PROBABILISTIC MODELS OF THE STRESS-RUPTURE
OF COMPOSITE MATERIALS

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ABSTRACT OF PROGRESS TO DATE

The development of time dependent models for the random strength and stress-rupture of unidirectional composites has been pursued. Emphasis has been placed on modeling accurately the lower tail behavior of the probability distributions of failure. Critical assumptions made in earlier studies to simplify the analyses were found to be invalid or to lead to questionable conclusions on lower tail behavior and specimen size effects. An assumption of previous models has been retained namely that of considering the composite as a chain of sections or bundles each having length δ , the so-called ineffective length. Also matrix influenced load concentration factors were generated for unbroken fibers neighboring one or more adjacent failed fibers. The point of departure from previous studies has been to obtain exactly the probability distribution of section strength concentrating on numerical accuracy in the lower tail. The case of five fibers was solved exactly and combinatorial and straightforward computational techniques are being developed to handle cases of much higher order on the computer. Already plotted results have been surprising and indicate that the matrix influence on lower tail behavior is severe enough to alter substantially failure predictions for long composite specimens. Work now is underway to include time dependent effects in the mechanical evaluation of the fiber load concentration factors, for use in the model.

1. Brief Review of Research Problem and Relevant Literature

In the original proposal, several probabilistic models of the tensile strength and stress-rupture of unidirectional composites were reviewed. For the most part, these models involved adaptations and modifications of the classic fiber bundle theory of Daniels [1]. For the (static) tensile strength of composites, models by Rosen [2], and Güçer and Gurland [3], Scop and Argon [4,5], Zweben [6] and Zweben and Rosen [7] were discussed. For the (time dependent) stress-rupture of composites, models by Coleman [8-14], Robinson [15], Lifshitz and Rotem [16] and a review by Lifshitz [17] were considered. The relationship between the static and time dependent models was discussed. It was pointed out that the state of development of the time dependent models was at the level of the early work of Rosen, (modified somewhat to include time dependent effects). Consequently local fiber interactions were not taken into account as in the later static models referenced above. The original proposal also contained a model proposed by the author. The point of departure was that the model did not require that the composite be partitioned conceptually into a series of layers (strips or links) of length δ , the "ineffective fiber length."

The main problem proposed for investigation involved the modification of the author's static model to allow time-dependent behavior, by assuming that the matrix is viscoelastic in nature. The lower tail behavior of the resulting distributions was to be investigated in terms of matrix and fiber flaw distribution properties. The potential satisfaction of a stochastic version of Miner's rule was to be investigated. Also to be determined was the extent to which power law or exponential breakdown rule stress-rupture behavior results from specific matrix viscoelastic properties or properties of the flaw

strength spectrum. The underlying belief was that the earlier static models [4-7] and the model proposed by the principal investigator were sufficiently advanced and realistic to warrant extension.

The second, third and fourth problems proposed were considered to be of lesser importance. However, the second problem has turned out to be far more important than originally thought. Indeed aspects of the first and second problem are proving to be inseparable. The important aspects of the second problem were 1) the investigation of the distributional characteristics of fiber flaw spacings within the composite and 2) the determination of the number of adjacent fiber failures, as a function of composite stress, required for unstable crack growth and subsequent composite fracture. Important results on both of these aspects have been obtained.

During the course of the investigation, additional literature has come to light - some of which contains disturbing material. Recent work by Bolotin [18] is not considered to be realistic though many of the assumptions are similar to those made in the earlier work cited above. However works by Argon [19] and Argon and Bailey [20] are significant. These works do not contain theoretical advances over the earlier models of Scop and Argon [4,5]. However they contain discussions of the predictions of the earlier models in light of recent experimental data. Details and disturbing implications of these predictions are discussed in a later section in connection with results obtained to date on the present contract. It suffices to say that in view of these results, some main assumptions and approximations made in earlier models [2-7] are now open to serious question. Of course some of these models only yielded predictions of mean strength for small composite specimens and the predictions were only moderately successful at that. Prediction of the size effect and scatter in composite strength was precluded. However the models of Scop and

Argon [4,5] do predict distributional characteristics of composite strength.

It is now believed that precise modeling of the lower tail of the probability distribution of strength for small composite specimens is essential for accurate forecasting of large composite structure behavior. The results of our studies to date indicate that the assumptions in the earlier models, which were thought to be innocuous, in fact lead to erroneous predictions. In view of this, the earlier cited aspects of the second research problem must be dealt with more thoroughly prior to proceeding in depth with the study of the first problem. We are making rapid progress on this and foresee completion of this aspect in the next few months. Computational techniques under current development are proving to be powerful. Details are considered in a later section.

2. Scope of the Investigation

In this early phase of the program, the scope of the investigation has been strongly influenced by the personnel assigned to the program. In September, no graduate student was available who had knowledge of stress analysis and viscoelasticity and even a moderate knowledge of probability and statistics. Consequently, Gary Harlow an applied mathematics graduate student with an undergraduate background in physics, was appointed as a Graduate Research Assistant. Gary had already completed a substantial portion of his Doctoral dissertation on extensions of the classic fiber bundle strength theory of Daniels [1] and Phoenix and Taylor [21]. Gary immediately became productive, being already familiar with much of the cited literature on the static strength of composites [1-7]. He has uncovered the additional significant studies [19,20] cited earlier.

Because of Gary's extensive background in probability theory, the principal investigator assigned Gary to the task of investigating the distributional characteristics of fiber flaw spacings in the composite. The main interest was in evaluating the validity of a main assumption in all earlier models [1-7] (including the model proposed by the principal investigator). This assumption is that zero-order breaks (due to the initial stress on the composite) occur sufficiently far away from each other so that subsequent interaction is probabilistically negligible. Initially it was thought that a Monte Carlo simulation approach would be required. However an extensive review of the literature on order statistics and combinatorial aspects of probability resulted in the formulation of a method of attack. Shortly thereafter a major result was proved. The essence of the result is that if the size effect in composites

is less than 1/2 as severe as it is for the constituent fibers, then the interaction of zero-order breaks in the composite fracture process must be taken into account in any realistic model. (Details of this result are considered later.)

Also during the early phase of the program, very recent results on the lower tail behavior of the strength distribution of classic fiber bundles cast doubt on the validity of another prime assumption of the more refined static models of composite strength [4-7] cited earlier. (These results were obtained by Professor Howard M. Taylor who has frequently consulted on the project free of charge.) To simplify calculations, it was assumed in these models that the number of adjacent fiber fractures sufficient for unstable crack growth was constant (usually two or three were assumed.)

Thus the number required was independent of the applied composite stress, but in reality the number required increases as the stress decreases.

The detrimental effects of this assumption on the predicted lower tail behavior of the strength distribution for small composite specimens are now believed to be substantial. Consequently the accuracy of the prediction of the strength behavior of larger composite structures is jeopardized.

Unfortunately if the above simplifying assumptions are not made, the computational difficulties expand tremendously. Nevertheless it was believed that extremely powerful combinatorial and computational techniques had yet to be brought to bear on the problem. Consequently Carl Diegert, a graduate student in Operations Research, was consulted on the possibility of computing distributional characteristics of composite strength believed not to be obtainable analytically (at least not after what has now been many man-years of effort). Carl is a graduate student of Howard M. Taylor and has a commanding background in numerical and computer techniques in

probability and applied stochastic processes. Carl worked with Gary Harlow on some models which were analytically formidable but relatively straight forward computationally. Very important results were obtained which substantiated earlier suspicions about the validity of the assumptions mentioned earlier. These results are discussed later in detail. Because of the positive results obtained, Carl Diegert was assigned to the project in January 1977 as a Graduate Research Assistant. Developments in computational techniques are continuing and are discussed later. Carl was productive even before being assigned officially to the project.

Funds were provided in the contract for a Post-Doctorate Associate for a six month period. The original plan was to appoint in January of 1977 a recent Ph. D. graduate with a background in solid mechanics (viscoelasticity, plasticity, elasticity) and some knowledge of probability theory. Even after extensive advertising in Mechanics and Mechanical Engineering Departments at leading Universities throughout the U.S. no suitable (or even marginally suitable) candidate was found. Consequently the viscoelastic aspects of the main stress-rupture (time dependent) problem have not been considered in depth. Some of this effort may well have been inconsequential in light of the recent developments discussed above. The current plan is to appoint in September 1977 a Graduate Research Assistant with a background in solid mechanics. Prospects in this regard are promising. In view of the difficulties in finding a Post-Doctorate Associate, Carl Diegert was assigned to the project as mentioned earlier. Since Gary Harlow will complete his Ph. D. thesis and final examination in early April 1977 he will be promoted to Post-Doctorate Associate for the remainder of the term. His contributions to the project have been outstanding and we hope to be able to appoint him for a suitable term commencing June 15, 1977.

In summary the scope has been to develop models of the composite failure process together with powerful computational techniques which do not rely on questionable assumptions. It is believed that these models will be extended in a straightforward manner to forecast the stress-rupture properties of composites as they are discussed in the original proposal. Progress on this is anticipated before the end of the current contract period.

