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Alternating-Phase Focusing: A Model to Study Nonlinear Dynamics

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Abstract. We discuss a new model to study alternating-phase focusing (APF). Our approach is based on representing the accelerating electric field with a continuous phase modulated traveling wave. The resulting nonlinear equations of motion can be solved analytically to predict the regions of stable APF motion. We also identify the key parameters which adequately describe the physics of APF. The model is believed to be applicable to low-β ion linacs with short independently-controlled superconducting cavities being developed at ANL.

1. Introduction

The basic idea of APF is to achieve stable beam transport in both axial and radial planes of motion by alternating the sign of equilibrium phase of the accelerating electric field. The advantages of realizing a three-dimensional beam focusing without the use of solenoids or quadrupole magnets must be weighed against the compromises in longitudinal acceptance one is forced to make. Previous works in the field [1, 2, 3, 4, 5] addressed APF in the context of a linac with a discrete number of accelerating gaps spaced in a predetermined manner to achieve a particular value of the synchronous phase in each gap (such as the case in the π-mode Wideroe linac and the Alvarez DTL). In contrast, the application we have in mind is a linac with superconducting accelerating cavities of the type described in [6]. These low-β cavities are short, can be independently controlled in adjusting both the phase and the amplitude of the electric field, and were shown to produce very high accelerating gradients [7]. The model presented in this paper is thought to be a good description of essential APF physics for the superconducting linacs and a starting point in trying to determine the practical limits of APF.

2. Beam dynamics in APF linacs

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2.1. Analytical model and assumptions

We assume that the electric field is described by a cylindrically symmetric traveling wave with a continuous phase modulation. Here, we choose the modulation to be sinusoidal:

$$E_z(r,z;t) = E_0 \cos\left[\omega t - \int_0^z k(z')dz' + \phi_0 + \phi_1 \sin(2\pi z/\Lambda)\right]. \qquad (1)$$

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where ω is the angular velocity and k is the wave number of the rf field, ϕ_0 is the equilibrium phase in the absence of APF, and Λ and ϕ_1 are the APF period and phase modulation amplitude respectively. For the central reference trajectory z_c, we choose

$$\omega t - \int_0^{z_c} k(z')dz' = 0 \tag{2}$$

In subsequent analysis, we will neglect the effect of the velocity change in one APF period; the reference particle is assumed to travel witn a constant β.

We can compute the average accelerating gradient by integrating eq. 1 for the reference trajectory:

$$< E >= \frac{E_0}{\Lambda}\int_0^{\Lambda} \cos\left[\phi_0 + \phi_1 \sin\left(2\pi z/\Lambda\right)\right] = E_0 \cos\phi_0 J_0(\phi_1). \tag{3}$$

2.2. Equations of motion

The equations of motion are

$$\frac{d^2 z}{dt^2} = \frac{q}{m} E_z(r, z; t), \tag{4}$$

$$\frac{d^2 r}{dt^2} = \frac{q}{m} E_r(r, z; t), \tag{5}$$

with E_z given by eq. 1 and E_r given to the first order in r by Maxwell's equations,

$$E_r(r, z; t) = -\frac{r}{2}\frac{\partial E_z}{\partial z}. \tag{6}$$

For an arbitrary longitudinal deviation from the reference trajectory given by $\Delta z = z - z_c$, the equation of motion becomes

$$\frac{d^2\Delta z}{dt^2} = \frac{qE_0}{m}\left\{\cos\left[\phi_0 - k\Delta z + \phi_1 \sin\left(2\pi(z_c + \Delta z)/\Lambda\right)\right] - \cos\left[\phi_0 + \phi_1 \sin\left(2\pi z_c/\Lambda\right)\right]\right\}. \tag{7}$$

We will first look at the APF linear motion.

