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PROGRESS REPORT

WAVE HEATING AND DIAGNOSTICS FOR FUSION PLASMAS

15 May 1975 - 14 May 1976

Contract AT(04-3)-767

with the

U.S. Energy Research and Development Administration

Division of Controlled Thermonuclear Research

California Institute of Technology
Pasadena, California

Roy W. Gould, Principal Investigator

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The following is a brief summary of progress during the period May 15, 1975 to May 14, 1976:

1. Research Tokamak Facility

   The toroidal plasma and associated power supplies or research tokamak has been completed. A 10 KW, 30 KHz discharge cleaning oscillator was constructed and is being used for discharge cleaning. Progress of the cleaning is monitored with a Varian residual gas analyzer and procedures recommended by Professor Taylor of UCLA for achieving low Z_{eff} plasmas have been initiated. Currents of about 5 KA have been achieved in normal tokamak operation for limited duration.

2. Poloidal Field Coil Design Technique

   A relatively simple numerical approach has been developed to design poloidal field coils, such as the ohmic heating, vertical, and horizontal windings of a tokamak when these windings are to be placed on a toroidal surface.

3. Externally Driven Linear Electromagnetic Modes of an Anisotropic Hot Plasma

   A computer program has been previously developed for the calculation of the roots of the dispersion relation for externally driven linear electromagnetic waves in a hot plasma. The plasma may consist of an arbitrary number of ion species. This program has now been applied in theoretical studies of several types of plasma waves which may be employed in rf heating of tokamaks: 1) in the case of the fast Alfven wave toroidal eigenmode scheme of rf heating, the fast Alfven wave is found to undergo partial wave conversion to and absorption by
an ion Bernstein mode. This occurs in regions of the tokamak where
the toroidal magnetic field is such that the wave frequency equals
locally a harmonic of the ion cyclotron frequency; 2) the fast Alfvén
wave toroidal eigenmode scheme has also been studied for the case of
plasmas having two ion components, the D-T plasma of a tokamak fusion
reactor in particular. In this case the fast Alfvén wave may undergo
partial mode conversion to and absorption by a hot plasma wave at the
location of the two-ion hybrid resonance; 3) in the case of the slow
Alfvén wave heating scheme in tokamaks wave conversion to an electro-
static, cold plasma type of wave is found in the region of the Alfvén
resonance. Both the slow Alfvén wave and the wave to which it con-
verts are subject to significant Landau damping by electrons.

4. Transient Response from a Small Antenna in an Anisotropic Plasma

The linear response of a cold anisotropic plasma to a point source
under impulse excitation has been shown to be directly related to the
response of the system under sinusoidal excitation. The transient res-
ponse is a function of angle of observation with respect to the dc
magnetic field, and its frequency spectrum displays maxima at the two
frequencies determined by the resonance cone condition and at the
upper hybrid frequency. A laboratory experiment has been performed
which demonstrates this effect.

5. Electrostatic Wave Propagation and Instabilities in a Magnetized
   Plasma

A computer program has been developed in order to compute complex
$\omega$ vs real $k$ roots of the electrostatic homogeneous, hot plasma,
linear dispersion relation. A class of instabilities can be treated and consists of those which result from drifts in the particle distribution functions in the direction parallel to the magnetic field. Specific instabilities which are being studied include the ion acoustic instability driven by a net shift in the Maxwellian velocity distribution of the electrons and instability of lower hybrid waves driven by a bump-in-tail electron distribution function. One objective is to determine which types of waves may yield observable emission from tokamak plasmas. Another type of instability, the collisional drift wave instability, is also being studied for this purpose but not by means of the dispersion relation computer program.

Another problem which is being studied by means of this computer program is the propagation from ion cyclotron harmonic waves in directions which are only slightly away from perpendicular to the dc magnetic field. Under this slightly oblique condition another ion cyclotron harmonic type of wave exists in addition to the relatively well understood ion Bernstein mode. While the ion Bernstein modes are known to play a significant role in the collisionless absorption by plasmas of rf power, it remains to be determined if that other cyclotron harmonic modes are of importance in rf heating.

6. Thompson Scattering Diagnostics

The Q-switched laser has been rehabilitated and rendered operational.

7. MHD Mode Survey

The transmitting and receiving antennas have been prepared.
Low frequency resonance cone structure in a warm anisotropic plasma

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(Received 11 March 1974; final manuscript received 13 March 1975)

Asymptotic analysis of the electrostatic Green's function for an antenna in a warm, anisotropic, nearly collisionless plasma reveals the presence of new resonance cone structures which exist in addition to the cold plasma resonance cone for frequencies below the lower hybrid frequency. These new cones, the ion acoustic resonance cones, occur for frequencies \( 0 \leq \omega \leq \min(\omega_{ci}, \omega_{ci}) \) and \( \max(\omega_{ci}, \omega_{ci}) \leq \omega \leq \omega_{ci} \); they exist only when the electron temperature is nonzero and are important when \( T_e/T_i < 1 \). The interference structure associated with this cone is quite sensitive to the ion temperature and could be the basis of an ion temperature diagnostic technique for plasmas that meet the conditions of the theoretical derivation: \( T_e/T_i < 1 \) and \( \omega_{ci} > \omega_{ci} \).

I. INTRODUCTION

There has been considerable experimental\textsuperscript{1-5} and theoretical\textsuperscript{6-9} interest recently in the resonance cone structure excited in a magnetoplasma by a small antenna. Except for the simplest cold plasma analyses, all of the previous work has assumed that the frequencies of interest are high enough so that the motion of the ions in the plasma can be neglected. Mobile ions must be included, of course, if one wishes to work at frequencies comparable to the ion plasma or ion cyclotron frequency. As will be shown in the following sections, consideration of ion motion leads to the presence of new cone structures. These are important for frequencies below the lower hybrid frequency if the electrons are much hotter than the ions.

The radiation pattern of a small antenna in a warm magnetoplasma will be investigated in the electrostatic approximation by asymptotically expanding the Green's function for the linearized problem. In Sec. II, the Green's function will be obtained, while Secs. III and IV will be devoted to obtaining the asymptotic expansion in various limits.

II. BASIC EQUATIONS

In most laboratory plasmas, especially at frequencies characteristic of the ions, the ratio of plasma dimensions to electromagnetic wavelength is such that the radiation pattern of a small antenna can be adequately described by solving for the quasi-static near-field antenna pattern. This quasi-static approximation is identical to the electrostatic approximation often made in warm plasma wave theory, and the complete problem can thus be consistently treated in the electrostatic limit.

It is to be expected that near the resonance cones the electrostatic approximation will still be valid, even though the electromagnetic wavelength may become so short that the near field approximation is no longer correct. This is because we are dealing with a resonance (wavenumber \( k \to \infty \)) where the solutions using the full set of Maxwell's equations still have an electrostatic character; i.e., \( E \) is parallel to \( k \).

To avoid dealing with boundary effects, assume an infinite, uniform warm plasma. Let it have one ionic type and assume it is in an externally imposed, static, uniform magnetic field. Let the \( z \) axis of a Cartesian coordinate system be oriented along the magnetic field, so that \( \mathbf{B}_0 = B_0 \mathbf{e}_z \). The equations that describe the plasma in the electrostatic limit are the Boltzmann equations for the electron and ion distribution functions,

\[
\frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \frac{\partial f_e}{\partial \mathbf{x}} - \frac{q_e}{m_e} \left( \frac{\partial \phi}{\partial \mathbf{x}} - \mathbf{v} \times \mathbf{B}_0 \right) \cdot \frac{\partial f_e}{\partial \mathbf{v}} = \frac{\partial}{\partial t} \left[ \frac{\partial f_e}{\partial \mathbf{v}} \right]_p ,
\]

and Poisson's equation for the electrostatic potential,

\[
\nabla^2 \phi = \frac{-\rho_E}{\epsilon_0} - \sum_{s=e,i} q_sn_s ,
\]

where the species index \( s \) denotes electrons (e) or ions (i). Here, \( q_s \) and \( m_s \) are the particle charge and mass, respectively, and the particle number density is given by

\[
n_s = \int d\mathbf{v} f_s(\mathbf{v}, t) .
\]

The antenna is the source of the external charge \( \rho_E \).

If we assume that collisions with neutrals are most important, then the collision term can be modeled as

\[
\frac{\partial f_s}{\partial t} = -v_s(f_s - f_s^{(0)}) ,
\]

where \( f_s^{(0)} \) is the equilibrium value of \( f_s \). This model has certain defects (e.g., it does not conserve particles locally), but as long as \( v_s \) is small, it should give reasonably correct results.

To obtain a mathematically tractable problem, one can linearize Eq. (1) about an equilibrium distribution by setting \( f_s = f_s^{(0)} + f_s^{(1)} \) and keeping only first-order terms. Once a linear system has been obtained, arbitrary antenna configurations can be handled, at least in principle, if the Green's function can be found. We are thus led to consider the linearized problem with \( \rho_E = q_0 \delta(x)e^{-i\omega t} \), where \( \delta(x) \) is the three-dimensional Dirac delta function.

The solution to the linearized problem can be effected by using Fourier transforms. Assuming that \( f_s^{(0)} \) is an isotropic Maxwellian and then utilizing the cylindrical symmetry about \( \mathbf{B}_0 \), the linearized solution can be written as

\[
\phi(\rho, z, t) = \frac{q_0 e^{-i\omega t}}{4\pi \epsilon_0} \int_{-\infty}^{\infty} d k_z e^{i k_z z} \int_0^\infty d k_1 \frac{k_1 J_0(k_1 \rho/\rho_s)}{D(k_1, k_z)} ,
\]

where \( J_0 \) is the Bessel function of the first kind, \( k_1 \) is the radial wavenumber, \( \rho_s \) is the electron Larmor radius, and \( D(k_1, k_z) \) is a dispersion function.
where \( \rho = (x^2 + y^2)^{1/2} \) is the cylindrical coordinate in the direction perpendicular to \( \mathbf{b}_n \). \( D(k_z, k_e) \) is the electrostatic dispersion relation and is given by :

\[
D(k_z, k_e) = k_e^2 + \sum_{\omega, s} k_{\omega s}^2 \left( 1 + \frac{i \nu_e}{\omega} \right) \left( 1 + i (\omega + i \nu_e) \right) \int_0^\infty dt \times \exp \left[ i (\omega + i \nu_e) t - \frac{k_e^2 v_e^2}{\omega_{\omega s}^2} \sin^2 \omega_{\omega s} \frac{2}{2} - \frac{k_e^2 v_e^2}{4} \right]
\]

and where

\[
k_{\omega s} = \frac{n q_s}{\epsilon_0 k^2 r_s^2}, \quad v_e = \frac{2 k_e T_e}{m_e}.
\]

Here \( n \) is the unperturbed particle number density and \( T_e \) is the temperature of the sth species; the cyclotron frequencies are given by \( \omega_{\omega s} = q_s |B| / m_s \). Finally, for the collision model to be reasonable, it is necessary that \( \nu_e/\omega < 1 \).

**III. ASYMPTOTIC EXPANSION**

In order to make clear the important features of the function defined in Eq. (3), one must either resort to numerical procedures or make further approximations. If we transform to spherical coordinates \( z = r \cos \theta, \rho = r \sin \theta \) and let \( r \to \infty \), we can develop an asymptotic approximation for the potential that is valid in that limit.

Formal expansions based on methods used by Burrell\(^8\) are possible for arbitrary values of the independent parameters. However, to keep the mathematics from obscuring the physics, restrictions on their values are useful. First, since we are interested in low frequencies, \( \omega_{\omega s} \gg \omega, \omega_{\omega s} \); hence, we may formally take the \( \omega_{\omega s} \to \infty \) limit in the dispersion relation, Eq. (4). Second, as a first cut at the problem, let \( T_e = 0 \). (This restriction will be relaxed in Sec. IV.)

Using these approximations, the dispersion relation becomes

\[
D(k_z, k_e) = k_z^2 + k_e^2 + \sum_{\omega, s} k_{\omega s}^2 \left( 1 + \frac{i \nu_e}{\omega} \right) \left( 1 + i (\omega + i \nu_e) \right) \int_0^\infty dt \times \exp \left[ i (\omega + i \nu_e) t - \frac{k_z^2 v_z^2}{\omega^2} \right]
\]

with

\[
k_z = 1 - \frac{\omega_{\omega s}^2}{\omega (\omega + i \nu_e)}
\]

and

\[
k_e = 1 - \frac{\omega_{\omega s}^2}{\omega (\omega + i \nu_e)}.
\]

In Eq. (5), the plasma frequency has been introduced, \( \omega_{\omega s} = n q_s^2 / \epsilon_0 m_s \). The integral over \( k_z \) in Eq. (3) can now be done explicitly,

\[
\Phi(\rho, z) = \frac{3}{2 \pi^2} \int_0^\infty dk \cos k_z K_0 (\rho P(k))
\]

with

\[
P(k) = \left( \frac{1}{K_1} \left[ k_z^2 - k_z^2 \left( 1 + \frac{i \nu_e}{\omega} \right) (\omega + i \nu_e) \right] \right)^{1/2}
\]

where the square root is defined so that \( \Re P(k) \geq 0 \).

Here, the symmetry of the dispersion relation has been used to reduce the range of integration to the positive real axis. After this is done, the equivalent expression for the electron part of the dispersion relation in terms of the plasma dispersion function, \( Z_{13} \) can be used. In the form stated, this equivalence is valid only for \( \Re k_z \leq 0 \).

One would like to be able to use the method of steepest descents\(^14\) to asymptotically expand the integral in Eq. (6) as \( r \to \infty \), but the integrand is not of the proper form. If \( P(k) \) were never zero and if we were to restrict \( \theta \) so that \( \sin \theta \neq 0 \), then for large \( r \) we could write

\[
K_0 (\rho P(k)) \sim \left( \frac{\pi}{2 \rho P(k)} \right)^{1/2} \exp \left[ -\rho P(k) \right]
\]

and the integrand would have the proper form. Unfortunately, the definition shows that \( P(0) = 0 \). Further, values of \( \theta \) such that \( \sin \theta = 0 \) are of interest.

These difficulties can be overcome by integrating once by parts in Eq. (6) and then adding and subtracting the cold plasma resonance cone formula, which is just the \( T_e = 0 \) limit of Eq. (6). The result is

\[
\Phi(\rho, z) = \left( \frac{3}{2 \pi^2} \int_0^\infty dk \cos k_z K_0 (\rho P(k)) \right)
\]

The potential \( \Phi(\rho, z) \) has been replaced here with \( \Phi(\rho, z, t) / q \), and \( K_0 = k_n - \omega_{\omega s} / \omega (\omega + i \nu_e) \) has been introduced.

The remaining integrand vanishes at \( k = 0 \); thus as \( r \to \infty \), the Bessel functions may be replaced by their asymptotic forms.

\[
\Phi(\rho, z) = \left( \frac{3}{2 \pi^2} \int_0^\infty dk \cos k_z P(k) \exp \left[ -\rho P(k) \right] \right)
\]

\[
\times \sin k_z \left( P(k) - P_{1/2}(k) \right) \exp \left[ -\rho P(k) \right]
\]

\[
\times \exp \left[ -\rho K_0 (\rho P(k)^{1/2}) \right]
\]

\[
\left[ \begin{array}{c}
\text{Eq. (8) will be valid when } \sin \theta = 0; \text{ it will be shown in Appendix B that they are valid there.}
\end{array}
\right]
\]

If we now split the integral into two, consideration of their forms shows that each is separately a convergent integral, one of which can be done exactly,

\[
\Phi(\rho, z) = \left( \frac{3}{2 \pi^2} \int_0^\infty dk \cos k_z K_0 (\rho P(k)) \right)
\]

\[
\times \sin k_z \left( P(k) - P_{1/2}(k) \right) \exp \left[ -\rho P(k) \right]
\]

Notice that if the first term has singularities, they are exactly canceled by the two terms in the brackets, and thus the procedure used to obtain Eq. (9) has introduced no new singularities into the problem.

For later reference, the point where the first term
in Eq. (9) has its maximum absolute value is defined to be the cold plasma resonance cone angle, \( \theta_c \). In the collisionless limit, it is given by \( \tan^2 \theta_c = -K_2/K_1, \).

The final term in Eq. (9) has most of the interesting physics in it. However, it is the dominant term only near the resonance cones. The addition and subtraction of the cold plasma resonance cone formula is quite essential to get the correct potential when \( \dot{\theta} \) is not near a resonance cone angle.

Defining the last term in Eq. (9) to be a new quantity, \( H(\rho, z) \), it may be rewritten in the form

\[
H(\rho, z) = \frac{P(k, \theta)}{iz} \int_0^\infty \frac{dk}{2\pi} P(k) \frac{P(k, \theta)}{P'(k)} \left\{ \exp[ih\rho - \rho P(k)] \right\}.
\]

We now have an integrand that is in the standard stationary phase form, as long as \( P(k, \theta) \) is assumed to be slowly varying. This latter condition is violated near \( k = 0 \) and \( \rho = 0 \). Using the asymptotic expansion of the plasma dispersion function, \( P(k) \approx \frac{1}{2\pi} \frac{3\omega_p^2}{(k_0^2 + \omega^2 \omega_p^2)} \),

where \( \omega_p \) is the plasma frequency, \( \omega \) is the electron plasma frequency, and the labeling of stationary points in Eq. (11) has been chosen so that the \( j = 1 \) solution goes with the above equation. Thus,

\[
h_j^2 \approx -i[\dot{\rho} + i(k_c/k_0)]/3 \alpha.
\]

If \( |k_c| \) is small enough so that the asymptotic expansion of the plasma dispersion function is valid [i.e., if \( \omega_p/\omega > 1 \)], then this is a proper solution. In the parameter ranges where the cold plasma resonance cone exists, and if \( \omega_c/\omega < 1 \), then \( \arg(k_c/k_0) \) is near \( \pi/2 \) and \( |k_c| \) can be small enough. (If \( \omega_c = 0 \), then \( k_c = 0 \) when \( \dot{\rho} = \dot{\theta}_c \).

