I. Introduction

The purpose of this talk is to give a short overview of some of the work that has been done in the area of general fluid dynamical models of heavy-ion collisions, focusing mainly on the very general calculations of three-dimensional collisions performed by the Los Alamos group \(^1\)\(^-\)\(^4\) and by our group at Oak Ridge.

A simple characterization of the differences between the two calculations, aside from the technical one of different numerical methods, might be given by stating that the Los Alamos group solves the relativistic Euler equations without full treatment of binding effects, whereas we solve the non-relativistic Navier-Stokes equations with correct treatment of binding. We intend to extend our code to relativistic situations, however.

II. Fluid Dynamics as a Branch of Continuum Mechanics

Before trying to describe nuclei in a fluid-dynamical model, let us take a glance at related branches of classical physics that might be useful for model descriptions of nuclei.

The most general theory encompassing the field is continuum mechanics, which considers all material systems that may be described by macroscopically smooth fields like density \(\rho\), velocity \(\vec{v}\), and energy density. The dynamics of these fields is governed by conservation laws, e.g. conservation of mass

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

and momentum

\[
\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = \nabla \cdot \vec{S},
\]
where \( \mathbf{S} \) is the stress tensor, and a set of constitutive equations depending on the properties of the materials, which serve to complete the macroscopic description. The most familiar example of these is the equation of state; for the two equations given above, we would have to give \( \mathbf{S} \) as a function of \( \rho \) and \( \mathbf{v} \) to complete the description.

Let us just mention in passing that there may be several interacting components in a continuum system, such as different chemical compounds or phases. This may be described easily by the use of several fields \( \rho, \mathbf{v}, \) etc. An example of a two-component system will be encountered later in the two-fluid model for nuclear collisions.

Continuum mechanics may be subdivided into three branches, depending on the properties of the stress tensor \( \mathbf{S} \):

- **Solid Mechanics**: \( \mathbf{S} \) depends on the deformation only. Example: elastic solids obeying Hooke's law.
- **Fluid Mechanics**: \( \mathbf{S} \) depends only on the rate of deformation. Example: Newtonian fluid.
- **Rheology**: This name covers everything not included in the two special branches mentioned. \( \mathbf{S} \) may depend on the deformation and its time change in any way and may also be modified by the history of the system.

In the following, we shall only be concerned with the special case of a fluid dynamical description. One should, however, always bear in mind that more general models are available in this field and might be considered for describing nuclei.

### III. Properties of the Nuclear Fluid

Let us assume for the moment that nuclear matter can be described as a classical fluid (the conditions for this will be discussed later). It is then of utmost interest to examine the properties of this fluid in a collision situation in order to decide how general a dynamical treatment is needed. Some of these decisions may be stated as simple alternatives:

1. **Compressible or Incompressible fluid?** — The criterion for this is whether flow velocities are comparable to the speed of sound. For nuclear matter with an incompressibility \( K \) (in MeV), the speed of sound is

\[
C_s = \left( \frac{K}{9m} \right)^{1/2}
\]

and the projectile energy per nucleon above the Coulomb barrier in the lab system needed to reach such a relative velocity in a
heavy-ion collision is

$$E/A = K/18,$$

so that for the typical range of estimates for $K$ between 150 and 300 MeV we obtain $C_\text{s}$ between 0.13$c$ and 0.19$c$ ($c$ is the speed of light) and $E/A$ between 8 MeV and 17 MeV. Apparently compressibility will be important at least in the "intermediate" and "high" energy ranges. It should be mentioned, however, that an incompressible fluid drop must have a sharp surface since the interior density is by definition constant; thus a realistic treatment of the nuclear surface requires a compressible fluid model at any speed.

2. Viscous or Inviscid? — The experimental data on fission and strongly-damped collisions seem to indicate that viscous effects play an important rôle in nuclear collisions. The introduction of damping into a fluid-dynamical model, however, is not without problems; a microscopic damping caused by one-body dissipation cannot be incorporated easily into the model; on the other hand, even an inviscid fluid shows strong energy dissipation in shock fronts. We shall disregard one-body dissipation and assume the presence of an explicit viscous force in the model equations. We then have to decide whether the fluid is

