \langle \Psi | O | \Psi' \rangle . \quad (3.22)$$

The center of SU(3) is

$$I, e^{i\frac{2\pi}{3}} I, e^{i\frac{4\pi}{3}} I , \quad (3.23)$$

where I is the 3×3 unit matrix. Thus, m in (3.15) must be an integer.

An argument completely analogous to that given in sec. 2 now yields the boundary conditions to be used in solving (3.19), (3.20):

$$\rho \rightarrow \infty: \quad \psi \rightarrow \sqrt{\mu^2/\lambda} , \quad (3.24)$$

$$A \rightarrow \frac{m}{3e^2\rho} , \quad (3.25)$$

$$\rho \rightarrow 0 \quad \psi \rightarrow a\rho^{m/3} , \quad (3.26)$$

$$A \rightarrow b\rho . \quad (3.27)$$

Figs. 4,5,6 present numerical results for the Eqs.(3.19)-(3.27) for the case $m = 1$ (one unit of flux). The curves are strictly analogous to Figs. 1,2,3. The differences are due to the changed boundary condition on ψ at $\rho \rightarrow 0$: $\psi \propto \rho$ for the Abelian case, $\psi \propto \rho^{1/3}$ for the non-Abelian case. This results in much narrower structures for both $\psi(\rho)$ and $B(\rho)$.

In Figs.7 and 8 we present the curves for $B(\rho)$ and $\psi(\rho)$ for different numbers of flux units ($m=1,4,7$) at a fixed $G=1.0$. As the number of flux units increases, $B(0)$ at the center of the flux tube stays nearly constant, but the flux tube becomes wider to accomodate the larger flux. The increased width of the flux tube is visible both in the $B(\rho)$ curves and the $\psi(\rho)$ curves. The direct cause of these variations is the variation of the value of m in the boundary condition (3.26) at the center of the flux tube. The energy per unit length is different for the three cases plotted in Figs.7 and 8. Using the formula (2.38) with $e \rightarrow e'$, $m \rightarrow m/3$, we find $B(0)$ almost constant and C roughly proportional to the number of flux units.

For 1,4,7 units of flux we find respectively $(e^2/\pi m_v^2)(E/L)$
 $= 0.675, 1.127, \text{ and } 1.565.$

We have called attention to these results because they appear to contradict statements made in the papers by Mandelstam ^{9/} and Ezawa and Tze ^{11/}. According to Mandelstam the number of units of flux is "only defined" modulo 3 for an SU(3) non-Abelian model. According to Ezawa and Tze the gauge fields corresponding to flux tubes with numbers of flux units differing by 3 can be "brought into" each other by a continuous gauge transformation. For the model discussed in the present paper it would appear that flux tubes with 1,4, and 7 units of flux are different physical objects, not related by gauge transformations.

On the other hand, Mandelstam's argument ^{9/} [for an SU(3) model] demonstrates that the asymptotic boundary conditions have solutions which change continuously from 7 to 4 to 1 units of flux as some external parameter is varied. This may indicate that the complete solutions (including boundary conditions at $r=0$) for 7,4, and 1 units of flux are separated by finite energy barriers and can decay into one another by quantum mechanical tunneling. ^{12,17,21,22/} No discussion has been given of the details of such a decay process, what is emitted in the decay, or the lifetime.

Finally we have some rather incomplete comments on how one might convert the somewhat special results above into a

more general confinement potential for a quark-antiquark system. The quark and antiquark are at the ends of the flux tube, and the potential energy is proportional to its length for appreciable separations. We can think of this flux tube as the result of the influence of the Higgs field on the magnetic Coulomb interaction between a monopole-antimonopole pair; the Higgs field channels the flux into a narrow tube. In the calculation above based on the ansatz (3.14), (3.15), an $a=8$ type monopole "emits" an $a=8$ type gauge boson which is "absorbed" by the anti-monopole. However, by a space independent rotation in $SU(3)$ we can turn the $a=8$ type sources and fields into $a=7$ types or $a=5$ types, etc. One possibility would be that the coupling between monopoles and fields is such as to correspond to octet exchange of the shielded gauge boson in the t -channel. A simple crossing matrix calculation shows that pure octet in the t -channel crosses to singlet and octet in the s -channel with coefficients $2/3$ and $-1/3$:

$$O_t = \frac{2}{3}S_s - \frac{1}{3}O_s \quad (3.28)$$

Thus the potential is attractive only in the singlet state of the s -channel. We have not attempted to write down this model as a full fledged field theory. We note, however, that it differs from the model considered by Mandelstam and Ezawa and Tze in that the color index on the gauge field is now the same as the usual color index for the quark field.