To cope with the origin, change the variable of integration in Eq. (10) to \( \eta = k \). Consequently,

\[
H(\rho, z) = \frac{1}{iz} \int_0^\infty \frac{d\eta}{\pi} \eta P'(\eta) P(k) \left\{ \exp[i\eta^2z - \rho P(\eta)] \right\} - \exp[-i\eta^2z - \rho P(\eta)]
\]

The stationary point conditions are now

\[
\eta(\pm i \dot{\rho} - P'(\eta)) = 0.
\]

The angle between the group velocity and the magnetic field, \( \psi \), is given by

\[
\tan \psi = \frac{dP/k_0}{dP/k_1}.
\]

Since the dispersion relation is symmetric in \( k_0 \), but the definition of \( P(k) \) is not, the above equation is equivalent to

\[
\tan \psi = \pm i/P'(k_0).
\]

This shows that the angle \( \theta \) at the stationary point is identical to \( \psi \).

In a magnetized plasma, the phase and group velocities need not be parallel; the angle between them can have any value. Since \( \theta \) determines the direction of group velocity, it is possible to propagate a signal in a direction in which, if it were the phase velocity direction, the dispersion relation would permit only strongly damped waves. The experimental results obtained by Gonfalone and Beghin illustrate this effect quite well.

In obtaining Eq. (11), it was assumed that all the stationary points are well separated and that \( P'(k)P^{-1/2}(k) \) is slowly varying. This latter condition is violated near \( k = 0 \) and \( \rho = 0 \). Using the asymptotic expansion of the plasma dispersion function, \( P(k) \approx \frac{1}{2\pi} \frac{3\omega_p^2}{(k_0^2 + \omega^2 \omega_p^2)} \),

where \( \omega_p \) is the plasma frequency, \( \omega \) is the electron plasma frequency, and the labeling of stationary points in Eq. (11) has been chosen so that the \( j = 1 \) solution goes with the above equation. Thus,

\[
h_j^2 \approx -i[\dot{\rho} + i(k_c/k_0)]/3 \alpha.
\]

If \( |k_c| \) is small enough so that the asymptotic expansion of the plasma dispersion function is valid [i.e., if \( \omega_p/\omega > 1 \)], then this is a proper solution. In the parameter ranges where the cold plasma resonance cone exists, and if \( \omega_c/\omega < 1 \), then \( \arg(k_c/k_0) \) is near \( \pi/2 \) and \( |k_c| \) can be small enough. (If \( \omega_c = 0 \), then \( k_c = 0 \) when \( \dot{\rho} = \dot{\theta}_c \).

To cope with the origin, change the variable of integration in Eq. (10) to \( \eta = k \). Consequently,

\[
H(\rho, z) = \frac{1}{iz} \int_0^\infty \frac{d\eta}{\pi} \eta P'(\eta) P(k) \left\{ \exp[i\eta^2z - \rho P(\eta)] \right\} - \exp[-i\eta^2z - \rho P(\eta)]
\]

The stationary point conditions are now

\[
\eta(\pm i \dot{\rho} - P'(\eta)) = 0.
\]

The angle between the group velocity and the magnetic field, \( \psi \), is given by

\[
\tan \psi = \frac{dP/k_0}{dP/k_1}.
\]

Since the dispersion relation is symmetric in \( k_0 \), but the definition of \( P(k) \) is not, the above equation is equivalent to

\[
\tan \psi = \pm i/P'(k_0).
\]

This shows that the angle \( \theta \) at the stationary point is identical to \( \psi \).

In a magnetized plasma, the phase and group velocities need not be parallel; the angle between them can have any value. Since \( \theta \) determines the direction of group velocity, it is possible to propagate a signal in a direction in which, if it were the phase velocity direction, the dispersion relation would permit only strongly damped waves. The experimental results obtained by Gonfalone and Beghin illustrate this effect quite well.
A procedure given by Chester et al. allows one to develop uniform asymptotic expansions in cases like this. Their procedure is applied in the present case in Appendix A, with the result that

$$H(p, z) = -\frac{\pi^{2/3} \pi}{2^{2/3} \cos \theta} \left( \frac{K'_{s}}{K_{s}} \right)^{1/4} \left\{ \left( \frac{\ell_{1}(\theta)}{(K_{s}/K')^{1/2} + i \cot \theta} \right)^{1/4} \right\}$$

$$\times A(X) \left\{ B(X) + i A(X) \right\} \left( \frac{\ell_{1}(\theta)}{(K_{s}/K')^{1/2} + i \cot \theta} \right)^{1/2}$$

$$\times A(X) \left\{ B(X) + i A(X) \right\} + \frac{1}{\tau} \sum_{n=1}^{N} \frac{\exp \left[ i \kappa_{n} \theta \right]}{\cos \theta + 2 \ell_{4} \theta^{2}}$$

(17)

where $A$ and $B$ are the Airy functions, $\ell_{1}$ are the solutions of the stationary phase condition, Eq. (12), that are small when $\theta = \theta_{c}$ (for +) or when $\theta = \pi - \theta_{c}$ (for -). Near each resonance cone

$$\ell_{1}(\theta) \approx -\left( \frac{\sin^{2} \theta}{6G} \right)^{1/3} \left[ \pm \cot \theta + \left( \frac{K'_{s}}{K_{s}} \right)^{1/2} \right] .$$

(19)

The above is valid for $\theta$ near $\theta_{c}$ (for +) and near $\theta = \pi - \theta_{c}$ (for -).

The presence of warm electrons has modified the cold plasma resonance cone structure, and this modified behavior is given by the Airy function terms in Eq. (17). If we had not included motion of the ions, the terms in the sum in Eq. (17) would be strongly damped, and only the Airy function terms would be important. However, inclusion of ion motion permits the propagation of nearly undamped ion acoustic waves, and some of the terms in the sum must also be considered.

For ion acoustic waves $|\omega + iv_{s}|/kv_{s} | \ll 1$, thus, $Z[(\omega + iv_{s})/kv_{s}] \approx -2$, and Eq. (12) has the approximate solution

$$\kappa_{s}^{2} = \kappa_{s}^{2}(1 + iv_{s}/\omega)\kappa_{s}^{2}(1 + (\kappa_{s}/K_{s}) \tan \theta)^{-1} ,$$

(20)

where the indexing in the sum has been chosen so that this is the $j = 2$ term. Since $\omega \ll \omega_{pe}$, one may easily verify that, except near $\theta = \pi/2$, the above solution does satisfy $|\omega + iv_{s}|/kv_{s} | \ll 1$.

The $j = 2$ term in the sum now becomes

$$\exp \left\{ -\kappa_{s}^{2} t \left[ (\omega + iv_{s})/\omega \left( \sin^{2} \theta/K_{s} + \cos \theta/\kappa_{s} \right)^{1/2} \right] \right\}$$

(21)

In the $\theta = 0$ limit, this term clearly describes the propagation along the magnetic field of an ion acoustic wave excited by a point source.

The important fact about this term is that, for $\nu_{s}/\omega \ll 1$, there are angles where the denominator can become quite small; for $\nu_{s} = 0$, the denominator vanishes for $\cot \theta = \cot \theta_{2} = \kappa_{s}'/K_{s}$. There are two frequency ranges where the denominator can be small: (1) $0 \leq \omega \leq \omega_{s}$ and (2) $\omega_{h} \leq \omega \leq \omega_{hl}$, where $\omega_{s} = \min(\omega_{pe}, \omega_{ce})$, $\omega_{h} = \max(\omega_{pe}, \omega_{ce})$, and $\omega_{hl}$ is the lower hybrid.

Consequently, there is a new cone structure, with apex at the point source, along which the magnitude of the potential can become quite large when $\nu_{s}/\omega \ll 1$. Qualitatively, this cone has the same features as the cold plasma resonance cone, but it can exist only when the electron temperature is nonzero. It owes its existence to the presence of almost undamped waves in a plasma with $T_{e} \neq 0$.

The resonance referred to in the cold plasma resonance cone case is a wave resonance $(k-\omega)$ found when plasma waves are described by the full set of Maxwell's equations and the cold fluid equations for electrons and ions. The wave resonance at this angle disappears when $T_{e} \neq 0$ regardless of whether the electron motion is modeled with the fluid equations or the Boltzmann equation. However, the resonance cone in the field pattern of a small antenna still exists near the cold plasma resonance cone angle when $T_{e} \neq 0$. This cone is given by the Airy function terms in Eq. (17). Since these terms are $O(\nu_{s}/\omega)$ when $\tan \theta \approx \tan \theta_{e}$, these terms are by far the largest in Eqs. (17) and (9). All other contributions to the potential are $O(\nu_{s}/\omega)$ at such values of $\theta$.

The new resonance cone, which could be called the ion acoustic resonance cone, is related to a resonance that occurs when $T_{e} \neq 0$. If the waves are described using Maxwell's equations and the linearized two-fluid equations, a dispersion relation results which exhibits a resonance if $T_{e} = 0$. An approximate dispersion relation given in Eq. (15)-(50) of Stix shows this resonance as well.

The essential features of this resonance are contained in the dispersion function in Eq. (11). In this, $\kappa_{s} = k \cos \theta$ and $\kappa_{i} = k \sin \theta$. In the collisionless case, there is a solution to $D(\kappa_{s}, \kappa_{i}) = 0$ with $k = \infty$ when $\tan \theta = \tan \theta_{A} \equiv \kappa_{i}/\kappa_{e}$. Just as in the cold plasma resonance cone case, the angle $\theta_{e}$ is the complement of $\theta_{A}$, the angle at which $\Phi(\rho, z)$ is singular. Further, since $\theta_{e}$ is also the group velocity direction, the phase and group velocities are at right angles to one another on the ion acoustic resonance cone.

The resonance will, of course, be affected by collisions and, as can be seen from the results of the next section, by nonzero ion temperature. Both of these will act to limit $k$ and $|\Phi(\rho, z)|$ to finite values. In analogy with the cold plasma resonance cone case, we expect that taking $T_{e} \neq 0$ will not automatically destroy the resonance cone. This expectation will be checked for one case in the next section.

IV. EXPANSION WITH $T_{e} \neq 0$

The obvious question is whether a nonzero ion temperature will seriously affect the ion acoustic resonance cone result that was obtained in the last section. To investigate this, it is easiest to take a different limit in the dispersion relation, Eq. (4). Let $T_{e}$ be arbitrary, but let $\omega_{ci} \gg \omega_{s}, \omega_{pe}$. This frequency range is chosen for mathematical convenience, although it could be of use in certain laboratory applications. The methods developed in Ref. 5 can be used to attack other frequency ranges.

The potential is again given by Eq. (6), but with $K_{s} = 1$ and $\omega_{ci} \gg \omega_{pe}, \omega_{se}$.
\[ P(i) = \left[ k^2 - \frac{h_i^2}{2} (1 + i \nu_i / \omega) Z(\omega + i \nu_i / k_i \omega) \right] \]
\[ - \frac{h_i^2}{2} (1 + i \nu_i / \omega) Z(\omega + i \nu_i / h_i \omega)^{1/2}. \]

With these same modifications, Eqs. (9) and (18) are still valid, and the Airy function terms in Eq. (17) are still correct. Equation (19) requires minor modification; the new definition of \( \alpha \) is
\[ \alpha = \frac{1}{2} k_i^{1/2} \left( \frac{\nu_i^2}{h_i^2 \omega (\omega + i \nu_i)} + \frac{\nu_i^2}{h_i^2 \omega (\omega + i \nu_i)} \right). \]

More extensive modifications must be made in the infinite sum in Eq. (17) and in Eqs. (20) and (21).

If we still assume that \(|\omega + i \nu_i|/k_i \omega_i | < 1\), but that \(|\omega + i \nu_i|/k_i \omega_i | \gg 1\), then the ion acoustic waves remain almost undamped. To be able to make these two assumptions, it is necessary that \( T_i/T_e \ll 1\). Expanding the analog of Eq. (12) under these assumptions and keeping terms to first order in \( T_i \) leads to
\[ k_i^2 (4 \nu_i \tan^2 \theta + 1) - k_i^2 \nu_i (\tan^2 \theta + 1) - h_i^2 (1 + i \nu_i / \omega) = 0, \]
where \( \gamma = 3 \omega_i^4 / h_i^2 \omega (\omega + i \nu_i)^3 \). There are two solutions to this equation:
\[ k_i^2 = -2 h_i^2 (1 + i \nu_i / \omega) \nu_i^2 (\tan^2 \theta + 1 + \Xi(\theta))^2 \]
and
\[ k_i^2 = -2 h_i^2 (1 + i \nu_i / \omega) \nu_i^2 (\tan^2 \theta + 1 - \Xi(\theta))^2, \]
where \( \Xi(\theta) = \left[ (k_i \tan^2 \theta + 1) + \beta^2 (4 k_i \tan^2 \theta + 1)^{1/2} \right] \) and \( \beta^2 = 4 k_i^2 (1 + i \nu_i / \omega) / \nu_i^2 \).

These solutions are not valid for all \( \theta \), since there are angles where one or both violate the original assumptions that \(|\omega + i \nu_i|/k_i \omega_i | < 1\) and \(|\omega + i \nu_i|/k_i \omega_i | \gg 1\). The first solution, \( k_2 \), is valid for all angles that are not too near \( \theta = \pi/2 \), while the second, \( k_3 \), is only valid for a range of angles about \( \theta = \theta_A \). (The precise boundaries of the regions of validity depend, of course, on one's definition of how big a quantity must be to be very much greater than unity.) Since \( T_i/T_e \ll 1\), then \( \beta^2 \ll 1\), and thus \( \Xi(\theta) \) can be quite small near \( \theta = \theta_A \). In fact, there are angles near \( \theta_A \), where \( |k_3 \tan^2 \theta + 1| \gg |\Xi(\theta)| \). Consequently, \( k_3 \) and \( k_2 \) can be almost equal; for \( \nu_i = 0 \), \( \Xi(\theta) \) can vanish at certain angles and then \( k_3 = k_2 \). This will require modification of the asymptotic expansion when \( \theta \) is near \( \theta_A \).

If \( \theta \) is not near \( \theta_A \), all of Eq. (17) is correct as it stands. Only the \( j \geq 2 \) term will contribute significantly to the infinite sum, and in that term \( k_3 \) will be given by the expression in Eq. (22). (Even though \( k_2 \) is not a valid solution to the stationary point condition for \( \theta \) near \( \pi/2 \), the terms in the sum are so small for these angles that the error does not matter.) For \( \tan^2 \theta = 0 \), the \( j = 2 \) term, just as in the \( T_e = 0 \) case, gives a nearly undamped ion acoustic wave propagating along the magnetic field.

For \( \theta \) near \( \theta_A \), the asymptotic expansion with two nearly coincident stationary points can be handled by the same technique used in Sec. III. The details will be carried out in Appendix A; the resulting contribution to \( H(\rho, z) \) from these two stationary points is
\[ \sqrt{\pi} \gamma^2 \left[ \left( \cos^2 \theta + \Xi(\theta) \sin^2 \theta \right)^{1/2} \right. \]
\[ + \left. \left( \cos^2 \theta + \Xi(\theta) \sin^2 \theta \right)^{1/2} \right] e^{-\alpha} \sin \phi \left( \gamma^2 / \sqrt{2} \right), \]

where
\[ a = \frac{1}{2} \left[ i \cos \theta (k_i + k_2) - \sin \theta (P(k_3) + P(k_2)) \right], \]
\[ \zeta = (\gamma^2 / \sqrt{2}) \left[ i \cos \theta (k_i - k_2) - \sin \theta (P(k_3) - P(k_2)) \right]^{1/3}. \]

As long as \( T_i/T_e \ll 1\), the terms given in Eq. (23) have an appreciable amplitude. When \( \zeta \) is small, \( |e^{\alpha}| \approx 1\); hence, the magnitude of the coefficient of \( \sin \phi \) is \( \sqrt{\pi / 12} P(k_3)\left[ P^{-1}(P(k_3)) \right]^2 \sin^2 \theta / \theta^{1/6} \). Neglecting factors of \( O(1) \), this is proportional to \( h_i^2 (T_i/T_e)^{5/6} \). Thus, the addition of a nonzero ion temperature to the theory keeps the magnitude of the electrostatic potential finite everywhere even when \( \nu_i = 0 \). When \( \zeta \) is small, all other contributions to the electrostatic potential are \( O(1/\gamma) \). Since \( h_i^2 \gamma \gg 1 \), there should be an appreciable peak in the magnitude of the potential here. Accordingly, the ion acoustic resonance cone is important even for \( T_i \neq 0 \).

For \( \theta \) near the ion acoustic resonance cone,
\[ \zeta \approx \left[ - (9 \theta^2 / 32 \nu_i^4 h_i^2 \gamma^2 (1 + i \nu_i / \omega))^{1/3} \right] \frac{\Xi(\theta)}{(k_i \tan^2 \theta + 1)^2}, \]
In the collisionless limit, since \( \beta^2 \ll 1 \), this can be re-expressed as
\[ \zeta \approx \left[ - (k_i \nu_i^2 \omega^2 / \nu_i \gamma^0) \right]^{1/3} \frac{\Xi(\theta)}{(k_i \tan^2 \theta + 1)^2}, \]
where \( \tan^2 \theta_A = (1 - \sqrt{3} \beta) / k_i \). The Airy function behavior revealed in Eq. (23) means that there is a spatial interference pattern associated with the ion acoustic resonance cone, analogous to the one associated with the temperature modified cold plasma resonance cone [see Eq. (17) and Ref. 3]. Except for a weak dependence on \( T_e \) through \( \theta_A \), the expression in Eq. (24) for \( \zeta \) is independent of the electron temperature. However, it depends strongly on the ion temperature. Consequently, experimental measurements of the interference structure could be used to measure the ion temperature in a plasma that meets the conditions of this derivation. When the interference structure can be seen, this will be one hot plasma wave effect where the effects of warm ions will not be masked by those of the much hotter electrons.