2.3. Linear stability

Let us define some dimensionless parameters which we are going to use throughout this paper,

$$\Delta\phi \equiv -k\Delta z = -2\pi\frac{\Delta z}{\beta\lambda}, \tag{8}$$

$$\tau \equiv \frac{z_c}{\Lambda} = \frac{\beta c t}{\Lambda}, \tag{9}$$

$$\nu \equiv \frac{\Lambda}{\beta\lambda}, \tag{10}$$

$$\eta \equiv \frac{qE_0\beta\lambda}{\frac{1}{2}mv^2}. \tag{11}$$

The linearized equations of motion are

$$\frac{d^2\Delta\phi}{d\tau^2} + \pi\eta\nu\left[(-\nu + \phi_1\cos 2\pi\tau)\sin(\phi_0 + \phi_1 \sin 2\pi\tau)\right]\Delta\phi = 0, \tag{12}$$

$$\frac{d^2 r}{d\tau^2} - \frac{\pi}{2}\eta\nu\left[(-\nu + \phi_1\cos 2\pi\tau)\sin(\phi_0 + \phi_1 \sin 2\pi\tau)\right]\Delta\phi = 0. \tag{13}$$

By expanding the linear coefficients in eq. 13 in a Fourier series, we get the familiar Mathieu-Hill equations:

$$\frac{d^2\Delta\phi}{d\tau^2} - 2\left[B + \sum_{n=1}^{\infty} C_n \sin\left(2\pi n\tau + \theta_n\right)\right]\Delta\phi = 0, \tag{14}$$

$$\frac{d^2 r}{d\tau^2} + \left[B + \sum_{n=1}^{\infty} C_n \sin\left(2\pi n\tau + \theta_n\right)\right] r = 0, \tag{15}$$

where

$$B = \frac{\pi}{2}\eta\nu^2 J_0(\phi_1)\sin\phi_0, \tag{16}$$

$$C_n = -\pi\eta\nu|J_n(\phi_1)|\begin{cases}\sqrt{\nu^2\cos^2\phi_0 + n^2\sin^2\phi_0} & \text{if } n \text{ odd}\\ \sqrt{\nu^2\sin^2\phi_0 + n^2\cos^2\phi_0} & \text{if } n \text{ even}\end{cases}, \tag{17}$$

$$\tan\theta_n = -\tan\phi_0\begin{cases}\frac{n}{\nu} & \text{if } n \text{ odd}\\ \frac{\nu}{n} & \text{if } n \text{ even}\end{cases}. \tag{18}$$

The equations are analogous to those obtained in ref. [5] using a discrete thin-lens approximation and a standing wave approach. Here, however, the beam dynamics variables B and C_n depend only on four independent parameters: ϕ_0, ϕ_1, ν, and η; moreover, the dependence is given in an explicit analytic form. Keeping just the $n = 1$ term, we obtain a well-known Mathieu equation for which we can compute the stable region boundaries. Fig. 1 shows the linear stable region. Fig. 2 shows stability boundaries

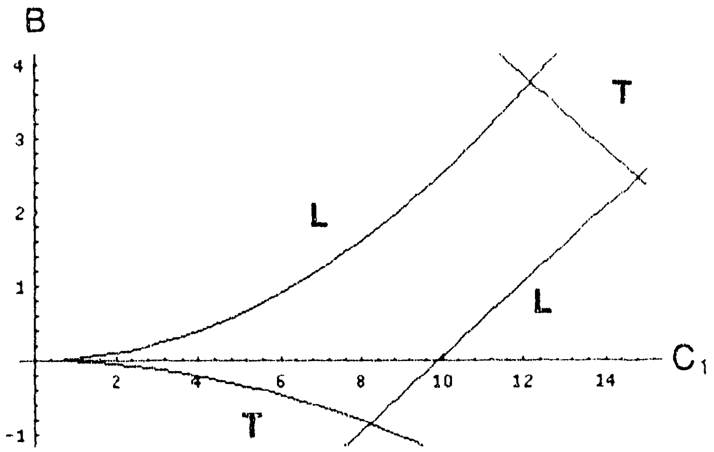

Figure 1. Transverse (**T**) and longitudinal (**L**) boundaries for the linear stability region.

in the $\phi_1 - \nu$ space for APF phase advances of less than 90° and $\phi_0 = 5°$, $\eta = 0.05$; fig. 3 shows the effect of increasing the "acceleration parameter" η to 0.25.