IV. DIPOLE CALCULATIONS

In this section we consider the solution of the Eqs.(2.11), (2.12) for a dipole consisting of positive and negative magnetic monopoles ($\pm g$) a distance $2b$ apart - see Fig.9. First we remind the reader of the vector potential for magnetic monopoles and dipoles in the absence of the Higgs field. For a single monopole g located at the origin the vector potential (in spherical coordinates)

$$A_\phi = - \frac{g}{4\pi r} \frac{\sin \theta}{1 - \cos \theta}, \quad A_\theta = A_r = 0, \quad (4.1)$$

gives a Coulomb field

$$B_r = \text{curl}_r \vec{A} = \frac{g}{4\pi r^2}, \quad B_\theta = B_\phi = 0. \quad (4.2)$$

The direction of the Dirac string is arbitrary. In (4.1) it lies along the line $\cos \theta = 1$. For the dipole of Fig.9 we have

$$\begin{aligned} A_\phi &= - \frac{g}{4\pi} \left[\frac{\sin \theta_1}{r_1(1 - \cos \theta_1)} - \frac{\sin \theta_2}{r_2(1 - \cos \theta_2)} \right] \\ &= - \frac{g}{2\pi b} \frac{\sqrt{1 - \eta^2}}{\sqrt{\xi^2 - 1}} \frac{\xi}{(\xi^2 - \eta^2)} \\ &= - \frac{1}{eb} \frac{\sqrt{1 - \eta^2}}{\sqrt{\xi^2 - 1}} \frac{\xi}{(\xi^2 - \eta^2)}. \end{aligned} \quad (4.3)$$

Here we have introduced elliptic coordinates

$$\xi = \frac{r_1 + r_2}{2b}, \quad \eta = \frac{r_1 - r_2}{2b}, \quad (4.4)$$

$$\rho = b\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \quad z = b\xi\eta, \quad (4.5)$$

and used the quantization condition (2.23), assuming $m=1$.

We now transform to elliptic coordinates in (2.11), (2.12).

Take $m=1$ in those equations and use as dependent variable

$$A_1 = A - \frac{1}{e\rho} . \quad (4.6)$$

Further, introducing dimensionless variables ψ' , A_1'

$$\begin{aligned} \psi &= \sqrt{\frac{\mu}{\lambda}} \psi' , \\ A_1 &= \frac{1}{eb} A_1' , \end{aligned} \quad (4.7)$$

and dropping the primes we obtain in place of (2.11) (2.12)

$$\begin{aligned} \frac{1}{\xi^2 - \eta^2} \left[\frac{\partial}{\partial \xi} \left\{ (\xi^2 - 1) \frac{\partial A_1}{\partial \xi} \right\} + \frac{\partial}{\partial \eta} \left\{ (1 - \eta^2) \frac{\partial A_1}{\partial \eta} \right\} - \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} A_1 \right] \\ = R^2 \psi^2 A_1 , \end{aligned} \quad (4.8)$$

$$\begin{aligned} \frac{1}{\xi^2 - \eta^2} \left[\frac{\partial}{\partial \xi} \left\{ (\xi^2 - 1) \frac{\partial \psi}{\partial \xi} \right\} + \frac{\partial}{\partial \eta} \left\{ (1 - \eta^2) \frac{\partial \psi}{\partial \eta} \right\} \right] \\ = \psi A_1^2 - \frac{1}{2} G R^2 (\psi - \psi^3) . \end{aligned} \quad (4.9)$$

There are two dimensionless parameters in these equations,

$R = m_v b$ and G , with m_v given by (2.28) and G by (2.32).