Finally, as further proof that the ion acoustic resonance cones do exist, Fig. 1 shows a plot of the magnitude and phase of \( \phi(\rho, z) \). This plot was generated by numerical integration of the defining equation. It is obvious from the plot that the main peak in \( \phi(\rho, z) \) occurs at an angle far removed from \( \theta_A \), but close to and inside \( \theta_A \). This location is what is predicted by Eq. (23).

V. CONCLUSION

It has been shown that if the ion temperature and collision frequencies are small enough so that almost undamped electrostatic waves can propagate in the plas-
expansion of integrals of the form
\[ G(\nu, \theta) = \int_0^M d\eta \eta^{1/4} (\eta^2)^{1/2} \exp[i\nu^2 \cos \theta - r \sin \theta (\eta^2)] , \]
where \( M = M(\theta) \) is chosen so that only the first two stationary points of the exponent are included in the range of integration.

The change of variable in this case is suggested by the small argument expansion of the exponent
\[ i\nu^2 \cos \theta - r \sin \theta (\eta^2) = \frac{1}{2} \eta^2 - \xi(\theta) S(\mu^2) \eta^2 , \]
where
\[ S(\xi) = \begin{cases} +1, & \text{Re} \theta > 0 \\ -1, & \text{Re} \theta < 0 \end{cases} . \]
With the inclusion of \( S(\mu^2) \), both the right- and left-hand sides of this equation have two stationary points for \( |\eta| \leq M \).

For the change of variable to be one to one, we require \( d\eta/du \) to be finite and nonzero in the range of interest. Since
\[ 2\eta[i \cos \theta - \sin \theta (\eta^2)] \frac{d\eta}{du} = u[u^4 - 2\xi(\theta) S(\mu^2)] , \]
this means that the points \( u = 0 \) and \( \mu^2 = (2\xi)^{1/2} \) must correspond to \( \eta = 0 \) and \( \eta^2 = \eta_0^2(\theta) \). [Recall that \( \eta_0^2 \) is the smallest solution to Eq. (12).] Consequently,
\[ \xi = -\frac{1}{2} \left[ 3 \sin \theta \left[ \cot \theta \eta_0 - iP(\eta_0^2) \right] \right]^{1/3} . \]
We can now rewrite Eq. (A1) as
\[ G(\nu, \theta) = \int_0^{\eta_0^2(\theta)} \eta \frac{d\eta}{du} P(\eta^2)^{1/2} \exp[r(\mu^2 - \mu)] . \]
Here, we have replaced the finite upper limit of the integral in Eq. (A1) by infinity; this produces a negligible error in the asymptotic limit. The factor \( \exp(i\pi/6) \) insures that the path of steepest descent is taken as \( |u| = \infty \).

If \( \theta \) is near \( \theta_0 \), then the two stationary points are close together. One of them is, of course, \( u = 0 \); hence, in the asymptotic limit the factor in front of the exponential may be evaluated there and taken out of the integral. Accordingly,
\[ G(\nu, \theta) = \left( \frac{K_0}{K_1} \right)^{1/4} \left( \frac{1}{i \cos \theta - \xi(\theta)} \right)^{1/2} \exp\left[ i\pi \eta_0^2(\theta)^{1/2} \sin \theta \right] \times r^{-1/6} \exp(i\pi/6) F(\pi^2/3\xi) , \]
where
\[ F(x) = \int_0^\infty d\lambda \exp\left[ -\frac{1}{2}\lambda^2 - x \exp(\pi/3)\lambda^3 \right] . \]
If \( \theta \) is not near \( \theta_0 \), one may quite properly wonder whether the above procedure is still valid. In this case, \( G(\nu, \theta) \) is dominated by the contribution due to the stationary point at \( u = 0 \), and this is accurately given by the above procedure even if \( \theta \) is not near \( \theta_0 \).

It is easy to show that
\[ F''(x) - 2xF'(x) - F(x) = 0 . \]
Under the change of variable \( x = 2^{1/3}X \) and \( F(X) = f(x) \), we have

\[
   f'''(X) - 4xf'(X) - 2f(X) = 0
\]

This is the differential equation satisfied by the product of Airy functions\(^9\); hence, its three linearly independent solutions are \( Ai(X) \), \( Ai(X)Bi(X) \), and \( Bi^2(X) \). By considering the values of \( F(0) \), \( F'(0) \), \( F''(0) \), one may show after a bit of algebra that

\[
   F(x) = 2^{1/6}x^{3/2} \exp(-ix/6)Ai(X)[Bi(X) + iAi(X)].
\]

This derivation is equivalent to one given by Kuehl,\(^9\) but the change of variable and introduction of the new parameter \( \xi(\theta) \) allows the inclusion of Landau damping in a very natural way. When Landau damping is important, \( \xi \) simply becomes complex.

Now turning to the stationary points given by Eq. (22), we need to consider integrals of the form

\[
   G(\nu, \theta) = \int_{k_1}^{k_2} dk P(k)P^{1/2}(k) \exp[ihk \cos \theta - r \sin \theta P(k)],
\]

where \( N_s \rightarrow N_s(\theta) \) are chosen to include only the two stationary points from Eq. (22) in the range of integration.

Integrals of precisely this type are considered by Chester et al.\(^10\). They show that the change of variable

\[
   ik \cos \theta - \sin \theta P(k) = \sqrt{u} - \xi(\theta)a + a(\theta)
\]

allows one to write, as \( \gamma \rightarrow \infty \),

\[
   G(\nu, \theta) = \frac{1}{2} \left[ P'(k_3)P^{1/2}(k_3) \left. \frac{dk}{du} \right|_{a=a_3} + P'(k_2)P^{1/2}(k_2) \left. \frac{dk}{du} \right|_{a=a_2} \right] \int_{-\epsilon}^{\epsilon} \frac{e^{i\nu \sqrt{u}/3}}{u} du \exp\left[ (\frac{3}{4}\nu^2 - \xi u + a) \right],
\]

where

\[
   a = \frac{1}{2} \left[ i \cos \theta(k_3 - k_2) - \sin \theta[P(3k_3) + P(k_2)] \right],
\]

\[
   \xi = \left( \frac{3}{4} \right)^{2/3} \left[ i \cos \theta(k_3 - k_2) - \sin \theta[P(3k_3) - P(k_2)] \right]^{2/3}.
\]

Since

\[
   Ai(x) = \frac{1}{2\pi i} \int_{-\epsilon+i\gamma}^{\epsilon+i\gamma} dv \exp(\frac{3}{4}v^2 - xv)
\]

the development of the asymptotic expansion in terms of known functions is easily finished.

This expansion is valid only while the stationary points in Eq. (A2) are reasonably close together.

**APPENDIX B**

At angles where \( \sin \theta = 0 \), the methods used to asymptotically expand Eq. (6) do not appear to be valid; however, the result obtained in the main text is actually correct there. This most easily seen by considering the form of \( \Phi(\rho, z) \) obtained by integrating Eq. (6) once by parts,

\[
   \Phi(\rho, z) = \frac{2p}{2\pi} \int_0^{\infty} dk \sin k \left. P' \right| \tau[P(\tau)].
\]

Taking the limit \( \sin k - 0 \) in Eq. (B1) yields

\[
   \Phi(0, z) = \frac{2}{\pi} \int_0^{\infty} dk \sin k \left. \frac{P'(k)}{P(k)} \right|.
\]

As is well known,\(^{18}\)

\[
   \lim_{\gamma \rightarrow \infty} \frac{\sin k}{k} = \pi \delta(k).
\]

Since \( hP'(k)/P(k) = 1 \) when \( k = 0 \), we see that as \( \gamma \rightarrow \infty \)

\[
   \Phi(0, z) = \frac{1}{\gamma} + O(e^{-\gamma}).
\]

If one combines Eqs. (9) and (11), this is precisely what the asymptotic expansion given in the text yields. In this context recall that all solutions to \( P(k) = 0 \) with \( k \neq 0 \) have \( k \) complex, thus insuring that there are no poles on the real axis in the integral in Eq. (B2). In addition, the sign of \( \text{Im}(k) \) will be such that all terms in the sum in Eq. (11) are of higher order than \( 1/\gamma \) due to Landau damping.

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Resonance cones in a warm plasma for finite magnetic fields

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(Received 19 March 1975)

The resonance cone structure excited by a localized source is studied theoretically and experimentally for a warm plasma with a finite, uniform magnetic field. The electrostatic Green's function is evaluated by asymptotically expanding it in the limit where the observation point is far from the source. Results are obtained that are valid for all angles and for all frequencies below the upper hybrid. The theory predicts the functional dependence of the angular location of the main resonance cone peak and the angular spacing between interference peaks on the physical parameters. Experimental parameter studies were made to check the predicted functional dependence; it was found to produce a good fit to the data. Both density and temperature can be found from these measurements; thus, resonance cone measurements are a useful diagnostic technique in any magnetoplasma in which antennas can be inserted.

I. INTRODUCTION

The discovery of the resonance cones excited by a small rf antenna in a magnetized plasma injected experimental measurements into what had been an exclusively theoretical field. The original theoretical derivation had been done for a cold plasma, but the observation of an interference structure near the cone demanded that warm plasma effects be included. This entails consideration of all the warm plasma waves which propagate in a magnetized plasma, including those that propagate obliquely to the magnetic field. The warm plasma theory was first done for the uni-axial plasma, but some of the predictions held even for plasmas in which the uni-axial condition was violated.

One purpose of the present work is to set forth a theoretical development of the potential excited by a source in a warm magnetoplasma when the imposed magnetic field is finite. The analysis derives from a suggestion by Kuehl that an approximate form for the potential could be obtained through an asymptotic expansion of the electrostatic Green's function for the problem. Kuehl has also developed such expansions himself.

The analysis given here is somewhat more formal than his, and is capable of more rapid generalization to parameter ranges where the resonance cones do not exist. (Thus, comparison with work such as Gonfa-loke's should be possible.) In addition, the results are in terms of known, tabulated functions, which eases evaluation of several constants needed for comparison with experiment. However, the dependence of cone location and structure on the plasma parameters predicted by each development is identical, even though several numerical constants differ slightly.

The second purpose of the work is to present new experimental data which illustrate good agreement between theoretical expectations and experimental reality. This confirmation of the theory opens the way for the use of resonance cone measurements as a diagnostic technique, which has been suggested by several authors.

The formal theoretical analysis is developed in Sec. II, while the theoretical predictions are detailed in Sec. III. Section IV is devoted to a description of the experimental apparatus, while the experimental results are presented in Sec. V.

II. FORMAL ASYMPTOTIC EXPANSION

A. Basic equations

In most laboratory plasmas, the ratio of plasma dimensions to electromagnetic wavelength is such that the radiation pattern of a small antenna can be adequately described by solving for the quasi-static near-field antenna pattern. This quasi-static approximation is identical, of course, to the electrostatic approximation often made in warm plasma wave theory, and the complete problem can thus be consistently treated in the electrostatic limit.

It is to be expected that near the resonance cones the electrostatic approximation will still be valid, even though the electromagnetic wavelength may become so short that the near-field approximation is no longer correct. This is because we are dealing with a resonance (wavenumber k = \infty) where the solutions using the full set of Maxwell's equations still have an electrostatic character, i.e., E is parallel to k.

To avoid dealing with boundary effects, assume an infinite, uniform warm plasma with an applied static, uniform magnetic field. Let the z axis of a Cartesian coordinate system be oriented along this field so that B_0 = B_0 \hat{z}. If we neglect ion motion, the equations that describe the plasma in the electrostatic limit are the Boltzmann–Vlasov equation for the electron distribution function and Poisson's equation for the electrostatic potential. The latter must contain a term \rho_e due to the charge on the rf antenna.

To obtain a mathematically tractable problem the Boltzmann equation must be linearized, thus restricting the analysis to small disturbances. Once a linear system has been obtained, arbitrary antenna configurations can be handled, at least in principle, if the Green's function can be found. We are thus led to consider the linearized problem with \rho_e = q \delta(x)e^{-i\omega t}, where \delta(x) is the three-dimensional Dirac delta function.

The solution can be easily effected using Fourier transforms. Assuming that the zero-order elec-
tron distribution function is an isotropic Maxwellian and utilizing the cylindrical symmetry about $B_0$, the Green's function is

$$
\phi(\rho, z, t) = \frac{qe^{-i\omega t}}{4\pi e_0} \int_0^\infty dk_n e^{i\omega t} \int_0^\infty dk_s \frac{k_s D(k_s, k_n)}{D(k_s, k_n)} ,
$$

(1)

where $\rho = (x^2 + y^2)^{1/2}$ is the cylindrical coordinate in the direction perpendicular to $B_0$, $D(k_s, k_n)$ is the electrostatic dispersion function\(^{12,14}\) given by

$$
D(k_s, k_n) = k_s^2 + k_n^2 + k_D^2(1 + iv/\omega),
$$

(2)

where $k_D^2 = ne^2/\epsilon_0 kT_e$ and $v_D^2 = 2kT_e/m$. Here, $T_e$ is the electron temperature, $m$ is the electron mass, $n$ is the unperturbed number density of electrons, and $\omega_{ce} = eB/m$ is the electron cyclotron frequency.

In Eq. (2) a phenomenological collision frequency $\nu \ll \omega$ has been introduced via a simple relaxation model. This will be set to zero at the end of the analysis, but its presence serves to prevent mathematical ambiguities at several points in the derivation.

The integrals in Eq. (1) can be done analytically only for the case $T_e = 0$. The result is formally identical to that given by Fisher and Gould:\(^3\)

$$
\phi(\rho, z, t) = \frac{qe^{-i\omega t}}{4\pi e_0(K_s^2 + K_n^2)^{1/2}} .
$$

(3)

However, the cold plasma dielectric tensor components $K_s = 1 - \omega^2/\omega_{ce}^2(1 + iv/\omega)/(\omega^2 + \omega_{ce}^2)$ and $K_n = 1 - \omega^2/\omega_{ce}^2(\omega^2 + \omega_{ce}^2)$ have been modified to include collisions.

The magnitude of the potential given in Eq. (3) has a pronounced peak at points where $\cot^2 \theta = \rho^2/z^2$ satisfies

$$
\cot \theta = \cot \theta_{c0} = -\frac{\text{Re} [K_s(K_n^* - K_n)]}{\text{Re} [K_s(K_n^* - K_n^*)]} .
$$

(4)

These points lie on a cone with apex at the source; consequently, $\theta_{c0}$ is referred to as the cold plasma resonance cone angle. For the cold plasma the maximum in $|\phi(\rho, z, t)|$ is $O(\omega/\nu)$, and thus is infinite in the collisionless case.

B. Large $r$ expansion

To evaluate Eq. (1) when $T_e \neq 0$, we must either return to numerical procedures, or make further approximations. The basic approximation is that the point of observation will be far from the source. More specifically, setting $\rho = r \sin \theta$ and $z = r \cos \theta$, we will assume $r \gg \lambda_{De}, \tau_{De}$, where $\lambda_{De} = k_D^{-1}$ is the Debye length and $\tau_{De} = e\Phi_0/\omega_{ce}$ is the Larmor radius. We will asymptotically expand the integral in Eq. (1) under this assumption.

A number of authors have worked on the asymptotic expansion of multidimensional Fourier integrals.\(^{15-17}\) Unfortunately, their results are only of marginal use here, since they are not uniformly valid for angles near the cold plasma resonance cone angle. One is thus forced to develop asymptotic expansions from first principles.

Consider the integral over $k_s$ in Eq. (2). Using the identities $J_n(z) = [H_n^{(1)}(z) + H_n^{(2)}(z)]/2$ and $H_n^{(1)}(z) = -H_n^{(2)}(ze^{i\theta})$, it can be done by formal contour integration, yielding

$$
\phi(\rho, z, t) = \frac{qe^{-i\omega t}}{4\pi e_0} \sum_{n \neq 0} s_n(\rho, z) ,
$$

(5)

where

$$
s_n(\rho, z) = \int_0^\infty dk_s e^{i\omega t} K_s[\rho P_n(k)] + R_n(k) .
$$

Here, $k_s = ip_n(k)$ is one of the solutions to the dispersion relation $D(k_s, k_n) = 0$ and

$$
R_n(k) = \frac{\partial D}{\partial k_s^2} \frac{1}{k_s^2 - p_n^2} .
$$

(7)

The quantity $P_n(k_n)$ is so chosen that $\text{Re} P_n(k_n) = 0$. [In Eq. (6), the subscript on $k_s$ has been dropped; from now on it will be left off, except in cases where ambiguity would result.]

If the integrand in Eq. (6) could be put into the proper form, the method of steepest descents\(^{18}\) could be used to asymptotically expand the integral. If the $P_n(k)$ were never zero and if $\sin \theta \neq 0$, then as $r \to \infty$, one could use the asymptotic form of the Bessel function

$$
K_0(z) \sim (\pi/2z)^{1/2} e^{-z} ,
$$

which would give an integrand of the right type.