We next turn our attention to the nonlinear problem of calculating the longitudinal acceptance for the APF linac.

2.4. Longitudinal acceptance

The full nonlinear equation of longitudinal motion is given by

$$\frac{d^2\Delta\phi}{d\tau^2} = -\pi\eta\nu^2\left\{\cos\left[\phi_0 + \Delta\phi + \phi_1\sin\left(2\pi\tau - \frac{\Delta\phi}{\nu}\right)\right] - \cos\left[\phi_0 + \phi_1\sin 2\pi\tau\right]\right\}. \tag{19}$$

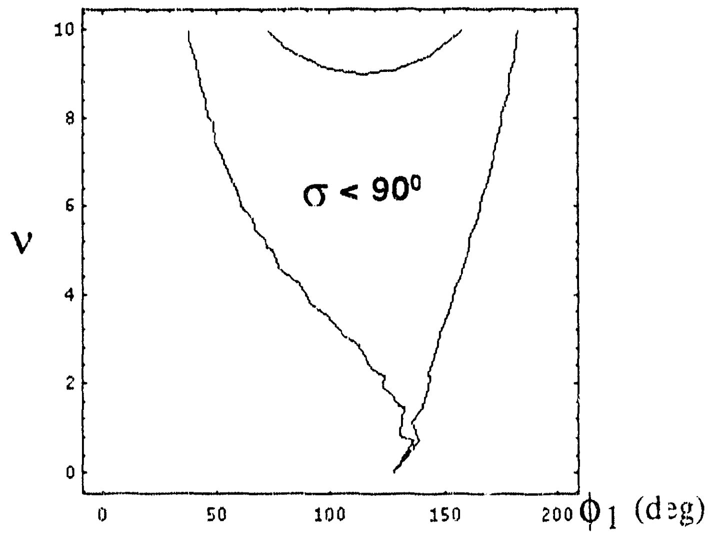

Figure 2. Stability boundaries for trajectories not exceeding 90° in either transverse or longitudinal phase advance with $\phi_0 = 5°$ and $\eta = 0.05$.

We can find the effective potential for eq. 19 by using the averaging method given in ref. [8] and applied to the problem of longitudinal acceptance in ref. [5]. We review the method here for completeness.

Consider an equation of the form

$$\frac{d^2x}{dt^2} = -\frac{\partial U_0}{\partial x} + f, \tag{20}$$

where

$$f = \sum_{n=1}^{\infty} \left[u_n \sin(n\Omega t) + v_n \cos(n\Omega t)\right]. \tag{21}$$

If the period of the motion caused only by the potential U_0 is T and $\Omega \gg 1/T$, the particle motion can be described by

$$x(t) = X(t) + \xi(t), \tag{22}$$

where $X(t)$ and $\xi(t)$ are caused by the slowly varying potential U_0 and the rapidly oscillating force f respectively. In this case, eq. 21 can be averaged to yield

$$\frac{d^2X}{dt^2} = -\frac{\partial U_{\text{eff}}}{\partial X}, \tag{23}$$

where the effective potential U_{eff} is given by

$$U_{\text{eff}} = U_0 + \frac{1}{4\Omega^2} \sum_{n=1}^{\infty} \frac{u_n^2 + v_n^2}{n^2}. \tag{24}$$

In the problem at hand, eq. 19 can be transformed to the canonical form of eq. 21 by using well-known Fourier expansions:

$$\cos(x \sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta); \tag{25}$$

$$\sin(x \sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin((2n-1)\theta). \tag{26}$$