3. Significant Results to Date

In the initial phase of our research we have assumed the basic discrete model of earlier investigators [2-7] instead of proceeding directly with the more continuous model outlined in the original proposal. This has allowed us to compare directly our findings with those of earlier investigators especially with respect to the questionable assumptions mentioned in the previous section. Second, the effect of these questionable assumptions is now believed to be far more severe than that of assuming the discrete model. The advantages of the continuous model will be manifest when a more refined analysis of local fiber failure and stress redistribution is carried out in a time dependent framework.

3.1 Basic assumptions. The first assumption is that the composite is considered as a series or chain of sections each having length δ , the ineffective length. Recall that δ is defined as the length of the interval near a break such that the stress in the fiber outside the interval is a prescribed high fraction of the fiber stress. See Figure 1. In the early models [2,3] each section was a Daniels bundle [1]. However in later models [4-7] local changes in the stress on fibers adjacent to broken fibers were taken into account. The increased stress in unbroken fibers was known to grow rapidly with the number of adjacent failed fibers.

Define K_r to be the stress increase factor for an unbroken fiber adjacent to r consecutive broken fibers. Clearly K_0 is defined as one. Hedgepeth [10] and Hedgepeth and Van Dyke [11] used a shear-lag analysis to determine K_r for certain types of regular configurations. However, H. Fukuda and K. Kawata [22] disagree with the earlier results, and they achieved a slightly different value for K_r . It is apparent that more work must be done in this area by stress analysts; however, it is believed that slight variations in the calculated value for K_r will alter

the stress-rupture and tensile strength results for a composite by a relatively small amount. Moreover the exact determination of K_r is an insignificant problem compared to that of accurately modeling the composite failure mechanism itself. Hence, an intuitive but conservative approximation for K_r to be used throughout this work will be given later. (The computational techniques being developed will accomodate more refined values of K_r as they become available.)

3.2. Mathematical development of the model. The model just described is quite simple to discuss heuristically, and to formulate mathematically but deriving useful results is another matter. General results are still unavailable but explicit results have been obtained for composites of five or fewer fibers. These results are profound and are discussed shortly. Computer power is considered great enough at present to extend these results to composites of a hundred or more fibers. Such a task is underway.

For the moment, suppose that the probability distribution for the strength of a section whose length is δ and which contains n fiber segments $H_{n,\delta}(\cdot)$ is known. It is very simple to obtain the probability distribution function for the strength of the whole composite; call it $H_n(\cdot)$. From weakest link theory we have the well known result

$$H_n(x) = 1 - [1 - H_{n,\delta}(x)]^M \text{ for } x \geq 0 \quad (3.1)$$

where M is the number of sections of length δ . Generally M is of high order so that $H_n(x)$ is greatly influenced by the lower tail behavior of $H_{n,\delta}(x)$.

In order to compute the probability distribution function $H_{n,\delta}(\cdot)$ for the strength of a section, the load concentration of the matrix must be considered. At this point, the stress concentration factors K_i must be considered because they almost exclusively determine the load-sharing rules for a composite. As an example, recall the extreme case where no matrix is present, the Daniels' model. If each fiber initially supports a

load of λ and if r fibers have failed out of a total of n , then each surviving fiber is subjected to a load of $n\lambda/(n-r)$. In other words, the stress factor for each unfailed fiber is $K_r = n/(n-r)$ in the Daniels model. For a composite the equal load-sharing rule is not at all applicable. The geometrical structure of the section is of utmost importance. The stress factor depends upon how many consecutively failed segments are adjacent to a surviving segment. A more detailed discussion can be found in Hedgepeth and Van Dyke [23]. Obviously for each unique geometrical structure there is a set of associated stress factors. Thus the problem of finding a general distribution without regard to the specific structure is impossible. At present we desire a representative case which closely approximates the real situation and yields representative results.

The configuration studied and presented in this section is a circular array of uniformly spaced segments. When a fiber breaks the load that it supported is transmitted by shear to its nearest surviving neighbors. Because a minuscule amount of load is transmitted beyond the nearest unbroken segment in any radial direction from the broken segment, then a logical rule is that the load is shared or transmitted equally between the surviving segments adjacent to the break. This simple description can be rationally extended to include almost any geometrical arrangement. Suppose that a load of λ per fiber is placed on the bundle. At any stage, the load supported by any unfailed segment is λ plus one half of λ times the total number of consecutively failed segments counted in each radial direction from the broken segment. For a circular array the only directions to be considered are the clockwise and counterclockwise directions. Let r be the number of broken segments immediately adjacent to an unbroken segment.

The stress concentration factor for the unbroken segment will be $K_r = 1 + r/2$. These values for K_r are larger than those reported by the authors cited. This allows for a more conservative model, and may be more useful in reliability studies.

The only remaining task, but most important to perform, is the determination of the probability distribution function for the strength of a section $H_{n,\delta}(\cdot)$. Due to the load-sharing rule that has been adopted, it is manifest that a composite with one, two, or three fibers will behave equivalently to a Daniels bundle with one, two, or three fibers respectively. Thus with respect to the behaviour of a single section, the case of four fibers in a composite is the first and simplest case where the two models differ analytically. In APPENDIX A, $H_{4,\delta}(\cdot)$ and $H_{5,\delta}(\cdot)$ are derived. Also in APPENDIX A are the associated Daniels results $G_{4,\delta}(\cdot)$ and $G_{5,\delta}(\cdot)$ for four and five fibers respectively. (The stress factors K_r were described in the example given earlier.) The laborious nature of the computations is immediately appreciated!

3.3 Comparison and significance of the results. Immediately it is appreciated that $H_{4,\delta}(l)$ and $H_{5,\delta}(l)$ given by (A1) and (A4), are formidable expressions. They are written in terms of $F_\delta(\cdot)$ the fiber strength distribution for a fiber segment of length δ . The corresponding Daniels versions $G_{4,\delta}(l)$ and $G_{5,\delta}(l)$ given by (A5) and (A6) are also complex but less so. Appreciate that in the Daniels case the number of additive terms in $G_{n,\delta}(l)$ is 2^{n-1} and the number of additive terms in $H_{n,\delta}(l)$ is much greater! Even $G_{10,\delta}(\cdot)$ has for example 512 terms! Three major questions arise as follows:

Question 1: Which terms contribute most to $H_{n,\delta}(\cdot)$ for the stress ranges of interest (such as the lower tail) and which terms dominate as n grown large?

Question 2: Which terms are retained or emphasized if the assumptions under question in earlier models [4-7] are made?

Question 3: How might the computation of $H_{n,\delta}(x)$ be automated for arbitrary K_r and $F_\delta(x)$ and larger n ?

We comment that Daniels' famous result [1] is that $G_{n,\delta}(x)$ is asymptotically normal for large n and we foresee no comparable result for $H_{5,\delta}(x)$.

Before dealing with these questions we compare numerically $G_{5,\delta}(\cdot)$ and $H_{5,\delta}(\cdot)$ for cases of importance. As is customary and realistic we assume the Weibull distribution for the strength ζ of a fiber segment, that is, assume

$$F_\delta(x) = 1 - e^{-\delta(x/\sigma_0)^\rho} \quad \text{for } x \geq 0. \quad (3.2)$$

The mean for (3.2) is

$$E[\zeta] = \sigma_0 \delta^{-1/\rho} \Gamma(1+1/\rho), \quad (3.3)$$

the standard deviation is

$$S.D.[\zeta] = \sigma_0 \delta^{-1/\rho} [\Gamma(1+2/\rho) - \{\Gamma(1+1/\rho)\}^2]^{1/2} \quad (3.4)$$

and the coefficient of variation is

$$CV = \left[\frac{\Gamma(1+2/\rho)}{\Gamma(1+1/\rho)^2} - 1 \right]^{1/2}. \quad (3.5)$$

Note that $\sigma_0 \delta^{-1/\rho}$ is a scale factor and ρ governs the relative "scatter" in strength as is seen by (3.5). No loss in generality occurs if we take $\sigma_0 \delta^{-1/\rho}$ as unity and scale accordingly after results have been obtained. The important variable is obviously the Weibull shape parameter ρ which governs the scatter in fiber strength. Our results are presented in terms of CV

since this quantity is more intuitive and available from fiber strength data. Table 3-1 gives CV in terms of ρ .

A comparison of the respective means and coefficients of variation for $H_{5,\delta}(\cdot)$, $G_{5,\delta}(\cdot)$ and $F_{\delta}(\cdot)$ appears in Table 3-2. For all practical purposes there is no difference in the values for the Daniels model ($G_{5,\delta}(\cdot)$) and the composite model ($H_{5,\delta}(\cdot)$). It is tempting to conclude that in reality the Daniels version is more than adequate when five fibers are involved. However we recall that the composite is M such sections in a chain, where M is generally large. Hence the lower tails of $G_{5,\delta}(\cdot)$ and $H_{5,\delta}(\cdot)$ govern the important behavior of the composite strength distribution $H_n(x)$ as is seen by (3-1). Graphs of the section strength distribution $H_{5,\delta}(\cdot)$ for the new composite model, $G_{5,\delta}(\cdot)$ for Daniels' model, and $F_{5,\delta}(\cdot)$ for a simple weakest link model are given in Figure 2 and Figure 3 for $\rho = 5$ and $\rho = 10$ respectively. The surprising fact is that in the lower tails there is a difference in the distribution functions. For certain ranges of the nominal strength the differences between the new composite model and the Daniels model is significant enough to alter reliability. Considering that in practice a composite is a large number of sections in series, then the difference in distribution functions has a multiplicative effect on the reliability of a long composite. Thus this difference is critical. (It is also manifest from these figures that a simple weakest link model would be inappropriate.)

