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An Investigation of Failure Mechanisms for HTGR Core Supports

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by

J. G. Bennett
F. D. Ju*
C. A. Anderson

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*Professor of Mechanical Engineering, University of New Mexico,
Albuquerque, NM 87131.

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NOMENCLATURE

The symbols used in this report are defined where first introduced and their meaning is clear in sections for which duplication may occur. The following is a partial list of their most common meanings:

- A, A_H = Cross-sectional area of a member; horizontal ground acceleration
- C, C_S = Damping coefficient; soil shear wave speed
- D = Diameter
- E, E_t = Young's modulus of elasticity; tangent modulus
- I, I_o = Area moment of inertia of a support post; mass moment of inertia of the core about a vertical central axis
- K_t, K_v = Equivalent core spring constants; torsional and vertical
- K_I, K_{IC} = Mode I stress intensity factor; critical value of Mode I stress intensity factor
- L = Length of a support post or column
- M = Moment of a resultant force or couple; mass of the core
- P, P_{cr} = Axial load in a support post; Euler critical load for a support post
- P_E = Engesser's critical load
- R, R_i = Radius of the core, or radius of a support post; radius of the i^{th} circle on which i support posts are located
- W = Weight of the core plus the resultant force due to the pressure drop through the core
- a = Net half width of a cracked section in the plane of flexure or net radius of cracked material for a circular cross section
- b = Nominal half width for a rectangular cross section
- c = Crack depth
- e = Eccentricity of loading on a column
- f = Frequency
- g = Acceleration due to gravity
- h = Thickness of rectangular column

i, j, k, m = Indices or subscripts
 t = Time
 u = Transverse displacement of a support column
 v = Vertical or axial displacement of a core column
 x = Axial coordinate along a core column
 y = Transverse coordinate for a core column
 U = Strain energy
 α = Nonlinear material parameter in the Ramberg-Osgood Law
 δ = Central deflection of support post column
 ϵ = Strain
 ζ, ξ = Integration variables
 η = Crack length parameter
 θ = Relative coordinate of rotational motion between the PCR and core
 κ = Curvature
 μ = Shear modulus of elasticity
 σ = Stress
 ϕ, Φ = Rotation of a support post, potential energy for a deformed nonlinear graphite column
 Ψ, Ψ_0 = Rotational coordinate for the core about a vertical central axis; rotational motion of the PCR
 ω = Natural circular frequency

AN INVESTIGATION OF FAILURE MECHANISMS FOR HTGR CORE SUPPORTS

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ABSTRACT

This report is concerned with potential instabilities of High-Temperature Gas-Cooled Reactor Cores supported by graphite columns. Two failure mechanisms are investigated in detail: that of torsional buckling of the entire core-column assemblage and that of column failure alone. A torsional model of the core-column assemblage is described and static buckling loads are calculated. Dynamic instability of the model to seismic loadings is also investigated. Individual column failure is examined using nonlinear graphite behavior and safety factors for static loading situations are given and compared to values given by conventional design formulas. A model of a cracked graphite column is given and buckling loads are computed for columns using a combined column and fracture mechanics analysis. Finally, a finite element analysis of a cracked graphite column is presented.

I. INTRODUCTION

The core of a High-Temperature Gas-Cooled Reactor (HTGR) is made up of prismatic blocks of graphite of a hexagonal cross section set in closely packed columns to form an approximately circular cylindrical core. Graphite core support blocks, which also act as reflector elements, each serve to support seven columns of active prismatic blocks and three graphite posts in turn support each core support block. The core is set inside a relatively rigid prestressed concrete reactor vessel (PCRv) that supports the core laterally through a system of spring packs that are tied into the side reflector elements. The

core assemblage (active and reflector elements) is typically ~ 9 m in diameter and 9 m in height for a large HTGR (1100 MWe power). It is claimed¹ that during normal reactor operating conditions the dead load and the pressure drop on a typical core support post will be about 55 000 N (12 000 lb) with a maximum of ~ 90 000 N (20 000 lb). From structural tests on graphite columns, a minimum safety factor of ~ 8 has been determined with column failure occurring locally in the seats of the column. These tests, together with substantiating calculations, have indicated that buckling or instability of the graphite columns occurs at loads in excess of those which cause local failure in the column seats. It is further claimed that the Operating Basis Earthquake (OBE) and Safe Shutdown Earthquake (SSE) loads are less than the normal operational loads and thus are not critical as far as column failure is concerned.¹

Failure of a column under axial compression occurs whenever the average axial stress exceeds the ultimate compressive stress of the material of the column or whenever the axial load reaches the buckling load for the column. For an elastic, perfect column simply supported on both ends and under axial loading, the buckling load (or Euler load) is given by

$$P_{cr} = \pi^2 EI / L^2, \quad (1)$$

where L is the column length, E is the material modulus, and I is the minimum cross-sectional moment of inertia about the axis of bending. The value of P_{cr} is an upper bound on the actual failure load observed for a column when real material behavior, column imperfections, and nonideal loading is taken into account. Real material behavior such as creep,² or plasticity,³ and column imperfections such as fractures or initial curvature can substantially reduce the buckling load below that given by the Euler value.

This report has two main aims: first, to establish safety factors on the torsional buckling load for an HTGR core-support post assemblage for both static and dynamic (seismic) loadings and second, to investigate the failure mechanisms and establish bounds on collapse loads for individual graphite columns. Section II contains the conclusions and recommendations from the investigation.

Section III describes a "table leg" model for an HTGR core on elastic posts. Critical torsional buckling loads were determined for the model and a large safety factor for torsional stability was seen to exist although the torsional natural frequencies lay in the frequency range of intense seismic motion. The equations of motion were then integrated and dynamic core response was computed from torsional ground acceleration time-histories obtained by numerically differentiating a horizontal accelerogram and dividing by the shear wave speed for hard and soft soil conditions.

In the remaining sections of the report are discussed the effects of nonideal material behavior and loading conditions on the failure loads for individual graphite columns. Section IV discusses buckling loads for perfectly straight columns where the graphite material behavior is described by a stress-strain law of the Ramberg-Osgood form. The effects of eccentricity of the load are examined. Section V of the report is concerned with an approximate treatment of the effect of a lateral crack on the carrying capacity of a graphite column with an eccentric axially directed load. Nonlinear material behavior and traditional methods of fracture mechanics are used. Finally in Section VI is presented a finite element analysis of the problem treated in Section V, with nonlinear graphite behavior and where no approximation is made in the stress fields produced by the eccentric loading and the presence of the crack. Comparisons are made with the results of the approximate method.

II. CONCLUSIONS AND RECOMMENDATIONS

The HTGR core, supported vertically by graphite columns and laterally by spring packs, has been shown to be statically stable to torsional motion with a safety factor of about 60. A method for evaluating the torsional seismic response of the HTGR core and supports is demonstrated in this report that gives, with realistic input parameters, a good representation of the core response. The maximum relative torsional displacement at the core boundary is shown to depend strongly on the soil conditions and the horizontal damping of the core block assemblage and spring packs. For a 0.5-g horizontal earthquake, soft soil conditions, and zero damping, a maximum relative torsional displacement of 61 mm (2.4 in.) at the core boundary was calculated - this exceeds the allowable clearance. Results for stiffer soil conditions or for the presence of 5% of critical damping yield "safe" torsional displacements. We recommend that the torsional damping mechanisms acting on the core be evaluated to ensure

that sufficient damping is present to attenuate torsional motion to safe values.

The buckling load of a perfect graphite column with length-to-diameter ratio typical of an HTGR core support post is reduced considerably when non-linear graphite properties are taken into account. Even so, large load eccentricities are required for column collapse under dead core loading. The buckling load computed using the finite element method is found to be in good agreement with Engesser's load - that is, based on Euler's formula, Eq. (1), with Young's modulus replaced by the tangent modulus at the critical buckling stress.

Collapse loads are determined by approximate engineering and fracture mechanics theories for imperfect graphite columns characterized by transverse cracks (mode I) at the section of maximum bending moment in combination with eccentric loading. For modest eccentricities, $e/R \leq 0.05$, the presence of a 10-mm transverse crack at the midplane of the column will reduce the collapse load by a maximum of 2.0% below Engesser's load. For large eccentricities the situation is greatly exaggerated as shown in Fig. 14 with large reductions in the collapse load. The approximate treatment predicts a maximum carrying load causing buckling if the load is applied along the column axis.

Oblique cracks (for example, mode II cracks) are not treated by the approximate analysis of this report. In addition, there is no experimental information on the critical stress intensity factors for either mode II or mode III cracks in graphite. We recommend that this experimental data be obtained for graphite, and the analysis of end-loaded graphite columns be reexamined for obliquely oriented cracks. Finally, the finite element approach demonstrated in Section VI merits consideration for attacking these more complicated problems, but a reliable crack extension criterion must be defined.