Newtonian or Non-Newtonian?, i.e., whether the viscous stress tensor is of the special form

$$S_{ij} = -p \delta_{ij} + \eta (\partial_j v_i + \partial_i v_j - \frac{2}{3} \nabla \cdot \mathbf{v} \delta_{ij} + \xi \nabla \cdot \mathbf{v} \delta_{ij})$$

where $p$ is the scalar pressure and $\eta$ and $\xi$ are the coefficients of viscosity. Since there is no experimental or theoretical information on this point, and since this form is the simplest possible one for a fluid, we shall adopt it for the time being. Whereas $\xi$ is completely unknown, there are estimates on $\eta$ based on fission calculations. The latest estimate\textsuperscript{5} is $\eta \approx 0.03 \, \text{fm}^{-3}$. This yields a Reynolds number of

$$\text{Re} = \frac{m \rho \nu d}{\eta} \approx 270 \frac{\nu}{c}$$

where $\rho \approx 0.17 \, \text{fm}^{-3}$, $d = 10 \, \text{fm}$ was used to characterize a nuclear collision. The classical limit for the transition to turbulent flow is for Reynolds numbers of the order of a thousand, so that we are well below this limit. Since a strong dependence of $\eta$ on temperature cannot be excluded, however, the possibility of turbulence cannot now be dismissed definitely.

3. Thermoconducting or not? — On the problem of the
thermoconductivity of nuclear matter, there is even less definitive knowledge than on the viscous properties. The calculation of the coefficient of thermoconductivity $\kappa$ based on a Fermi gas model yields, in the low temperature limit ($kT \ll \epsilon_F$),

$$\kappa = \frac{7}{48 \pi \sqrt{2}} \frac{\epsilon_F^{3/2}}{m_o^{1/2} T} \frac{1}{Q_1}, \quad Q_1 \approx 2.17 \text{ fm}^2$$

$$\eta = \frac{4}{15 \sqrt{2} \pi^3} \frac{\sqrt{m_o / \rho} \epsilon_F^{5/2}}{kT^2} \frac{1}{Q_2}, \quad Q_2 \approx 1.06 \text{ fm}^2.$$  

It should be noted that both quantities become infinite near $T = 0$, which is improbable for nuclei because of their superconducting behaviour near the ground state. (For a more detailed discussion of these problems, see ref. 7.) To get an idea of the importance of thermal conduction, one may compute the Prandtl number

$$Pr = \frac{C_p \eta}{\kappa} = \frac{35}{32} \frac{Q_2}{Q_1} = 2$$

where the heat capacity at constant pressure $C_p = \frac{\pi^2}{2} kT^2 / m_o \epsilon_F$ for a Fermi gas was used. The product of the Reynolds and Prandtl numbers measures the ratio of convective heat transfer to thermal conduction. In our case we have

$$Re \cdot Pr \approx 500 \frac{v}{c},$$

so that at moderate velocities convective heat transfer should dominate, at least as far as the pure Fermi gas model is accurate. Since the situation is by no means conclusive, and to allow for heat conduction in model calculations is not expensive, it seems best to provide for that possibility.

IV. Validity of the Fluid Model

There has been some discussion in recent years as to the validity of a fluid dynamical description for the collision of two pieces of nuclear matter. All of the arguments found, whether presented in favor of or against the fluid model, seem to indicate that the physical quantities concerned are just at the borderline of validity of fluid dynamics. Because of this and of the possibility of additional physical effects like pion condensation, which may change the arguments entirely, it may be best to try to use a comparison of experimental data with the model results to obtain information on the validity of the assumptions made. Let us therefore just glance at the basic criteria
and elementary estimates of their validity:

1) **Validity of continuum description**

"There should be a large number of microscopic degrees of freedom in each volume of macroscopic dimensions" — In a heavy-ion reaction we may have several hundred nucleons, but only on the average one in a cube of size \((2 \text{ fm})^3\). It thus seems doubtful whether a fluid dynamical description could give details of that scale.

2) **Local equilibrium**

"The mean free path of the microscopic particles should be small compared to the macroscopic scale" — Estimates using the free nucleon-nucleon cross section\(^3\) indicate a mean free path of the order of 3 fm, which is only five to ten times smaller than the size of the compound system. This value is for ~ 250 MeV per nucleon; because of Pauli exclusion effects, it increases for lower energies and because of the diminishing nucleon-nucleon cross section also at higher energies.