Since the new model for composite tensile strength considers how the true failure mechanism affects the strength whereas the Daniels version does not, then the new model should be studied in depth. From the difference in the lower tail behavior observed in Figures 2 and 3, even though

both models yield the same mean strength and variance, it is strongly implied that the Daniels model is inadequate as a model for composites. It appears that more pronounced differences will exist in the larger composites, once several consecutive fibers break, the cascading effect from matrix shear considered in the new model is much greater than the equal load sharing in the Daniels fracture model. Obviously, this must receive more study. To determine the strength distribution under the true failure mechanism it will necessarily require computer assistance. Even for ten fibers there are $9!$ configurations of the ordered fiber strengths to be considered. Thus an outstanding problem is to develop either a new technique to derive the distribution or a systematic method making efficient use of computer techniques.

The lower tails of $H_{5,\delta}(\cdot)$ and $G_{5,\delta}(\cdot)$ in Figures 2 and 3 appear to be almost parallel. In the way $H_{5,\delta}(\cdot)$ and $G_{5,\delta}(\cdot)$ were derived, there is sufficient similarity that the lower tail behavior may indeed have the same order of magnitude. Since reliability is most concerned with lower tail behavior, it may be possible to find the lower tail of the strength distribution for large composites without using an involved combinatorial argument. This would certainly be a worthwhile project to conduct.

Returning to Question 1 posed earlier, there is no basis for neglecting any of the terms in A4. As n grows large there is no obvious reason why terms of a certain form should dominate. Argon [19] has argued for certain limiting forms and has discussed their implications. These forms were obtained earlier by Scop and Argon [5] using heuristic arguments and amount to the retention of only a few terms in the large n analogs of A1 and A4. We believe that such approximations are misleading in their

predictions especially with respect to the size effect in composites. This question is being pursued further. With respect to Question 2 we have no definitive answer as yet but we believe that certain important terms will be left out. We return to this later. With respect to Question 3, we foresee the automation in the computation of $H_{n,\delta}(\cdot)$ in the next few months. The procedure will involve a modification of the enumeration technique developed in APPENDIX A.

3.4 Location of zero-order breaks. As mentioned earlier previous investigators [2-7] have assumed that zero-order breaks occur sufficiently far apart that their probabilistic interaction is negligible. This simplifies the analysis in that initial multiple failure groups are neglected. Here we investigate the validity of the assumption.

For the computation we assume that the composite is in the form of a planar tape. We consider the situation in one section of length δ having n fiber segments. Under the initial composite stress σ , each fiber segment may fail. The failure of the segments may be viewed as independent and identically distributed Bernoulli trials with $F_\delta(\sigma)$ being the probability that a given segment fails under the nominal composite stress. We assume the Weibull form for $F_\delta(\sigma)$ given by (3.2). In view of the fact that the composite will exhibit a size effect as the number of fibers n is increased, we allow σ to be a decreasing function of n of the form

$$\sigma(n) = \sigma_0 \left[\frac{1}{\delta} \left(\frac{\lambda}{n} \right)^k \right]^{1/\rho} \quad \text{for } k \geq 0 \quad (3.6)$$

and λ is any positive constant. In essence $k = 1(>1)(<1)$ corresponds to the size effect in the composite being the same as (greater than) (less than) the size effect in the fibers. Thus we combine (3.6) and (3.2) to obtain

$$F_{\delta}(\sigma(n)) = 1 - e^{-\left(\frac{\lambda}{n}\right)^k}. \quad (3.7)$$

Consider the event A_1 that at least one nonbroken fiber is between any two zero-order breaks. If there are n fiber segments in the section and N is the number of zero-order breaks, we obtain by a simple conditioning argument

$$\Pr\{A_1\} = \sum_{j=0}^n \Pr\{A_1 \mid N=j\} \Pr\{N=j\}. \quad (3.8)$$

In order to evaluate $\Pr\{A_1 \mid N=j\}$, a simple counting argument is needed to count the number of ways that j zero-order breaks can occur, out of the n fibers, so that any two broken fibers will have at least one nonbroken fiber between them. Thus we obtain

$$\Pr\{A_1 \mid N=j\} = \frac{\binom{n+1-j}{j}}{\binom{n}{j}}, \quad j = 0, 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor \quad (3.9)$$

= 0 otherwise.

Also

$$\Pr\{N=j\} = \binom{n}{j} F_{\delta}(\sigma)^j (1-F_{\delta}(\sigma))^{n-j},$$

so that

$$\begin{aligned} \Pr\{A_1\} &= \sum_{j=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1-j}{j} F_{\delta}(\sigma)^j (1-F_{\delta}(\sigma))^{n-j} \\ &= (1-F_{\delta}(\sigma))^n 2^{-n-2} \alpha^{-1} [(1+\alpha)^{n+2} - (1-\alpha)^{n+2}] \end{aligned} \quad (3.10)$$

where

$$\alpha = \sqrt{1 + \frac{4F_{\delta}(\sigma)}{1-F_{\delta}(\sigma)}} \quad (3.11)$$

The last equality follows from problem 7, page 75 in Combinatorial Identities by Riordan [24].

Making the above substitution for $F_{\delta}(\sigma(n))$ and taking a limit as $n \rightarrow \infty$ yields the following:

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr\{A_1(n)\} &= 0 && \text{if } 0 \leq k < 1 \\ &= e^{-\lambda^2} && \text{if } k = 1/2 \\ &= 1 && \text{if } k > 1/2 \end{aligned} \quad (3.12)$$

Thus for $0 \leq k < 1/2$ zero-order breaks will occur in adjacent fibers for large composites. Thus, if the stress as a function of n decreases at a rate lower than the Weibull rate $n^{-1/(2p)}$, then the location of zero-order breaks becomes critical for large composites. In this case some multiple failure groups occur with certainty, and the associated assumption in previous models would be inappropriate. Putting it another way, if the size effect of the composite is less than half as severe as the size effect for single fibers then zero-order failure interactions are important and must be taken into account in any realistic analysis. Experimentally, it is known that the size effect of the composite is in fact less than half of that for the constituent fibers, as the Weibull shape parameter p (experimental) for composites tends to be more than double that for the individual fibers.

If a rational basis exists for determining approximate results for large composites it has yet to be revealed. Certainly earlier approximations

[2-7] and their implications cannot be relied upon. The situation will become even more complex when viscoelastic effects are taken into account.

3.5 Controlled Strain Model. A controlled strain model was developed for composites which extended the work done by Phoenix and Taylor [21]. The new model assumed that composites can be viewed as a series of sections consistent with weakest link theory. With this model an attempt was made to capture the first order effects of the composite failure mechanism; however, more work must be done to refine the model to conform more closely to the total composite failure mechanism.

If α represents the fraction of load that a broken fiber can hold via the shear transfer from the matrix, then the asymptotic strength distribution of a section is normal with mean

$$\mu_{\max}(\alpha) = \sup_{e \geq 0} e(1-F(e))(1+\alpha F(e))$$

where $F(\cdot)$ is the distribution for the breaking strain of a fiber. Notice that this model predicts an asymptotic mean strength which is slightly higher than in the Phoenix and Taylor model.

4. Work Underway and Anticipated for Remainder of Term

A number of avenues of investigation are being pursued for the remainder of the term;

1. Automation of the computation of $H_{n,\delta}(\cdot)$ for $n > 5$ is being pursued using combinatorial and computer techniques. There is strong encouragement that accurate results can be obtained for $n=100$. This should be sufficient to establish the major trends of interest with respect to lower tail behavior and size effects of the distribution of composite strength. Secondly, terms in $H_{n,\delta}(x)$ will be scrutinized numerically in order to determine which if any may be neglected. As refinements of the stress increase factors K_T become available, these will be incorporated. Later K_T will become time dependent as matrix viscoelastic effects are taken into account. Ultimately stress-rupture failure will be modeled using the techniques developed.

2. The works of Argon and Bailey [20], Argon [19] and Scop and Argon [5] are being studied further with respect to the approximations presented. The work of Argon and Bailey is particularly significant because it contains experimental results on a 5-layer laminate. We wish to relate these results to the exact 5-fiber composite results discussed earlier and in APPENDIX A. Preliminary calculations indicate that the approximations in Argon [20] yield curves which lie a rather large distance away from the curves on Figures 2 and 3, and the deviations are exaggerated in the lower tail. This is not conclusive since 5 is not large n , but doubt is cast on the validity of the approximations.

3. Efforts are being made to secure summer 1977 personnel to investigate viscoelastic effects around failed fibers. The resulting refinements will be useful.