III. TORSIONAL STABILITY AND TORSIONAL SEISMIC RESPONSE OF THE HTGR CORE

This section is directed at answering questions regarding the torsional stability and torsional seismic response of the HTGR core. If the current HTGR designs are examined critically, one possible mode of core structural failure is for the core to collapse through a torsional mode. Since the permanent reflector blocks and the core support blocks are designed to be essentially locked together, the core may be visualized as a relatively rigid right circular cylinder that rests on a considerable number of elastic two-force members. The torsional restraint for the core is supplied through the side support

structure that has been primarily designed to resist lateral seismic motions and radial thermal expansion. Thus, the questions concerning the torsional static stability of the core and the possibility of torsional motions creating large relative motions between the core and the Prestressed Concrete Reactor Vessel (PCRVR) are valid. Relative motions large enough between the PCRVR and the core will cause contact between the side support springs and their housings. This contact can take the form of impact, or it can create a nonlinear torsional spring restoring torque. In either case, because of the very large rotational inertia of the core, it is possible that the side support structure can sustain a considerable amount of damage if relative motion large enough for this contact occurs.

The analytical model in this study is intended to be a relatively simple but accurate model such that the possibility of such failures can be examined. The advantage of a simple model of this type in seismic studies is that time history calculations of seismic motions are economically feasible. For example, less than 30 s of central processing time on a CDC 7600 is required to examine the combined torsional and vertical response of the model excited by a full 15-s 1/2-g earthquake. This computation represents an estimated cost of less than \$4.00 at the LASL computing facility.

A. Table Leg Model for Describing the Combined Torsional and Vertical Motion of the Core

Referring to Fig. 1, the core is modeled as a rigid right-circular cylinder of mass M , supported by an elastic center post and n elastic posts on a circle of radius R_n . The core support posts are modeled to carry only axial force. The coordinates $\psi_0(t)$ and $v_0(t)$ are the rotational and vertical motions of the PCRVR arising from seismic excitation. Presumably these motions can be supplied from a lumped mass seismic analysis of the plant and associated structures. If desired, the coordinates ψ_0 and v_0 can be further resolved into coordinates which describe relative motion between the bottom head of the PCRVR and sides and top heads without introducing any serious complications with regard to the present model. However, relative motions between these components will be negligible with respect to overall relative motion between the core and PCRVR.

The single torsional spring represents the combined effect of the 252 spring packs which position and restrain the core from contact with the sides

Fig. 1. Core and support post model for studying torsional stability.

of the PCR. The properties of the single spring for a typical plant will be discussed in the following section. Similarly, the single viscous damper represents the total energy dissipation due to the friction, material hysteresis, and other energy loss mechanisms that may be operating.

From the free body diagram in the lower half of Fig. 1, the following equations of motion can be derived:

$$I_o \ddot{\psi} + K_t(\psi - \psi_o) + C_t(\dot{\psi} - \dot{\psi}_o) - nRP(v) \sin \phi = 0, \quad (2)$$

and

$$M\ddot{v} + C_v(\dot{v} - \dot{v}_o) + W - nP(v) \cos \phi - P(v) = 0, \quad (3)$$

where

$$P(v) = \frac{W}{n+1} + \sigma A .$$

In addition to the terms and coordinates already described, a dot over a coordinate means time derivation, I_0 is the mass moment of inertia of the core about the vertical central axis, W is the weight of the core plus the resultant forces arising from pressure drop through the core vertically, and ϕ is the rotational coordinate for a support post. The quantity σ is the dynamic normal stress in a typical post whose properties are A , the cross-sectional area; E , the modulus of elasticity for the post material; and L , the post length.

The equations can be written in a somewhat simpler form using the following scheme:

(a) Introduce the relative coordinates,

$$\theta = \psi - \psi_0 ; \quad \dot{\theta} = \dot{\psi} - \dot{\psi}_0 ; \quad \ddot{\theta} = \ddot{\psi} - \ddot{\psi}_0$$

$$v = v - v_0 ; \quad \dot{v} = \dot{v} - \dot{v}_0 ; \quad \ddot{v} = \ddot{v} - \ddot{v}_0$$

(b) Introduce the definitions,

$$K_v = \frac{AE}{L} , \quad P_{cr} = \frac{K_t L}{nR^2} , \quad \omega_v^2 = (n+1) \frac{AE}{LM}$$

$$\omega_\theta^2 = \frac{K_t}{I_0}$$

(c) Restrict the relative angular deformations to being small so that

$$\sin \theta \approx \theta ; \quad \cos \theta \approx 1 .$$

After some algebra, the equations of motion can be written as follows:

$$\ddot{\theta} + \left(\frac{C_t}{I_0} \right) \dot{\theta} + \omega_\theta^2 \left[1 - \frac{P(v)}{P_{cr}} \right] \theta = \ddot{\psi} + G(v) \psi_0$$

$$\ddot{v} + \left(\frac{C_v}{M} \right) \dot{v} + \omega_v^2 v = - \ddot{v}_0 ,$$

(4)

where now

$$P(v) = \frac{W}{n+1} - K_v v$$

and

$$G(v) = nR^2 \frac{P(v)}{LI_o}$$

These equations describe a model that we refer to as the "table leg" model. Extension of this model to an actual core, which has no center post but has a number of post circles, is discussed in a later section of this report.

It should be noted that the assumption of small relative angular deformation is not restrictive for the purposes of this model because these assumptions are fairly good to at least 0.3 rad, for which the relative motion between the core and PCRV for a core radius of 4.6 m (191 in.) will exceed 1.3 m (54 in.).

There is a tacit assumption in these equations that each post carries the same average load $P(v)$. This assumption requires that at any time the distributed post load can be decomposed into an average load $P(v)$, as given, plus any additional load which is "sign" antisymmetric. Thus, the model does not preclude any "rocking" mode, but supposes that this rocking motion can be superposed onto the vertical results. In short, any rocking of core does not affect the torsional response that is of interest here. It should also be noted that the fact that the core is only "relatively rigid" probably insures that the average core support post load is given by $P(v)$. Thus, this model when extended to include the effects of all support posts is believed to give a very accurate description of the torsional response provided that the torsional input motion can be accurately described and that the damping can be accurately estimated.

B. Torsional Static Stability

Referring to the free body diagram in Fig. 1, it may be visualized that there are n posts on a circle of radius R_n , m posts on a circle of radius R_m , etc. Let ψ and v be virtual displacements, π be the total potential, U be the strain energy, and W_e be the work done by the external forces acting on the core mass M . For the core mass M , we form the expression $\pi = U - W_e$ for the total potential as:

$$\begin{aligned}
\pi = & \frac{1}{2} K_t \psi^2 + n P_n L \cos \phi_n (1 - \cos \phi_n) + m P_m L \cos \phi_m (1 - \cos \phi_m) \\
& + \dots - n P_n L \sin^2 \phi_n - m P_m L \sin^2 \phi_m + \dots \\
& + \dots - (n P_n \cos \phi_n) v = (m P_m \cos \phi_m) v + \dots - Wv.
\end{aligned} \tag{5}$$

From classical mechanics, we know that for neutral equilibrium associated with the critical load that the expression for the second variation of π is equal to zero, where in examining expressions for the change $\Delta\pi$, we must keep terms of second order in the Taylor's expansion. Observe that kinematically,

$$L \phi_n = R_n \psi$$

$$L \phi_m = R_m \psi, \text{ etc.}$$

or

$$\phi_i = \frac{R_i \psi}{L} \quad i = n, m, \dots$$

Then Eq. (5) can be written as

$$\begin{aligned}
\pi = & \frac{1}{2} K_t \psi^2 + \sum_{i=n, m, \dots} i P_i L \cos \left(\frac{R_i \psi}{L} \right) \left[1 - \cos \left(\frac{R_i \psi}{L} \right) \right] \\
& - \sum_{i=n, m, \dots} i P_i L \sin^2 \left(\frac{R_i \psi}{L} \right) + \sum_{i=n, m, \dots} \left[i P_i \cos \left(\frac{R_i \psi}{L} \right) \right] v - Wv,
\end{aligned} \tag{6}$$

where the summation extends over all post circles. Expanding the trigonometric functions in a Taylor series and keeping terms of order $O(\psi^2)$ and $O(v^2)$ and some small amount of algebra gives

$$\pi = \frac{1}{2} K_t \psi^2 - \sum_{i=n,m,\dots} i P_i \frac{R_i^2 \psi^2}{2L} + \sum_{i=n,m,\dots} i P_i v - W_v .$$