A convenient way to incorporate the boundary conditions on A_1 is to factor out the dipole form (4.3). Thus we write (in dimensionless variables)

$$A_1 = - \frac{\sqrt{1 - \eta^2}}{\sqrt{\xi^2 - 1}} \frac{\xi}{\xi^2 - \eta^2} F(\xi, \eta) . \quad (4.10)$$

The boundary conditions are now

$$\xi \rightarrow \infty : F(\xi, \eta) \rightarrow 0 , \quad (4.11)$$

$$\psi(\xi, \eta) \rightarrow 1 , \quad (4.12)$$

$$\xi \rightarrow 1 : F(\xi, \eta) \rightarrow 1 , \quad (4.13)$$

$$\psi(\xi, \eta) \rightarrow 0 . \quad (4.14)$$

Actually, the relaxation method of solving the differential equation will not converge unless one specifies the manner of variation near the singular line $\xi \rightarrow 1$, particularly near the singular points $\xi \rightarrow 1, \eta \rightarrow \pm 1$, where the monopoles are located-- this was also necessary in the one dimensional case; see (2.17), (2.18), (A.19). It is also convenient to spread out these singular regions by a change of variables

$$x = \sqrt{\xi^2 - 1} , \quad y = \sqrt{1 - \eta^2} , \quad (4.15)$$

such that

$$\begin{aligned} \rho &= bx , \quad |z|=0 && \text{when } y = 1 , \\ \rho &= 0 , \quad |z|=b\sqrt{1+x^2} && \text{when } y = 0 , \\ \rho &= 0 , \quad |z|=b\sqrt{1-y^2} && \text{when } x = 0 , \\ \rho &= bxy, \quad |z|=bx\sqrt{1-y^2} && \text{when } x \rightarrow \infty . \end{aligned} \quad (4.16)$$

In terms of these variables, Eqs.(4,8),(4,9) become

$$\begin{aligned} (x^2+1)\frac{\partial^2 F}{\partial x^2} - \left(\frac{1}{x} + \frac{2x(2+x^2-y^2)}{x^2+y^2}\right)\frac{\partial F}{\partial x} + (1-y^2)\frac{\partial^2 F}{\partial y^2} \\ + \left(-\frac{1}{y} + \frac{4x^2}{x^2+y^2}\frac{1-y^2}{y}\right)\frac{\partial F}{\partial y} = (x^2+y^2)K^2\psi^2 F, \end{aligned} \quad (4.17)$$

$$\begin{aligned} (x^2+1)\frac{\partial^2 \psi}{\partial x^2} + \left(2x+\frac{1}{x}\right)\frac{\partial \psi}{\partial x} + (1-y^2)\frac{\partial^2 \psi}{\partial y^2} + \left(-2y+\frac{1}{y}\right)\frac{\partial \psi}{\partial y} \\ = \frac{x^2+1}{x^2}\frac{y^2}{x^2+y^2}F^2\psi - \frac{1}{2}GR^2(x^2+y^2)(\psi-\psi^3). \end{aligned} \quad (4.18)$$

For x and y both small the leading terms in these equations are

$$xy(x^2+y^2)\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) - \left[y(x^2+y^2) + 4x^2y\right]\frac{\partial F}{\partial x} + \left[-x(x^2+y^2) + 4xy^2\right]\frac{\partial F}{\partial y} = 0, \quad (4.19)$$

$$x^2y^2(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] + xy(x^2+y^2)\left[y\frac{\partial \psi}{\partial x} + x\frac{\partial \psi}{\partial y}\right] = y^4\psi, \quad (4.20)$$

and the solutions of these satisfying the boundary conditions (4.13),(4.14) are

$$x, y \rightarrow 0 : F \rightarrow 1 - cx^2(x^2+y^2), \quad (4.21)$$

$$\psi \rightarrow ax(x^2+y^2)^\alpha, \quad \alpha = -1 + \frac{1}{2}\sqrt{3}, \quad (4.22)$$

with a and c unknown constants.

These are the boundary conditions for both x and y small.

For only x small we have

$$x \rightarrow 0 : F \rightarrow 1 - f(y)x^2 , \quad (4.23)$$

$$\psi \rightarrow d(y)x , \quad (4.24)$$

with $d(y)$ and $f(y)$ unknown functions of y .

The constant c in (4.21), which must be evaluated in the course of the numerical solution of the differential equations, is closely related to the force between the monopole-antimonopole pair. The ξ -component of the magnetic field is

$$\begin{aligned} B_{\xi} &= \frac{-1}{eb^2} \frac{1}{\sqrt{\xi^2 - \eta^2}} \frac{\partial}{\partial \eta} \left\{ \sqrt{1 - \eta^2} A_1 \right\} \\ &= \frac{1}{eb^2} \frac{1}{\sqrt{\xi^2 - \eta^2}} \frac{\xi}{\sqrt{\xi^2 - 1}} \frac{\partial}{\partial \eta} \left\{ \frac{1 - \eta^2}{\xi^2 - \eta^2} F(\xi, \eta) \right\} . \end{aligned} \quad (4.25)$$