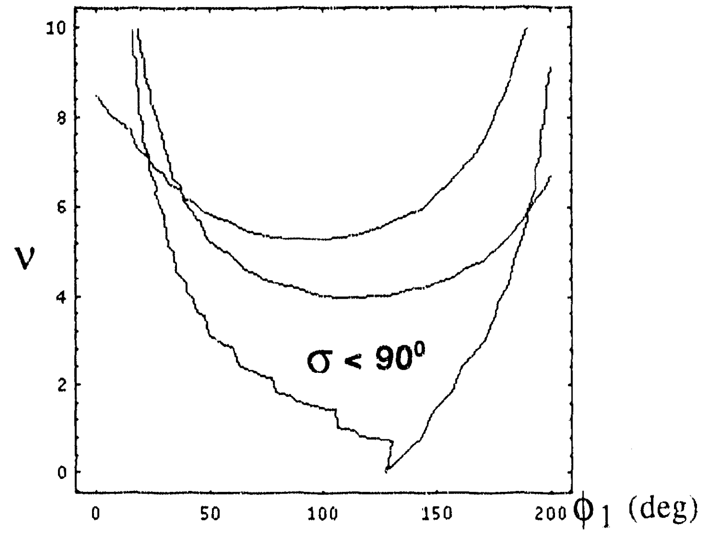

Figure 3. Stability boundaries for trajectories not exceeding 90° in either transverse or longitudinal phase advance with $\phi_0 = 5°$ and $\eta = 0.25$.

We find the effective potential to be given by

$$U_{\rm eff} = U_0 + \sum_{n=1}^{\infty} U_n \tag{27}$$

with

$$U_0 = \pi\eta\nu^2 J_0(\phi_1)\left[\sin(\phi_0 + \Delta\phi) - \Delta\phi\cos\phi_0 - \sin\phi_0\right], \tag{28}$$

$$U_n = \left(\frac{\eta\nu^2}{2}\right)^2 J_n^2(\phi_1)\frac{S_n}{n^2}, \tag{29}$$

where

$$S_n = \begin{cases} \sin^2(\phi_0 + \Delta\phi) + \sin^2\phi_0 - 2\sin(\phi_0 + \Delta\phi)\sin\phi_0\cos\left(\frac{n}{\nu}\Delta\phi\right) & \text{if } n \text{ odd} \\ \cos^2(\phi_0 + \Delta\phi) + \cos^2\phi_0 - 2\cos(\phi_0 + \Delta\phi)\cos\phi_0\cos\left(\frac{n}{\nu}\Delta\phi\right) & \text{if } n \text{ even} \end{cases}. \tag{30}$$

Checking the validity of the assumption that $\Omega \gg 1/T$, we find $\Omega = 2\pi$, $2\pi/T = \sqrt{|\pi\eta\nu^2 J_0(\phi_1)\sin\phi_0|}$, and the assumption is satisfied if $\eta\nu^2|J_0(\phi_1)\sin\phi_0| \ll 16\pi^3$. For any practical application, the requirement is $\nu \ll 150$.

Given the effective potential, we can calculate the equation for the separatrix in the $(\Delta\phi, \Delta W/W)$ space and the total longitudinal acceptance. The separatrix is given by

$$\frac{\Delta W}{W} = \pm\frac{1}{\pi\nu}\sqrt{2\left[\Delta U - U_{\rm eff}(\Delta\phi)\right]}, \tag{31}$$

where

$$\Delta U = U_{\rm eff}(\Delta\phi_c) \tag{32}$$

and $\Delta\phi_c$ is the unstable fixed point of the motion given by

$$\left.\frac{\partial U_{\rm eff}}{\partial\Delta\phi}\right|_{\Delta\phi=\Delta\phi_c} = 0, \quad \left.\frac{\partial^2 U_{\rm eff}}{\partial^2\Delta\phi}\right|_{\Delta\phi=\Delta\phi_c} < 0. \tag{33}$$