We thus need to investigate whether $P_n(k)$ can ever be zero for real $k$. If one considers the dispersion function, Eq. (2), then $D(k_s, k_n) = 0$ and $k_s = \pm P_n(k_n)$ imply that

$$
k_s^2 + k_D^2(1 + iv/\omega) \left[ 1 + i(\omega + iv) \right] \int_0^\infty dt \exp[i(\omega + iv)t]
$$

$$
- \frac{1}{\omega_{ce}^2} \left[ \frac{k_s^2 v_D^2 \sin^2 \frac{1}{2} \omega_{ce} t^2} {\omega_{ce}^2} \right] = 0 .
$$

(8)

For $\nu = 0$, this is the well-known electrostatic plasma wave dispersion relation first obtained by Landau.\(^20\) He showed that for all real $k_s \neq 0$, the only solutions to this equation demand $\omega$ complex and $\text{Im} \omega < 0$. Since $\nu \neq 0$ just adds more damping, there are still no solutions with real $\omega$ and $k_s \neq 0$. However, $k_s = 0$ is clearly a solution to Eq. (8). Accordingly, we have $P_n(k_n) = 0$ for real $k$ if and only if $k_n = 0$.

To see how many of the $P_n(k_n)$ can be zero when $k_n = 0$, let us investigate $D(k_s, k_n) = 0$ under the condition $k_n = 0$. We then obtain

$$
k_s^2 + k_D^2(1 + iv/\omega) \left[ 1 + i(\omega + iv) \right] \int_0^\infty dt \exp[i(\omega + iv)t]
$$

$$
- \frac{1}{\omega_{ce}^2} \left[ \frac{k_s^2 v_D^2 \sin^2 \frac{1}{2} \omega_{ce} t^2} {\omega_{ce}^2} \right] = 0 .
$$

(9)

This is the cyclotron harmonic wave dispersion relation, first obtained by Bernstein\(^4\) for the case $\nu = 0$. In the form stated, $k_s = 0$ is a solution to Eq. (9) no matter...
what the value of the plasma parameters. As long as \( \nu \neq 0 \), expansion of the left-hand side of Eq. (9) in a series in \( k^2 \) will demonstrate that this is the only such solution.

Taking the \( \nu \to 0 \) limit in Eq. (9) requires care. The approach to the limit is not uniform, so interchange of integration and series expansion must be done with caution. Fortunately, solutions to Eq. (9) have been given by Tataronis and Crawford. They that Eq. (9) has additional \( k^2 = 0 \) solutions for \( \omega = \omega_m \) \( = (\omega_m^2 + \omega_h^2)^{1/2} \) and for \( \omega = \nu \omega_m \) \( (n = 2, 3, 4, \ldots) \). However, as shown by Shkarofsky and Johnson, both the nonrelativistic Boltzmann equation and the electrostatic approximation are not valid near the cyclotron frequency or its harmonics. We will thus limit our investigation to frequencies \( \omega \approx \omega_m \) \( (n = 1, 2, 3, \ldots) \).

Consequently, only one \( P_n(0) \) is zero for all \( \omega \); in addition, one other has \( P_n(0) = 0 \) when \( \omega + \nu = \omega_h \). The \( P_n \)’s will be ordered so that the former is \( P_0 \) while the latter is \( P_1 \). Work with the cold plasma result\(^2\) shows that the resonance cones exist only for \( \omega < \omega_m \). Since experimental data are quite difficult to obtain when \( \omega = \omega_h \), we will confine our attention to \( \omega < \omega_m \). This simplifies the mathematics but is still sufficiently general for our later applications. (By treating the \( P_1 \) term as we shall presently treat the \( P_0 \) term, this restriction can be relaxed; however, the results are then unnecessarily complicated.)

Accordingly, for all \( n \geq 1 \) we have

\[
\phi_n > (2^np_1) \pi \int_0^{\infty} d\nu \ \exp\left(i(kz - \rho \nu)\right) R_n(k) P_n(k).
\]

As \( \nu \to \infty \), one can use the method of steepest descents\(^19\) to show that

\[
\phi_n > (2^np_1) \pi \sum_{\nu \to \infty} \left[ A_n^2 \right]^{1/2} R_n(k) P_n(k).
\]

Here, the \( k_m \) are the solutions of the stationary point condition

\[
P_n(k_m) = f(k_m)
\]

and we have allowed for the possibility that there may be more than one \( k \) which will solve Eq. (12) for a given \( n \). The function \( D_n(k_m) \) is the derivative of \( D_n(k) \)

\[
D_n(k) = \frac{1}{2} \frac{d}{dk} P_n(k).
\]

In obtaining Eq. (11), the identity \( P_n''(k_m) P_n(k_m) = \cot \theta + D_n(k_m) \) has been used.

In making this expansion, we have implicitly assumed that \( R_n(k_m) 
eq 0 \) and that \( P_n''(k_m) 
eq 0 \). This latter condition is equivalent to saying that \( \cos \theta + D_n(k_m) 
eq 0 \) or to saying that each stationary point is isolated. This is not always the case and, as can be shown (Ref. 12, Appendix C), there are parameter values near which the form of the asymptotic expansion must be changed. However, as will be shown presently, the asymptotic expansion is dominated by the \( n = 0 \) term; further consideration of the \( n = 1 \) terms will be left for later work.

Let us now turn to the most difficult term, \( n = 0 \). For this case, the Bessel function in Eq. (19) may not be automatically replaced with its asymptotic form because \( P_1(0) = 0 \). To cope with this, we need to investigate how \( P_1(k_m) \) behaves for small \( k \). Expanding the dispersion function, Eq. (5), for small \( k \) and \( n \), we obtain

\[
D(k, k_m) = K_0 k^2 + K_1 k^4 + \cdots,
\]

which is actually the cold plasma result. Hence, for small \( k_m \)

\[
P_1(k_m) \approx \left( k^2 K_0 / K_1 \right)^{1/2},
\]

where the square root has positive real part. For future reference, Eq. (14) also shows that \( R_0(0) = K_1 \).

These limiting forms suggest one way to take care of the fact that \( P_1(0) = 0 \); Add and subtract the cold plasma result in Eq. (6),

\[
\phi_1 > (2^p) \pi \int_0^{\infty} d\nu \ \exp\left(i(kz - \rho \nu)\right) R_0(k) P_1(k).
\]

The integrand in Eq. (16) is zero at \( k = 0 \); thus, as \( \nu \to \infty \), the integral is asymptotic to

\[
\phi_1 > (2^p) \frac{\pi}{K_1} \left[ (z^2 + \rho^2) / K_1 \right]^{1/2} - (2\rho)^{1/2} / K_1.
\]

If we now split the above integral into two, consideration of their forms shows that each is a convergent integral, one of which can be done exactly. Accordingly,

\[
\phi_1 > (2^p) \frac{\pi}{K_1} \left[ (z^2 + \rho^2) / K_1 \right]^{1/2} - (2\rho)^{1/2} / K_1,
\]

\[
\times \left[ \left[ (z^2 + \rho^2) / K_1 \right]^{1/2} + (\rho / K_1) \right] - (2\rho)^{1/2} / K_1
\]

\[
+ (2\rho) \int_0^{\infty} d\nu \ \exp\left[i(kz - \rho \nu)\right] R_0(k) P_1(k).
\]

The term in very large parentheses in Eq. (17) is finite and \( O(1) \) for all \( z \) and \( \rho \neq 0 \), even when \( \nu = 0 \). Thus, the procedure used to obtain Eq. (17) has not introduced any singularity at the cold plasma resonance cone angle.

The final term in Eq. (30) has most of the interesting physics in it. Defining \( k \) to be a new quantity, \( H(\rho, z) \), it can be rewritten in the form

\[
H(\rho, z) = \left( \frac{2^p}{\pi} \right)^{1/2} \int_0^{\infty} d\nu \ \exp\left[i(kz - \rho \nu)\right] R_0(k) P_1(k).
\]
tributions from the endpoints (Ref. 19, p. 30). If \( P_q(k) \) were nonzero at \( k=0 \) and if it had a continuous first derivative there, then the endpoint contributions from the \( \eta \) terms in Eq. (31) would cancel each other. However, \( P_q(0) = 0 \) and Eq. (15) shows that

\[
\lim_{k \to 0} P_q(k) = - \lim_{k \to 0} P_q'(k).
\]

(19)

To handle this manifestation of the fact that \( P_q(0) = 0 \), change the integration variable in Eq. (18) to \( \eta \), where \( \eta^2 = k \). This yields

\[
H(\rho, z) = \left( \frac{2\pi}{\rho} \right)^{1/2} \left( \int_0^\infty d\eta \frac{\eta}{R_0(\eta)} P_q(\eta \rho^2) \exp[\eta^2 z - \rho P_0(\eta)] \right) + \left( \int_0^\infty d\eta \frac{\eta}{R_0(\eta)} P_q(\eta \rho^2) \exp[-\eta^2 z - \rho P_0(\eta)] \right).
\]

(20)

The factor multiplying the exponential in each integrand in Eq. (20) is now finite for all \( \eta \).

When we asymptotically expand the integrals in Eq. (20), the stationary point condition becomes

\[
\eta [i \rho \cos \theta - \sin \eta P_0(\eta^2)] = 0,
\]

(21)

where \( \sigma = +1 \) for the first integral and \( \sigma = -1 \) for the second. One set of solutions to this is

\[
P_0'(\eta^2) = \cot \theta, \quad \text{Re} \eta^2 \geq 0,
\]

(22)

which is just what Eq. (12) becomes under the change of variable. The other solution is \( \eta = 0 \), which is a new stationary point located at the lower limit of integration.

If all the solutions of Eq. (22) also satisfy \( \eta^2 P_q'(\eta^2) \neq 0 \), then

\[
H(\rho, z) \sim \left( \frac{2\pi}{\rho} \right)^{1/2} \left( \int_0^\infty d\eta \frac{\eta}{R_0(\eta)} P_q(\eta \rho^2)^{1/2} \exp[-i \rho \eta \cos \theta - \rho \sin \eta P_0(\eta^2)] \right)^{-1/2} \left( \int_0^\infty d\eta \frac{\eta}{R_0(\eta)} P_q(\eta \rho^2)^{-1/2} \exp[-i \rho \eta \cos \theta - \rho \sin \eta P_0(\eta^2)] \right)^{-1/2} + \frac{\pi}{\rho} \sum \exp \{ i \rho \eta \cos \theta - \rho \sin \eta P_0(\eta^2) \} \exp \{ i \rho \eta \cos \theta - \rho \sin \eta P_0(\eta^2) \} \int_0^\infty d\eta \frac{\eta}{R_0(\eta)} P_q(\eta \rho^2)^{-1/2} \exp[-i \rho \eta \cos \theta - \rho \sin \eta P_0(\eta^2)] \int_0^\infty d\eta \frac{\eta}{R_0(\eta)} P_q(\eta \rho^2)^{-1/2} \exp[-i \rho \eta \cos \theta - \rho \sin \eta P_0(\eta^2)] \}
\]

(23)

The quantities \( \eta^2 \) are the solutions to Eq. (22).

At first glance, we seem to have re-acquired singularities at the cold plasma resonance cone angle when \( \nu = 0 \). Equation (23), however, is not valid for \( \cot \theta = \cot \theta_0 \). To understand this, consider the fact that \( P_0(k) \) is an even function of \( k \) and is proportional to \( k^2 \) for small \( k \) [see Eq. (15)]. Thus, near \( k = 0 \), we must have

\[
P_0(k) = \left( \frac{\nu^2}{\alpha} \right)^{1/2} \left[ 1 + \left( \frac{\alpha}{\nu^2} \right) k^2 + \cdots \right].
\]

(24)

Consequently, for \( \text{Re} k \neq 0 \) we find

\[
P_0(k) = \left( \frac{\nu^2}{\alpha} \right)^{1/2} \left[ 1 + \left( \frac{\alpha}{\nu^2} \right) k^2 + \cdots \right].
\]

(25)

This means that \( P_0'(k) = 0 \). In this form for \( \eta^2 = k \) (i.e., \( \eta^2 = 0 \)) is a solution to Eq. (2), \( \eta \cot \theta = \cot \theta_0 \), and \( \nu = 0 \). Thus, Eq. (23) is valid for angles sufficiently far from the cold plasma resonance cone angle. In other words, this asymptotic form is not uniformly valid in \( \theta \) and \( \nu \) as \( r \to \infty \).

Mathematically, the problem is that one of the \( \eta_j \) approaches the stationary point at \( \eta = 0 \) as \( \cot \theta - \cot \theta_0 \). Physically, this means that there are two waves in the plasma with nearly the same group and phase velocity, and they are interfering with one another. Accordingly we need a method of asymptotic expansion which can handle integrals with coalescing stationary points. One such has been worked out by Chester, et al. It consists of changing variables in the integral to obtain more analytically tractable forms while preserving the nature of the stationary points. The present problem requires modification of the details of their analysis, but the basic idea remains the same.

In general, for \( \sigma = +1 \), there is one \( \eta_j \) that approaches zero as \( \theta = \theta_0 \), while for \( \sigma = -1 \) another \( \eta_j \) approaches zero in the same manner when \( \theta = \pi - \theta_0 \). Consequently, the contribution due to the first two stationary points in each integral in Eq. (20) will require modification at certain angles. The contribution of all other stationary points is still given by the sum in Eq. (23).

We are thus led to consider the large \( r \) expansion of integrals of the form

\[
G_s(r, \theta) = \left( \frac{2\pi}{r} \right)^{1/2} \int_0^\infty d\eta \frac{\eta}{R_0(\eta^2)} P_q(\eta^2)^{1/2} \exp[i r \eta \cos \theta - \eta \sin \eta P_0(\eta^2) + \text{terms in Eq. (31)}],
\]

(26)

where \( M_s(\theta) \) is chosen so that only the first nonzero stationary points of the exponent are included in each range of integration. Let us name these two stationary points \( \eta_j \). In Eq. (23) we already have the asymptotic expansions of \( G_s(r, \theta) \) when the \( \eta_j \) are not near \( \eta = 0 \). Consequently, we only need consider the cases where one of them is small.

The form of the change of variable is suggested by the small expansion of the exponents in Eq. (35), but its validity is not dependent on having \( \eta \) small. Let

\[
\pm i \rho^2 \cos \theta - \sin \eta P_0(\eta^2) = \frac{1}{\rho^2} u^2 - \xi_j(\theta) S(u^2) u^2,
\]

(27)

where \( S(x) = 1 \) if \( x > 0 \) and \( -1 \) if \( x < 0 \). With the inclusion of \( S(u^2) \), both the right and left-hand sides of Eq. (27) have two stationary points for \( |\eta| \leq M_s \).

For the change of variable to be one to one, we require \( d\eta/du \) to be finite and nonzero in the range of interest. Since

\[
2 \eta \left[ \pm i \cos \theta - \sin \eta P_0(\eta^2) \right] d\eta/du = u^4 - 2 \xi_j(\theta) S(u^2),
\]

this demands that the points \( u = 0 \) and \( u^2 = (2 \xi_j)^{1/2} \) must correspond to \( \eta = 0 \) and \( \eta^2 = \eta_j^2(\theta) \). Consequently,

\[
\xi_j(\theta) = -\frac{1}{2} \left[ 3 \sin \left( \pm \cot \eta_j^2 + i P_0(\eta_j^2) \right) \right]^{1/2}.
\]

(28)

We can now write Eq. (26) as

\[
G_s(r, \theta) = \left( \frac{2\pi}{r} \right)^{1/2} \int_0^\infty d\eta \frac{\eta}{R_0(\eta^2)} P_q(\eta^2)^{1/2} \exp[i r \eta \cos \theta - \eta \sin \eta P_0(\eta^2) + \text{terms in Eq. (31)}].
\]

(29)

Here, we have replaced the finite upper limit of the integral in Eq. (26) by infinity. This produces a negligible error in the asymptotic limit. The factor \( \exp[i r \eta \cos \theta - \eta \sin \eta P_0(\eta^2)] \) inlet.
In Fig. 1, the magnitude and phase of \( \text{Ai}(x)[\text{Bi}(x) + i \text{Ai}(x)] \) are plotted for real \( x \). Note that the magnitude plot has qualitative features similar to the data given by Fisher and Gould\(^1\): a main peak followed by a set of subsidiary peaks due to the spatial interference of two waves. This combination of Airy functions will give us the mathematical description of the interference structure in the asymptotic limit.

The Airy functions do not give the whole story, of course. The complete expansion will never have points where the magnitude of the potential is exactly zero. Near the points where \( |F(x)| = 0 \), the \( O(r^{-1}) \) terms in the expansion give important contributions. This fact will also affect the phase. However, the location of the maxima in the interference structure will be given by \( |F(2^{-1/3}r^{2/3} \xi)| \). In this context, it should be noted that the \( \xi \) are, in general, complex, due to Landau damping. Only near \( \xi = 0 \) will they be predominantly real.