Along the line $\eta=1$, above the monopole ($-g$) (see Fig. 9), this reduces to

$$B_{\xi} = - \frac{2}{eb^2} \frac{\xi}{(\xi^2 - 1)^2} F(\xi, 1) . \quad (4.26)$$

Subtracting out the self field of the monopole ($-g$),

$$B_{\xi}^{\text{self}} = \frac{-1}{2eb^2} \frac{1}{(\xi - 1)^2} , \quad (4.27)$$

we find for the field due to the monopole ($+g$) and the surrounding Higgs field

$$B_{\xi} - B_{\xi}^{\text{self}} = \frac{1}{2eb^2} \frac{1}{(\xi - 1)^2} \left[1 - \frac{4\xi}{(\xi + 1)^2} F(\xi, 1) \right] \quad (4.28)$$

along the line $\eta=1$. The limit of this expression as we approach the location $\xi=1$ of the monopole ($-g$) is

$$\begin{aligned}
 B_{\xi} - B_{\xi}^{\text{self}} &\xrightarrow{\xi \rightarrow 1} \frac{1}{8eb^2} \left[1 - 2 \frac{\partial^2 F(\xi, 1)}{\partial \xi^2} \right]_{\xi=1} \\
 &= \frac{1}{8eb^2} \left[1 - \frac{2}{3} \frac{\partial^4 F}{\partial x^4} \right]_{x=0} \\
 &= \frac{1}{8eb^2} [1 + 16c] ,
 \end{aligned} \tag{4.29}$$

in terms of the constant c in (4.21).

We can also use the formula for B_{η} ,

$$B_{\eta} = \frac{1}{eb^2} \frac{\sqrt{1-\eta^2}}{(\xi^2-\eta^2)^{3/2}} \left[\frac{\xi^2+\eta^2}{\xi^2-\eta^2} F - \xi \frac{\partial F}{\partial \xi} \right] , \tag{4.30}$$

which, along the line $\xi=1$, below the monopole ($-g$), yields

$$B_{\eta} = \frac{1}{eb^2} \frac{1}{(1-\eta^2)} \left[\frac{1+\eta^2}{1-\eta^2} - \frac{\partial F}{\partial \xi} \right]_{\xi=1} . \tag{4.31}$$

Subtracting out the self field,

$$\begin{aligned}
 B_{\eta} - B_{\eta}^{\text{self}} &= \frac{1}{eb^2} \left[\frac{1}{2(1+\eta)^2} - \frac{1}{(1-\eta^2)} \frac{\partial F}{\partial \xi} \right]_{\xi=1} , \\
 &= \frac{1}{eb^2} \left[\frac{1}{2(1+\eta)^2} - \frac{1}{y^2} \frac{\partial^2 F}{\partial x^2} \right]_{x=0}
 \end{aligned} \tag{4.32}$$

we obtain again

$$B_{\eta} - B_{\eta}^{\text{self}} \xrightarrow{\eta \rightarrow 1} \frac{1}{8eb^2} [1 + 16c] . \tag{4.33}$$

The force between the monopoles is thus

$$\text{Force} = -g(B-B^{\text{self}}) = -\frac{g^2}{4\pi} \frac{1}{(2b)^2} [1+16c] , \quad (4.34)$$

where we have used the quantization condition (2.23) with $m=1$. In (4.34), the force is given as the sum of the Coulomb force between the monopoles and a correction due to the shielding effect which produces the flux tube. For large separations $2b$ the contribution involving c in (4.34) should reduce to the energy per unit length calculated in sec.2, Eq.(2.24) or Eq.(2.37).

In Figs.10 and 11 we plot the fields $F(x,y)$ and $\psi(x,y)$ obtained with $G=2$ and $R=4$. The fields are slowly varying in y ; the relaxation method discussed in the Appendix would be impractical if this were not so. Fig.12 shows the magnetic field (4.30) in the mid-plane of the dipole ($y=1$) for $G=2$ and various values of R . The formation of the flux tube discussed in sec.2 appears to be essentially complete for $R=4$, as evidenced by the fact that $B(0)$ approaches its asymptotic value and $B(x)$ its proper shape.