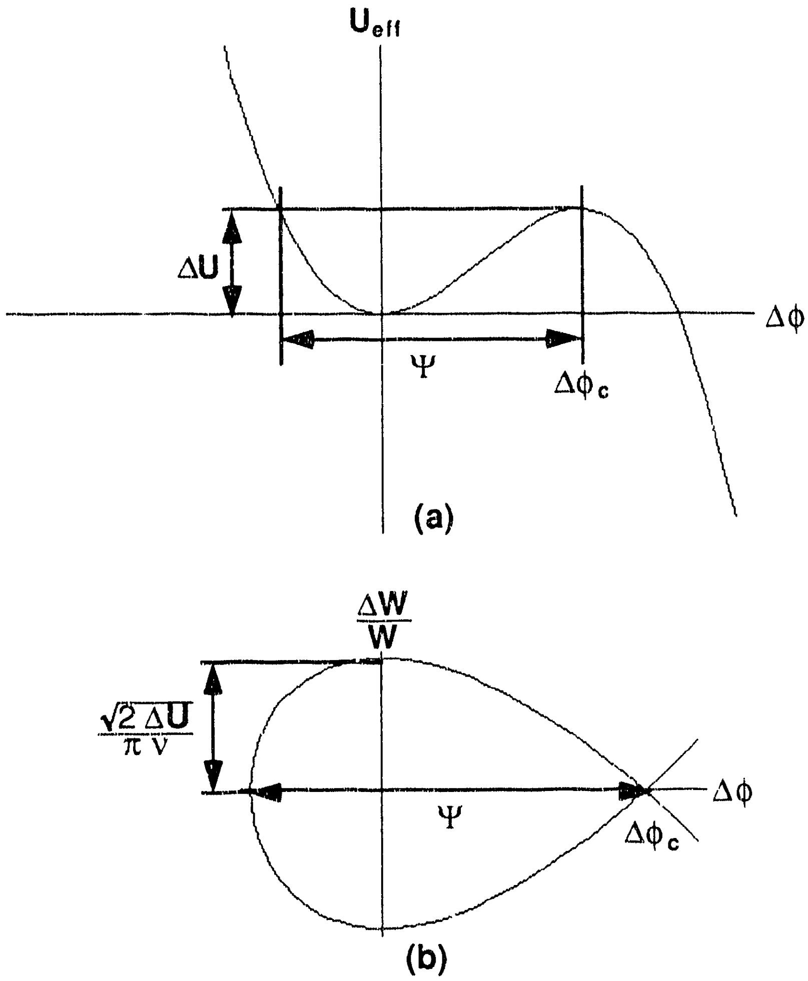

Figure 4. Relationship between (a) the effective potential $U_{\rm eff}(\Delta\phi)$ and (b) the stable region in the $(\Delta\phi, \Delta W/W)$ phase space.

Fig. 4 illustrates the relationship between the potential well $U_{\rm eff}(\Delta\phi)$ and the stability boundaries in the $(\Delta\phi, \Delta W/W)$ plane of motion. The width of the separatrix Ψ is simply the distance between the values of $\Delta\phi$ at which $U_{\rm eff}(\Delta\phi) = \Delta U$ (cf. fig. 4). The height of the separatrix is given by

$$\left(\frac{\Delta W}{W}\right)_{\max} = \frac{\sqrt{2\Delta U}}{\pi\nu}. \tag{34}$$

The area of the stability region in the $(\Delta\phi, \Delta W/W)$ phase space (longitudinal acceptance) is

$$\alpha_L = 2\left(\frac{\Delta W}{W}\right)_{\max} \int_{\Delta\phi_c-\Psi}^{\Delta\phi_c} \sqrt{1 - \frac{U_{\rm eff}}{\Delta U}}\, d(\Delta\phi)\,. \tag{35}$$

Below we give an explicit solution for α_L accurate to the second order in $\Delta\phi$, i. e. we consider terms up to $O(\Delta\phi^3)$ only in the Taylor expansion of $U_{\rm eff}$.