Barring unknown effects like pion condensation, etc., the fluid dynamical model thus should describe at most the larger-scale effects in a collision. However, one should not forget that the basic quantum mechanical description bears a close resemblance to fluid mechanics\(^\text{11)}\), and suitable constitutive equations could conceivably make fluid dynamics a viable description in spite of its possible non-validity in the classical sense presented above.

One final aspect of this is whether, if the collision situation does not lead to almost instantaneous equilibration of the two colliding fluids to form one thermally excited mixed fluid, where all the directed motion of the initial system has been transformed into random thermal motion, one might not be better off describing the two ions as made up of two distinguishable kinds of fluid that are allowed to interpenetrate interacting via a drag force that will eventually force formation of one single fluid at equilibrium. Some work done by the Los Alamos group in this "two-fluid" model\(^\text{10)}\) will be discussed later. One should note, however, that the one-fluid model does not require instantaneous equilibration with infinitely narrow shock fronts; it is known empirically that the macroscopic Navier-Stokes equations work surprisingly well in the description of the internal structure of shock fronts spread over several mean free path lengths.

V. **Definition of the Model**

In the non-relativistic case the equations of motion used for the fluid-dynamical model are the Navier-Stokes equations, which describe the conservation of particle number,
\[ \frac{\partial \mathbf{p}}{\partial t} + \nabla \cdot (\mathbf{p} \mathbf{v}) = 0, \]
momentum,
\[ \frac{\partial}{\partial t} (mpv^+) + \nabla \cdot (mpvv) = -\nabla \cdot S^+ - \rho vV \]
and energy,
\[ \frac{\partial}{\partial t} (\rho E) + \nabla \cdot (\rho E\mathbf{v}) = \nabla \cdot (\kappa VT) - \nabla \cdot (S^+ \mathbf{v}) - \omega^+ vV. \]

To complete this set of equations, a number of constitutive equations are needed, viz. the stress tensor \( S \), the temperature \( T \), and the potential \( V \) have to be given in terms of \( \rho(\mathbf{r},t) \), \( \mathbf{v}(\mathbf{r},t) \), and \( E(\mathbf{r},t) \).

To this purpose, we investigate the internal energy per particle of nuclear matter, \( \omega(\rho,\sigma) \), where \( \sigma(\mathbf{r},t) \) is the entropy density. \( \omega \) is related to the total energy per particle \( E \) via
\[ E = \frac{1}{2} m v^2 + \omega(\rho,\sigma). \]
Thus \( E \) determines the magnitude of \( \omega \), and standard thermodynamic relations then give the pressure
\[ p = \rho^2 \left( \frac{\partial \omega}{\partial \rho} \right)_{\sigma}, \]
and the temperature
\[ T = \left( \frac{\partial \omega}{\partial \sigma} \right)_{\rho}. \]
The stress tensor is then completed with the Newtonian viscosity as given in Chapter III.

It still remains to fix the functional form of \( \omega(\rho,\sigma) \). Since little is known about its behaviour for \( \rho \) distinct from the equilibrium density of infinite unclear matter, and for any non-zero \( \sigma \), one has to proceed with a simple assumption about \( \omega \). The choice made by both the Los Alamos and Oak Ridge groups is to split up \( \omega \) into a ground state (zero-entropy) part \( \omega_0 \) and a thermal part,
\[ \omega(\rho,\sigma) = \omega_0(\rho) + \omega_1(\rho,\sigma). \]
\( \omega_0 \) is expressed as a polynomial in \( \rho^{1/3} \):
\[ \omega_0(\rho) = \frac{n}{a_1^i \rho^{i/3}}. \]

In our case, \( n \) was taken to be 5, and this allows the determination of the \( a_i^i \)'s in terms of the equilibrium density, the equilibrium binding energy per particle, and the incompressibility at equilibrium.\(^{12}\)

For \( \omega_1(\rho, \sigma) \), the expression for a Fermi gas at low temperatures is used:

\[ \omega_1(\rho, \sigma) = \left( \frac{6}{4\pi} \right)^{2/3} \frac{\hbar \sigma m}{2 c^2} \frac{a^2}{2} \rho^{2/3}. \]

The functional form of \( \omega_1 \) already determines the maximum non-relativistic shock compression ratio as 4. Relativistically, however, there is no limit on compression ratios obtainable in a shock front\(^{13,14}\).