5. Deviation of Technical Effort from Original Proposal

The progress to date has exceeded that reasonably hoped for by the principal investigator, in spite of the fact that a suitable Post-Doctorate Associate was not found. Motivation for the probabilistic aspects of the progress is in large measure the result of the help and guidance of Professor Howard M. Taylor of Cornell University who is a renown authority in Applied Probability and Stochastic Processes. Not only has he consulted extensively at no cost to the project but two of his Ph.D. students, Gary Harlow and Carl Diegert have performed exceptionally well as Graduate Research Assistants. With respect to the probabilistic aspects of the problem, the sophistication and computational expertise of the personnel greatly exceeds that brought to bear on the problem in past efforts.

With respect to the solid mechanics and viscoelastic aspects of the problem, the situation is rapidly being approached where refined mechanical modeling of the local fiber failure process will be necessary. Specifically, time dependent effects must be taken into account. The progress in this respect has been slower than originally hoped for because of the lack of a suitable Post-Doctorate Associate. The appointment of a Graduate Research Assistant with a background in solid Mechanics should alleviate the difficulty.

The planned visits to Lawrence Livermore Laboratory, where critical stress-rupture experiments are being carried out by T.T. Chiao and associates, have been delayed. At the commencement of the project, the principal investigator was not an American Citizen and hence was not permitted entrance to the Livermore facility. U.S. citizenship was applied for in

November 1976 and is anticipated in April 1977 according to latest estimates. Visits to Lawrence Livermore Laboratory and other facilities on the West coast of the U.S. are hoped for in late May or early June. These visits will result in the establishment of a substantial data base together with information on the mechanical aspects of the fracture mechanisms.

In summary no significant deviations from the technical effort, as originally proposed, have occurred. Shifts in emphasis have occurred however in light of recently uncovered literature and the talents of available personnel.

6. Project Personnel and Time Spent

Principal Investigator. Dr. S. Leigh Phoenix, Assistant Professor in Mechanical and Aerospace Engineering has directed the project. Since receiving his degree in Theoretical and Applied Mechanics from Cornell University in 1972, Dr. Phoenix has conducted research in the application of probability and statistics to materials fracture and fatigue problems. In particular he has advanced the literature substantially in the area of the stochastic strength and fatigue of fiber bundles. Dr. Phoenix devoted one month to the project in the summer of 1976. The time was spent organizing and recruiting personnel for the project and completing two earlier works on the stochastic strength and fatigue of fiber bundles. While the fiber load sharing features of these models differed from that induced by the existence of a matrix in the bundle; the approach has proved to have features relevant to the present problem. During the academic year about 15% of the total time of the principal was devoted to the project. The above mentioned works were completed and daily interaction with other personnel on the project took place.

Graduate Research Assistant. In September 1977 Gary Harlow was assigned to the project. He has devoted 50% of his time (15 hours per week) to the project. Gary is a graduate student in Applied Mathematics and is completing his Ph.D. Thesis on "Probabilistic Models for the Tensile Failure of Composite Materials" under the direction of Professor Howard M. Taylor. He has a background in Physics at the B.A. level. Gary expects to complete his thesis and final examination in early April 1977, after which he will be promoted to Post-Doctorate Associate on the project.

Graduate Research Assistant. Carl Diegert was appointed to the project in January 1977. He has devoted 50% of his time (20 hours per week) to the project. Carl is a graduate student in Operations Research and is now selecting a thesis topic in the area of computational techniques in Probability and Stochastic Processes. Carl's numerical and computational expertise is crucial to carrying out the complex numerical analysis of the models under study.

Consultant. Professor Howard M. Taylor has consulted at least 25 hours to the project. His guidance with respect to relevant literature and methods of attack at the research level has been invaluable.

Post-Doctorate Associate. The chairpersons of major Mechanics and Mechanical Engineering departments at Universities in the U.S. were notified of the Post-Doctoral Position. A background in solid mechanics, viscoelasticity, and probability was listed as desirable. The response was limited and no suitable candidate was found. Hence Carl Diegert was appointed as a Graduate Research Assistant and Gary Harlow will be promoted to Post-Doctoral Associate in April because of his outstanding performance to date.

7. Articles Prepared for Publication.

Less than nine months have elapsed since the commencement of the program. Consequently no recent results have been submitted for publication as yet. The first article that will contain results of the current research will be submitted for publication in July or August of 1977. In March of 1977 an article by the principal investigator entitled "Stochastic Strength and Fatigue of Fiber Bundles" was submitted to the International Journal of Fracture. This article contains relevant material generated prior to the contract period. Also a pertinent manuscript entitled "Stochastic Failure of Fiber Bundles" was completed in January 1977. This article is very advanced mathematically and is being reviewed by several renown mathematicians. Some of the theorems proved therein will be relevant to the current research at a later time. This article will be submitted to a mathematics journal in the summer of 1977. Finally, some of the research of the current contract period will appear in the Ph.D. thesis of Gary Harlow entitled "Probabilistic Models for the Tensile Failure of Composite Materials."

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TABLE 3.1: Single Fiber CV as a Function of ρ

ρ	CV
1.0000	1.00
1.1128	0.90
1.2582	0.80
1.4513	0.70
1.7171	0.60
2.1013	0.50
2.6956	0.40
3.7138	0.30
5.7974	0.20
6.0000	0.19
7.0000	0.17
8.0000	0.15
9.0000	0.13
10.0000	0.12
12.1530	0.10
20.0000	0.06
50.0000	0.03

TABLE 3.2: Comparison of Daniels' Bundle and Composite Model for Five Fibers.

CV	Composite mean	Daniels mean	Composite CV	Daniels CV
1.0	0.635	0.639	0.474	0.472
0.9	0.605	0.610	0.430	0.428
0.8	0.583	0.589	0.388	0.387
0.7	0.570	0.577	0.349	0.348
0.6	0.567	0.577	0.312	0.311
0.5	0.577	0.589	0.276	0.273
0.4	0.602	0.617	0.239	0.233
0.3	0.647	0.664	0.202	0.192
0.2	0.722	0.736	0.160	0.145
0.1	0.841	0.844	0.097	0.089

Figure 1: Ineffective length model for composite. Composite is modeled as a chain of M sections each of length δ .

Figure 2: Comparison of section strength distributions $H_{5,\delta}(x)$, $G_{5,\delta}(x)$ and a simple weakest link result $F_{5,\delta}(x)$ for $\rho=5$.

Probability $\times 100$

Figure 3: Comparison of section strength distributions $H_{5,\delta}(x)$, $G_{5,\delta}(x)$ and a simple weakest link result $F_{5,\delta}(x)$ for $\rho=10$.

Determination of $H_{4,\delta}(\cdot)$, $H_{5,\delta}(\cdot)$, $G_{4,\delta}(\cdot)$ and $G_{5,\delta}(\cdot)$

Consider a composite comprised of four fibers arranged in a circular array. Since attention is restricted to only one of the independent and identically distributed sections, then no special notation will be needed to denote that fact. Postulate that the independent and identically distributed random variables $X_1, \dots, X_4$ represent the strengths of the four fiber segments. Let $F_\delta(\cdot)$ be the distribution function for X_i ,

$$F_\delta(x) = \Pr\{X_i \leq x\} \text{ for } x \geq 0,$$

where $F_\delta(0) = 0$. Let $Y_1 \leq \dots \leq Y_4$ be the order statistics for $X_1, \dots, X_4$.

In a circular array there are six different configurations in which the four segments could be placed with respect to ordered strength. Table A1 displays these configurations. Given any such configuration, the quantity desired is the probability that a section will fail under an applied load of ℓ per segment. It is necessary to determine a condition which will guarantee failure. Consider Ia, if

$$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq 2\ell; Y_4 \leq 4\ell$$

is satisfied, then failure will result. Note that if just one inequality is not satisfied, then the section would hold. This type of condition is needed for each configuration in Table A1. Due to geometrical symmetry and load-sharing symmetry, some conditions are applicable to more than one configuration making such configurations probabilistically equivalent.

TABLE A1: Configurations for a Section with Four Fiber Segments

	a	b
1	$Y_1 \ Y_2$	$Y_1 \ Y_3$
	$Y_3 \ Y_4$	$Y_2 \ Y_4$
2	$Y_1 \ Y_2$	$Y_1 \ Y_4$
	$Y_4 \ Y_3$	$Y_2 \ Y_3$
3	$Y_1 \ Y_3$	$Y_1 \ Y_4$
	$Y_4 \ Y_2$	$Y_3 \ Y_2$

TABLE A2: Equivalent Configurations and Breaking Conditions for a Section with Four Fiber Segments

CONFIGURATION	CONDITION
1a, 1b, 2a, 2b	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq 2\ell; Y_4 \leq 4\ell$
3a, 3b	$Y_1 \leq \ell; \ell < Y_2 \leq \frac{3}{2}\ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 4\ell$ or $Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq 4\ell$

TABLE A3: Reformulation of Breaking Conditions

CONFIGURATION	CONDITION
1a, 1b, 2a, 2b	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq 2\ell; Y_4 \leq 4\ell$
3a, 3b	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq 4\ell$ plus $Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 4\ell$ minus $Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 4\ell$

These facts are indicated in Table A2. Notice that for 3a the conditions are disjoint, and they can be reformulated as in Table A3. The advantage to the last format is that each subcondition is of the same form and has a probability given by the joint distribution function of $Y_1, \dots, Y_4$.