The quantity $B-B^{\text{self}}$ is evaluated for $G=2$ at the location of the monopole ($-g$), Eq.(4.29), and displayed in Fig. 13 as a function of R , the separation of the monopoles in dimensionless units. The force between the monopoles is proportional (Eq. 4.34) to the quantity plotted, and the result from sec.2 for this force (valid in the limit $R \rightarrow \infty$) is shown as

a dashed line. The calculated force is very near its asymptotic value for $R \geq 2$. Also plotted is a curve of the form $\beta + \alpha/R^2$, which is consistent with the commonly used phenomenological potential Eq.(2.40). The force between the dipoles for $G=2$ is somewhat different from this form in the intermediate region $R \approx 1$. A different choice of G would change the shape of this curve; there probably exists some value of G which results in a potential almost identical to the phenomenological form (2.40).

The magnetic field $B-B^{\text{self}}$ was calculated along the z -axis, Eq.(4.16), using (4.28) for $z/b \geq 1$ and (4.32) for $z/b \leq 1$, and the results are shown in Fig.14 for three values of R . We remind the reader that only the self field of the monopole ($-g$) at $z/b = +1$ has been subtracted out; the quantity $B-B^{\text{self}}$ is singular at $z/b = -1$. There were some numerical difficulties encountered in the evaluation of $\left. \frac{d^2 F}{dx^2} \right|_{x=0}$ in (4.32) which caused scatter on the 3-5% level in the values of $B-B^{\text{self}}$ calculated for $z/b \leq 1$; the curve drawn has been smoothed by eye. We are confident, however, that such features as the bump near $z/b=1$ are not spurious, although the width and height of this structure are not well determined. A completely satisfactory solution to this problem (as well as a resolution of the difficulty discussed in the Appendix) would probably involve the use of a significantly more sophisticated numerical technique^{23/}. However, the physical interest of such fine details of the particular model discussed here did not seem sufficient to warrant the extra effort at this time.

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APPENDIX

The relaxation method of Henyey, et al, (Ref.15) is a rather general iterative scheme by which one can solve systems of ordinary differential equations with boundary conditions specified at both endpoints. (An extension to partial differential equations will be mentioned presently).

Given a system of M differential equations such as

$$Q = f(x, \phi) \frac{d^2 \phi}{dx^2} + g(x, \phi) \frac{d\phi}{dx} + h(x, \phi) = 0, \quad (A.1)$$

one wishes to impose boundary conditions

$$\phi(x_L) = \phi_L, \quad \phi(x_U) = \phi_U. \quad (A.2)$$

In these expressions $Q(x)$, $\phi(x)$, ϕ_L , ϕ_U , and $h(x, \phi)$ are column vectors

$$\phi(x) = \begin{pmatrix} \phi^1(x) \\ \phi^2(x) \\ \vdots \\ \phi^M(x) \end{pmatrix}, \text{ etc.}, \quad (A.3)$$

and f and g may be $M \times M$ matrices.

The equation $Q=0$ can be converted into a set of difference equations,

$$Q_j = Q(x_j, \phi) = f(x_j, \phi_j) \frac{d^2 \phi}{dx^2} \Big|_{x_j} + g(x_j, \phi_j) \frac{d\phi}{dx} \Big|_{x_j} + h(x_j, \phi_j) = 0, \quad (A.4)$$

using approximations

$$\frac{d^2 \phi}{dx^2} \Big|_{x_j} \approx \frac{1}{\Delta^2} (\phi_{j+1} - 2\phi_j + \phi_{j-1}), \quad (A.5)$$

$$\left. \frac{d\phi}{dx} \right|_{x_j} \cong \frac{1}{2\Delta} (\phi_{j+1} - \phi_{j-1}) , \quad (A.6)$$

with

$$x_j = (j-1)\Delta + x_L, \quad j = 1, N , \quad (A.7)$$

$$\phi_j = \phi(x_j) , \quad j = 1, N . \quad (A.8)$$

Starting with an initial guess for the solution to (A.6), $\phi = \check{\phi}(x)$, one can improve the extent to which the MxN equations (A.4) are satisfied by using Newton's method,

$$\check{\phi}(x_j) + \phi(x_j) = \check{\phi}(x_j) + \delta\phi(x_j) , \quad (A.9)$$

where $\delta\phi_j \equiv \delta\phi(x_j)$ is determined by

$$0 = Q_j + B_j \delta\phi_j + C_j \delta\phi_{j-1} + D_j \delta\phi_{j+1} . \quad (A.10)$$

The MxM matrices B_j, C_j , and D_j are given by

$$B_j^{ik} = \frac{\delta Q_j^i}{\delta \phi_j^k} , \quad (A.11)$$

$$C_j^{ik} = \frac{\delta Q_j^i}{\delta \phi_{j-1}^k} ,$$

$$D_j^{ik} = \frac{\delta Q_j^i}{\delta \phi_{j+1}^k} .$$

Here and throughout the lower index refers to the point x_j and the upper indices to the space of functions $\phi^1, \dots, \phi^M$.