Figure 1 shows the binding energy per nucleon, the pressure, and the speed of sound as functions of the nucleon density for an equation of state of the type considered, and at zero entropy. Note that, since below equilibrium density nuclear matter tends to contract rather than to expand, the pressure is negative in this range. Also, sound propagation is possible only at densities where \( \frac{dP}{d\rho} \bigg|_\sigma > 0 \), i.e. where matter is stable with respect to density perturbations. Thus, there is no speed of sound at low densities.

Finally, the potential \( V \) in the equations of motion is a sum of a Coulomb potential \( V_c \) determined via

\[ NV_c(\mathbf{r}) = -4\pi \left( \frac{2e}{A} \right)^2 \rho(\mathbf{r}) \]

from the nucleon density (equal spatial distributions of protons and neutrons assumed), and a Yukawa part \( V_y \), given by

\[ NV_y(\mathbf{r}) = a^2 V_y(\mathbf{r}) = -4\pi \beta \rho(\mathbf{r}). \]

The Yukawa potential has the important advantage of allowing ground states with a non-sharp surface. In the absence of such a potential, equilibrium is guaranteed by the zero pressure at equilibrium matter density, which agrees with the surrounding vacuum pressure on a sharp surface. The Yukawa potential smoothen the surface density. A realistic surface thickness was obtained with the parameters \( a = 2.1 \text{ fm}^{-1} \) and \( \beta = -280 \text{ MeV fm} \).

In relativistic calculations, such as done by the Los Alamos group,
the inclusion of potentials causes severe problems, most notably the retardation effect, which would complicate the calculation immensely. Therefore, potentials are not used in those calculations. Since their main advantage is to create a smooth surface, their omission is not expected to be critical at bombarding energies in the 100 MeV to 1 GeV per nucleon range.

VI. Remarks on Numerical Solutions

The use of a purely numerical finite-difference solution to the fluid flow equations may at first seem somewhat of an overkill with its large requirements of computer and programmer time. However, there are important advantages to be gained:

1) Numerical accuracy is the only restriction on the type of solution allowed. There are no additional assumptions about the dynamical behaviour of the system aside from the basic equations of motion and constitutive equation. In particular, there is no restriction on the surface shape, local compression zones, geometry of shock waves, etc.

2) The model can be changed easily; e.g., it is extremely easy to vary the equation of state.

The main drawbacks are that the procedure is very expensive and that the advantages of more analytic solutions are lost; e.g., the dependence of the collision result on parameters such as viscosity or even the impact parameter is not transparent. However, one of the most effective uses of the fully numerical solution might be to find more restricted descriptions like a selection of the most important surface modes. Only a full study of the numerical solution in a wide range of collision situations will provide enough data to extract such information.

VII. Comparison with TDHF

Since a comparison of fluid dynamics and TDHF has been given in an earlier paper in these proceedings\textsuperscript{15}), I shall only add some later results here, which may help to clarify the situation.

As fluid dynamics is based on the assumption of instantaneous local equilibrium, it is interesting to see whether TDHF approaches local equilibrium during a collision. For this purpose, we have studied the Wigner function for a one-dimensional TDHF collision. In terms of the one-particle density matrix

\[ \rho(\vec{r},\vec{r}') = \sum_{i=1}^{A} \psi_i(\vec{r}) \psi_i^*(\vec{r}') \]
one may define the associated Wigner function as

\[ \omega(\mathbf{r}, \mathbf{k}) = C \int d^3 r' e^{-i\mathbf{kr}'} \rho(\mathbf{r} + \frac{1}{2} \mathbf{r}', \mathbf{r} - \frac{1}{2} \mathbf{r}') \]

with a constant C (arbitrary for our purpose). Its properties may be summed up as follows:

1) \( \omega(\mathbf{r}, \mathbf{k}) \) is a real function of \( \mathbf{r} \) and \( \mathbf{k} \). It is the closest quantum-mechanical counterpart to the classical phase-space distribution function. Because of the uncertainty principle, it is not quite a probability density itself (\( \omega \) may become negative!) but its integral over a volume in phase space big enough not to violate the uncertainty restriction may be interpreted as a probability. \( \int f \omega \, d^3 r \, d^3 k \) is the expectation value of any function \( f \) of \( \mathbf{r} \) or \( \mathbf{k} \) alone, whereas e.g. \( \int \mathbf{r} \cdot \mathbf{k} \omega \, d^3 r \, d^3 k \) yields the average of the quantum-mechanical expectation values of \( \mathbf{r} \cdot \mathbf{k} \) and \( \mathbf{k} \cdot \mathbf{r} \).