The expression for $\delta^2 \pi = 0$ then becomes

$$\left(K_t - \sum_{i=n,m,\dots} \frac{i P_i R_i^2}{L} \right) (\delta \psi)^2 = 0$$

$$\left(\sum_{i=n,m,\dots} i P_i - W \right) (\delta \psi)^2 = 0 .$$

Recalling that ψ and v are virtual displacements, we arrive at

$$K_t = \frac{\sum_{i=n,m,\dots} i P_i R_i^2}{W} \quad (7)$$

$$\sum_{i=n,m,\dots} i P_i = W . \quad (8)$$

Again, we require each post to carry the same average load P and combine Eqs. (7) and (8) to arrive at the expression for the minimum spring constant for torsional stability

$$K_t \geq \frac{\sum_{i=n,m,\dots} i R_i^2}{\sum_{i=n,m,\dots} i} \frac{W}{L} . \quad (9)$$

Or, from Eq. (7) the critical post load P becomes

$$P_{cr} = \frac{K_t L}{\sum_{i=n,m,\dots}^i R_i^2} \quad (10)$$

We note that from Eq. (9), the expression $\frac{\sum_{i=n,m,\dots}^i R_i^2}{\sum_{i=n,m,\dots}^i}$ is the expression

for $\bar{R}^2$, the equivalent radius at which all posts can be located on the rigid core to have the same effect as being on different circles.

C. Equations of Motion for the Table Leg Model to Include the Effect of all Posts

With the above analysis as a guide, it is not difficult to arrive at the modified equations of motion for a core containing n posts on R_n , m posts on R_m , etc. as follows:

Vertical,

$$\ddot{v} + \left(\frac{C_v}{M} \right) \dot{v} + \omega_v^2 v = - \ddot{v}_o.$$

Torsional,

$$\ddot{\theta} + \left(\frac{C_t}{I_o} \right) \dot{\theta} + \omega_\theta^2 \theta = - \left(\frac{P(v) \sum_{i=n,m,\dots}^i R_i^2}{L I_o} \right) \theta = - \ddot{\psi}_o + G(v) \psi_o,$$

where

$$G(v) = \frac{\sum_{i=n,m,\dots}^i R_i^2}{L I_o} P(v)$$

$$P(v) = \frac{W}{\sum_{i=n,m,\dots}^i} - K_v v. \quad (11)$$

In practice, we replace $\sum_{i=n,m,\dots} i R_i^2$ with $n \bar{R}^2$ where n is the total number

of posts, and $\bar{R}$ is the equivalent post radius.

Having obtained expressions for the minimum torsional spring constant and equations of motion, we turn our attention to the task of realistic evaluation of core designs.

D. Analysis of Spring Packs for Torsional Stiffness

A current HTGR core design as described in Ref. 1 calls for a side support structure composed of a vertical series of seven spring packs on 10° radial lines around the core. The 252 spring packs separate the permanent side reflector blocks from the core cavity liner and PCRV and essentially serve to position the core within the PCRV cavity.

References 1 and 4 indicate that each spring pack is composed of a group of helical compressive springs with a combined full-scale spring rate of $3.5 \times 10^6 \text{ N/m}$ (20 000 lb/in.). The mean coil diameter is $\sim 76.2 \text{ mm}$ (3 in.) and the free length is 222 mm (8.75 in.). Reference 1 also indicates that for normal operation at the beginning of core life, the springs are compressed 3.4 mm (0.12 in.) and at the end of core life, 10.7 mm (0.42 in.). The ratio of axial to lateral stiffness for steel springs of round wire is given in Ref. 5 by the following expression:

$$\frac{k_x}{k_y} = 1.44 C_L \left(0.204 \frac{h^2}{D^2} + 0.265 \right), \quad (12)$$

where

k_x is the axial stiffness of the coil,

k_y is the lateral stiffness of the coil,

h is the compressed spring length,

D is the mean coil diameter, and

C_L is a factor depending on the ratios of compressed to free length and free length to coil diameter.

Values of C_L are given in Ref. 5 and for the spring pack coils as described, C_L is very close to one. For the spring coil dimensions and compressed spring lengths of Ref. 1, we have found from Eq. (5) that the ratio of axial to lateral stiffness ranges from 2.95 to 2.79, which in turn give lateral stiffness that vary from $1.2 \times 10^6 \text{ N/m}$ (6780 lb/in.) to $1.3 \times 10^6 \text{ N/m}$ (7200 lb/in.). Since the contribution to the torsional stiffness is given by $k_t = R^2 K_x$ where R is the core radius, we calculated a total torsional spring constant for the core which ranged from $6.2 \times 10^9 \text{ N-m/rad}$ ($5.6 \times 10^{10} \text{ in. lb/rad}$) to $6.7 \times 10^9 \text{ N-m/rad}$ ($6.0 \times 10^{10} \text{ in. lb/rad}$).

E. Equivalent Post Radius and Minimum Required Torsional Spring Constant

Using a plan view of a typical core from Ref. 1 that shows the support post locations, the results given in Table I were developed. The results in Table I are only approximate since some judgement must be used in deciding how many circles to draw and which posts are close enough to a circle to assume that they be on it.

The equivalent post radius is evaluated from Table I as being $\bar{R} = 3.43 \text{ m}$ (135 in.). Using this value of the equivalent post radius, letting W be the weight of the core plus the force due to the pressure drop vertically, that is, $W = 1.61 \times 10^7 \text{ N}$ ($3.62 \times 10^6 \text{ lb}$); and taking a post length of 2 m, we get from Eq. (9),

$$K_t = 9.5 \times 10^7 \text{ N-m/rad} \quad (8.4 \times 10^8 \text{ in.lb/rad}).$$

This value is the minimum overall torsional spring constant for static stability for the reference core. This value of K_t is a factor of between 60 and 70 less than the value calculated from the spring packs in Sec. D; and hence under normal reactor operating conditions, the core and core support posts have a factor of safety of ~ 60 for torsional instability.

With such a large safety factor for torsional stability, the question could be asked whether it is necessary to examine the equation of motion for seismic response at all. However, the range of natural torsional frequencies for the core is given by

$$f_\theta = \frac{1}{2\pi} \sqrt{\frac{K_t}{I_o}}$$

TABLE I
SUPPORT POST ARRANGEMENT USED FOR
STATIC STABILITY ANALYSIS

Circle Number	Radius (m)	Number of Posts n	nR^2 (m ²)
1	0.52	9	2.4
2	0.90	9	7.3
3	1.28	18	29.5
4	1.69	18	51.4
5	1.97	9	34.9
6	2.11	18	80.1
7	2.42	9	52.7
8	2.59	18	120.0
9	2.73	18	134.0
10	2.94	18	156.0
11	3.32	36	397.0
12	3.49	27	329.0
13	3.80	27	390.0
14	4.01	36	579.0
15	4.18	27	472.0
16	4.39	36	694.0

and for the values in Sec. D, is found to be

$$3.6 H_z > f_\theta > 3.8 H_z .$$

It is noteworthy to recall that the typical earthquake exhibits intense motion around this frequency range and that some amplification of any torsional motion input at this frequency will arise. As pointed out in Sec. III.B, very little relative torsional displacement is required to produce large actual relative displacements at the core PCRV boundary.

F. Torsional Motion of the PCRV

Equation (11) can be solved numerically if a reasonable method of determining the forcing functions can be found. The vertical direction offers no problem, since an artificial earthquake can be used directly to simulate the vertical motion of the PCRV. (The conservativeness of this approach will be commented on in the discussion of the results.) The torsional direction, however, is not as easily accommodated.

Newmark considers the torsion of symmetric buildings in Ref. 6. Using the expression for the rotation vector from the theory of elasticity, and assuming that the torsional component in an earthquake results from the shear wave, he concludes that the expression for the torsional component is

$$\ddot{\theta} = \frac{\dot{A}_H}{C_s}, \quad (13)$$

where $\ddot{\theta}$ is the ground rotational acceleration about a vertical axis, A_H is the horizontal ground acceleration and C_s is the shear wave speed in the soil. Reference 1 gives planned plant soil conditions in the following range:

$C_s = 5180 \text{ m/s}$ (17 000 ft/s) for hard rock,

$C_s = 152 \text{ m/s}$ (500 ft/s) for soft soil.

Assuming the PCRV moves as a rigid body on the soil, time histories for $\ddot{\psi}_O(t)$ can be obtained by numerically differentiating a horizontal accelerogram and dividing by the wave speed. Figure 2 shows the torsional component obtained in this manner for the soft soil condition. This record was obtained from a 1/2-g earthquake that generates a response spectrum that envelopes the required NRC-response spectrum for 7% damping.