As a simple example, if

$$\phi = \begin{pmatrix} A \\ \psi \end{pmatrix} \quad (\text{A.12})$$

$$Q = \begin{pmatrix} \frac{d^2 A}{dx^2} + \frac{1}{x} \frac{dA}{dx} + f(x, A, \psi) \\ \frac{d^2 \psi}{dx^2} + \frac{1}{x} \frac{d\psi}{dx} + g(x, A, \psi) \end{pmatrix},$$

then

$$Q_j = \begin{pmatrix} A_{j+1} - 2A_j + A_{j-1} + \frac{\Delta}{2x_j}(A_{j+1} - A_{j-1}) + \Delta^2 f_j \\ \psi_{j+1} - 2\psi_j + \psi_{j-1} + \frac{\Delta}{2x_j}(\psi_{j+1} - \psi_{j-1}) + \Delta^2 g_j \end{pmatrix}, \quad (\text{A.13})$$

$$C_j = \begin{pmatrix} 1 - \frac{\Delta}{2x_j} & 0 \\ 0 & 1 - \frac{\Delta}{2x_j} \end{pmatrix},$$

$$D_j = \begin{pmatrix} 1 + \frac{\Delta}{2x_j} & 0 \\ 0 & 1 + \frac{\Delta}{2x_j} \end{pmatrix},$$

$$B_j = \begin{pmatrix} -2 + \Delta^2 \frac{\delta f_j}{\delta A_j} & \Delta^2 \frac{\delta f_j}{\delta \psi_j} \\ \Delta^2 \frac{\delta g_j}{\delta A_j} & -2 + \Delta^2 \frac{\delta g_j}{\delta \psi_j} \end{pmatrix},$$

where a factor of Δ^2 has been absorbed into Q, C, B , and D .

Having disposed of these preliminaries, a description of the Henyey relaxation method is now straightforward. Since the initial guess, $\phi(x)$, is taken to satisfy the boundary conditions at x_L and x_U , we already know that

$$\delta\phi_1 = 0, \quad \delta\phi_N = 0. \quad (\text{A.14})$$

But the $j=2$ part of (A.10) can now be solved for $\delta\phi_2$ in terms of $\delta\phi_3$,

$$\delta\phi_2 \equiv L_3\delta\phi_3 + K_3, \quad (\text{A.15})$$

$$L_3 = -B_2^{-1}D_2, \quad K_3 = -B_2^{-1}Q_2, \quad (\text{A.16})$$

and the M^2+M components of L_3 and K_3 can be computed and saved for future reference. Similarly, $\delta\phi_3$ can be expressed in terms of $\delta\phi_4$, and, in general,

$$\delta\phi_j = L_{j+1}\delta\phi_{j+1} + K_{j+1}, \quad (\text{A.17})$$

$$L_{j+1} = -(B_j + C_j L_j)^{-1} D_j, \quad (\text{A.18})$$

$$K_{j+1} = -(B_j + C_j L_j)^{-1} (Q_j + C_j K_j).$$

These recursion relations are used to calculate and store L_j and K_j for $j=3,4,\dots,N$. At this point, however, we know that $\delta\phi_N=0$, and thus (A.17), (A.9), and (A.4) can be used to compute $\delta\phi_j$, ϕ_j , and Q_j for $j=N-1, N-2, \dots, 2$. This should represent an improved solution, in the sense that $Q_j=0$ should be more nearly satisfied for the new ϕ_j than it was for $\tilde{\phi}_j$.

If all the values of Q_j are not satisfactorily small, ϕ_j can be used as a starting point for another relaxation.

If it converges at all, this iterative method usually converges extremely rapidly, requiring only a few iterations even if the initial guess is very bad. However, the singular points of the equation must be treated carefully. If $x_L=0$ is such a point, then both singular and non-singular solutions to $Q=0$ will exist near $x=0$ (e.g. $\phi(x) \propto x^{\pm\alpha}$). The scheme outlined above will then fail to converge; a few iterations will yield a $\phi(x)$ which is very large near $x=0$, jumping discontinuously to the boundary condition ϕ_L at the point $x=0$. One can prevent the singular solution from creeping in by constraining the first three points to vary like $\phi(x) \propto x^\alpha$, replacing L_3 and K_3 in (A.16) by

$$L_3 = \left(\frac{1}{2}\right)^\alpha = \frac{\phi(\Delta) - \phi(0)}{\phi(2\Delta) - \phi(0)}, \quad (\text{A.19})$$

$$K_3 = 0.$$

More points can be constrained if desired, but three is usually enough.