Figure 2 shows the results for a collision of two slabs of thickness 2 fm\(^{-2}\). The lower part of each small graph shows the density at a given time, and the upper part contains the Wigner function as a function of \( \mathbf{k} \) at the center of the collision. Only at the beginning of the collision we show instead the Wigner function at the center of each slab, giving an idea of the equilibrium shape and the relative displacement due to the motion of the slabs. To guide the eye, this curve is then repeated as a dashed line in each graph, but centered at \( \mathbf{k} = 0 \), since the center of a symmetric collision should have an equilibrium distribution around \( \mathbf{k} = 0 \). Clearly, the distribution comes closer to something resembling equilibrium in the course of the collision (because of higher density and excitation, it should be wider than the dashed line, though), albeit there is always a division into two humps. However, in the final stage the sides of the distribution have been depleted, because the fast wave functions literally have moved out to the sides to form the fragments. Although these data do not cover all aspects of the equilibration process yet, it seems clear that TDHF does not exhibit strong enough equilibration to resemble the fluid behaviour. The most striking case is the head-on collision of two heavy ions, where TDHF lets the two nuclei pass through each other at higher energy, whereas fluid dynamics will always generate a hot shock zone at the center of the collision that is a very effective dissipation mechanism.

VIII. Relativistic Fluid Dynamics

Let us now take a look at the important results of the Los Alamos group for the relativistic case\(^3\). The relativistic equations of motion are practically identical with those given in Chapter V without viscosity and thermoconduction; it is mainly in the constitutive equations that relativity plays a role. The equation of state is defined in the local rest frame and takes the form
\[ E = m_0 + a \left( \frac{\rho}{\rho_0} \right)^{2/3} - b \left( \frac{\rho}{\rho_0} \right) + c \left( \frac{\rho}{\rho_0} \right)^{5/3} + W_1(\rho, \sigma) \]

with \( W_1 \) a Fermi-gas thermal energy similar to the one given in Chapter V. The pressure can be defined straightforwardly, and the transformation to the lab frame (primed) is as follows:

\[
\rho' = \gamma \rho
\]
\[
\vec{h}' = \gamma^2 (\rho E + p) \vec{v}'
\]
\[
\rho' E' = \gamma^2 (\rho E + p) - p.
\]

Here \( \gamma = (1 - v^2/c^2)^{-1/2} \), as usual. The need to go forth and back between lab and rest frame introduces some numerical difficulties but no essential problems.

The numerical method employed to solve the equations is the Particle in Cell (PIC) method. In this method the fluid density is represented by point particles of given mass (no relation to the nucleon mass) moving in a finite-difference mesh, represented by cells in space, under the influence of the fluid pressure. The fluid aspect enters the method through the averaging over one cell, e.g. the density in a cell is simply obtained by counting the number of particles in the cell, and this density then determines the pressure. The point particles are thus purely fictitious numerical devices, and one should bear this in mind when interpreting the results that may give the impression of a classical interacting particle model as reported elsewhere in this conference.

Figure 3 shows the time dependence of the density as represented by the distributions of the numerical particles, as a function of time for different impact parameters in the \(^{20}\text{Ne} + \text{^{238}U}\) system at 250 MeV per nucleon. The plots are done in the lab system, so that the incident \(^{20}\text{Ne}\)-nucleus appears relativistically contracted. Also, because of the absence of potentials, both nuclei have sharp surfaces and a constant density inside.

For the near-central collision (labeled 0.1) the \(^{20}\text{Ne}\) nucleus penetrates into the target nucleus and sets off a strong shock wave (clearly visible at \(5.1 \times 10^{-23}\) sec). Subsequently, most of the energy of the projectile is thermalized and the nucleus expands in response to the high thermal pressure. The calculations do not allow for the formation of a residual nucleus since there is no attractive potential and also negative pressures are set equal to zero. Tests allowing for negative pressure yielded small clusters in the final state, but these did not appear to be physically meaningful.
For the intermediate impact parameter (0.5) the behaviour is similar. The main difference is that the projectile is not completely absorbed in the target, but part of it flies off, giving a far from isotropic behaviour in the final state. Both projectile and target have been excited so strongly that they expand rapidly.