2.4.1. Second-order solution Consider a potential function $U(x)$ described by

$$U(x) = \frac{a}{2}x^2 - \frac{b}{3}x^3 \tag{36}$$

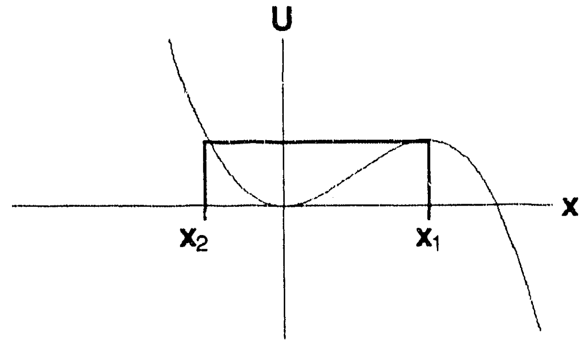

Figure 5. A general potential $U(x)$ described by a cubic polynomial in x.

and the stable motion confined to $x_2 < x < x_1$ as shown in fig. 5. The turning points x_1, x_2, potential well depth $U(x_1)$, and the area of the stable (x, x') region are calculated to be

$$x_1 = \frac{a}{b}, \tag{37}$$

$$x_2 = -\frac{1}{2}\frac{a}{b}, \tag{38}$$

$$U(x_1) = \frac{1}{6}\frac{a^3}{b^2}, \tag{39}$$

$$\alpha = 2\int_{x_2}^{x_1}\sqrt{2\left[U(x_1) - U(x)\right]} = \frac{6}{5}\frac{a^{5/2}}{b^2}. \tag{40}$$

In the problem at hand, the effective potential U_{eff} given in eq. 27 can be Taylor expanded to $O\left(\Delta\varphi^3\right)$ to yield

$$U_{\text{eff}}(\Delta\phi) = \frac{a}{2}\Delta\phi^2 - \frac{b}{3}\Delta\phi^3 + \cdots, \tag{41}$$

where a is the square of the linear motion's phase advance σ_L,

$$a = \sigma_L^2 = 2B + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left(\frac{C_n}{n}\right)^2 \tag{42}$$

and b is given by

$$b = \frac{\pi}{2}\eta\nu^2 J_0(\phi_1)\cos\phi_0 + \frac{3}{8}\eta^2\nu^2\sin 2\phi_0\sum_{n=1}^{\infty}(-1)^n J_n^2(\phi_1)\left(1 - \frac{\nu^2}{n^2}\right). \tag{43}$$

Then, the width of the separatrix Ψ and the $(\Delta\phi, \Delta W/W)$ acceptance α_L are given by

$$\Psi = \frac{3}{2}\frac{a}{b}, \tag{44}$$

$$\alpha_L = \frac{6}{5\pi\nu}\frac{a^{5/2}}{b^2}. \tag{45}$$

Fig. 6 shows the results of acceptance calculations for $\phi_0 = 5°$, $\eta = 0.1$ using eq. 45 and keeping just the $n = 1$ term in eqs. 42, 43. Computer simulations indicate that for most practical cases the second-order acceptance approximation is accurate with an error of less than 10%.