G. Results from the Torsional Model

Equation (11) for the torsional model is solved numerically using a 1/2-g vertical accelerogram and torsional components derived from 1/2-g horizontal accelerograms for the two extreme soil conditions. Table II lists the other parameters used in solving the equations. Figure 3 shows the vertical response of the core that is the same for all results reported here. It should be noted that the effect of using the vertical ground acceleration as the PCRV motion can be examined here. The maximum vertical acceleration for the core mass acting as a single degree of freedom system from Fig. 3 is

Fig. 2. Torsional component of a 1/2-g earthquake.

Fig. 3. Vertical acceleration response of the core mass.

TABLE II
PARAMETERS USED IN SOLUTIONS OF EQUATION TEN

I_O	=	$1.19 \times 10^7 \text{ N-ms}^2$ ($1.05 \times 10^8 \text{ in.-lb s}^2$)
M	=	$1.13 \times 10^6 \text{ kg}$ ($6436 \text{ lb s}^2/\text{in.}$)
$\bar{R}$	=	$\sqrt{\frac{\sum_{i=n,m,\dots}^i R_i^2}{\sum_{i=n,m,\dots}^i}} = 3.43 \text{ m}$ (135 in.)
K_t	=	$6.7 \times 10^9 \text{ N-m/rad}$ ($6 \times 10^{10} \text{ in.-lb/rad}$)
L	=	2 m (78.7 in.)
ω_v	=	193 rad/s
ω_θ	=	23 rad/s
$\sum_{i=n,m,\dots}^i$	=	333 posts

0.4 g. Reference 1 has the reactor's bottom head vertical response spectra for a safe shutdown earthquake of 0.25 g. For the natural vertical frequency (30 Hz) of the system used here, Ref. 1 indicates a maximum vertical response of 0.6 g. It is seen then that the model here is slightly underpredicted vertically, even though a 1/2-g earthquake is used as a forcing function, the difference being that we are neglecting the amplification of vertical motion through the PCRv.

Figure 4 shows the undamped relative angular displacement (ψ) between the PCRv and core for both the hard and soft soil conditions. A maximum value of $\psi = 3.8 \times 10^{-4}$ rad was recorded at $t = 0.5 \text{ s}$ for the hard soil condition. This value gives a relative displacement of 0.18 cm (0.07 in.) between the PCRv and outer reflector blocks at the core boundary. This motion will not be enough to cause any contact between the springs and their housing.

Fig. 4. Relative torsional undamped response of core for hard and soft soil conditions.

For the soft soil condition, however, the maximum value of ψ is 1.3×10^{-2} rad which gives a value of 6.1 cm (2.4 in.) of relative displacement between the core boundary and the PCR. The clearance between the side support springs and their housing is estimated to be less than 2.54 cm (1 in.) from the figures in Ref. 1. This much motion would present some problems with contact between the springs and their housings resulting at best in some type of nonlinear restoring torque, and at worst, damage to all springs and subsequent torsional collapse.

Fortunately, damping has a strong influence on the solution of Eq. (11). Figure 5 illustrates the response of the core for a damping value that would represent 5% of critical if the equations were linear (left-hand scale). A maximum relative angular displacement of 0.004 rad was recorded at $t = 4.4$ s for this computer run. This value results in a relative displacement at the boundary of 1.8 cm (0.72 in.), which is probably tolerable for no spring damage. Figure 5 also shows the response for 10% of critical damping (right-hand scale) with the maximum value being 0.003 rad. This value gives a relative displacement at the core boundary of 1.3 cm (0.5 in.), also probably acceptable.

Fig. 5. Relative torsional core response for soft soil condition, 5% and 10% damping.

IV. COLLAPSE LOADS FOR PERFECT GRAPHITE COLUMNS

This section of the report is concerned with the behavior of single graphite columns under static compressive loads. As seen from Eq. (1) the buckling load for end-loaded perfectly elastic columns depends on the product of the cross-sectional moment of inertia and the elastic modulus divided by the square of the column length L with a constant of proportionality depending on the end conditions of the column. Assumptions inherent in the Euler formula, Eq. (1), are that the material behaves linearly and elastically up to the critical load, that the end load is axially applied through the centroid of the cross section, and that the column is initially straight and without imperfections. For the 1100 MW(e) HTGR, for instance, we have computed by means of Eq. (1) a buckling load of $\sim 0.44 \times 10^6 \text{ N}$ (100 000 lb) for its graphite columns. This represents a factor of safety of about 8 over the dead load of the core on the support posts,¹ (for a 15.2-cm-diam post).

Unirradiated graphite seems to obey an empirical nonlinear stress-strain ($\sigma - \epsilon$) relation of the Ramberg-Osgood form,

$$\epsilon = \sigma/E + \alpha(\sigma/E)^2, \quad (14)$$

where E is the modulus at zero stress and α is a material constant which indicates the magnitude of the nonlinear effect.⁷ The effect of nonlinear material behavior on the critical buckling load is to reduce the critical load below the Euler load particularly for small length-to-diameter (L/D) ratios. Since the HTGR core support posts have small (L/D) ratios we have initiated a calculation of the safety margin on the buckling load for a material obeying a stress-strain curve of the form of Eq. (14).

Figure 6 shows the situation we are modeling. A circular graphite rod of radius R and length L is acted on by an eccentrically applied axial load P . Under the action of this end load the rod deforms both in bending and axial compression; for L/D ratios greater than five the Bernoulli assumption that plane sections remain plane and perpendicular to the centroidal axis after deformation is valid. The relationship between the curvature κ of the rod and the bending moment M at a section for a material obeying the stress-strain relationship of Eq. (2) can be shown to be

$$M(\kappa) = \frac{16}{3\pi} EI\kappa \int_0^{\pi/2} \frac{\cos^4 \theta d\theta}{\sqrt{1 + 4\alpha R \sin \theta}}. \quad (15)$$

In Eq. (15) I is the moment of inertia of the cross section about a diameter. With the moment-curvature relationship given by Eq. (15) the deflected shape of the equilibrium configuration $u(x)$ can be found by taking the minimum of the potential energy Φ :

$$\Phi = \int_0^L dx \int_0^{\kappa} M(\kappa) d\kappa - \frac{1}{2} P \int_0^L \left(\frac{du}{dx} \right)^2 dx - eP \left[\left(\frac{du}{dx} \right)_{x=0} - \left(\frac{du}{dx} \right)_{x=L} \right] \quad (16)$$

and the curvature κ is given by the second derivative of $u(x)$.

Equation (16) was evaluated using cubic shape functions to represent $u(x)$ in individual finite elements with dependent variables being deflections

Fig. 6. Model of circular graphite rod in combined bending and compression.

(u_i, u_j) and rotations (θ_i, θ_j) at the nodes (i, j) with coordinates $(0, \Delta L)$. Thus, in any element

$$u(x) = u_i + \theta_i x + \left[\left(\frac{3}{\Delta L^2} \right) (u_j - u_i) - \left(\frac{1}{\Delta L} \right) (2\theta_i + \theta_j) \right] x^2 + \left[\left(\frac{1}{\Delta L^2} \right) (\theta_i + \theta_j) - \left(\frac{2}{\Delta L^3} \right) (u_j - u_i) \right] x^3, \quad (17)$$

where ΔL is the length of the finite element. Minimization of Φ with respect to the nodal dependent variables $(u_1, \theta_1, u_2, \theta_2, \dots, \theta_N)$ together with enforcement of pinned end conditions gives a system of nonlinear equations that serve to determine the dependent variables of the problem. All integrals were evaluated by Gauss integration formulae using four or six integration points. The resulting nonlinear equations were solved numerically by using an incremental tangent stiffness method (see Chapter 19 of Ref. 8) with

stiffness reformulation for all iterations during a given axial load increment. As the critical buckling load is approached, the system of approximating linear equations becomes ill conditioned and divergence of the iteration method was observed. Whenever divergence occurred, interval halving of the applied load was used to continue the calculation toward critical buckling much as described in Ref. 9. With this method we were able to predict the Euler buckling load within 1% using only two finite elements to represent an elastic column. Typical results in the form of a load-deflection curve are shown in Fig. 7 for an elastic column [$\alpha = 0$ in Eq.(14)] and for a graphite column ($\alpha = 200$). For nonzero α the load deflection curve continues beyond the critical deflection and there are two deflected shapes of the column for any load less than the critical load. We have not computed the "softening" portion of the load-deflection curve.

Fig. 7. Load-deflection curves for elastic and graphite columns.