The formal asymptotic expansion is now complete. A host of intermediate functions were defined to ease the derivation; however, by simple manipulations, everything can be put in terms of the dispersion function. The result is

\[
\phi(\rho, z, t) = \frac{\sin \theta}{4 \pi} \int \frac{d\lambda}{\lambda^2 - 2 \lambda^2 \exp(i \pi/3 \lambda^2)}.
\]

It is easy to show that

\[
F''(x) - 4xF'(x) - 2F(x) = 0.
\]

This is the differential equation satisfied by the product of Airy functions\(^2\); hence, its three linearly independent solutions are \( \text{Ai}(x) \), \( \text{Ai}(x) \text{Bi}(x) \), and \( \text{Bi}(x) \). By considering the values of \( F(0) \), \( F'(0) \), and \( F''(0) \), one may show after a bit of algebra that

\[
F(x) = (r^{3/2} \gamma^{1/6}) \text{Ai}(x) \left[ \text{Bi}(x) + i \text{Ai}(x) \right].
\]

This function will control the primary contribution to the asymptotic expansion near the resonance cone. As long as \( F(2^{-1/3}r^{2/3} \xi) \) is \( O(1) \), then \( G_1(r, \theta) \) are \( O(r^{-2/3}) \). Contrast this with all the other terms in the expansion, which are at most \( O(r^{-1}) \), and may be much smaller due to damping. Thus, this part of the term \( \delta(x, z) \) is dominant near the resonance cone.
\[
G_\ast(r, \theta) = 2^{1/2} \pi^{-1} (K_\ast/1/2) \frac{\xi_\ast(\theta)}{(K_{n_\ast}/K_\ast)\cot \theta + i \cot \theta} \times Ai(X_\ast)[Bi(X_\ast) + i Ai(X_\ast)],
\]

where
\[
X_\ast(\theta) = 2^{1/2} r^{2/3} \xi_\ast(\theta) = - \left( \frac{4}{3} r \cdot k_n \right)^{2/3}.
\]

In Eqs. (32)-(38) we thus have the asymptotic expansion of the Green's function for large \( r \), derived under the assumptions \( Q(k_{n_\ast}, k_n) \neq 0, \omega < \omega_\ast \), and \( \sin \theta \neq 0 \). However, the result is finite as \( \sin \theta = 0 \). The terms in the braces in Eq. (33) which appear to give an infinite result are exactly canceled by two corresponding terms in \( G_\ast(r, \theta) \). It can be shown (Ref. 12, Appendix D) that the result is correct for all \( \theta \).

III. FURTHER THEORETICAL CONSIDERATIONS

A. Physical predictions from the asymptotic expansion

The results in Eqs. (32)-(39) are still of a rather formal nature. To get the physics out of them, we need more explicit information on the solutions to Eqs. (35) and (36). If one needed all the solutions to these equations, he would have to resort to numerical procedures. Special cases, however, can be done by explicit analysis.

One qualitative prediction can be made rather easily. Due to Landau damping, only solutions to Eqs. (35) and (36) with small \( k_n \) will contribute significantly to the potential in the asymptotic limit. Consider \( k_n = 0 \) to see what terms these might be. In the lower branch where the cold plasma cone exists \( [\omega \leq \min(\omega_\ast, \omega_\ast)] \), the only solutions to the dispersion relation that have \( k_\ast \) real when \( \theta = 0 \) are \( (k_{n_\ast}, k_{n_\ast}) \) at \( \theta = \theta_\ast \) and \( (k_{n_\ast}, k_n) \) at \( \theta = \pi - \theta_\ast \). Consequently, one of the terms in the sum in Eq. (33) is significant. In the upper branch of the cold plasma cones \( [\max(\omega_\ast, \omega_\ast) \leq \omega \leq \omega_\ast] \), in addition to the previous solution, the Bernstein mode solution exists. Since this has \( k_\ast \) real and nonzero for \( k_n = 0 \), at least one of the terms in the sum in Eq. (40) can contribute appreciably to the potential \([i.e., \sim O(d^-1)]\).

To make more detailed predictions, it is necessary to have an approximate form for the \( \xi_\ast \) when they are small. This requires an approximate solution to the stationary point condition, Eq. (21), for small \( \eta \). Expanding \( P_\ast \) as
\[
P_\ast(\eta) = (K_n/K_\ast)^{1/2} \eta^{1/2} + \frac{1}{2} P_\ast''(0) \eta^{1/2} + \ldots,
\]
the solution yields
\[
\xi_\ast(\theta) = \left[ \frac{i \sin \theta}{P_\ast''(0)} \right]^{1/2} + \cot \theta + i \left( \frac{K_{n_\ast}}{K_\ast} \right)^{1/2}.
\]

Since \( k_\ast = i P_\ast(k_n) \) is one solution of the dispersion relation, \( P_\ast''(0) \) can be found by differentiating Eq. (35) and then using Eq. (2). Fortunately, all the resulting integrals can be done in the limit where \( k_{n_\ast} \to 0 \) and \( k_n \to 0 \). yields
\[
X_\ast = 2^{1/2} r^{2/3} \xi_\ast(\theta) = \left[ \frac{K_{n_\ast}}{K_\ast} \right]^{1/2} \left[ \frac{i v_\ast^2 \omega_\ast \sin \theta}{\omega_\ast^2 (1 + i v_\ast/w_\ast) v_\ast^2} \right]^{1/3} \times \left[ \frac{\pm \cot \theta + i (K_{n_\ast}/K_\ast)^{1/2}}{} \right],
\]

where
\[
\Delta = 1 - \frac{(\omega + iv_\ast)^2}{3(\omega + iv_\ast)^2 - \omega_\ast^2} \left( \frac{3(\omega + iv_\ast)^2 - \omega_\ast^2}{(\omega + iv_\ast)^2 - \omega_\ast^2} \right) \left( \frac{(\omega + iv_\ast)^2 - \omega_\ast^2}{(\omega + iv_\ast)^2 - \omega_\ast^2} \right).
\]

The combination \( K_n K_{n_\ast}^{1/2} \) has been left in that form so that the proper sign can be obtained in the \( \theta = 0 \) limit. For each sign, Eq. (42) is valid only for \( |\xi_\ast(\theta)| = 0 \).

Having Eqs. (42) and (43), we can predict where the maxima in \( |F(\rho, z, t)| \) will occur. These are controlled solely by one or the other of the \( G_\ast(r, \theta) \) terms in the asymptotic expansion; thus, the maxima occur whenever \( |Ai(X_\ast)|Bi(X_\ast) + i Ai(X_\ast)| \) have relative maxima. For \( \nu = 0 \), the values of \( \xi_\ast \) that give the maxima are real. The first seven values, computed by a numerical search procedure, are given in Table I.

By differentiating the potential, one can get the electric field components. For future reference, the values of \( \xi_\ast \) that give the maxima in the magnitude of the electric field components are also listed in Table I again for the case \( \nu = 0 \).

Notice from Table I (or from Fig. 1) that all the maxima occur for \( X_\ast < 0 \). Since \( X_\ast = 0 \) on the cold plasma resonance cone, it is more useful to discuss the location of the peaks relative to the cold cone. Define \( \Delta_\ast \theta_\ast = \theta_\ast - \theta_\ast \), where \( \theta_\ast \) is the angular position of the \( n \)th maximum closest to \( \theta_\ast \) as \( n = 1 \). Using this, as \( r \to \infty \), we have for the case \( \nu = 0 \),
\[
\Delta_\ast \theta_\ast = - \frac{x_\ast (v_\ast / \tau \omega_\ast)^{1/3}}{A},
\]

where
\[
A = \left( \frac{9 v_\ast^2 \omega_\ast^2 \sin \theta \Delta_\ast}{(\omega_\ast^2 \sin \theta) - (K_n K_{n_\ast} / 17)} \right)^{1/3}.
\]

Equation (44) will be used extensively in Sec. V, so it is worth discussing in some detail. First, notice that each \( \Delta_\ast \theta_\ast \) has the same dependence on the plasma parameters. All the \( \Delta_\ast \theta_\ast \) just differ by a scale factor. This leads to an interesting prediction: The ratio \( \Delta_\ast \theta_\ast / \Delta_\ast \theta \ast \) should be a constant, independent of the plasma conditions, whose value depends only on \( m \) and \( n \). Second, the dependence on temperature and radius is particularly simple: \( \Delta_\ast \theta_\ast \) is proportional to \( T_\ast^{1/3} \) and to \( \nu^{1/3} \). The dependence on \( \omega \), \( \omega_\ast \), and \( \omega_\ast \) is a good deal more complicated.

The complicated frequency dependence is embodied
Consider whether a more realistic antenna model might be useful. In the present work, we know that $P_n(k)$ is discontinuous at $k=0$; hence, it is worth defining the analog of $H(p, z)$ as

$$H(x, z) = \int_0^\infty dk \frac{\exp[i k x - x P_0(k)]}{R_0(k)} + \int_0^\infty dk \frac{\exp[-i k x - x P_0(k)]}{R_0(k)}.$$  

(47)

From our previous work, we know that $P_n(k)$ is discontinuous at $k=0$; hence, it is worth defining the analog of $H(p, z)$ as

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(47)

We now have the predictions which we must have to compare with the experimental investigations discussed in Secs. IV and V. Before turning to the data, we should consider whether a more realistic antenna model might change any of our conclusions.

B. The line source antenna

any experimental investigation, the actual antenna of course, not a point source. One can regard it as only if the antenna dimensions are much smaller than the wavelength radiated. The antennas used in the present work are long, thin wires, $0.25$ mm in diameter and either $35$ mm or $75$ mm in length. The wavelengths involved are usually greater than $1$ mm, so the antennas can be suitably modeled as a finite length line source. However, the easiest line source to treat mathematically is one that is infinitely long; even this model has its difficulties.

Take the antenna to be an oscillating line source of charge density $\sigma(x) \delta(z) e^{-i \omega t}$. However, just as with the line source in electrostatics, the potential in this case will not approach zero as $r \to \infty$. This leads to Fourier integrals which are defined only as generalized functions, and these are difficult to expand asymptotically. On the other hand, the integrals for the electric field are well defined in the classical sense, and we will work with them instead.

Using the representation in Eqs. (5) and (6) for the electrostatic Green's function, the electric fields for the present case can be found by differentiating with respect to $x$ or $z$ and then integrating over the new charge distribution. The results are

$$E_j(x, z, t) = \frac{e^{-i \omega t}}{4 \pi \epsilon_0} \sum_{n=0}^\infty \int_{-\infty}^\infty dk \alpha_j(k) \frac{\exp[i k x - x P_0(k)]}{R_0(k)},$$  

(46)

where $j=x$ or $z$, $\alpha_x=1$, and $\alpha_z=-ik P_n(k)$. Since $\alpha_j$ is well behaved near $k=0$, the asymptotic expansion near the resonance cone can be done without worrying about its $k$ dependence. Thus, if we can do the expansion for $E_x$, that for $E_z$ follows by a simple multiplication.

In expanding the Green's function, we learned that the interesting physics comes out of the $n=0$ term. Since the expansion of all other terms can be done by standard steepest descents, we will concentrate on that term.

From our previous work, we know that $P_0(k)$ is discontinuous at $k=0$; hence, it is worth defining the analog of $H(p, z)$ as

$$H(x, z) = \int_0^\infty dk \frac{\exp[i k x - x P_0(k)]}{R_0(k)} + \int_0^\infty dk \frac{\exp[-i k x - x P_0(k)]}{R_0(k)}.$$  

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We now have the predictions which we must have to compare with the experimental investigations discussed in Secs. IV and V. Before turning to the data, we should consider whether a more realistic antenna model might change any of our conclusions.

B. The line source antenna

any experimental investigation, the actual antenna of course, not a point source. One can regard it as only if the antenna dimensions are much smaller than the wavelength radiated. The antennas used in the present work are long, thin wires, $0.25$ mm in diameter and either $35$ mm or $75$ mm in length. The wavelengths involved are usually greater than $1$ mm, so the antennas can be suitably modeled as a finite length line source. However, the easiest line source to treat mathematically is one that is infinitely long; even this model has its difficulties.

Take the antenna to be an oscillating line source of charge density $\sigma(x) \delta(z) e^{-i \omega t}$. However, just as with the line source in electrostatics, the potential in this case will not approach zero as $r \to \infty$. This leads to Fourier integrals which are defined only as generalized functions, and these are difficult to expand asymptotically. On the other hand, the integrals for the electric field are well defined in the classical sense, and we will work with them instead.

Using the representation in Eqs. (5) and (6) for the electrostatic Green's function, the electric fields for the present case can be found by differentiating with respect to $x$ or $z$ and then integrating over the new charge distribution. The results are

$$E_j(x, z, t) = \frac{e^{-i \omega t}}{4 \pi \epsilon_0} \sum_{n=0}^\infty \int_{-\infty}^\infty dk \alpha_j(k) \frac{\exp[i k x - x P_0(k)]}{R_0(k)},$$  

(46)

where $j=x$ or $z$, $\alpha_x=1$, and $\alpha_z=-ik P_n(k)$. Since $\alpha_j$ is well behaved near $k=0$, the asymptotic expansion near the resonance cone can be done without worrying about its $k$ dependence. Thus, if we can do the expansion for $E_x$, that for $E_z$ follows by a simple multiplication.

In expanding the Green's function, we learned that the interesting physics comes out of the $n=0$ term. Since the expansion of all other terms can be done by standard steepest descents, we will concentrate on that term.

From our previous work, we know that $P_0(k)$ is discontinuous at $k=0$; hence, it is worth defining the analog of $H(p, z)$ as

$$H(x, z) = \int_0^\infty dk \frac{\exp[i k x - x P_0(k)]}{R_0(k)} + \int_0^\infty dk \frac{\exp[-i k x - x P_0(k)]}{R_0(k)}.$$  

(47)
If we now define \( z = r \cos \theta \) and \( x = r \sin \theta \), \( H(x, z) \) can be expanded for \( r \to \infty \).

Just as in the former case, we shall attempt to use the method of steepest descents. However, we know that when \( \theta \) is near the resonance cone, one of the stationary point conditions \( P_0''(k) = \pm i \cot \theta \) will have a solution with \( k \) small. Accordingly, \( P_0''(k) \) will also be quite small, and the usual steepest descents formula will no longer apply.

Scoffer\(^2\) has developed a method to handle exponents of the type that occur here. His arguments are based on the method of stationary phase, and so, implicitly assume that the arguments of the exponentials are purely imaginary on the path of integration (i.e., the waves involved are undamped.) Since this will be very nearly true for angles near the resonance cone, we can take over his results directly. Accordingly, for \( \theta \) near \( \theta_c \) (for +) or for \( \theta \) near \( \pi - \theta_c \) (for -), we have

\[
\int_0^\infty dk \left[ R_0(k) \right]^{-1} \exp \left\{ i \left[ k \cos \theta - \sin \theta P_0(k) \right] \right\}
\sim \frac{\pi}{R_0(k_\theta)} \left( \frac{-2i}{r \sin \theta P_0''(k_\theta)} \right)^{1/2} e^{i \Phi \left[ A_\theta(k_\theta) + i G_\theta(k_\theta) \right]}.
\]

(48)

Here, \( k_\theta \) are the solutions to the stationary point conditions \( P_0''(k_\theta) = \pm i \cot \theta \), \( G_\theta(x) \) is a relative of the Airy function,\(^1\) and

\[
\beta_\theta = - \left[ ik_\theta \cos \theta - \sin \theta \left( P_0(k_\theta) + \frac{1}{3} \left[ P_0''(k_\theta) \right]^2 \right) \right],
\]

\[
X_\theta = \left( \frac{r \sin \theta}{2} \right)^{2/3} \left[ \frac{P_0''(k_\theta)}{i P_0''''(k_\theta)} \right]^{1/3}.
\]

Using the approximate form for \( P_0(k) \) given in Eq. (40), we may rewrite the right-hand side of Eq. (48) as

\[
\left( \frac{-2i}{r \sin \theta P_0''(0)} \right)^{1/3} \left[ A_\theta(k_\theta) + i G_\theta(k_\theta) \right],
\]

where

\[
X_\theta = \left( \frac{2r^2 \sin^2 \theta}{P_0''(0)} \right)^{1/3} \left[ \pm \cot \theta + i (K_\theta/k_\theta)^{1/2} \right].
\]

(50)

The quantities \( X_\theta \) that were defined in the expansion of the Green's function have exactly the same functional form as the ones given in Eq. (50), but the previous ones are smaller by a factor of \( 2^{2/3} \). Since the functional form is the same, all of the discussion of how \( X_\theta \) depend on the physical parameters holds here, too.

To illustrate what the interference structure looks like in this case, the magnitude and phase of \( A_\theta(x) + i G_\theta(x) \) are plotted in Fig. 4. The qualitative features are much the same as the Green’s functions. Notice, however, that the difference in height between a maximum in the magnitude and the adjacent minima is much smaller here than it was in the previous case. The nonzero length of the antenna in the \( y \) direction is the cause of this.

The angular interference spacing can again be specified by defining \( \Delta \theta_n \) to be the spacing between the cold plasma cone angle and the \( n \)th interference peak. In terms of our previously defined functions, \( \theta \to \infty \) we obtain

\[
\Delta \theta_n = - y_n (u_\infty/u_{\omega_0})^{2/3} \Lambda,
\]

(51)

where \( y_n = 2^{2/3} x_n \), and where \( x_n \) are the values of \( x \) at which \( A_\theta(x) + i G_\theta(x) \) has a relative maximum. The first seven \( x_n \) and \( y_n \) are given in Table II.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( -x_n )</th>
<th>( -y_n = -2^{2/3} x_n )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4</td>
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</tr>
<tr>
<td>7</td>
<td>15.3204</td>
<td>9.65127</td>
</tr>
</tbody>
</table>

[In Ref. 12 it is stated that Eq. (48) is true for all \( \theta \); however, due to neglect of damping, this is not the case.]

To compare the line source results with the previous point source prediction, the \( y_n \) in Table II should be compared to the \( x_n \) in Table I. (To compare the \( y_n \) with the \( x_n \) in Table I is to compare the incommensurable, since the potential does not have its maximum modulus at the same point that the electric field components do.) Except for the first element in each list, the \( x_n \) and \( y_n \) are approximately equal, differing by at most 10%. More importantly, the first differences of the elements agree to better than 3%, again excepting the ones involving the first elements. These differences
also agree with those taken among the $x_n$ of Table I. Since the interpretation of the experimental measurements depends on these differences, it will take quite accurate measurements to see which model fits the data best.