The peripheral collision (0.9) shows a different behaviour. A small part of the projectile is sheared off by the contact with the target nucleus and sets waves running through it, which, after thermalization, will again cause the target to expand. The projectile itself is also excited. (Its darker shading at \( t = 13.5 \times 10^{-23} \text{ sec} \) does not indicate a higher density; initially the numerical particles were aligned and several of them appeared as one dot only when viewed from above.)

To make a comparison with experiment, one may look at the final state, which always contains an expanding cloud of nucleons, and compute the expected angular distribution, interpreting the fluid density as a probability density for nucleons. Averaging over impact parameters, one then obtains a differential cross section for outgoing nucleons. Under the assumption of equal behaviour of protons and neutrons, a comparison can finally be made to proton-inclusive cross sections (fig. 4). Apparently the general trend of the data is well reproduced, but there is a definite quantitative discrepancy both at very forward angles, where the theoretical cross section is too small, and at backward ones, where it is too large.

This shift of the theoretical yield to backward angles compared to experiment seems to indicate that there is "too much interaction", i.e. too much forward momentum is converted into transverse momentum. Since the assumption of local equilibrium implies a fast conversion of directed into random momentum, this assumption may have to be modified, at least for this energy range.

Thus, the Los Alamos group recently did some two-fluid calculations\(^4\). The basic concept of the two-fluid model is to describe the two nuclei as distinct fluids which are allowed to interpenetrate without losing their separate identities, but do interact via a drag force, which is proportional to the relative velocity with a strength factor depending on its magnitude and on the density. The strength is adjusted such as to reproduce the stopping length expected from free nucleon-nucleon cross sections.

One problem with this approach is that one would like to have a transition to the regular one-fluid description, whenever the relative velocity of the two fluids becomes small. In this case, for example, there should be only one pressure depending on the total density instead of the two partial pressures. In the calculations this was achieved by interpolation in the relative velocity, but if comparison to experiment makes more refined two-fluid calculations desirable, this aspect deserves some study for a more cogent solution.
The first results of the model are encouraging though they do not appear to completely solve the problems. The cross section in the backward direction is lowered to much better agreement with experiment, but in the forward direction there is no definite improvement. It should be kept in mind, however, that the parameters of the model have not been fitted to these data, but have been taken over from different areas of nuclear physics, so this result may be regarded essentially as a parameter-free calculation.

IX. Low-Energy Results

The calculations by our own group in Oak Ridge have just gotten off the ground, so that we can show only sample results at present.

We utilize the full set of equations given in Chapter V, including viscosity and potentials, but for the non-relativistic case. The numerical method employed is the "flux-corrected transport" method developed by Boris and Book at NRL\(^1\). Although this method eliminates the most severe problem facing the numerical fluid dynamicist, instability, even in the presence of very steep shock waves, there were still many technical problems, notably the transition to zero density in the surrounding vacuum and the short range of the Yukawa force of \(< 0.\) fm, which forced us to go to a spatial mesh finer than that limit. In fact, many test calculations were done with a coarser mesh of 0.8 fm separation, and these turned out to be quite adequate in qualitative results, but showed unphysical effects such as squarish deformations in detail. Since these calculations are still quite cheap, they may well be useful for exploring the general behaviour of the system as a function of the various parameters.

For now, however, let us take a glance at the first high-accuracy results obtained with a newly developed code\(^2\). It works on a cartesian mesh of 64 x 64 x 33 points with a spacing of 0.4 fm, and the physical situation corresponds to a collision of \(^{16}O + ^{16}O\) at a relative velocity of 0.2c (corresponding to 18.8 MeV per nucleon in the lab), with an impact parameter of 2 fm. The type of nucleus does not really enter drastically, only the relative surface thickness and the Coulomb effects will change smoothly with mass and/or charge number. The equation of state employed corresponded to an incompressibility of 134 MeV, so that the initial Mach number for this collision is 1.54 for the regions of equilibrium density. It is thus expected that the flow will show the typical phenomena of supersonic flow — strong compressive effects, shock waves, and the lack of upstream propagation of information.

Figure 5 shows the system close to the initial state. The lower two subgraphs show the density and velocity field viewed facing the collision plane (the density graph actually shows the integral of the density in that direction to give an impression of total thickness, whereas the velocity vectors are plotted in the scattering plane. The two additional graphs show the integrated density from above (upper
left) and from the right (upper right) relative to the scattering plane plot already discussed.