Define $P(\alpha, \beta, \gamma, \nu)$ by

$$P(\alpha, \beta, \gamma, \nu) = \Pr(Y_1 \leq \alpha, Y_2 \leq \beta, Y_3 \leq \gamma, Y_4 \leq \nu);$$

this can be evaluated by a direct computation. Thus

$$\begin{aligned} P(\alpha, \beta, \gamma, \nu) &= \int_0^\alpha \int_0^\beta \int_0^\gamma \int_0^\nu 4! dF_\delta(x_4) dF_\delta(x_3) dF_\delta(x_2) dF_\delta(x_1) \\ &= 24F_\delta(\nu)F_\delta(\gamma)F_\delta(\beta)F_\delta(\alpha) - 12F_\delta(\nu)F_\delta(\gamma)F_\delta^2(\alpha) \\ &\quad - 12F_\delta(\nu)F_\delta^2(\beta)F_\delta(\alpha) + 4F_\delta(\nu)F_\delta^3(\alpha) - 12F_\delta^2(\alpha)F_\delta(\beta)F_\delta(\alpha) \\ &\quad + 6F_\delta^2(\gamma)F_\delta^2(\alpha) + 4F_\delta^3(\beta)F_\delta(\alpha) - F_\delta^4(\alpha). \end{aligned}$$

Enough background has been cited so that the distribution for the strength of a section can be computed. Recall that

$$\begin{aligned} H_{4,6}(t) &= \Pr(\text{strength of a section} \leq 4t) \\ &= \Pr\{\text{a section fails when subjected to a load of } t \text{ per member}\} \\ &= \sum \Pr(\text{strength} \leq 4t | \text{one configuration}) \Pr\{\text{configuration occurs}\} \end{aligned}$$

where the sum is over all nonequivalent configurations. Since each configuration is equally likely, then $\Pr\{\text{configuration occurs}\}$ is

simply the number of equivalent configurations divided by six. Thus

$$\begin{aligned}
 H_{4,\delta}(\cdot) &= \frac{2}{3} P(\ell, \frac{3}{2} \ell, 2\ell, 4\ell) + \\
 &\quad \frac{1}{3} \{P(\ell, \ell, 2\ell, 4\ell) + P(\ell, \frac{3}{2} \ell, \frac{3}{2} \ell, 4\ell) - P(\ell, \ell, \frac{3}{2} \ell, 4\ell)\} \\
 &= 16F_{\delta}(4\ell)F_{\delta}(2\ell)F_{\delta}(\frac{3}{2} \ell)F_{\delta}(\ell) - 4F_{\delta}(4\ell)F_{\delta}(2\ell)F_{\delta}^2(\ell) \\
 &\quad - 4F_{\delta}(4\ell)F_{\delta}^2(\frac{3}{2} \ell)F_{\delta}(\ell) + 4F_{\delta}(4\ell)F_{\delta}^3(\ell) - 8F_{\delta}^2(2\ell)F_{\delta}(\frac{3}{2} \ell)F_{\delta}(\ell) \\
 &\quad + 2F_{\delta}^2(2\ell)F_{\delta}^2(\ell) - 8F_{\delta}(4\ell)F_{\delta}(\frac{3}{2} \ell)F_{\delta}^2(\ell) + 4F_{\delta}^2(\frac{3}{2} \ell)F_{\delta}^2(\ell) - F_{\delta}^4(\ell);
 \end{aligned} \tag{A1}$$

the last equality results from the proper substitutions and simplification.

Thus $H_{4,\delta}(\cdot)$ has been explicitly determined.

The derivation of $H_{5,\delta}(\cdot)$ is very similar to that of $H_{4,\delta}(\cdot)$, but the magnitude of the computations is much greater. Let $X_1, \dots, X_5$ be the independent and identically distributed random variables for the strengths of the fiber segments. Again, let $F_{\delta}(\cdot)$ be the distribution for X_i ,

$$F_{\delta}(x) = \Pr\{X_i \leq x\} \text{ for } x \geq 0.$$

Also denote the order statistics of $X_1, \dots, X_5$ by $Y_1 \leq \dots \leq Y_5$.

For a circular array there are 24 ways in which the five segments could be arranged. In adding just one segment the number of possible arrays has quadrupled. The arrays are displayed in Table A4. As before, a bounding condition is needed for each array. Fortunately only seven distinct classes must be considered. This information is given in Table A5. Notice that the classes with multiple conditions are such that all the conditions can be expressed in a similar format; see Table A6.

TABLE A4: Arrays for a Section with Five Fiber Segments

	a	b	c	d
1	$\begin{array}{ccc} & Y_1 & \\ Y_5 & & Y_2 \\ & Y_4 & Y_3 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_2 & & Y_3 \\ & Y_5 & Y_4 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_3 & & Y_4 \\ & Y_2 & Y_5 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_4 & & Y_5 \\ & Y_3 & Y_2 \end{array}$
2	$\begin{array}{ccc} & Y_1 & \\ Y_4 & & Y_2 \\ & Y_5 & Y_3 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_5 & & Y_3 \\ & Y_2 & Y_4 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_2 & & Y_4 \\ & Y_3 & Y_5 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_3 & & Y_5 \\ & Y_4 & Y_2 \end{array}$
3	$\begin{array}{ccc} & Y_1 & \\ Y_3 & & Y_2 \\ & Y_5 & Y_4 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_2 & & Y_3 \\ & Y_4 & Y_5 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_5 & & Y_4 \\ & Y_2 & Y_3 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_4 & & Y_5 \\ & Y_2 & Y_3 \end{array}$
4	$\begin{array}{ccc} & Y_1 & \\ Y_5 & & Y_2 \\ & Y_3 & Y_4 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_4 & & Y_3 \\ & Y_2 & Y_5 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_2 & & Y_4 \\ & Y_5 & Y_3 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_2 & & Y_5 \\ & Y_4 & Y_3 \end{array}$
5	$\begin{array}{ccc} & Y_1 & \\ Y_3 & & Y_2 \\ & Y_4 & Y_5 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_4 & & Y_3 \\ & Y_5 & Y_2 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_5 & & Y_4 \\ & Y_3 & Y_2 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_2 & & Y_5 \\ & Y_3 & Y_4 \end{array}$
6	$\begin{array}{ccc} & Y_1 & \\ Y_4 & & Y_2 \\ & Y_3 & Y_5 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_5 & & Y_3 \\ & Y_4 & Y_2 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_3 & & Y_4 \\ & Y_5 & Y_2 \end{array}$	$\begin{array}{ccc} & Y_1 & \\ Y_3 & & Y_5 \\ & Y_2 & Y_4 \end{array}$

TABLE A5: Equivalent Arrays and Breaking Conditions for a Section with Five Segments

ARRAY	CONDITION
1a, 2a, 3a, 5a, 5d, 2c, 3b, 1b	$Y_1 \leq l; Y_2 \leq \frac{3}{2}l; Y_3 \leq 2l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$
4a, 6a, 4d, 4c	$Y_1 \leq l; Y_2 \leq \frac{3}{2}l; l < Y_3 \leq 2l; Y_4 \leq 2l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; Y_3 \leq l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$
2b, 2d	$Y_1 \leq l; Y_2 \leq l; Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; \frac{3}{2}l < Y_3 \leq 2l; Y_4 \leq 2l; Y_5 \leq 2l$ or $Y_1 \leq l; l < Y_2 \leq \frac{3}{2}l; Y_3 \leq \frac{3}{2}l; Y_4 \leq 2l; Y_5 \leq 5l$
4b, 6c	$Y_1 \leq l; Y_2 \leq l; Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; \frac{3}{2}l < Y_3 \leq 2l; Y_4 \leq 2l; Y_5 \leq 5l$ or $Y_1 \leq l; l < Y_2 \leq \frac{3}{2}l; Y_3 \leq \frac{3}{2}l; Y_4 \leq 2l; Y_5 \leq 5l$
5b, 6b, 1c, 6d	$Y_1 \leq l; l < Y_2 \leq \frac{3}{2}l; Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; Y_3 \leq 2l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$
3c, 1d	$Y_1 \leq l; l < Y_2 \leq \frac{3}{2}l; Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{3}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; l < Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; \frac{3}{2}l < Y_3 \leq 2l; Y_4 \leq 2l; Y_5 \leq 2l$ or $Y_1 \leq l; Y_2 \leq l; Y_3 \leq l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$
5c, 3d	$Y_1 \leq l; l < Y_2 \leq \frac{3}{2}l; Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{3}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; l < Y_3 \leq \frac{3}{2}l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; \frac{3}{2}l < Y_3 \leq 2l; Y_4 \leq 2l; Y_5 \leq 5l$ or $Y_1 \leq l; Y_2 \leq l; Y_3 \leq l; Y_4 \leq \frac{5}{2}l; Y_5 \leq 5l$