In summary, consideration of a line source rather than a point source has not modified any of the qualitative features of the resonance cone structure, nor has it affected the way these features depend on the plasma parameters. Only a few of the numerical details have changed.

IV. EXPERIMENTAL EQUIPMENT

A. Plasma generation and confinement

The experiment was performed in the afterglow of a repetitively pulsed, argon plasma which was generated by radio frequency breakdown. The pulsed plasma has several advantages over a steady state discharge, even though the associated electronics are somewhat more complex. First, the strong electric fields that are associated with plasma generation are absent and cannot affect the wave propagation that we wish to study. Second, as the plasma decays in the afterglow, the velocity distribution function of the particles quickly becomes isotropic and nearly Maxwellian, thus facilitating comparison with the theory. Third, a whole range of plasma densities is easily available simply by taking data at different times in the afterglow. Fourth, such plasmas have low electron temperature, which makes the wavelength of the warm plasma waves very short, thus making it easier to experimentally attain the asymptotic limit used in the theory.

The electron density in the pulsed plasma was around $10^8$ cm$^{-3}$, and the electron temperature was roughly 1000 K. Using the measured electron collision cross section for argon, the electron-neutral collision frequency was about $1 \times 10^6$ sec$^{-1}$. The electron Coulomb or self-collision frequency, as computed from the collision time given by Spitzer, was also of the same order. Since the working frequency was greater than 80 MHz, the plasma waves propagate in an essentially collisionless medium.

The plasma was produced in the apparatus shown in Fig. 5. It was contained in a glass vacuum vessel assembled of standard sections of Pyrex glass pipe (inside diameter $\approx$ 15 cm). Base pressure was about $5 \times 10^{-7}$ Torr. While the plasma was being produced, a continuous, slow flow of argon through the system was maintained. Argon pressure was in the range $1 \times 10^{-5}$ to $2 \times 10^{-2}$ Torr, with most work being done at $5 \times 10^{-3}$ Torr.

The external magnetic field was produced by a pair of coils in Helmholtz configuration (the glass pipe was aligned along the symmetry axis of the coils.) The voltage drop across a resistor in series with the coils was used as a measure of the magnitude of the magnetic field. The original calibration was done by using a nuclear magnetic resonance unit. The voltage could be read accurately to within 0.2%, the uniformity of the field was not much better than this.

The plasma was generated by gas discharge breakdown in a 30 MHz radio frequency electric field. Originally, the radio frequency field was coupled to the plasma by means of the two copper bands around the glass tube (see Fig. 5). In an attempt to obtain more plasma at lower argon pressures, the set of aluminum grids shown in Fig. 5 was inserted in the tube and the middle member of each triplet was connected to one output of the radio frequency oscillator. This failed to produce more plasma, but it led to a quieter, more reproducible one. The oscillator was gated on once every 7 msec for about 100 $\mu$s. No experimental measurements were made while the oscillator was on.

B. The probe system

As indicated in Fig. 5, both transmitting and receiving probes are mounted opposite each other on the same rotating structure; thus, they each rotate about a common center at equal distances from it. In contrast to probe configurations having one centrally located fixed probe with another probe rotating about it, this probe configuration allows the probes to be twice as far apart and still remain in the region of relatively uniform density near the center of the plasma.

The whole probe carriage is mounted in the side arm of a glass cross. (The main stem of the cross is the cylinder in Fig. 5.) The arm is long enough so that only the probes reach the main body of the plasma.

In addition to rotating through a full 360°, the carriage can vary the probe separation continuously from 1.5 to 11.0 cm. Potentiometers are driven by the gears that rotate the assembly and by those that alter the probe separation; thus, an analog voltage proportional to these two quantities can be obtained. Since the carriage is rotated continuously from stop to stop when data are taken, backlash in the gears is no problem; hence, the voltage analog to the angular location is accurate to within $0.2^\circ$. The probe carriage and its supporting structure are shown in Fig. 5. In an attempt to obtain more plasma at lower argon pressures, the set of aluminum grids shown in Fig. 5 was inserted in the tube and the middle member of each triplet was connected to one output of the radio frequency oscillator. This failed to produce more plasma, but it led to a quieter, more reproducible one. The oscillator was gated on once every 7 msec for about 100 $\mu$s. No experimental measurements were made while the oscillator was on.
within the linearity of the potentiometer, 0.25%. Backlash does limit the determination of probe separation to within 0.1 cm.

The probes themselves are made of 2.2 mm o. d. semi-rigid coaxial cable with a portion of the inner conductor exposed. The usual center conductor in the transmitters was replaced with 0.25 mm o. d. hollow stainless steel tubing, in an attempt to obtain a smaller diameter antenna that would not bend under its own weight. The two transmitters that were used at various times had either 35 or 75 mm of the center conductor exposed, while the receiving probe had just 4 mm of the usual 0.5 mm inner conductor uncovered. The outer conductor of all probes was sheathed in teflon tubing to prevent sputtering.

C. Experimental electronics

A block diagram of the experimental electronics is given in Fig. 6. The master pulse generator controls the repetition rate of the experiment and duration of plasma generation by gating the radio frequency oscillator on and off. It also triggers the sample-hold module which, after waiting for a certain interval, samples the received signal. Samples can be taken at any point in the afterglow.

The input signal for the transmitting probe is provided by a 60 to 500 MHz frequency variable oscillator. Input frequency is measured with a 500 MHz counter. The 20 dB directional coupler and 20 dB attenuator enable the power meter to read the input power directly.

It was found empirically that an input power of more than 20 μW changed the appearance of the resonance cones; however, their shape was independent of input power for any level below that. During data taking, the input power was held below 10 μW, and was usually 6 μW. (One should expect nonlinear effects to enter when the power level is too high, and this should bring with it a whole new class of phenomena.)

The received signal was detected either with the superheterodyne receiver shown in the main body of Fig. 6 or with the Hewlett Packard network analyzer shown in the inset. Both detectors are sensitive to the amplitude of the received signal. Most of the data were taken with the superheterodyne receiver. The 500 kHz bandwidth of its intermediate frequency amplifier ensures more than enough frequency response. The 10 kHz bandwidth of the network analyzer is adequate, but some smoothing of rapidly varying features was noticeable.

Without plasma in the system, the signal passing between the probes was attenuated by about 60 dB. Although the level with plasma is some 20 dB higher, thermal noise in the receivers was always somewhat of a problem.

After the detector output is sampled, it is applied to the Y axis of an X-Y recorder whose X axis is driven by the voltage analog of the angular location of the probes. One thus obtains a graph of received signal amplitude vs angle relative to the magnetic field.

Not shown in Fig. 6 is the oscilloscope used to monitor the experiment. By using a dual trace scope and multitrace plug-in units, continuous monitoring of all important signals was possible. The oscilloscope was also used to measure when the received signal was sampled in the afterglow.
V. EXPERIMENTAL RESULTS

A. Resonance cone location

As illustrated in Fig. 7, by far the most prominent feature in the experimental data is the resonance cone peak itself. The interference peaks are never as large as the theoretical development would suggest (see Figs. 1 and 4); in fact, as shown in Fig. 7(c), on some experimental graphs they are invisible. This mismatch is due to a defect in our model of the antenna: the neglect of the plasma sheath around the antenna. Since sheath dimensions are of the order of the wavelength of the warm plasma waves, they can be strongly affected by it while the near-field electromagnetic waves are entirely unaltered. This would alter the relative amplitude of the main resonance cone peak and the interference peaks. A theoretical investigation of the radio frequency properties of the sheath is far beyond the scope of this work; consequently, we will confine our attention to those aspects of the data which should be insensitive to the detailed antenna characteristics.

FIG. 7. Typical experimental data showing the received signal as a function of probe angle relative to the magnetic field. Graphs (a) and (c) are for frequency $\omega$ in the lower branch, while $\gamma$ was obtained with $\omega$ in the upper branch. Notice the absence of interference in (c), even though they are clearly visible in (a) and (b). Antenna separation $\gamma$, transmitter length $l$, working frequency $\omega$, and cyclotron frequency $\omega_c$ are given for each case.

These include the angular resonance cone location and the angular interference spacing.

The sheath around the receiving antenna produces another uncertainty in interpretation of the data. Since the signal must transit the sheath before reaching the antenna, it is not clear whether the antenna is sensitive to the local potential or electric field. Leuterer has presented evidence that antennas oriented parallel to $B$ respond to the potential. The antennas in the present work are, however, normal to $B$, which makes his evidence suggestive but not conclusive. Consequently, at points where there is a difference, the theoretical results computed both for the potential and for the electric field will be compared with the experimental results.

Turning finally to the data, we first consider the dependence of the main peak location on $\omega/\omega_c$ and $\omega_pe/\omega_c$. As was first shown experimentally by Fisher and Gould, most of the dependence can be accounted for by the collisionless cold plasma formula, Eq. (4). The results of the present work confirm that statement.

In Fig. 8, four sets of data are presented, showing the measured resonance cone angle as a function of $\omega/\omega_c$. Within each set, the plasma conditions were kept constant; accordingly, $\omega_pe$ should be constant. The solid lines in the figure are computed from Eq. (4) with $r = 0$; it is clear that the theoretical curves and the experimental points correspond rather well. Fisher and Gould have shown that the density inferred from plots such as this one agrees with microwave interferometer measurements of the density.

The density that is found by inserting the measured
cone angle in the cold plasma formula can be used as another check on the consistency of our theoretical interpretation of the data. Differing densities can be obtained by taking data at different times in the afterglow. In an afterglow plasma, simple diffusion theory predicts that the density should decay exponentially in time. We can check to see that the density we infer from the measured cone angle has this behavior.

Two different plots of the inferred density vs time in the afterglow are given in Fig. 9. If only propagation of measurement errors is considered, the error in the density is about the size of the symbols used in the plot. The density decays exponentially with time, as expected. The two sets of data have different slopes because they were taken with different background pressures. (Gofalone has reported similar density measurements using the resonance cones.)

Figures 8 and 9 indicate that as long as one knows \( \omega \) and \( \omega_{ce} \), a measurement of resonance cone angle is a useful way to determine the plasma density. This is the method that was used throughout the experiment.

One should, however, be aware of two sources of systematic error that can affect this method. First, if the density between the probes is not uniform, the WKB method indicates that the measured cone angle will give a result that underestimates the peak density between the probes. Second, as the theory for warm uniform plasmas shows, the actual cone angle also depends on the separation \( r \) between the transmitting and receiving probes. (This will be dealt with in more detail presently.) These systematic effects, and any random errors as well, can have serious consequences if one is trying to find the density when \( \omega_{pe}/\omega_{ce} > 1 > \omega/\omega_{ce} \). A glance at Fig. 8 shows that the cone angle is quite insensitive to the density in this parameter region. Small errors can thus have a large effect on the inferred density.

Now we are in a position to begin comparing the data with the warm plasma theory developed in Secs. II and III. Since that is an asymptotic theory, we need to verify how well its basic assumption \( r \gg \lambda_{pe}, r/L_e \) is met. Since \( r \approx 1,5 \text{ cm and } T_e = 300 \text{ K, and since } \omega_{pe}/2\pi \text{ and } \omega_{ce}/2\pi \text{ are around } 300 \text{ MHz, we have } r/r_L > 290 \text{ and } r/\lambda_{pe} > 400 \). The basic assumption is indeed well met.

From Sec. III, theory predicts that the main resonance cone peak should occur at \( \theta_1 = \theta_r - \Delta \theta_1 \). The good agreement between theory and experiment demonstrated in Figs. 8 and 9 indicates that \( \Delta \theta_1 \) must be small; however, certain measurements should reveal its presence. Since \( \theta_r \) does not depend explicitly on \( r \), while \( \Delta \theta_1 \propto r^{-2/3} \), any dependence of \( \theta_1 \) on \( r \) could be ascribed to \( \Delta \theta_1 \).

In Fig. 10, the main resonance cone location \( \theta_1 \) and the location of the first interference peak, \( \theta_2 \), are plotted as a function of \( r^{-2/3} \). (Recall \( \theta_2 = \theta_r - \Delta \theta_2 \) and \( \Delta \theta_2 \propto r^{-2/3} \). The solid lines through the data points are lines of best fit as determined by a linear least-squares procedure. The linear fit is consistent with the data, thus confirming the prediction \( \Delta \theta_1 \propto r^{-2/3} \). Therefore, \( \Delta \theta_1 \) may be small, but it is nonzero.

Additional confirmation comes from the fact that, as predicted by theory, \( \theta_1 - \theta_r \) as \( r \to \infty \). The least-squares fit yields intercepts \( \theta_1 = 30.4 \pm 0.4 \degree \) and \( \theta_2 = 30.6 \pm 0.7 \degree \). Further, the signs of the slopes of the lines and the relative magnitudes agree with the prediction. Since \( \omega/\omega_{ce} = 0.497 \) for the data, Figs. 2 and 3 indicate that \( \Delta > 0 \), and consequently the slopes of the graphs of \( \Delta \theta_1 \) vs \( r^{-2/3} \) should be negative, as they are. The ratio of the slopes should either be given by \( x_2/x_1 \) or \( x_1/x_2 \)
FIG. 11. Measured values of the resonance cone angle at various values of antenna separation $r$ and taken at different times in the afterglow. Note the change in slope as the time increases. The data were taken using the 35 mm transmitting antenna.

FIG. 12. Angular interference spacing $\Delta \theta_n$ as a function of $r$ for $n=1(1)5$. Each point plotted is the mean of four measurements; the error is the standard deviation of the mean or the basic instrumental error, whichever is greater. The solid lines are the lines of best fit, which were constrained to go through the origin. The $l=55$ mm transmitting antenna was used for these data.

<table>
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<th>Point Source</th>
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<td>(potential)</td>
</tr>
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</tr>
<tr>
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<td>$2.3810$</td>
<td>$1.6496$</td>
</tr>
<tr>
<td>$1.6901$</td>
<td>$2.3810$</td>
<td>$1.6496$</td>
</tr>
</tbody>
</table>

from Table I, or $y_2/y_1$ from Table II. The experimental value is $5.4 \pm 0.8$ while $x_2/x_1 = 4.1058$, $x_2/x_1 = 2.23745$, and $x_2/y_1 = 2.93848$.

The data in Fig. 10 were taken under conditions that make it difficult to find the density with any accuracy from the cold plasma formula. Needlelessly applying that formula yields $\omega_{pi}/2\pi = 894$ MHz and $\omega_{pe}/\omega_{ce} = 4.47$, which is in the parameter region where $\theta_c$ is very nearly independent of $\omega_{pe}$. This choice of operating region was deliberate. Langmuir probe measurements showed that the density in the region between the antennas varied somewhat with antenna separation. As Fig. 10 demonstrates, the variation of $\theta_c$ due to $\Delta \theta_1$ is slight. If the data had been taken with $\omega_{pe}/\omega_{ce} \leq 1$, that density variation would have caused $\theta_c$ to change so much that the variation in $\theta_1$ due to $\Delta \theta_1$ would have been undetectable.

A graphic illustration of what happens as density dr.
is given in Fig. 11. Early in the afterglow when the density is high, the slope of the data is negative, in agreement with the theoretical prediction. However, as the density drops, the slope shifts from negative to positive. Since the antennas act as sinks for plasma, one would expect the density between antennas to decrease somewhat with antenna separation, thus increasing $\theta_e$. The change in density required to account for the measured shift in $\theta_e$ agrees reasonably well with the density change measured by the Langmuir probes.

In summary, if the density were to remain constant as $r$ varies, then the warm plasma theory accounts well for the changes in cone angle. The shifts due to $\Delta \theta_1$ are small, which explains why the cold plasma formula worked so well in Figs. 8 and 9. The present work thus explains why the cold plasma formula fits the data of other investigators so well, while illustrating that the warm plasma effects do enter. By using the formula for $\Delta \theta_1$, one can check whether, in a given situation, the warm plasma corrections are important. One can then assess possible systematic errors that may occur in inferring the plasma density from the measured cone angle using the cold plasma cone angle formula.

B. Angular interference spacing

The presence of the interference structure near the resonance cone is the most striking warm plasma effect in the experimental data. The aspect of the data most easily related to the theoretical predictions is the angular spacing between the various interference peaks and the main resonance cone peak. These should be governed by the $\Delta \theta_m$ defined in Eq. (44) or Eq. (51). A series of parameter studies was undertaken to verify the theoretical predictions.

First, consider the variation with $r$. According to the theory, the $\Delta \theta_m$ are proportional to $r^{-2/3}$ if all other
The parameters are constant as \( r \) changes. However, as we saw in the last section, the plasma density drops somewhat as the antennas approach one another. Fortunately, the resultant shift in plasma frequency is at most 30\% and, as shown in Fig. 3, there is a large range of parameter values where a 30\% variation in \( \omega_{pe} \) will have a small effect on the theoretically predicted interference spacing. Accordingly, if \( \omega_{pe}/\omega_{ce} \geq 1 \), a simple plot of the observed angular interference spacing vs \( r^{-2/3} \) can be used to check the theoretically predicted radial dependence.

Figure 12 illustrates the variation of the angular spacing between the main resonance cone peak and the first five interference peaks with \( r^{-2/3} \). Since the lines of best fit match the data so well, the assumed theoretical dependence of \( r^{-2/3} \) is the correct one. The data for the first three interference peaks give an especially good test of the theory, since the error is smaller for these data. Whether the slopes of the lines of best fit have the proper magnitude will be dealt with presently.

Second, consider whether all the spacings vary together as predicted. The interference spacings considered in Fig. 12 can be expressed in terms of the \( \Delta \theta_n \) as

\[
\Delta \theta_n = \Delta \theta_{n+1} - \Delta \theta_1 , \quad n \geq 1 .
\]

Theory predicts that all the \( \Delta \theta_n \) are proportional, i.e., there exist numbers \( \alpha_{m,n} = \Delta \theta_n/\Delta \theta_m \) which should be constant. This prediction can be tested.