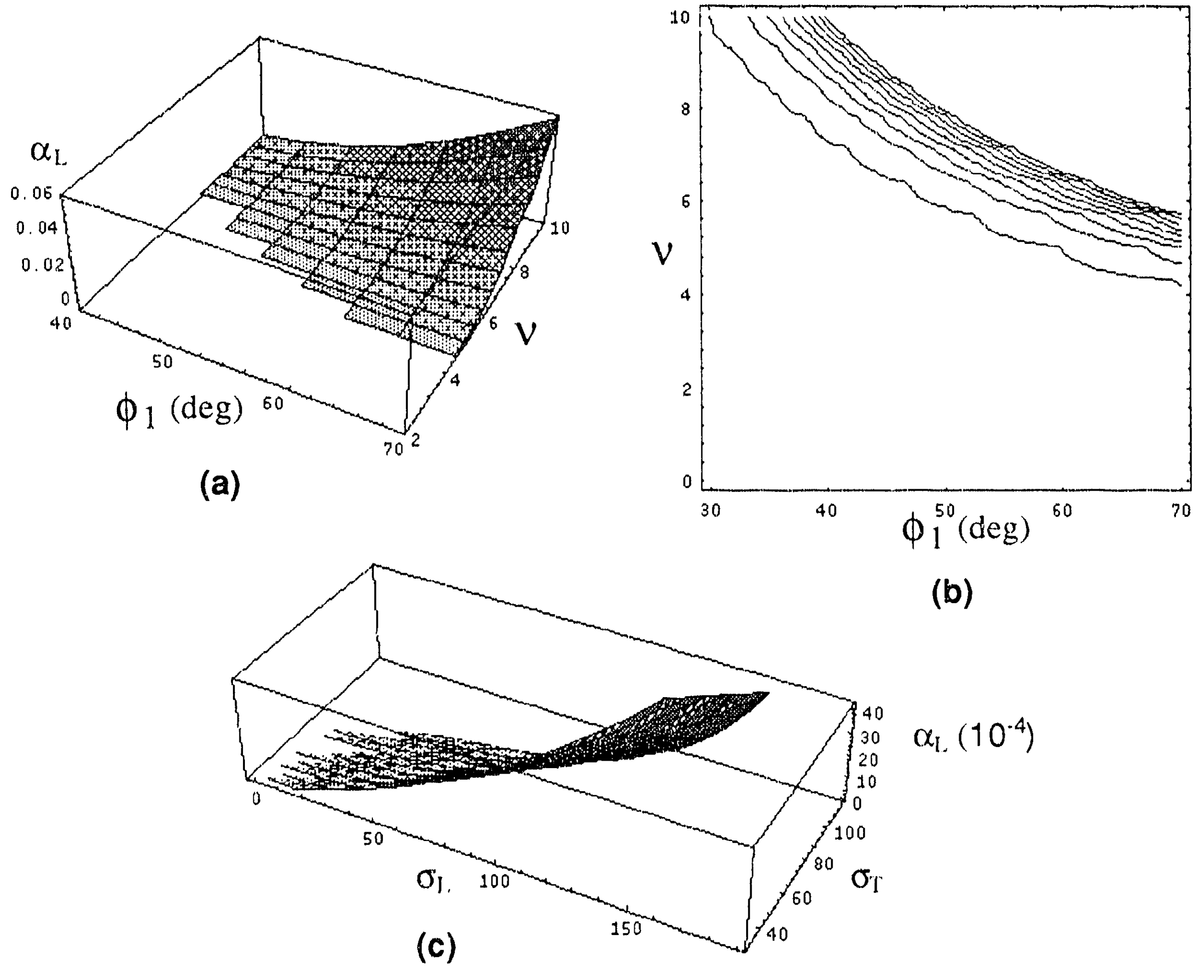

Figure 6. Plots of the longitudinal acceptance α_L for $\phi_0 = 5°$, $\eta = 0.1$. (a) Plot of α_L as a function of ϕ_1 and ν. (b) Contour plot representation of (a). (c) Plot of α_L as a function of σ_L and σ_T, the longitudinal and transverse phase advances respectively.

3. Conclusions and future work

The model of the traveling wave with continuous phase modulation presented in this paper gives quantitative predictions to the problem of longitudinal stability in APF linacs. The model describes the physics of APF with four parameters and yields analytic solutions for the effective potential and the acceptance for the longitudinal motion to any order in $\Delta\phi$.

Future work on the model will include investigations of practical limits in linacs with independent superconducting cavities, space-charge current limits, and ways to improve the acceptance by modulating both the phase and the amplitude of the accelerating field.

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