Figure 8 summarizes the results of this calculational method when applied to unirradiated graphite columns using values of $E = 6900$ MPa and $\alpha = 200$ (approximately those given in Ref. 7 for H-451 graphite). Three values of end load eccentricity e and L/D ratios of 5, 10, 20, 30, 40, 60, and 80 were used to generate the data in this figure. The results shown in Fig. 8 are as expected. For large values of L/D the critical stress (the buckling load divided by the cross-sectional area) is low and the effective modulus is closely approximated by E ; consequently the buckling load is close to the Euler buckling load. For smaller values of L/D , though, the critical stress at buckling is high, the effective modulus is significantly less than E , and there is a substantial lowering of the critical buckling load below the Euler value.

The preceding analysis fails to take into account the influence of axial force on the inelastic buckling of columns. Due to the presence of significant axial stress, the neutral axis is displaced from the centroidal axis of the columns which affects significantly the buckling load of an inelastic column (the axial stress does not influence elastic buckling since the axial

Fig. 8. Critical stress in a graphite column vs length-to-diameter ratio.

stress distribution is separable into that of bending and that of direct compression). Referring to Fig. 6 the position of the neutral axis (that is, that fiber of the column which is unstrained) is determined for a circular cross section by

$$-P = 2 \int_{-R}^{+R} \sqrt{R^2 - y^2} \sigma(y) dy,$$

which can be forced into the following equation:

$$\frac{P}{\pi R^2 E} = \frac{-1}{2} \left[\frac{\sigma(-1)}{E} + \frac{\sigma(+1)}{E} \right] + R |\kappa| \int_{-1}^{+1} \left[\frac{\xi \sqrt{1-\xi^2} + \sin^{-1} \xi}{1 + 2\alpha \left(\frac{\sigma}{E} \right)} \right] d\xi. \quad (18)$$

Given the curvature κ the axial strain distribution according to the Bernoulli hypothesis is

$$\epsilon(y) = -\epsilon_0 + y\kappa,$$

from which the axial stress distribution can be determined from Eq. (14) provided ϵ_0 is known (and hence the position of the neutral axis). However, ϵ_0 is determined by the requirement that Eq. (18) is valid. The equations determining ϵ_0 are nonlinear and only numerical methods can provide solutions. Once the value of ϵ_0 is determined in terms of the curvature κ , an equation analogous to Eq. (15) is obtained,

$$M(\kappa) = \frac{8EI}{3\pi} \kappa \int_{-1}^{+1} \frac{(1-\xi^2)^{3/2}}{1 + 2\alpha \left(\frac{\sigma}{E} \right)} d\xi, \quad (19)$$

from which the strain energy in bending can be determined.

A computer program implementing these analysis procedures has been written to handle general Ramberg-Osgood materials [that is, arbitrary exponent in Eq. (14)]. In addition to the features mentioned previously (for example, cubic shape functions, use of the incremental tangent stiffness

method, interval halving, etc.), the program is capable of determining, by iteration, the neutral axis for each load increment as well as the column stiffness and dependent variables for the column with displaced neutral axis.

Figure 9 illustrates the results of analyses for buckling loads for graphite columns with $L/D = 13$ and various axial load eccentricities. The ordinate is the ratio of the computed buckling load to the Euler load given in Eq. (1) and e/R is the eccentricity as a fraction of the radius. The upper curve represents the situation where the neutral axis and centroidal axis are assumed to coincide whereas the lower curve illustrates the results for a displaced neutral axis. Again two finite elements and six integration points were used to represent the column and evaluate the stiffness respectively.

Fig. 9. Buckling loads for graphite columns as a function of eccentricity ($L/D = 13$).

As can be seen from Fig. 9, taking into account the influence of the axial load on displacing the neutral axis from the centroidal axis has a significant effect in reducing the buckling load of a column with length to diameter ratio typical of HTGR core support posts. The computed buckling loads for small eccentricity were found to be in good agreement with Engesser's load P_E ,

$$P_E = \pi^2 E_T I / L^2, \quad (20)$$

where E_T is the tangent modulus at the critical buckling stress given from Eq. (14) by

$$E_T = \frac{E}{1 + 2\alpha(\sigma/E)}. \quad (21)$$

V. LOAD CARRYING CAPACITY OF IMPERFECT GRAPHITE COLUMNS

In the previous section, the analysis for buckling loads of graphite columns was based on the condition of perfect columns. By perfect, we mean that the material is homogeneous and that there is no initial curvature nor interior or exterior flaws in the column. In the present section the effect of an exterior flaw on the maximum carrying load of the column will be analyzed. Although at installation the problem of flaws can be eliminated through close control in fabrication and acceptance inspection, cracks may develop during operation particularly after such unfavorable loading conditions as severe temperature variations and seismic shocks. The load carrying capacity of the column, especially if eccentrically loaded, will be reduced by the presence of the cracks. The present analysis will show this effect for one crack type.

The analysis will assume, a priori, the existence of the crack. The geometry of the crack will determine whether the crack is loaded by extension, by in-plane shear, by out-of-plane shear or by a combination of these. In terms of the fracture configuration they are the Mode I, the Mode II, and the Mode III cracks or their combinations. The circumferential Mode I (open mode) crack is possibly the worst case under flexural configuration. It is, therefore, considered in the present section.

A. Fracture Mechanics in Columns

1. Basic Concepts of Fracture Mechanics

There are three fundamental modes of cracks as shown in Fig. 10. Both Modes I and II have force in the plane of the crack (c); P is perpendicular to the crack (tensile) in Mode I while in Mode II P is parallel to the crack (c). In Mode III the shear force P is perpendicular to the plane containing the crack (c). For linearly elastic materials the stress field in the neighborhood of the Mode I crack tip (not corrected for geometry) in crack tip coordinates (r, θ) will generally have the form,¹⁰

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \begin{Bmatrix} 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \\ 1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \end{Bmatrix}, \quad (22)$$

Fig. 10. Fundamental fracture modes. (Note that the reference plane containing the crack C is vertical.)

where the constant K_I is a Mode I stress-intensity factor having the units of MPa - $\sqrt{m}$ (or psi - $\sqrt{in.}$). The associated displacements are

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_I}{2\mu} \left(\frac{r}{2\pi} \right)^{\frac{1}{2}} \begin{Bmatrix} \cos \left(\frac{\theta}{2} \right) \left[\kappa - 1 + 2 \sin^2 \left(\frac{\theta}{2} \right) \right] \\ \sin \left(\frac{\theta}{2} \right) \left[\kappa + 1 - 2 \cos^2 \left(\frac{\theta}{2} \right) \right] \end{Bmatrix}, \quad (23)$$

where $\kappa = 3 - 4\nu$ for plane strain, μ is the shear modulus and ν is Poisson's ratio. The critical stress-intensity factor K_{IC} , is a constant depending on material and mode of loading. For other modes of cracks similar equations have been developed. The definition of the stress intensity factor is that

$$\sigma_{\max} = \lim_{r \rightarrow 0} \frac{K_I}{\sqrt{2\pi r}}.$$

2. Column Theory

As mentioned in Sec. I, Euler's equation was developed for linearly elastic columns. For nonlinear materials or those loaded in the plastic range without unloading, it has been found that Engesser's Eq.(20) using the tangent modulus E_T is adequate to predict the collapse load (see Sec. IV).

When there is no eccentricity in loading the circumferential Mode I crack will simply be closed for forces less than the Engesser's load. The theoretical buckling load will then be conservative for a column with or without such cracks.

When the load is displaced from the centroidal axis, a bending moment is always present and the postulation of a straight configuration in stability analysis no longer holds. Indeed, the problem is basically one of structural equilibrium of a beam-column rather than that of stability of a simple column. Under such a flexural state then, a crack can propagate. In looking for the worst case one would readily identify that of the crack on the convex side of the column and at the midsection of a simple column. The analysis in Sec. IV has given an upper limit when the crack depth approaches zero.

If δ is the midspan lateral deflection then we have

$$M(x) = P[e + u(x)],$$

and hence

$$M_{\max} = P[e + u(L/2)] = P(e + \delta).$$

The midspan deflection δ of an eccentrically loaded simple column is given in Sec. IV. However, in order to facilitate analysis instead of the iteration method called for in Sec. IV, an approximate expression will be used here,

$$\delta = e \left[\sec \left(\frac{L}{2} \sqrt{\frac{P}{E_T I}} \right) - 1 \right],$$

where e is the eccentricity. The maximum moment is then

$$M_{\max} = Pe \sec \left(\frac{L}{2} \sqrt{\frac{P}{E_T I}} \right). \quad (24)$$

It is postulated that for a single crack, the relaxation of stiffness on the convex side is predominantly a local phenomenon in the neighborhood of the cracked section without much effect on the overall stiffness of the column. The midspan deflection is therefore used for both cracked and uncracked columns.