TABLE A6: Reformulation of Breaking Conditions

ARRAY	CONDITION
1a, 2a, 3a, 5a, 5d, 7c, 3b, 1b	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq 2\ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5\ell$
4a, 6a, 4d, 4c	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq 2\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5\ell$
2b, 2d	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq 2\ell; Y_5 \leq 2\ell$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 2\ell; Y_5 \leq 2\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$
4b, 6c	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq 2\ell; Y_5 \leq 5\ell$
5b, 6b, 1c, 6d	$Y_1 \leq \ell; Y_2 \leq \frac{3}{2}\ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5$ minus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq \frac{3}{2}\ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5\ell$ plus
	$Y_1 \leq \ell; Y_2 \leq \ell; Y_3 \leq 2\ell; Y_4 \leq \frac{5}{2}\ell; Y_5 \leq 5\ell$

3c, 1d	$Y_1 \leq 1; Y_2 \leq \frac{3}{2} 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{3}{2} 1; Y_5 \leq 51$	minus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{3}{2} 1; Y_5 \leq 51$	plus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{5}{2} 1; Y_5 \leq 51$	minus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{5}{2} 1; Y_5 \leq 51$	plus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq 21; Y_4 \leq 21; Y_5 \leq 21$	minus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq 21; Y_5 \leq 21$	plus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq 1; Y_4 \leq \frac{5}{2} 1; Y_5 \leq 51$	
5c, 3d	$Y_1 \leq 1; Y_2 \leq \frac{3}{2} 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{3}{2} 1; Y_5 \leq 51$	minus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{3}{2} 1; Y_5 \leq 51$	plus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq \frac{5}{2} 1; Y_5 \leq 51$	minus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq 1; Y_4 \leq \frac{5}{2} 1; Y_5 \leq 51$	plus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq 21; Y_4 \leq 21; Y_5 \leq 51$	minus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq \frac{3}{2} 1; Y_4 \leq 21; Y_5 \leq 51$	plus
	$Y_1 \leq 1; Y_2 \leq 1; Y_3 \leq 1; Y_4 \leq \frac{5}{2} 1; Y_5 \leq 51$	

Define $P(\alpha, \beta, \gamma, v, \epsilon)$ by

$$P(\alpha, \beta, \gamma, v, \epsilon) = \Pr\{Y_1 \leq \alpha; Y_2 \leq \beta; Y_3 \leq \gamma; Y_4 \leq v; Y_5 \leq \epsilon\}$$

which can be explicitly given by

$$\begin{aligned} P(\alpha, \beta, \gamma, v, \epsilon) &= \int_0^\alpha \int_{x_1}^\beta \int_{x_2}^\gamma \int_{x_3}^v \int_{x_4}^\epsilon 5! dF_\delta(x_5) dF_\delta(x_4) dF_\delta(x_3) dF_\delta(x_2) dF_\delta(x_1) \\ &= 120F_\delta(\epsilon)F_\delta(v)F_\delta(\gamma)F_\delta(\beta)F_\delta(\alpha) - 60F_\delta(\epsilon)F_\delta(v)F_\delta(\gamma)F_\delta^2(\alpha) \\ &\quad - 60F_\delta(\epsilon)F_\delta(v)F_\delta^2(\beta)F_\delta(\alpha) + 20F_\delta(\epsilon)F_\delta(v)F_\delta^3(\alpha) \\ &\quad - 60F_\delta(\epsilon)F_\delta^2(\gamma)F_\delta(\beta)F_\delta(\alpha) + 30F_\delta(\epsilon)F_\delta^2(\gamma)F_\delta^2(\alpha) \\ &\quad + 20F_\delta(\epsilon)F_\delta^3(\beta)F_\delta(\alpha) - 5F_\delta(\epsilon)F_\delta^4(\alpha) \\ &\quad - 60F_\delta^2(v)F_\delta(\gamma)F_\delta(\beta)F_\delta(\alpha) + 30F_\delta^2(v)F_\delta(\gamma)F_\delta^2(\alpha) \\ &\quad + 30F_\delta^2(v)F_\delta^2(\beta)F_\delta(\alpha) - 10F_\delta^2(v)F_\delta^3(\alpha) + 20F_\delta^3(\gamma)F_\delta(\beta)F_\delta(\alpha) \\ &\quad - 10F_\delta^3(\gamma)F_\delta^2(\alpha) - 5F_\delta^4(\beta)F_\delta(\alpha) + F_\delta^5(\alpha). \end{aligned} \quad (A2)$$

Assuming that initially an applied load of ℓ per member is placed on the composite, $H_{5,\delta}(\cdot)$ can be computed. Thus

$$\begin{aligned} H_{5,\delta}(\ell) &= \Pr\{\text{strength of a section} \leq 5\ell\} \\ &= \Pr\{\text{a section fails when subjected to a load of } \ell \text{ per member}\} \\ &= \sum \Pr\{\text{strength} \leq 5\ell | \text{one array}\} \Pr\{\text{array occurs}\} \end{aligned}$$

where the sum is over all nonequivalent classes of arrays. Since all

arrays are equally likely to occur, $Pr(\text{array occurs})$ is the number of arrays in one class divided by 24. Thus

$$\begin{aligned}
 H_5(l) &= \frac{1}{3} P(l, \frac{3}{2} l, 2l, \frac{5}{2} l, 5l) \\
 &+ \frac{1}{6} \{P(l, l, 2l, 2l, 5l) - P(l, l, l, 2l, 5l) + P(l, \frac{3}{2} l, 2l, 2l, 5l) \\
 &\quad - P(l, l, 2l, 2l, 5l) + P(l, l, l, \frac{5}{2} l, 5l)\} \\
 &+ \frac{1}{12} \{P(l, l, \frac{3}{2} l, \frac{5}{2} l, 5l) + P(l, l, 2l, 2l, 2l) - P(l, l, \frac{3}{2} l, 2l, 2l) \\
 &\quad + P(l, \frac{3}{2} l, \frac{3}{2} l, 2l, 5l) - P(l, l, \frac{3}{2} l, 2l, 5l)\} \\
 &+ \frac{1}{12} \{P(l, l, \frac{3}{2} l, \frac{5}{2} l, 5l) + P(l, l, 2l, 2l, 5l) - P(l, l, \frac{3}{2} l, 2l, 5l) \\
 &\quad + P(l, \frac{3}{2} l, \frac{3}{2} l, 2l, 5l) - P(l, l, \frac{3}{2} l, 2l, 5l)\} \\
 &+ \frac{1}{6} \{P(l, \frac{3}{2} l, \frac{3}{2} l, \frac{5}{2} l, 5l) - P(l, l, \frac{3}{2} l, \frac{5}{2} l, 5l) + P(l, l, 2l, \frac{5}{2} l, 5l)\} \\
 &+ \frac{1}{12} \{P(l, \frac{3}{2} l, \frac{3}{2} l, \frac{3}{2} l, 5l) - P(l, l, \frac{3}{2} l, \frac{3}{2} l, 5l) + P(l, l, \frac{3}{2} l, \frac{5}{2} l, 5l) \\
 &\quad - P(l, l, l, \frac{5}{2} l, 5l) + P(l, l, 2l, 2l, 5l) \\
 &\quad - P(l, l, \frac{3}{2} l, 2l, 5l) + P(l, l, l, \frac{5}{2} l, 5l)\} \\
 &= \frac{1}{3} \{P(l, \frac{3}{2} l, 2l, \frac{5}{2} l, 5l) - P(l, l, \frac{3}{2} l, 2l, 5l)\} \\
 &+ \frac{1}{6} \{P(l, l, 2l, 2l, 5l) - P(l, l, l, 2l, 5l) + P(l, \frac{3}{2} l, 2l, 2l, 5l) \\
 &\quad + P(l, l, l, \frac{5}{2} l, 5l) + P(l, l, \frac{3}{2} l, \frac{5}{2} l, 5l) + P(l, l, 2l, 2l, 2l) \\
 &\quad - P(l, l, \frac{3}{2} l, 2l, 2l) + P(l, \frac{3}{2} l, \frac{3}{2} l, 2l, 5l) + P(l, \frac{3}{2} l, \frac{3}{2} l, \frac{5}{2} l, 5l) \\
 &\quad + P(l, l, 2l, \frac{5}{2} l, 5l) + P(l, \frac{3}{2} l, \frac{3}{2} l, \frac{3}{2} l, 5l) - P(l, l, \frac{3}{2} l, \frac{3}{2} l, 5l)\}. \quad (A3) \dots
 \end{aligned}$$