The theoretical values of the \( \alpha_{m,n} \) are easily obtained from the numbers in Tables I and II; they are given in Table III for \( m, n \leq 5 \). Since \( \alpha_{m,n} = 1/\alpha_{n,m} \), only the upper off-diagonal elements of the matrix are given. Due to the relation \( \alpha_{m,n} = \alpha_{lm}/\alpha_{ln} \) (all \( l \)), there are only four independent elements in each part of Table III. The different predictions cover the alternatives: a transmitting antenna that acts either as a line source or a point source and a receiving antenna that is sensitive either to the potential or the electric fields. (One is missing, since the potential excited by a line source was analytically intractable.)

The data in Fig. 12 can be fit well with a set of straight lines; thus, the \( \Delta \theta_n \) for \( n \leq 5 \) remain proportional as \( r \) varies, as predicted. The experimental values of the \( \alpha_{m,n} \) can be found by taking ratios of the slopes of the lines of best fit. These are given, along with their errors, in Table IV. They agree reasonably well with both the theoretical predictions for the potential of a point source and the electric field of a line source. They do not agree nearly as well with the predictions based on a point source transmitter and a receiver sensitive to the electric field.

To check the proportionality as \( \omega_{pe} \) varies, data were taken at various times in the afterglow. In Fig. 13, the \( \Delta \theta_n \) are plotted vs \( \omega_{pe} \) for \( m \leq n \leq 4 \) and \( m = 1, 2, 3 \). The lines of best fit are constrained to go through the origin as theory demands. (Please note that the origin of the plot has been shifted for the \( m = 2 \) data.) Clearly, the

\[
\text{FIG. 14. Demonstration of the proportionality of } \Delta \theta_1 \text{ and } \Delta \theta_2 .
\]

For clarity, the error bars (which ranged from 0.4° to 1.2°) have been suppressed. The data were taken using the 35 mm transmitting antenna.
various $\Delta \theta_n$ are proportional. The experimentally
determined proportionality constants, the $\alpha_m$, are given in Table V.

For these data, $r$ and $\omega_{pe}$ were held constant; the
change in each $\Delta \theta_n$ is caused by the variation in the plasma frequency
$0.56 \leq \omega_{pe}/\omega_{ce} \leq 1.12$. In addition,
the longer, 75 mm transmitting antenna was used.
Thus, it is not too surprising that the $\alpha_m$ for a line source fit the data best.

Finally, to ascertain whether the $\Delta \theta_n$ remain propor
tional as $\omega_{pe}/\omega_{ce}$ varies, the plot in Fig. 14 of $\Delta \theta_1$ vs
$\Delta \theta_1$ was made. Since these two are proportional as $r$ and $\omega_{pe}$ vary, no attempt was made to restrict the data to sets with constant values of these parameters. (In fact, almost all experimental measurements of these two quantities ever taken with the 35 mm antenna were used for this plot.) Over the range of the data, $0.32
\leq \omega_{pe}/\omega_{ce} \leq 1.24$, $\Delta \theta_1$ and $\Delta \theta_2$ are obviously proportional.
The slope of the line of best fit through the origin gives $\alpha_1 = 1.692 \pm 0.013$, which compares quite well with the line source prediction. Indeed, since the 35 mm antenna was used for the data, the agreement is somewhat startling.

The data in Figs. 12–14 have demonstrated that all the $\Delta \theta_n$ remain proportional to each other as $r$, $\omega_{pe}/\omega_{ce}$, and $\omega_{pe}/\omega_{ce}$ vary. Accordingly, all subsequent investigations of the dependence of the $\Delta \theta_n$ on these parameters can, without loss of generality, be restricted to $\Delta \theta_1$, which is the easiest to measure.

Now turning to the third parameter study, consider
the dependence of $\Delta \theta_1$ on $\omega_{pe}/\omega_{ce}$ and $\omega_{pe}/\omega_{ce}$. The theoretical prediction for this dependence is embodied in the function $\Lambda$, Eq. (45).

The data used to investigate the dependence on $\omega_{pe}/
\omega_{ce}$ were originally taken in an attempt to see how the plasma temperature decayed with time in the afterglow. By assuming the theory to be correct, one can invert Eq. (44) or (51) and find the temperature. However, the derived temperature always came out near 300 °K. If we assume that it is constant, as this procedure indicates, then we can use the data to investigate the de
pendence of $\Delta \theta_1$ on $\omega_{pe}/\omega_{ce}$.

The data are presented in Fig. 15. The solid line through the data is the theoretical function, with the temperature adjusted for the best fit. Recall that the temperature enters only as a multiplicative scale factor; the curve shape is independent of $T_o$. As was shown in the discussion on Fig. 13, the constants (the $\gamma_n$) for the line source calculation are the most appropriate for the long antenna. Using this, the temperature that gives the best fit is $330 \pm 31$ °K.

Several different methods were tried in an attempt to obtain an independent measurement of the temperature. All failed. They involved determining the temperature from the dispersion curves for various types of previously studied electrostatic warm plasma waves. These included Bernstein or cyclotron harmonic waves, which propagate perpendicular to the magnetic field, and Landau waves and ion acoustic waves, which propagate parallel to it. All proved to be undetectable. The failure is, at least, consistent with the low temperature inferred from the interference spacing. If $T_o
= 300$ °K, the wavelength of the Bernstein and Landau waves would be so short ($\leq 1$ mm) that detection would be difficult. The ion acoustic waves on the other hand, should be very strongly damped in a plasma with equal electron and ion temperatures.

In the absence of independent measurements, one can say that it is quite believable that a temperature this close to room temperature can persist for the interval of 4 msec in the middle of the afterglow during which the data were taken. (Recall that the e-folding time for the density decay is around 2 msec, while the total length of the afterglow is 7 msec.)

Other sets of data were taken to see how $\Delta \theta_1$ varies with $\omega_{pe}/\omega_{ce}$. One interesting theoretical prediction, first given by Kuehl, is that the interference structure can appear outside the resonance cone ($\Delta \theta_n$, $\Lambda > 0$) even in the lower branch if $\omega_{pe}/\omega_{ce}$ is large enough and if $\omega_{pe}/\omega_{ce}$ is chosen properly. The plot in Fig. 16 demonstrates that this is indeed the case.

In this figure, the solid line is the theoretical line of best fit. A nonlinear least squares fit was used to find the best fit values $T_o = 600 \pm 100$ °K and $\omega_{pe}/\omega_{ce} = 10^{13}$. The absurdly large value of $\omega_{pe}$ occurs because the density is sufficiently high so that the interference spacing is quite insensitive to its value. (The density is sufficiently large so that the main cone angle cannot be used to measure it, either.)
FIG. 16. Variation of $\Delta \theta_1$ with $\omega/\omega_{ce}$ for frequency $\omega$ in the lower branch. The solid line is the theoretical prediction with the parameters adjusted to give the best fit. The 35 mm transmitting antenna was used here.

The temperature is found by using the constants (the $y_s$) for the line source. If the point source results were used, the computation for the potential would result in a temperature smaller by a factor of 1.2696, while that for the electric field would give one larger by 1.3867. These, of course, do not affect the fit in Fig. 16.

The upper branch is much more challenging experimentally, since $A_{el}$ changes more rapidly with $\omega/\omega_{ce}$ and $\omega_p/\omega_{ce}$. As shown in Fig. 17, reasonably good data were obtained here also. The theoretical lines fitted to the data yield the parameter values given in Table VI. The temperature was again computed using the constant appropriate for a line source. The internal agreement of the best fit parameters provides a check on the self-consistency of our theoretical interpretation of the data.

One last check can be made by comparing the values of $\omega_p/\omega_{ce}$ in Table VI with that derived from the main cone angle using the cold plasma cone angle formula. Since $\omega$ is in the upper branch, the cone angle should give a good measure of $\omega_p/\omega_{ce}$. A least-squares fit to the measured angle as a function of frequency yields $\omega_p/\omega_{ce} = 0.989 \pm 0.008$. These two measurements of $\omega_p/\omega_{ce}$ are independent; their agreement helps confirm that the theoretical development is correct.

In conclusion, we have verified that the interference spacing depends on the antenna separation as $r^{-2/5}$, as predicted by theory; that all the spacings remain proportional as $r$, $\omega_p/\omega_{ce}$, and $\omega/\omega_{ce}$ vary, with proportionality constants that agree with theoretical predictions; and that the spacings change with $\omega_p/\omega_{ce}$ and $\omega_p/\omega_{ce}$ in a manner consistent with the theory.

VI. SUMMARY AND EVALUATION

A detailed theoretical and experimental investigation has been presented of the resonance cone pattern excited by an antenna in a warm, magnetized plasma.

The warm plasma theory, valid for arbitrary $\omega < \omega_{pe}$, was developed for a uniform plasma in the limit that $r/\lambda_p \gg 1$ and $r/r_{ce} \gg 1$. This led to predictions about the functional dependence of the angular interference spacing on the physical parameters $r$, $\omega$, $\omega_p$, $\omega_{ce}$, and $T_e$.

The experimental work verified the theoretical picture. The main cone angle and interference spacing were shown to depend on $\omega$, $\omega_p$, $\omega_{ce}$, and $r$ in a manner consistent with the theory. From the data, values of $\omega_p$ could be obtained in more than one way, and the results were consistent. The temperature obtained from the data was appropriate for the afterglow plasma employed in the experiment.

<table>
<thead>
<tr>
<th>Antenna separation (cm)</th>
<th>$\omega_p/\omega_{ce}$</th>
<th>$T_e$(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>0.931 ± 0.053</td>
<td>868 ± 82</td>
</tr>
<tr>
<td>6.5</td>
<td>0.945 ± 0.067</td>
<td>771 ± 91</td>
</tr>
<tr>
<td>8.0</td>
<td>0.962 ± 0.095</td>
<td>796 ± 138</td>
</tr>
</tbody>
</table>
The results of this work should facilitate use of resonance cone measurements as a plasma diagnostic. The constants required to obtain $T_e$, from the data, which had only been known approximately, are given exactly for several cases. In principle, those necessary for other antenna configurations can be found using the Green's function of Sec. II. In addition, the systematic error made in determining $\omega_p$, from the measured cone angle using the cold plasma formula can be assessed by considering the size of $\Delta \beta_1$. Since both $\omega_p$ and $T_e$ can be found from one experimental trace, resonance cone measurements are an important diagnostic technique for any magnetoplasma in which antennas can be inserted.

ACKNOWLEDGMENTS

I would like to thank Professor R. W. Gould for his advice and guidance throughout the course of this work. I am indebted to Dr. H. H. Kuehl for several cogent suggestions.

This work was supported by the U. S. Energy Research and Development Administration Contract AT(04-3)767.

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8H. H. Kuehl (private communication).
Propagation and collisionless absorption of linearly converted lower hybrid plasma waves

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(Received 18 February 1975; final manuscript received 24 July 1975)

Mode conversion and collisionless absorption of externally driven waves in the lower hybrid frequency range in an inhomogeneous plasma are described through numerical analysis of the locally evaluated linear dispersion relation for a homogeneous plasma. The effects of finite temperature and inhomogeneity of the plasma are emphasized since, according to the linear theory, these effects are responsible for the eventual absorption of the waves by the plasma. The exact forms of the dispersion relations, derived first from the two-fluid equations with an isotropic pressure law and then from the Vlasov equation, are analyzed for plasma parameters applicable to present day tokamaks. The two-fluid theory predicts that a mode conversion to an acoustic wave takes place near the lower hybrid resonance density layer. The Vlasov theory, in addition, predicts that this may be followed by a second mode conversion at a lower density. The results of the Vlasov theory also include electron Landau damping and ion cyclotron harmonic damping.

I. INTRODUCTION

Cold plasma analysis of the propagation of waves near the lower hybrid resonance frequency in an inhomogeneous plasma indicates that this frequency range may allow for very effective transport of rf power from an antenna, located just outside of a plasma column, to the lower hybrid resonance layer. Through proper adjustment of the driving frequency this layer may be located near the center of the plasma. For conditions under which damping by collisions is negligible, the further evolution of the rf power is, according to the linear theory, determined by finite temperature effects. Linear mode conversion in the presence of plasma inhomogeneity has a profound effect on the propagation, while electron Landau damping and collisionless damping directly by the ions near a harmonic of the ion cyclotron frequency account for the absorption of the wave by the plasma. For the goal of achieving selective heating of the ions of low beta plasmas through the application of rf power in the lower hybrid frequency range, this absorption directly by the ions plays a key role. Some general characteristics of the conversion and absorption of the externally driven oscillation within the nonuniform plasma can be described, in a WKB sense, through an analysis of the linear dispersion relation for waves in a homogeneous plasma of finite temperature. Such an analysis is the subject of this paper.

In Sec. II of this paper there will be studied the dispersion relation derived within the theory of the two-fluid equations including finite electron and ion temperature and a scalar pressure law, no zero-order drifts, and no zero-order electric field. The presentation is not intended to serve as a comprehensive analysis of lower hybrid waves under the two-fluid theory. However, it will provide a useful background for Sec. III, which discusses the Vlasov theory of the waves involved in the process of linear mode conversion near the lower hybrid frequency. A direct comparison will indicate that the simpler two-fluid theory may be useful for describing some properties of these waves. Of course, it fails to describe the wave-particle resonant effects which are responsible for collisionless absorption. The dispersion relation for lower hybrid waves under the two-fluid theory has been previously investigated by Glagolev. Emphasis was upon an approximate analytical treatment. The approach here will be to present numerical results obtained from the exact form of the dispersion relation derived from this theory. Most of the results will be in keeping with the conclusions of Glagolev. However, the present analysis of the exact form of the dispersion relation reveals the interesting behavior of finite temperature effects when the lower hybrid resonance accessibility condition is not satisfied. Also, unlike the analysis by Glagolev, the present study, where appropriate, will make some use of the electrostatic approximation in describing the modes of interest in the lower hybrid wave conversion problem. Where this approximation is an accurate one it can simplify the analysis considerably.

The dispersion relation for lower hybrid waves in the Vlasov theory has been previously investigated by Glagolev and later by Pěšić. The treatment by Glagolev was solely within the straight line ion orbit approximation which will be discussed briefly in Sec. IV, and which will be shown to have serious deficiencies in describing the ion-kinetic effects for these waves. Also, the work by Glagolev treated only the problem of damping in time, \( \omega(k \text{ real}) \), while for the driven wave problem, damping in space, \( k_x (\omega \text{ and } k_y \text{ real}) \), is more directly applicable. As will be shown, the Vlasov theory predicts the relatively familiar effect of linear mode conversion near the lower hybrid resonance density. However, this theory also predicts that this may be followed by a second mode conversion at a lower density. In the work by Pěšić this second conversion region was not properly identified. The analysis failed to make the critical distinction between the process of evanescence of waves near this second mode conversion region, a process which is associated with no wave absorption, and the process of collisionless damping of the waves. As a result the important new mode generated at this second mode conversion region was not described in detail. The present analysis should clarify these points.
II. THE TWO-FLUID DESCRIPTION

A. Full set of Maxwell’s equations

Plane wave solutions of the form \( \exp[i(k_x x + k_y y + k_z z - \omega t)] \) will be considered within a rectangular coordinate system in which the dc magnetic field is along the \( z \) direction. The analysis in this section is based on the linearized continuity and momentum equations for a homogeneous electron-ion plasma and no collision term is included.

\[
-i\omega n_1 + ik \cdot n_{20} v_{s1} = 0,
\]

\[
-im \cdot n_{20} v_s = n_{20} q_s(E_1 + v_{s1} \times B_0) - ik P_{s1},
\]

where

\[
k = k_x 1_x + k_y 1_y + B_0 1_z.
\]

The subscript \( s \) indicates particle species; \( 0 \) and \( 1 \) indicate zero and first order variables, respectively. The linearized scalar pressure law is assumed to be of the standard form,

\[
P_{s1} = \gamma_s kT_x n_{s1}.
\]

Although \( \gamma_s \) could be determined through a best fit with the kinetic theory, for simplicity, \( \gamma_s = 3/2 \) will be taken arbitrarily in the numerical evaluations. Since \( \gamma_s \) and \( T_x \) each appear only as a product of the two, the effect of varying \( \gamma_s \) can be determined by varying the parameter \( T_x \). In addition to these fluid equations Maxwell’s equations are required, although under the electrostatic approximation to be used later only Poisson’s equation is necessary. The actual dispersion relation may be formulated from the set of linearized algebraic equations by straightforward means. Only the results of the numerical analysis of the dispersion relation will be presented here. Several parameters used in this presentation are defined as follows:

\[
n_z = \frac{k_z c}{\omega}, \quad n_x = \frac{k_x c}{\omega}, \quad \alpha_z = \frac{k_z \nu_{hi}}{\omega}, \quad \alpha_x = \frac{k_x \nu_{hi}}{\omega},
\]

\[
\lambda_x = \frac{2kT_x}{m_x}, \quad \lambda_z = \frac{k_z^2 q_z^2}{2\nu_{cs}}.
\]

\[
\omega_{cs} = \left| \frac{q_z e}{m_z} \right|, \quad \omega_{ps} = \frac{n_{20} q_z^2}{m_z \nu_{si}}, \quad \eta = \frac{\omega}{\omega_{e1}}, \quad \Pi_z = \frac{\omega_{pz}}{\omega_{ps}}.
\]

In order to emphasize the effects of the density inhomogeneity on the driven wave in slab geometry, \( k_z \) (in terms of \( n_x \) or \( \alpha_x \) ) values obtained from the dispersion relation are plotted vs density (in terms of \( \Pi_z^2 \) ) for constant real values of \( \omega \) and \( n_x \), constant \( B_0 \), \( T_e \), \( T_i \), etc. The value of unity on the horizontal scale of the figures for this section corresponds to the density of the lower hybrid resonance as determined by cold plasma theory. The symbol \( x \) on the vertical scale indicates the \( k_z \) value at which \( \lambda_z = 1 \) i.e., where the \( x \)-directed wavelength equals the average ion Larmor radius (to within a constant factor on the order of unity).