3. Existing Methods of Analysis

Fracture mechanics has made many significant advancements in the past decade. Its application has extended from the early field theory problems to include many structural members such as prismatic bars of rectangular or circular cross sections subjected to tension, bending or torsion. Applications to columns, however, are few. A comprehensive review of recent work was done by Liebowitz¹¹ in 1969. Four fundamental approaches to fracture in structural members are described in Ref. 11, and two of these are described below. One is the semi-analytical energy method, employing the

Griffith criterion and experimentally obtained data for energy computation. Here one uses

$$U_{cr} = U_A - (U_B + U_C),$$

where U_{cr} is the crack energy, U_A is the applied energy, and $(U_B + U_C)$ is the strain energy through bending and compression. The second method is based on the stress intensity factor at the crack tip. The stress intensity factor of a Mode I crack in the column is assumed to be

$$K_I = K_{IB} - K_{IT}, \quad (25)$$

where K_{IB} is the stress intensity factor of pure bending for a Mode I crack and K_{IT} is that of tension. At impending failure, $K_I = K_{IC}$.

Since the fracture phenomenon is basically a local effect, any global analysis, such as that using the energy method, can only provide estimates of failure loads. Indeed, experiments on aluminum columns conducted by Liebowitz showed wide dispersion (see Ref. 11). Because its failure criterion is based on the local state of stress at the crack tip, the method of the stress intensity factor is more accurate in the prediction of failure loads. There is still the problem of whether the stress intensity factor obtained from certain mathematical models represents that of the actual physical problem or not. In Liebowitz's paper, Neuber's solutions¹² for hyperbolic notches were mainly used. Some solutions by Bueckner¹³ and Bowie¹⁴ have been correlated to experimental data on aluminum columns. These results as shown by graphs and tables in Ref. 11 were actually better than those from the Neuber equations. The analytical method used in the present section will use principally the method of stress intensity factor.

4. Effect of Geometrical Configuration

It has been pointed out that centrally loaded straight columns with a circumferential crack will fail by buckling at Engesser's load. Eccentricity in loading and/or initial curvature will introduce a flexural state at any load. If a circumferential crack exists on the convex side of the column, then the effect of stress concentration is accentuated. The depth of the circumferential crack plays both the role of influencing the

stress concentration effect and changing the nominal stress due to a change in the uncracked area at the cross section. The effect on the maximum load capacity is further influenced by the slenderness ratio (L/R_g), where R_g is the radius of gyration of the cross-sectional area with respect to the central bending axis. The data by Liebowitz using 12.7 mm x 12.7 mm (1/2 in. by 1/2 in.) 7075-T6511 aluminum stock are reproduced in Fig. 11, in which the effect of crack depth at low slenderness ratios is clearly shown.

B. Analysis of Graphite Column with a Circumferential Crack

In this analysis we will develop a set of working equations best suited for graphite columns. The crack configuration will be selected on the basis of simplicity of analysis, but still retaining the basic features of a cracked column. The column geometry is shown in Fig. 12.

The HTGR supporting columns are manufactured by Union Carbide under the trade name of ATJ. Currently, the production process is to press powdered graphite particles into a mold under specified pressure. The density of the resulting product varies somewhat, which results in a statistical distribution of mechanical properties. Mechanical properties of graphite are given in the General Atomics Corporation (GAC) report by R. J. Price.⁷ Whenever properties of ATJ were not available, those of a nuclear grade graphite H-451 were substituted. No quantitative information is available for temperature or irradiation effects. However, since most studies have shown an enhancement of Young's modulus from irradiation or temperature, designs based on the ambient properties may be considered conservative. The critical stress intensity factor of ATJ for Mode I fracture is contained in Ref. 7. No work has been done for either Mode II or Mode III fractures. For the case of with-grain-loading and across-grain-crack, the critical value of K_I is

$$K_{IC} = 0.89 \text{ MPa} \cdot \sqrt{\text{m}} \quad (810 \text{ psi} \cdot \sqrt{\text{in.}}) .$$

The stress-strain curve for ATJ is not available, and consequently the non-linear stress strain curve of Eq. (14) will be used because it appears to fit H-451 graphite.

The stress intensity factors can be computed by using the correction factors $f(\eta)$, sometimes called the dimensionless stress intensity factors,

Fig. 11. Load carrying capacity of notched column vs slenderness ratio. From Ref. 11.

Fig. 12. Nonlinear graphite material stability model and columns containing cracks.

such that

$$K_{IT} = \sigma_{NT} \sqrt{\pi c} f_T(\eta), \quad K_{IB} = \sigma_{NB} \sqrt{\pi c} f_B(\eta), \quad (26)$$

where σ_{NT} and σ_{NB} are the nominal stresses at the crack tip for tensile and bending cases respectively; $f_T(\eta)$ and $f_B(\eta)$ are the correction factors in tension and in bending respectively, and η is the crack length parameter.

For circular cross sections ($\eta = a/R$), then

$$f_T(\eta) = \frac{1}{2} \sqrt{\eta} \left(1 + \frac{\eta}{2} + \frac{3}{8} \eta^2 - 0.363 \eta^3 + 0.75 \eta^4 \right)$$

$$f_B(\eta) = \frac{3}{8} \sqrt{\eta} \left(1 + \frac{\eta}{2} + \frac{3}{8} \eta^2 + \frac{5}{16} \eta^3 + \frac{35}{128} \eta^4 + 0.53 \eta^5 \right). \quad (27)$$

The expressions for the correction factors are interpolated from the limiting cases of $\eta \rightarrow 0$ and $\eta \rightarrow 1$, obtained through asymptotic methods.¹⁵

For rectangular cross sections $\eta = \frac{a}{b}$, then (see Ref. 15 and 16)

$$f_T(\eta) = \sqrt{\eta} (0.637 - 0.804 \eta + 0.258 \eta^2 + 0.091 \eta^3)$$

$$f_B(\eta) = \sqrt{\eta} \left[\frac{4}{3\pi} \left(1 + \frac{1}{2} \eta + \frac{3}{8} \eta^2 + \frac{5}{16} \eta^3 \right) - 0.470 \eta^4 + 0.663 \eta^5 \right]. \quad (28)$$

After application of Eq. (25), we have for the limiting value of K_{IC} ,

$$K_{IC} = \sqrt{\pi c} \left[\sigma_{NB} f_B(\eta) - \sigma_{NT} f_T(\eta) \right]. \quad (29)$$

The relation of the nominal bending stress to the flexural moment is elementary for linearly elastic materials. However, for nonlinear materials, the analytical expression for their relationship is not easily obtained and such is the case with graphite. If we assume that plane sections remain plane, the strain will then be linear. For the case of beam columns, the strain on the convex side is small compared to that of the compressive side. The stress distribution curve has a slope inflection point close to the convex side. The flexural tensile stress in the beam-column may be approximated in the present case by the linear relationship. Equation (29) hence becomes,

$$K_{IC} = P \sqrt{\pi c} \left[\frac{e}{Z_o} f_B(\eta) \sec \left(\frac{L}{2} \sqrt{\frac{P}{E_T I}} \right) - \frac{1}{A_o} f_T(\eta) \right], \quad (30)$$

where A_o is the cross-sectional area and Z_o is the section modulus at the cracked section. The tangent modulus E_T is an approximation of the average stiffness of the column over the cross section and is given by Eq. (21) in terms of the average stress on the column. Alternatively,

$$E_T = \frac{E^2 A}{EA + 2\alpha P}, \quad (31)$$

where P is the axial load and A is the nominal cross-sectional area. Equation (30), when adapted to the quadratic Ramberg-Osgood stress-strain relation, becomes

$$g(P) = P \sqrt{\pi c} \left[\frac{e}{Z_0} f_B(\eta) \sec \left(\frac{L}{2EAR_g} \sqrt{P(EA + 2\alpha P)} \right) - \frac{1}{A_0} f_T(\eta) \right] - K_{IC} = 0. \quad (32)$$

Equation (32) may then be expressed for the columns with circular or rectangular cross sections with respective values of Z_0 , A_0 , A and R_g . The correction factors $f_B(\eta)$ and $f_T(\eta)$ must use the expressions in Eq. (27) and Eq. (28) respectively for the different cross sections. Hence, for the circular cross-section columns

$$g(P) = \frac{P}{a^2} \sqrt{\frac{c}{\pi}} \left[\frac{4e}{a} f_B(\eta) \sec \left(\frac{L}{\pi ER^3} \sqrt{P(\pi ER^2 + 2\alpha P)} \right) - f_T(\eta) \right] - K_{IC} = 0. \quad (33)$$