$$\begin{aligned}
H_{5,\delta}(l) = & 40F_{\delta}(5l)F_{\delta}(\frac{5}{2}l)F_{\delta}(2l)F_{\delta}(\frac{3}{2}l)F_{\delta}(l) - 10F_{\delta}(5l)F_{\delta}(\frac{5}{2}l)F_{\delta}(2l)F_{\delta}^2(l) \\
& - 10F_{\delta}(5l)F_{\delta}(\frac{5}{2}l)F_{\delta}^2(\frac{3}{2}l)F_{\delta}(l) - 10F_{\delta}(5l)F_{\delta}^2(2l)F_{\delta}(\frac{3}{2}l)F_{\delta}(l) \\
& + 5F_{\delta}(5l)F_{\delta}^2(2l)F_{\delta}^2(l) + 10F_{\delta}(5l)F_{\delta}(2l)F_{\delta}^3(l) \\
& - 30F_{\delta}(5l)F_{\delta}(2l)F_{\delta}(\frac{3}{2}l)F_{\delta}^2(l) + 5F_{\delta}(5l)F_{\delta}^2(\frac{3}{2}l)F_{\delta}^2(l) \\
& + 10F_{\delta}(5l)F_{\delta}(\frac{3}{2}l)F_{\delta}^3(l) - 5F_{\delta}(5l)F_{\delta}^4(l) \\
& - 20F_{\delta}^2(\frac{5}{2}l)F_{\delta}(2l)F_{\delta}(\frac{3}{2}l)F_{\delta}(l) + 5F_{\delta}^2(\frac{5}{2}l)F_{\delta}(2l)F_{\delta}^2(l) \\
& + 5F_{\delta}^2(\frac{5}{2}l)F_{\delta}^2(\frac{3}{2}l)F_{\delta}(l) + 5F_{\delta}(2l)F_{\delta}^2(\frac{3}{2}l)F_{\delta}^2(l) \\
& - 5F_{\delta}^2(2l)F_{\delta}^3(l) + 10F_{\delta}^2(2l)F_{\delta}(\frac{3}{2}l)F_{\delta}^2(l) \\
& - 5F_{\delta}^2(\frac{3}{2}l)F_{\delta}^3(l) + F_{\delta}^5(l);
\end{aligned} \tag{A4}$$

the last equality results from the proper substitutions and a laboriously long algebraic simplification.

Using the recursion formula for the Daniels' model [1], it is easy to derive $G_{4,\delta}(\cdot)$ and $G_{5,\delta}(\cdot)$ explicitly in terms of $F_{\delta}(\cdot)$ and the load concentration coefficients resulting from the equal load-sharing assumption. This allows for a term by term comparison with $H_{4,\delta}(\cdot)$ and $H_{5,\delta}(\cdot)$. Thus

$$\begin{aligned}
G_{4,\delta}(l) = & 24F_{\delta}(l)F_{\delta}(\frac{4}{3}l)F_{\delta}(2l)F_{\delta}(4l) - 12F_{\delta}(l)F_{\delta}(\frac{4}{3}l)F_{\delta}^2(2l) \\
& - 12F_{\delta}(l)F_{\delta}^2(\frac{4}{3}l)F_{\delta}(4l) + 4F_{\delta}(l)F_{\delta}^3(\frac{4}{3}l) \\
& - 12F_{\delta}^2(l)F_{\delta}(2l)F_{\delta}(4l) + 6F_{\delta}^2(l)F_{\delta}^2(2l) + 4F_{\delta}^3(l)F_{\delta}(4l) - F_{\delta}^4(l)
\end{aligned} \tag{A5}$$

and

$$\begin{aligned}
 G_{5,6}(x) = & 120F_6(x)F_6\left(\frac{5}{4}x\right)F_6\left(\frac{5}{3}x\right)F_6\left(\frac{5}{2}x\right)F_6(5x) \\
 & - 60F_6(x)F_6\left(\frac{5}{4}x\right)F_6\left(\frac{5}{3}x\right)F_6^2\left(\frac{5}{2}x\right) - 60F_6(x)F_6\left(\frac{5}{4}x\right)F_6^2\left(\frac{5}{3}x\right)F_6(5x) \\
 & + 20F_6(x)F_6\left(\frac{5}{4}x\right)F_6^3\left(\frac{5}{3}x\right) - 60F_6(x)F_6^2\left(\frac{5}{4}x\right)F_6\left(\frac{5}{2}x\right)F_6(5x) \\
 & + 30F_6(x)F_6^2\left(\frac{5}{4}x\right)F_6^2\left(\frac{5}{2}x\right) + 20F_6(x)F_6^3\left(\frac{5}{4}x\right)F_6(5x) \\
 & - 5F_6(x)F_6^4\left(\frac{5}{4}x\right) - 60F_6^2(x)F_6\left(\frac{5}{3}x\right)F_6\left(\frac{5}{2}x\right) \\
 & + 30F_6^2(x)F_6\left(\frac{5}{3}x\right)F_6^2\left(\frac{5}{2}x\right) + 30F_6^2(x)F_6^2\left(\frac{5}{3}x\right)F_6(5x) \\
 & - 10F_6^2(x)F_6^3\left(\frac{5}{3}x\right) + 20F_6^3(x)F_6\left(\frac{5}{2}x\right)F_6(5x) \\
 & - 10F_6^3(x)F_6^2\left(\frac{5}{2}x\right) - 5F_6^4(x)F_6(5x) + F_6^5(x).
 \end{aligned}
 \tag{A6}$$

Notice that the concentration constants, given previously, are derived by an equal load-sharing rule, and this is reflected in the above distribution functions.

It has been determined that the Weibull distribution is the most applicable distribution for the strength of a fiber, see References [2-7]. Suppose that the distribution of a fiber is found to be

$$F(x) = 1 - e^{-\sigma x^\rho}, \quad \sigma \geq 0, \quad \rho \geq 0,$$

then the strength distribution of a segment is

$$F_6(x) = 1 - e^{-\frac{\sigma}{N} x^\rho},$$

which is still a Weibull distribution function.

There are two ways to compute $H_{5,6}(\cdot)$ for this Weibull distribution;

one by directly substituting into the general form (A4) and the other by performing the entire calculation again. It appears that each are equally unpleasant, and arbitrarily the second way was chosen. Set

$$P_w(\alpha, \beta, \gamma, \nu, \epsilon) = \int_0^\alpha \int_{x_1}^\beta \int_{x_2}^\gamma \int_{x_3}^\nu \int_{x_4}^\epsilon 5! \left(\frac{\sigma}{W}\right)^5 \rho^5 \prod_{i=1}^5 x_i e^{-\frac{\sigma}{W} x_i^p} dx_5 dx_4 dx_3 dx_2 dx_1,$$

the proper modification of (A2) ; also set $W(x) = e^{-\frac{\sigma}{W} x^p}$. Then

$$\begin{aligned} P_w(\alpha, \beta, \gamma, \nu, \epsilon) = & 1 - W^5(\alpha) - 5W^4(\beta) + 5W^4(\beta)W(\alpha) - 10W^3(\gamma) \\ & + 10W^3(\gamma)W^2(\alpha) + 20W^3(\gamma)W(\beta) - 20W^3(\gamma)W(\beta)W(\alpha) - 10W^2(\nu) \\ & + 10W^2(\nu)W^3(\alpha) + 30W^2(\nu)W^2(\beta) - 30W^2(\nu)W^2(\beta)W(\alpha) + 30W^2(\nu)W(\gamma) \\ & - 30W^2(\nu)W(\gamma)W^2(\alpha) - 60W^2(\nu)W(\gamma)W(\beta) + 60W^2(\nu)W(\gamma)W(\beta)W(\alpha) \\ & - 5W(\epsilon) + 5W(\epsilon)W^4(\alpha) + 20W(\epsilon)W^3(\beta) - 20W(\epsilon)W^3(\beta)W(\alpha) \\ & + 30W(\epsilon)W^2(\gamma) - 30W(\epsilon)W^2(\gamma)W^2(\alpha) - 60W(\epsilon)W^2(\gamma)W(\beta) \\ & + 60W(\epsilon)W^2(\gamma)W(\beta)W(\alpha) + 20W(\epsilon)W(\nu) - 20W(\epsilon)W(\nu)W^3(\alpha) \\ & - 60W(\epsilon)W(\nu)W^2(\beta) + 60W(\epsilon)W(\nu)W^2(\beta)W(\alpha) - 60W(\epsilon)W(\nu)W(\gamma) \\ & + 60W(\epsilon)W(\nu)W(\gamma)W^2(\alpha) + 120W(\epsilon)W(\nu)W(\gamma)W(\beta) \\ & - 120W(\epsilon)W(\nu)W(\gamma)W(\beta)W(\alpha). \end{aligned}$$