Another difficulty occurs if $\phi(x) \propto x^\alpha$ with $\alpha < 1$, as is the case in the SU(3) model of sec.3 where $\alpha = 1/3$. Here the relaxations converge, but to an answer which does not satisfy $Q_j=0$ very well near $x=0$. This happens because the difference equations are not accurate near a singularity of $\frac{d\phi}{dx}$. The problem can be avoided by making the substitution $u=x^\beta$, where $\beta \leq \alpha$, so that $\phi(u) \propto u^{\alpha/\beta}$.

The Henyey relaxation method can be extended to solve a partial differential equation in x and y simply by treating $\phi(x_j, y_i)$ as distinct functions of x for each y_i . If L grid points are used in the y direction, then the matrices and vectors in (A.10) and (A.17) are dimensioned $M \times L$; it is clear that this method is impractical if rapid variations in y force one to choose L large. In sec.4 the singular part of A has been factored out, so that F and ψ are slowly varying in y .

It is crucial in sec.4 that the linearized equations (4.19), (4.20) are solved to yield (4.21) and (4.22). These expressions must be used in place of (A.19) for the region where both x and y are small. Slightly different values of a and c will result for each value of y_i , but these may be averaged and F and ψ recomputed in this region.

Careful attention to the above points does not, however, ensure that a satisfactory solution to the partial differential equation will be found. The difference equations resulting from (A.4), (A.5), and (A.6) may not limit smoothly into the differential equation as $\Delta_x, \Delta_y \rightarrow 0$ ^{24/}. The equations of sec.4, (4.17) and (4.18), suffer from this problem. For example, the small-argument solution for F , (4.21), satisfies the differential equation (4.17) to fourth order in x and y . Conversion of (4.17) to a difference equation using (A.5) and (A.6) spoils this property; the difference equation fails to reproduce the x -derivatives. A better difference equation may be

derived for which

$$F = 1 - \epsilon x^2 (x^2 + y^2) \quad (\text{A.20})$$

is a solution to fourth order. We introduce a new parameter $\beta(x, y)$ in the difference formula for the first derivative,

$$F_{xx}^D(x_j) \equiv \frac{1}{\Delta_x^2} (F_{j+1} - 2F_j + F_{j-1}) \quad (\text{A.21})$$

$$F_x^D(x_j) \equiv \frac{1}{\Delta_x} \left[\beta (F_{j+1} - F_j) + (1-\beta) (F_j - F_{j-1}) \right] \quad (\text{A.22})$$

and, writing (4.17) as

$$F_{xx} + g(x, y) F_x = h(x, y) \quad (\text{A.23})$$

require that

$$(F_{xx}^D - F_{xx}) + g(x, y) (F_x^D - F_x) = 0 \quad (\text{A.24})$$

be satisfied to fourth order. This determines β ,

$$\beta = - \frac{(F_{xx}^D - F_{xx}) + g(x, y) \left[\frac{1}{\Delta_x} (F(x, y) - F(x - \Delta_x, y)) - F_x \right]}{g(x, y) F_{xx}^D \Delta_x} \quad (\text{A.25})$$

$$= \frac{Y^4 + Y^2 + 5X^2 - 2Y^2X - 18X^3 - 11Y^2X^2 + 30X^4}{2Y^4 + 2Y^2 + 10X^2 + 22Y^2X^2 + 60X^4} \quad (\text{A.26})$$

with

$$X \equiv \frac{x}{\Delta_x} \quad , \quad Y \equiv \frac{y}{\Delta_x} \quad (\text{A.27})$$

The difference formulas (A.21), (A.22) are tailored for use in (4.17), and are greatly superior to (A.5), (A.6).

This technique is not as successful when applied to the ψ -equation (4.18). Singularities appear in the analogue of (A.25) corresponding to zeroes of $g(y)$ and ψ_{yy}^D , and the difference formulas are not improved. As a result, derivatives of ψ are not well determined near the point $x=y=0$. Fortunately, F and its derivatives are physically more interesting than derivatives of ψ , and the F -equation (4.17) decouples from (4.18) in the region where the problem occurs. A more satisfactory treatment of this problem would almost certainly have little or no effect on the results presented in sec.4.