For the rectangular cross-section columns

$$g(P) = \frac{P}{2ha} \sqrt{\pi c} \left[\frac{3e}{a} f_B(\eta) \sec \left(\frac{\sqrt{3} L}{4Eb^2 h} \sqrt{P(2bhE + 2\alpha P)} \right) - f_T(\eta) \right] - K_{IC} = 0. \quad (34)$$

For deep cracks in the rectangular column, Neuber's equation for hyperbolic notches may be used^{12,17}

$$K_{IT} = \frac{1}{2} \lim_{\rho \rightarrow 0} \sqrt{\pi \rho} \left[\frac{\frac{4}{3} \left(\frac{a}{\rho} + 1 \right) \sqrt{\frac{a}{\rho}}}{\left(\frac{a}{\rho} + 1 \right) \tan^{-1} \sqrt{\frac{a}{\rho}} + \sqrt{\frac{a}{\rho}}} \right] \sigma_{NT} = 2 \sqrt{\frac{a}{\pi}} \sigma_{NT}$$

$$K_{IB} = \frac{1}{2} \lim_{\rho \rightarrow 0} \sqrt{\pi \rho} \left[\frac{\frac{4}{3} \left(\frac{a}{\rho} \right) \sqrt{\frac{a}{\rho}}}{\sqrt{\frac{a}{\rho}} + \left(\frac{a}{\rho} - 1 \right) \tan^{-1} \sqrt{\frac{a}{\rho}}} \right] \sigma_{NB} = \frac{4}{3} \sqrt{\frac{a}{\pi}} \sigma_{NB},$$

where ρ is the notch root radius. The resulting equation for the maximum carrying load is solved as

$$g(P) = \frac{P}{h\sqrt{\pi a}} \left[\frac{2e}{a} \sec \left(\frac{L}{2} \sqrt{\frac{P}{E_T I}} \right) - 1 \right] - K_{IC} = 0. \quad (35)$$

C. Carrying Load for ATJ Columns with Symmetric Crack

1. Circular Cross Section

It was pointed out previously that the maximum load at zero eccentricity is the buckling load with or without the circumferential Mode I crack. The tangent modulus formula readily yields Engesser's loads for columns with circular cross section of $R = 76 \text{ mm}$ (3 in.), $L = 1.98 \text{ m}$ (78 in.) typical of large HTGRs and of graphite properties $E = 6900 \text{ MPa}$ and $\alpha = 200$. However, the buckling load would have been the failure load, provided the column did not fail first through material limitation. In other words, the column will fail before the buckling load is reached if the outer fiber of the column in the flexural state attains the maximum tensile stress on the convex side of the column or the maximum compressive stress on the concave side, whichever occurs first. Nominally the beam columns are subjected to higher compressive stresses than tensile, but compensating for this is the fact that graphite exhibits higher compressive strength than the tensile strength. The failure criterion is then the lower value of P from the solutions of the following two equations:

$$g(P) = \frac{P}{A} \left[\frac{ed_t}{R_g^2} \sec \left(\frac{L}{2} \sqrt{\frac{P}{E_T I}} \right) - 1 \right] - \sigma_{ut} = 0$$
$$g(P) = \frac{P}{A} \left[\frac{ed_c}{R_g^2} \sec \left(\frac{L}{2} \sqrt{\frac{P}{E_T I}} \right) + 1 \right] - \sigma_{uc} = 0, \quad (36)$$

where d_t and d_c are the centroidal distances to the outer tensile and compressive fibers respectively, and σ_{ut} and σ_{uc} are the tensile and compressive strengths of the graphite. For H-451 graphite these values are

$$\sigma_{ut} = 18.5 \text{ MPa (2686 psi)}$$

$$\sigma_{uc} = 57.3 \text{ MPa (8300 psi)}$$

taken from Ref. 7. For very large values of σ_{ut} and σ_{uc} the maximum load P in

Eq. (36) will have its limiting value of Engessor's load. However, the tensile or compressive strength is generally finite. The maximum load for eccentrically loaded columns is therefore in general less than the Engessor's load, depending on the material property and the amount of eccentricity. We therefore may compute maximum loads for columns based upon: (a) a centrally loaded configuration for which columns will fail by buckling as discussed in the previous section, (b) an eccentrically loaded configuration with very shallow cracks for which columns will fail essentially by overloading the outer fibers beyond the limit strength and (c) the same eccentrically loaded configuration as (b) but with cracks of reasonable depth for which the column will fail by crack extension. All three failure modes are represented in Figs. 13 and 14 using data based upon ATJ graphite or H-451 graphite whenever the former property is not available. The stress-strain relationship is that of the quadratic Ramberg-Osgood, Eq. (14).

Fig. 13. Maximum load of fractured column as a function of crack depth.

Fig. 14. Maximum load of fractured column as a function of loading eccentricity.

Figure 13 shows maximum carrying loads of the particular column as functions of crack depth for various loading eccentricities. The deteriorating effect of Mode I crack on the carrying strength of columns is very much in evidence. Two limit conditions are demonstrated. In one limit condition (at zero eccentricity) Engessor's load ($254.4 \times 10^3 \text{ N}$) is the limiting load for all crack depths. In the other, for the range of very small crack depths (in the present case between 0 and 1 mm), the trend would have been to the limiting buckling load. However, the limit strength of the outer fiber bends the load curves toward a value much less than the buckling load. Here the cracks may be merely scratches and the failure of column may be due to crack extension or may well be due to fiber failure by exceeding the limit strength. The end result will be the same. As an illustration, based upon the crack extension Eq. (33), the maximum carrying loads of the graphite column with a 0.5-mm-deep crack for 10 mm- and 20-mm eccentricities are 211 200 N and 177 200 N respectively. Yet, based upon the fiber failure Eq. (36), the maximum carrying loads of the same column without any cracks are 209 500 N and 174 200 N respectively for the given eccentricities. The broken lines within this region represent interpolated data for mixed failure models.

In Fig. 14, the same sets of data represent maximum carrying loads of the column as functions of loading eccentricity for various crack depths. The figure illustrates fully the rapid reduction of load carrying capacity of the column due to eccentricity of load, especially in the presence of Mode I cracks. The limit conditions in this figure are (a) the converging point at the Engessor's load at zero eccentricity and (b) the two curves for zero crack depths. Indeed, the eccentrically loaded column is subjected to higher compressive stress at fibers on the concave side of the column. Since the compressive strength of the graphite (H-451) is three times its tensile strength, the maximum load based upon the maximum tensile stress is lower than that based upon the maximum compressive stress. The former values are therefore used as a criteria to determine the load capacity of eccentrically loaded graphite columns with "very shallow cracks".

The two figures may also be used as a quick reference in design of specific columns. For example, at 2.76 MPa (400 psi) design stress the column should support a working load of 50 300 N (11 300 lb). With reference to Fig. 13 and assuming that a maximum of 38-mm (half-radius) eccentricity is to be expected, we may conclude that a maximum crack depth of 15 mm may be

sustained. Conversely, by referring to Fig. 14, for the same working load in order to sustain a crack depth of 20 mm, the load eccentricity should not be more than 27.5 mm.

2. Square Cross Section

Figures 13 and 14 apply to the solutions of the equations for ring-shaped edge cracks and properly apply to case (b) of Fig. 12. To investigate the effects of differing geometrical cross section, the solution of Eq. (34) was obtained and is shown in Fig. 15. This equation applies to the symmetric edge crack of case (a) of Fig. 12. The slenderness ratio is 52 and the cross-sectional area is taken equal to that of the column of Fig. 14, which has a square cross section. In general, the effect of the

Fig. 15. Load carrying capacity given by Eq. (34) for a square section column of slenderness ratio 52 containing symmetric edge cracks.

square cross section is to lower the predicted crack extension load for eccentricities less than 50 mm; the curves are very close for $e = 50$ mm with a slightly higher load carrying capacity predicted for the square cross section column. This effect is to be expected since in order to maintain the same carrying capacity for the uncracked columns, the square cross section must be slightly longer. Crack extension under this loading is caused by the tensile strain produced by the maximum moment at the midspan cross section that in turn is a function of both column length and eccentricity of the loading. Thus, the decrease in load carrying capacity for the longer length column is expected, but as the eccentricity increases, the effect of added length becomes less pronounced.

D. Carrying Load for ATJ Columns with Single-Edge Crack

In the next section of this report we consider a finite element analysis of a cracked column for which the crack is a single-edge crack. The extension of the structural theory presented above to the case of a square cross-section column with a single-edge crack on the convex side is relatively straightforward, and is presented below.