Upon making the proper substitutions into (A3) and performing the laborious algebra, $H_{5,6}(\cdot)$ for the Weibull case is obtained. Hence

$$\begin{aligned}
H_{5, \delta}(x) = & 1 - \exp\left[-5 \frac{\sigma}{M} x^p\right] - 10 \exp\left[-2 \frac{\sigma}{M} \left(\frac{5}{2} x\right)^p\right] \\
& + 5 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + 2\left(\frac{3}{2} x\right)^p\right)\right] - 5 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + 2\left(\frac{3}{2} x\right)^p + x^p\right)\right] \\
& + 15 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + (2x)^p\right)\right] - 5 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + (2x)^p + 2x^p\right)\right] \\
& - 20 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + (2x)^p + \left(\frac{3}{2} x\right)^p\right)\right] + 20 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + (2x)^p + \left(\frac{3}{2} x\right)^p + x^p\right)\right] \\
& - 5 \exp\left[-\frac{\sigma}{M} (5x)^p\right] + 5 \exp\left[-\frac{\sigma}{M} ((5x)^p + 4x^p)\right] + 5 \exp\left[-\frac{\sigma}{M} ((5x)^p + 2(2x)^p)\right] \\
& - 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + 2(2x)^p + \left(\frac{3}{2} x\right)^p)\right] + 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + 2(2x)^p + \left(\frac{3}{2} x\right)^p + x^p)\right] \\
& + 20 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p)\right] - 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + 2\left(\frac{3}{2} x\right)^p)\right] \\
& + 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + 2\left(\frac{3}{2} x\right)^p + x^p)\right] - 30 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + (2x)^p)\right] \\
& + 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + (2x)^p + 2x^p)\right] + 40 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + (2x)^p + \left(\frac{3}{2} x\right)^p)\right] \\
& - 40 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + (2x)^p + \left(\frac{3}{2} x\right)^p + x^p)\right] + 5 \exp\left[-\frac{\sigma}{M} (2(2x)^p + 3x^p)\right] \\
& - 5 \exp\left[-\frac{\sigma}{M} (2(2x)^p + x^p)\right] + 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + (2x)^p + x^p)\right] \\
& - 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + (2x)^p + 3x^p)\right] + 5 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + x^p\right)\right] \\
& - 5 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + 2x^p\right)\right] + 10 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + \left(\frac{3}{2} x\right)^p\right)\right] \\
& - 10 \exp\left[-\frac{\sigma}{M} \left(2\left(\frac{5}{2} x\right)^p + \left(\frac{3}{2} x\right)^p + x^p\right)\right] + 5 \exp\left[-\frac{\sigma}{M} ((5x)^p + 2\left(\frac{3}{2} x\right)^p)\right] \\
& - 5 \exp\left[-\frac{\sigma}{M} ((5x)^p + 2\left(\frac{3}{2} x\right)^p + 2x^p)\right] - 10 \exp\left[-\frac{\sigma}{M} ((5x)^p + \left(\frac{5}{2} x\right)^p + 2x^p)\right]
\end{aligned}$$

$$\begin{aligned}
& -20\exp\left[-\frac{\sigma}{H}\left((5x)^p + \left(\frac{5}{2}x\right)^p + \left(\frac{3}{2}x\right)^p\right)\right] + 20\exp\left[-\frac{\sigma}{H}\left((5x)^p + \left(\frac{5}{2}x\right)^p + \left(\frac{3}{2}x\right)^p + x^p\right)\right] \\
& + 10\exp\left[-\frac{\sigma}{H}\left((2x)^p + 2\left(\frac{3}{2}x\right)^p + x^p\right)\right] - 5\exp\left[-\frac{\sigma}{H}\left((2x)^p + 2\left(\frac{3}{2}x\right)^p + 2x^p\right)\right] \\
& + 10\exp\left[-\frac{\sigma}{H}\left(2(2x)^p + \left(\frac{3}{2}x\right)^p + x^p\right)\right] + 10\exp\left[-\frac{\sigma}{H}\left((5x)^p + (2x)^p + \left(\frac{3}{2}x\right)^p\right)\right] \\
& + 30\exp\left[-\frac{\sigma}{H}\left((5x)^p + (2x)^p + \left(\frac{3}{2}x\right)^p + 2x^p\right)\right] - 10\exp\left[-\frac{\sigma}{H}\left(2(2x)^p + \left(\frac{3}{2}x\right)^p + 2x^p\right)\right] \\
& - 40\exp\left[-\frac{\sigma}{H}\left((5x)^p + (2x)^p + \left(\frac{3}{2}x\right)^p + x^p\right)\right] - 5\exp\left[-\frac{\sigma}{H}\left((5x)^p + 2(2x)^p + 2x^p\right)\right] \\
& - 10\exp\left[-\frac{\sigma}{H}\left(2\left(\frac{5}{2}x\right)^p + (2x)^p + x^p\right)\right] + 20\exp\left[-\frac{\sigma}{H}\left((5x)^p + \left(\frac{5}{2}x\right)^p + (2x)^p + x^p\right)\right] \\
& - 5\exp\left[-\frac{\sigma}{H}\left(2\left(\frac{3}{2}x\right)^p + 2x^p\right)\right] + 10\exp\left[-\frac{\sigma}{H}\left((5x)^p + \left(\frac{3}{2}x\right)^p + 2x^p\right)\right] \\
& + 5\exp\left[-\frac{\sigma}{H}\left(2\left(\frac{3}{2}x\right)^p + 3x^p\right)\right] - 10\exp\left[-\frac{\sigma}{H}\left((5x)^p + \left(\frac{3}{2}x\right)^p + 3x^p\right)\right] \\
& - 10\exp\left[-\frac{\sigma}{H}\left((5x)^p + \left(\frac{5}{2}x\right)^p + x^p\right)\right] - 5\exp\left[-\frac{\sigma}{H}\left((2x)^p + 2\left(\frac{3}{2}x\right)^p\right)\right]. \tag{A7}
\end{aligned}$$

It is manifest that this expression is very hard to manipulate; however, using numerical computing techniques it is possible to find the mean, variance, and graph of $H_{5,\delta}(\cdot)$ as a function of the parameters.

In a similar manner, $C_{5,\delta}(x)$ may be derived for the case where

$$F_{\delta}(x) = 1 - e^{-\frac{\sigma}{H}x^p}$$

by making the proper substitutions into equation A6.

Thus, assuming $\frac{\sigma}{H}$ to be identically equal to one,

$$\begin{aligned}
G_{5,6}(x) = & 1 - \exp[-5x^p] - 5\exp[-4(\frac{5}{4}x)^p] + 5\exp[-(4(\frac{5}{4}x)^p + x^p)] \\
& - 10\exp[-3(\frac{5}{3}x)^p] + 10\exp[-(3(\frac{5}{3}x)^p + 2x^p)] + 20\exp[-(3(\frac{5}{3}x)^p + (\frac{5}{4}x)^p)] \\
& - 20\exp[-(3(\frac{5}{3}x)^p + (\frac{5}{4}x)^p + x^p)] - 10\exp[-2(\frac{5}{2}x)^p] \\
& + 10\exp[-(2(\frac{5}{2}x)^p + 3x^p)] + 30\exp[-(2(\frac{5}{2}x)^p + 2(\frac{5}{4}x)^p)] \\
& - 30\exp[-(2(\frac{5}{2}x)^p + 2(\frac{5}{4}x)^p + x^p)] + 30\exp[-(2(\frac{5}{2}x)^p + (\frac{5}{3}x)^p)] \\
& - 30\exp[-(2(\frac{5}{2}x)^p + (\frac{5}{3}x)^p + 2x^p)] - 60\exp[-(2(\frac{5}{2}x)^p + (\frac{5}{3}x)^p + (\frac{5}{4}x)^p)] \\
& + 60\exp[-(2(\frac{5}{2}x)^p + (\frac{5}{3}x)^p + (\frac{5}{4}x)^p + x^p)] - 5\exp[-(5x)^p] \\
& + 5\exp[-((5x)^p + 4x^p)] + 20\exp[-((5x)^p + 3(\frac{5}{4}x)^p)] \\
& - 20\exp[-((5x)^p + 3(\frac{5}{4}x)^p + x^p)] + 30\exp[-((5x)^p + 2(\frac{5}{3}x)^p)] \\
& - 30\exp[-((5x)^p + 2(\frac{5}{3}x)^p + 2x^p)] - 60\exp[-((5x)^p + 2(\frac{5}{3}x)^p + (\frac{5}{4}x)^p)] \\
& + 60\exp[-((5x)^p + 2(\frac{5}{3}x)^p + (\frac{5}{4}x)^p + x^p)] + 20\exp[-((5x)^p + (\frac{5}{2}x)^p)] \\
& - 20\exp[-((5x)^p + (\frac{5}{2}x)^p + 3x^p)] - 60\exp[-((5x)^p + (\frac{5}{2}x)^p + 2(\frac{5}{4}x)^p)] \\
& + 60\exp[-((5x)^p + (\frac{5}{2}x)^p + 2(\frac{5}{4}x)^p + x^p)] - 60\exp[-((5x)^p + (\frac{5}{2}x)^p + (\frac{5}{3}x)^p)] \\
& + 60\exp[-((5x)^p + (\frac{5}{2}x)^p + (\frac{5}{3}x)^p + 2x^p)] \\
& + 120\exp[-((5x)^p + (\frac{5}{2}x)^p + (\frac{5}{3}x)^p + (\frac{5}{4}x)^p)] \\
& - 120\exp[-((5x)^p + (\frac{5}{2}x)^p + (\frac{5}{3}x)^p + (\frac{5}{4}x)^p + x^p)].
\end{aligned}
\tag{A8}$$

Notice that A7 is comprised of 49 terms while A8 is comprised of only 32 terms.

From the length of A7 and A8, it is difficult to simply observe which are the predominate terms for each equation. Consequently, approximations cannot be considered as reasonable choices in composite material theories until detailed analysis is done.