Figure 6 shows the situation at t = 21.6 fm/c. At the center a shock front has formed with very small velocities, and matter has already started to flow out to the sides from there. The rest of the initial nuclei, however, is quite undisturbed, as is characteristic of supersonic flow (there are small internal motions caused by the imperfection of the initial equilibrium).

Figure 7 at 37.1 fm/c has a small part of the initial nuclei still streaming into the collision zone, but there is also now a strong current to the sides. Note that matter from both initial nuclei mixes in each outgoing direction. At the same time, however, as is visible in the top two graphs, the systems expand perpendicular to the scattering plane and thus seem to flatten into a pancake shape.

This development is quite striking in the final fig. 8 at 76.7 fm/c. The system is now very elongated in the scattering plane with the main direction of motion outward. If one had only this viewpoint, one might expect it to fission soon, but the fully 3-dimensional representation shows that it is rather a very flat disk.

At this point, we had to stop the calculation. Whether it will rebound into a prolate shape (as observed in ref. 7), or has lost so much energy that it may rather break up in the oblate form, cannot be decided yet. The upper left graph in fig. 8 seems to show a slight necking-in, but it cannot yet be said whether this will lead to fission.

This, however, leads to an important point on the interpretation of the final state. It is well known from fission calculations\textsuperscript{7,20} that the fluid-dynamical model likes to form very long necks. Although it is expected that in the present compressible calculation low-density areas will break up spontaneously because of the negative pressures, the final state may still be quite diffuse and require a long time before final breakup. If this should happen in the calculations, we shall have to try to add some extraneous arguments to be able to extract the final masses, scattering angles, etc. Also, one would have to distrust calculational results that show, e.g., a flat disk with thickness a fraction of a nucleon diameter. Thus, all results should be scrutinized using physical intuition before being accepted as predictions of the model.

The results of a few additional collisions at different impact parameters and relative velocities are shown in table 1. $E^*$ is the thermal energy reached in the final state (in those cases where the system had not separated again, this number is probably still meaningful, because most of the dissipation of energy takes place in the shock waves, and only a few MeV are dissipated later). The ratio of densities gives the maximum compression reached; these numbers are, of course, below the theoretical maximal ratios given in ref. 14.
X. Conclusion

The results presented in this paper have demonstrated that three-dimensional fluid-dynamical calculations are feasible and can be done with sufficient accuracy. The Los Alamos group has already done a comparison to experimental data for proton-inclusive spectra, whereas our calculations are not yet at that stage. Additional experimental data that may provide useful testing grounds are the fusion cross sections and, as far as they are reducible to classical deflection functions, differential cross sections for deep-inelastic collisions at lower energies. One problem one may encounter is that all of these data may not give sufficient information to distinguish between models; e.g., the fusion cross section really tells only the critical impact parameter and can certainly be fitted with the freedom available in fluid dynamics.

In our own calculations, one of the first things we would like to try in the future is to use a more complicated equation of state with a density isomer, and to see how that affects the collision. Among other possible extensions one might think about treating protons and neutrons, or the different spin states, separately, thus again going over to a two- or multi-fluid theory. Also, it is relatively easy to describe the production of additional particles, such as pions, or the excitation of nucleons, again by a many-fluid approach.

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Table 1

Summary of Sample Collision Studies

<table>
<thead>
<tr>
<th>( v_{\text{init}}/c )</th>
<th>( b )</th>
<th>( E_{\text{kin}}^{(\text{init})} )</th>
<th>( E^* )</th>
<th>( (\rho/\rho_0)_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0</td>
<td>37 MeV</td>
<td>28 MeV</td>
<td>1.3</td>
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<td></td>
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<td>13 MeV</td>
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Fig. 1. The Equation of State
Fig. 2. The Wigner Function in TDHF
Results from Ref. 3
$v = 0.1c, \ t = 6.9\text{fm}/c, \ b = 2.0$

Fig. 5
$v = 0.1c, \quad t = 21.6\text{fm/c}, \quad b = 2.0$

Fig. 6
\( v = 0.1c, \ t = 37.1 \text{fm/c}, \ b = 2.0 \)

**Fig. 7**
$v = 0.1c, \; t = 76.7 \text{fm/c}, \; b = 2.0$