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FIGURE CAPTIONS

1. The magnetic field $B(\rho)$ as a function of radius in the Abelian model with $m=1$ (1 unit of flux) for several values of the coupling parameter $G = \lambda/e^2$. $m_V^2 = 2e^2\mu^2/\lambda$.
2. The Higgs field $\psi(\rho)$ as a function of radius in the Abelian model with $m=1$ (1 unit of flux) for several values of the coupling parameter $G = \lambda/e^2$. $m_V^2 = 2e^2\mu^2/\lambda$.
3. The energy per unit length of the flux tube in the Abelian model with $m=1$ (1 unit of flux) as a function of the coupling parameter $G = \lambda/e^2$. The quantities $B(0)$ and $B(0) + C$ are plotted--see Eqs. (2.37), (2.38). $m_V^2 = 2e^2\mu^2/\lambda$.
4. The magnetic field $B(\rho)$ as a function of radius in the SU(3) model with $m=1$ (1 unit of flux) for several values of the coupling parameter $G = \lambda/e'^2 = 3\lambda/e^2$. $m_V^2 = 2\mu^2e'^2/\lambda = 2\mu^2e^2/3\lambda$.
5. The Higgs field $\psi(\rho)$ as a function of radius in the SU(3) model with $m=1$ (1 unit of flux) for several values of the coupling parameter $G = \lambda/e'^2 = 3\lambda/e^2$. $m_V^2 = 2\mu^2e'^2/\lambda = 2\mu^2e^2/3\lambda$.
6. The energy per unit length of the flux tube in the SU(3) model with $m=1$ (1 unit of flux) as a function of the coupling parameter $G = \lambda/e'^2 = 3\lambda/e^2$. The quantities $B(0)$ and $B(0) + C$ are plotted--see Eqs. (2.37), (2.38) with $e^2 \rightarrow e'^2 = e^2/3$, $m \rightarrow m/3$. $m_V^2 = 2\mu^2e'^2/\lambda = 2\mu^2e^2/3\lambda$.

7. The magnetic field $B(\rho)$ as a function of radius in the SU(3) model for flux tubes with three different values for the numbers of units of flux ($m=1,4,7$). $m_v^2 = 2\mu^2 e'^2 / \lambda = 2\mu^2 e^2 / 3\lambda$, $G=1.0$.
8. The Higgs field $\psi(\rho)$ as a function of radius in the SU(3) model for flux tubes with three different values for the numbers of units of flux ($m=1,4,7$). $m_v^2 = 2\mu^2 e'^2 / \lambda = 2\mu^2 e^2 / 3\lambda$, $G=1.0$.
9. Geometric quantities for the magnetic dipole. Monopoles of strength ($\pm g$) are separated by a distance $2b$.
10. The vector field $F(x,y)$ as a function of x for three values of y in the Abelian model of the dipole with $G = \lambda/e^2 = 2.0$, $R = m_v b = 4.0$.
11. The Higgs field $\psi(x,y)$ as a function of x for three values of y in the Abelian model of the dipole with $G = \lambda/e^2 = 2.0$, $R = m_v b = 4.0$.
12. The magnetic field $B(\rho)$ in the midplane of the dipole, $y=1$, for $G = \lambda/e^2 = 2.0$ and various values of $R = m_v b$.
 a) $B(\rho)/B(0)$ as a function of ρ ; b) $B(0)$ as a function of R . The value of $B(0)$ for the infinite vortex of sec. 2 is shown as a dashed line.
13. $B - B^{\text{self}}$ at the location of the monopole ($x=y=0$) for $G = \lambda/e^2 = 2.0$ as a function of $R = m_v b$. Shown as dashed lines are $(B(0) + C)/2$ from sec.2, and $\beta + \alpha/R^2$. The

values of α and β are chosen so that the two curves coincide at $R = 0.25$ (not plotted) and $R = 4.0$.

14. $B - B^{\text{self}}$ along the z-axis for $G = \lambda/e^2 = 2.0$ and several values of $R = m_v b$.

Fig 1

Fig 2

Fig 3

Fig 4

Fig 5

Fig 6

Fig 7

Fig 8

Fig 9

Fig 10

Fig 11

Fig 12a

Fig 12b

Fig 13

Fig 14