As in Sec. V.B we let $\eta = a/b$ where b is now the total section width and a is the net section width after the crack length has been subtracted out. With these definitions the asymptotic expressions for the dimensionless correction factors for both tension $f_T(\eta)$ and bending $f_B(\eta)$ are given in Ref. 15. Because of the large slope in K_I as a function of η as η approaches unity, a polynomial expression is deemed inappropriate as an interpolation function. Instead, a conic section interpolation function is suggested. Thus, the data of Ref. 15 for the functions $f_T(\eta)$ and $f_B(\eta)$ were fit with a function of the form

$$f^2 + B\eta^2 + C\eta + Df + E\eta + F = 0$$

with the constants B through F numerically given in Table III.

From Eq. (32) for rectangular cross sections and with a , b , and c as defined above, the expression for the load carrying capacity of a rectangular column containing a single-edge crack on the convex side becomes

TABLE III
COEFFICIENTS FOR CONIC SECTION INTERPOLATION FUNCTION

	$f_T(\eta)$	$f_B(\eta)$
B	-4.299685	0.0815943
C	169.8663	77.22194
D	-210.9024	-105.8534
E	-29.65439	-9.413302
F	78.73761	39.07528

$$g(P) = \frac{P\sqrt{\pi c}}{ah} \left[\left(\frac{6e}{a} \right) f_B(\eta) \sec \left(\frac{\sqrt{3L}}{Eb^2h} \sqrt{P(bhE + 2\alpha P)} \right) - f_T(\eta) \right] - K_{IC} = 0. \quad (37)$$

Figure 16 shows the solution of this equation for various eccentricities as a function of crack depth. Again, this solution is given for the equivalent HTGR support column defined previously. The effect of a single edge crack is to increase the load carrying capacity of the column over that for a column with symmetric edge cracks.

VI. FINITE ELEMENT ANALYSIS OF A CRACKED GRAPHITE COLUMN

The previous section describes a structural theory for examining failure of graphite columns containing a Mode I crack. The complexity of this problem and the assumptions that were necessary to formulate even the simplest case for crack location, orientation, mode, and loading may preclude easily extending this type of theory to other cases.

Another approach is to attack this problem with the finite element method.^{8,18,19} However, such an approach calls for a special type of finite element code that can accommodate the following features necessary to describe cracked graphite columns:

Fig. 16. Load carrying capacity given by Eq. (35) for a square cross-section column (slenderness ratio 52) containing a single-edge crack.

- (a) The primary material obeys a nonlinear stress-strain law.
- (b) The crack surface is capable of transmitting compressive stress, but not tensile stress.
- (c) The central deflection of the column may become relatively large as the axial load approaches the Engessor value.
- (d) The stress field contains a $r^{-1/2}$ singularity near the crack tip.

The finite element code (NONSAP) used in this study is one that has been modified at the Los Alamos Scientific Laboratory to analyze the behavior of prestressed concrete reactor vessels. Besides nonlinear material and geometry capabilities, other desirable features of the NONSAP code are:²⁰

- (a) a nonlinear truss element that may be used to enforce the time-(or load-) dependent displacement boundary condition on the crack surface, (b) an incremental loading capability necessary to enable the failure load for crack

extension to be determined, and (c) an isoparametric element that can be used to approximate numerically the stress field singularity at the crack tip.²¹

A. Mesh Description

The problem investigated here is a plane strain version of the problem investigated in Sec. V.D. Figure 17 shows the mesh used to represent the upper half of a column whose slenderness ratio is 52. Figure 18 shows the

Fig. 17. Finite element mesh of a column containing a crack.

mesh and boundary conditions used to model a crack. All continuum elements in this mesh are either the 7-node or 8-node isoparametric element. The mesh was produced using a general purpose mesh generating code.²²

The truss elements in Fig. 18 are defined to have only compressive stiffness of such magnitude that in compression, the U_z displacements along the crack are of the order of 10^{-10} mm. With this model, the column can be loaded axially and with a given eccentricity. Also, the model can be loaded incrementally until failure occurs through either crack extension, tensile material failure through exceeding the fiber tensile strength, compressive material failure or crushing, or column instability or buckling.

B. Material Model

An elastic-plastic strain hardening material model with a Von Mises initial yield condition was taken to represent nonlinear graphite behavior in this study. In effect this is a bilinear approximation to the uniaxial stress-strain law given by the Ramberg-Osgood relation, Eq. (14), used

Fig. 18. Finite element model of the crack region.

previously for graphite. Our studies indicated that the load carrying capacity of the columns depended more upon the crack extension criterion than upon the nonlinear material model, and since the crack extension criterion for graphite is based on a number of ad hoc assumptions that need experimental verification, use of the more realistic graphite material model was felt to be unjustified for this study.

C. Crack Extension Criterion

As was mentioned above, one of the principal difficulties in applying the finite element method to crack problems is the definition of the crack extension criterion. The critical stress criterion as discussed in Sec. V.A is the commonly used crack extension criterion, but the implication here is that the stress field is known accurately in the elements near the crack tip. However, since in the displacement approach to the finite element method it is the displacements which are known most accurately, a crack extension criterion based on critical crack displacement would seem to be preferred. Two such criteria are discussed below.

For the first criterion we assume that fracture is caused by the local tensile stress field surrounding the crack tip, and that the local displacement field is described by Eq. (23). These equations apply to linear elastic materials only and are uncorrected for geometry. We assume that the Eq. (23) can be corrected for geometry by correcting K_{IC} in accordance with the solution for a strip containing a single edge crack under tension.¹⁵ Under these assumptions Eq. (23) gives the critical crack opening displacement,

$$U_{\text{CRIT}} = \frac{2\gamma_{\text{IC}}}{\mu} \sqrt{\frac{r}{2\pi}} (1-\nu) \quad (38)$$

where all terms have been defined previously except for γ , which is the dimensionless stress intensity geometry correction factor. The form of the stress field - in particular the singular behavior of $r^{-1/2}$ as $r \rightarrow 0$ - was taken into account by moving the midside nodes of the elements at the crack tip as shown in Fig. 19 and as described in Ref. 20.

Fig. 19. Mesh around crack tip illustrating modifications used to account for $r^{-0.5}$ singularity in the stress field.

Another critical crack opening displacement criterion can be established based on the elastic, plastic solution of a sheet containing an edge crack.²³ It can be shown, using the expressions derived in Ref. 23 for the small scale yielding assumptions, that

$$U_{\text{CRIT}} = \frac{(\gamma K_{\text{IC}})^2 (1-\nu^2)}{2E\sigma_y}, \quad (39)$$

where γ is the dimensionless geometry correction factor, K_{IC} is the critical stress intensity factor for the equivalent elastic problem, σ_y is the yield stress for the material, and the other symbols are as defined previously.

D. Comparison of Finite Element and Structural Theories for ATJ Columns

Calculations with NONSAP were made for the square cross-section column of slenderness ratio 52 and with the load carrying capacity defined by the crack extension criteria given by Eqs. (38) and (39). Figure 20 shows typical deformed meshes for both the column and the crack region. Calculations were made for two different eccentricities of load and for varying crack lengths. Figure 21 shows the results of this calculation of the load carrying capacity using the crack extension criteria given by Eqs. (38) and (39) together with the load carrying capacity given by Eq. (37).

Regardless of how crack extension is defined, the detrimental effect of a Mode I crack on the load carrying capacity of a graphite column is evident. It is interesting to observe that for crack lengths of less than 15 mm the predicted load carrying capacity from all solutions varies by less than 7%. In general the finite element solutions predict load carrying capacities that do not deteriorate as rapidly with crack depth as those predicted by the structural theory. Indeed, for the smaller eccentricity case, the load carrying capacity is actually predicted to increase slightly with increasing crack depth. Since this prediction is physically unrealistic, it must be presumed that the critical crack opening displacement criteria used is not valid for these crack depths and eccentricities.

In conclusion there is good agreement between the results of the structural theory of Sec. V and the finite element treatment of Sec. VI - particularly for shallow cracks. Improvements to the finite element treatment can come through a more realistic material model such as described in

Fig. 20. Deformed meshes — column under eccentric loading and local crack mesh.

Fig. 21. Load carrying capacity for a square cross-section column of slenderness ratio 52 predicted by finite element and structural theories.

Ref. 24 but, more importantly, through further theoretical and experimental research into the critical crack opening displacement criterion. The structural theory could be improved by taking into account the additional deformation produced by the crack opening such as was done in Ref. 25. Presumably the improved theories would be in agreement, especially for deeper cracks.

Finally the finite element method, by virtue of its generality, could easily be applied to a graphite column with a Mode II crack or a mixed mode crack provided a theoretical or experimentally measured crack extension criterion were available.

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