Under the full set of Maxwell’s equations, the solutions to the dispersion relation may be expressed as the roots of a fourth order, real coefficient polynomial in the variable \( n_x^2 \). Figure 1 shows, for representative plasma parameters, the four \( n_x \) vs density branches and shows their behavior as the parameter \( n_x \) is varied. The uppermost of the four branches has, in this frequency range, pure imaginary values of \( n_x \) and is relatively independent of \( n_x \). This branch is one which exists as a consequence of finite plasma temperature. In the case of zero magnetic field and zero frequency, it would represent the phenomenon of Debye shielding for which

\[
k_x^2 = -\left( \frac{\omega_{ps}^2 + \omega_{pe}^2}{\gamma_s kT_x + \gamma_i kT_i} \right).
\]

In the present case of a finite dc magnetic field and finite frequency, this expression does not accurately describe this uppermost branch. However, this branch is of no concern to the subject of this paper.

The presence of finite temperature in the two-fluid theory does not alter the description by the cold plasma theory of the smaller \( n_x \) (longer wavelength) portions of the two lowest branches. For exact 90° propagation \( (n_x = 0) \), these are the ordinary and extraordinary modes of cold plasma theory. As in the cold plasma theory, for a small value of \( n_x \) there is a range of density below the lower hybrid resonance density in which these two branches take on complex conjugate values. Then, as \( n_x \) is increased beyond the minimum required by the accessibility condition \( (n_x^2 \geq 1 + \omega_{ps}^2/\omega_{pe}^2 \) evaluated at the lower hybrid resonance layer), \( n_x^2 \) is no longer complex in this region. In the cold plasma theory the lower hybrid resonance density layer is then “accessible” by means of a propagating wave incident from the low density region. However, for \( n_x > 1 \) there is always a narrow region adjacent to the vacuum boundary at which \( n_x^2 \) is real but negative for these two branches. In the cold
plasma theory, one of these two branches undergoes a resonance \((n_e^* = \infty)\) at the lower hybrid resonance layer for any value of \(n_e\). In the presence of finite temperature in the two-fluid theory, this branch no longer exhibits this resonance. Instead, the real \(n_e\) portion of what would be the resonant branch in the cold plasma theory joins, at large \(n_e\) values, with a short wavelength acoustic mode, the fourth mode predicted by the two-fluid theory. As shown in Figs. 1 (a–f) this new propagating mode is relatively independent of \(n_e\) at low densities. Near the lower hybrid resonance density layer it joins onto the cold plasma mode which is highly dependent upon \(n_e\).

When \(n_e\) meets the accessibility condition as in Figs. 1 (d–f), the conditions exist which are of interest for the conventional lower hybrid wave conversion problem. The acoustic mode, a forward wave in the \(x\)-directed group velocity sense, and the cold plasma mode, a backward wave, join at some density to be termed the wave conversion density. This density is always less than the lower hybrid resonance density. Beyond the conversion density, their wavenumbers comprise a complex conjugate pair, the waves are evanescent, and propagation by these two modes ceases. Still has shown the important result that under this situation, in an inhomogeneous plasma, rf power incident upon the mode conversion region by one of these two branches converts entirely onto the other branch and is directed back away from this region.

**B. Electrostatic approximation**

The remaining figures to be presented for the two-fluid theory are obtained within the simplifying electrostatic approximation in which the dispersion relation reduces from fourth to third order in \(k_x^2\). In this approximation the velocity of light \(c\) is an extraneous parameter in the dispersion relation. The four separate parameters \(n_e^*\), \(n_e\), \(T_e\), and \(T_i\) previously required in specifying the problem can be replaced by three, \(\alpha_e\), \(\alpha_i\), and \(T_e/T_i\), for example. However, in order to facilitate a direct comparison of these results with those of Sec. IIA obtained under the full set of Maxwell’s equations, one possible set of \(n_e^*, T_e,\) and \(T_i\), consistent with the fundamental \(\alpha_e\) and \(T_e/T_i\) parameters will be included in parentheses on the figures, along with a corresponding vertical \(n_e\) scale. The reader must understand that for the results under the electrostatic approximation the \(n_e\) scale applies only to the particular \(n_e^*, T_e,\) and \(T_i\) set given, while the more fundamental \(\alpha_e\) scale applies to any set of \(n_e^*, T_e,\) and \(T_i\) consistent with the given \(\alpha_e\) and \(T_e/T_i\) values.

In this third-order system derived under the electrostatic approximation, the uppermost pure imaginary branch of the previous set of four is still present essentially unchanged. For clarity, it is excluded from the figures which follow. The lowest of the previous four branches has no counterpart in the electrostatic approximation. It is the absence of this mode which accounts for the reduction of the order of the system from fourth to third. The remaining two, those which are of interest in the lower hybrid wave conversion problem, are present. Figure 2 shows the dependence of these two branches of interest upon the parameter \(\alpha_e\). A careful comparison with the curves of Fig. 1 will indicate that the results from the electrostatic approximation are consistently in close agreement for most of the upper branch, which is relatively independent of \(k_x\). For the lower branch there is close agreement only for values of the parameter \(n_e\) sufficiently large. Obviously, this minimum value of \(n_e\) required for close agreement must be at least that required to meet the accessibility condition. It can be shown, in general, that for any plasma parameters, the electrostatic approximation accurately describes this lower branch for values of \(n_e\) sufficiently large. The lower branches of Fig. 2 are, for density values somewhat below the conversion density, described closely by the cold plasma, electrostatic dispersion relation whose functional dependence upon \(k_x\) and \(k_z\) is solely through their ratio \(k_x/k_z\). The \(k_x\) dependence of these modes is to be stressed since a practical lower hybrid rf heating scheme would have an antenna which inevitably launches a \(k_x\) spectrum of waves. This finite width of the \(k_x\) spectrum is known to have a marked effect upon the lower hybrid oscillation.

In Fig. 3, with some sacrifice in generality, results are plotted relative to a fixed \(n_e\) vertical scale and the parameter \(T_i\) is varied while \(n_e\) and \(T_e\) are fixed. This is in contrast to the more general presentation relative to a fixed \(\alpha_e\) scale while the parameter \(\alpha_i\) is varied. This figure is intended to indicate directly the absolute \(x\)-wavelength dependence of the modes as \(T_i\) is varied. The portion of the lower branch away from the mode conversion density is described closely by cold plasma theory and is independent of \(T_i\). The upper branch, however, is directly dependent upon \(T_i\). It can be shown analytically that, in the limit of zero density, this branch intersects the ordinate at the value \(\alpha_i^2 = 2/\gamma_i\), i.e., where the phase velocity of the wave approaches the slowness of the ion thermal velocity. It can be expected that the kinetic theory description of these waves should, in this region, significantly differ from that of the present two-fluid treatment. Figures 2 and 3 indicate clearly that the density at the point of wave conversion decreases from the lower hybrid resonance density as either \(k_x\) or \(T_i\) is increased. Finally, Fig. 4 illustrates the very weak dependence of these results upon \(T_e\), the electron temperature. This suggests that analytical treatment under the two-fluid theory of the lower hybrid wave conversion problem may be simplified through the approximation of treating the electrons as being cold. In this limit the system reduces from third to second order in \(\alpha_i^2\), and the purely imaginary uppermost branch, excluded from these figures for clarity of presentation, actually disappears.

In the previous figures and in those to follow which are obtained from the Vlasov theory, the wavenumber \(k_y\) imposed in the \(y\) direction, the direction perpendicular to \(B_0\) and to the density gradient, is taken identically zero. However, it should be recognized that it is possible to use these figures to determine exactly the effect of nonzero \(k_y\) on the \(k_z\) vs density plots by making use of the fact that the functional dependence of
the dispersion relations on $k_x$, $k_y$, and $k_z$ is through the two parameters $k^2_x - k_y^2$ and $k_z$ only. This implies the following rule to determine the $k_x$ value in the case of a finite imposed $k_z$:

$$k_x^2 |_{k_y=0} = k_y^2 |_{k_x=0} - k_z^2. $$

An important implication is that the lower hybrid wave conversion density, while directly dependent upon the imposed $k_y$, is independent of the imposed $k_x$. Also, it follows that the accessibility condition on the magnitude of $n_e$ required to avoid a region of complex $k^2_x$ below the lower hybrid resonant density, is not changed when finite $k_y$ is included. However, the narrow low density region in which $k^2_x$ is real but negative will increase in width as $k^2_y$ is increased.

III. THE KINETIC THEORY DESCRIPTION

In this section a similar study will be made of the modes in the lower hybrid frequency range as they are described by the dispersion relation derived within the kinetic theory through the Vlasov equation. A homogeneous, magnetized plasma with Maxwellian electron and ion velocity distributions with no zero-order drifts is considered. The electrostatic approximation is used, as supported by the previous analysis within the two-fluid theory, and no collision term is included. The form of the dispersion relation is

$$D = k_x^2 + k_y^2 + \sum_{s=t,e} \sum_{m=n_e} \frac{2\omega^2}{v_{ts}^2} \exp(-\lambda_s) I_s(\lambda_s)$$

$$\times \left[ 1 + \frac{\omega}{k_x v_{ts}} \frac{Z}{k_{y,ts}} \left( \frac{\omega - i\omega_{pe}}{k_{x,ts}} \right) \right] = 0,$$

where $Z$ and $I_s$ are, respectively, the plasma dispersion function and the modified Bessel function of the first kind. This dispersion relation is treated as an expression of the functional dependence of $k^2_x$ upon all of the other parameters, and it is analyzed numerically by standard complex root finding techniques. Details of the numerical treatment of the problem may be found in Ref. 10. The present analysis will provide a more exact description, over that of the two-fluid theory, of the finite temperature effects on driven lower hybrid waves, and the description will include the important wave-particle resonant effects of collisionless damping.

In the $k_x$ vs density plots for lower hybrid waves under the kinetic, or hot plasma, theory a new propagating branch appears which generally joins with the acoustic, or warm plasma, branch which is present in the two-fluid theory. The details of this new third branch have strong dependence on the proximity of the driving frequency to a harmonic of the ion cyclotron frequency, and the dependence of this branch upon the parameter $\eta$ is nearly periodic with a period of unity. As will be shown later, the upper two of the three branches are actually ion Bernstein modes modified by the addition of a small but finite value of $k_z$. It is well known that the $\omega$ vs $k_x$ ($k_z=0$) diagrams for the ion Bernstein modes are, over a range of $\omega$, in a sense somewhat periodic in nature with the parameter $\eta$.

Figures 5(a)-(f) are presented for otherwise identical parameters except that $\eta$ is varied from 40.5 to 41.5 in increments of 0.2. Since the $\Pi^2_x$ scales are identical in each of these figures, as $\eta$ is varied they may be looked upon as modeling identical plasma and driving source conditions except for small changes in the magnitude of the dc magnetic field. Since the $\Pi^2_x$ scales are identical on Figs. 5(a)-(f), the small changes in magnetic field cause small differences in the value of $\Pi^2_x$ at which the cold plasma lower hybrid resonance layer is located. However, on each of these figures this resonance layer is located at a value within approximately 1% of unity on the abscissa. In Sec. II dealing with the two-fluid theory, only the complex magnitude of $\alpha_x$ was plotted vs density. Here, where damping may be present, the real and imaginary parts of $\alpha_x$ are shown separately. The three imaginary part branches are labeled a, b, or c indicating that they are associated with the lowest, middle, or uppermost real part branch, respectively. While the signs of the imaginary parts are not indicated on the figures, for a, b, and c they are respectively always opposite, like, and opposite in sign to the real part. This indicates respectively backward, forward, and backward waves, in the $x$-directed group velocity sense. As on the figures shown for the two-fluid theory, an $n_e$ vertical scale is marked for each imaginary branch and $e$, $T_e$, and $T_i$ values which might be chosen from the set of two
parameters $\alpha_s$ and $T_e/T_i$. Since the results of the electrostatic approximation are certainly not valid on the smaller $k_e$ values of these curves when $n_e$ is smaller than that required by the accessibility condition, the curves shown labeled $n_e=0.9$ are not entirely valid when applied for this value of $n_e$. However, the very same curves would apply for the case where the temperatures are divided by a sufficiently large factor, nine for example, in which case $n_e$ is multiplied by its square root, 3, to become 2.7 thereby rendering $\alpha_s$ unchanged. The $n_e$ scale must also be multiplied by this factor of three while the $\alpha_s$ scale remains unchanged.

Figure 5(a) begins the series with $\eta=40.5$ where the frequency lies midway between adjacent ion cyclotron harmonics. The real parts of the lowest and of the second branch, for values of $k_e$ up to the region in which the second branch joins the third branch, have a behavior similar in nature to results of the two-fluid theory under the same electrostatic approximation. The new uppermost branch joins the middle branch at some density at which their imaginary parts begin to increase from very small values to values having the same order of magnitude as the real part. Since also the middle branch represents a forward wave and the third branch represents a backward wave in the x-directed group velocity sense, it follows that the region in which the uppermost two branches join is another wave conversion region analogous to that where the lowest two branches join. This situation implies that the rf power-
directed from the low density edge toward the center of the plasma via the mode represented by the lowest branch, a backward wave, converts at the first conversion region onto the middle branch, a forward wave. This power is then directed back in the direction of decreasing plasma density via the second branch until the second mode conversion region is reached. There conversion onto the uppermost branch, a backward wave, takes place. By mean of the third branch, and when damping is absent, the power may be directed toward increasing plasma density to beyond the lower hybrid resonance density. Through this process of propagation and mode conversion in the inhomogeneous plasma, the local wavelength of the fixed frequency oscillation decreases monotonically from relatively long wavelengths, by which the rf power is launched at the outside edge of the plasma, to ultimately very short wavelengths. When the frequency is sufficiently close to a harmonic of the ion cyclotron frequency the short wavelength oscillation can be readily damped by the ions, as will be shown.

The real part of the uppermost branch on each of these curves is very weakly dependent upon \( k_z \) while an increase in \( k_z \) leads to a reduction in the width of the propagation region of the second branch. In fact, in Figs. 5(b) and 5(c), for the largest \( k_z \) values, the negatively sloped second branch is not present and the first branch joins smoothly to the third in a unidirectional manner.

The three branches in Fig. 5(a) for the two smaller \( k_z \) cases have propagating regions where the imaginary part of \( k_z \) is orders of magnitude less than its real part and is well off of the scale of the figures. In such cases where the damping in the propagation regions is negligible, two \( k_z \) branches join each other at a value of density, the wave conversion density, where effectively their real parts are equal and imaginary parts are zero. This point is a double root in the dispersion relation function \( D \). Beyond this density is an evanescent region where propagation via those two modes ceases and where the two branches take on essentially complex conjugate values with real and imaginary parts of the same order of magnitude. This behavior has been displayed in the single wave conversion region of the previously discussed two-fluid theory in which no damping mechanism was included.

The presence of damping modifies this behavior significantly. For example, for sufficiently large values of imposed \( k_z \) such that the phase velocity of the wave along the direction of the dc magnetic field approaches the slowness of the electron thermal velocity, as in the \( \alpha_z = 0.004 \) case of Fig. 5(a), a significant imaginary part to \( k_z \) is present in the propagating region of the lowest branch. This damping may be identified as electron Landau damping when \( \alpha_z \) is compared with Fig. 6. Parameters for the two figures are identical except for what is effectively a decrease in electron temperature in the latter. The real parts of the curves are essentially unchanged by the difference in electron temperature. However, the imaginary part of the lowest branch is greatly reduced.

An expression can be readily obtained which determines the electron Landau damping of the lowest branch when that branch displays otherwise cold plasma behavior which it so does before the conversion region,

\[
k_z^2 = \frac{2}{s} \frac{k_z^2}{\omega} + \frac{1}{3} \frac{2 \pi^{1/2}}{\alpha_z} \frac{\omega_e \omega}{\alpha_z k_z} \exp \left[ -\left( \frac{\omega}{\omega_{le}} \right)^2 \right],
\]

where \( S \) and \( P \) are components of the cold plasma dielectric tensor in the notation of Stix.\(^9\) It should be recognized that the first term on the right-hand side is precisely that of the cold plasma electrostatic dispersion relation term.

The damping of the third branch of Fig. 5(a) for \( \alpha_z = 0.004 \) appears to be a combination of electron Landau damping and ion cyclotron harmonic damping which will be discussed in detail. In the presence of either type of damping, there is a fundamental difference in the detail of the curves around the wave conversion regions. When damping is present, two branches no longer take on precisely equal and real \( k_z \) values at some real density value. The double root occurrence is moved to a complex value of density. Also in the evanescent regions located beyond the wave conversion region, the branches are no longer a complex conjugate pair. This behavior is clearly evident in the diagrams only in examples where the collisionless damping is particularly great as in Fig. 5(c) and 5(d). It has been shown by Moore and Oakes\(^12\) that these effects of damping lead to a reduction in mode conversion efficiency. Instead of the situation where all of the rf power incident upon the mode conversion region converges entirely onto the next branch, only a fraction converts and the difference is dissipated around the mode conversion region.

In Fig. 5(b) and 5(c) \( \eta \) is increased by small increments to approach the harmonic \( \eta = 41 \) from below. The small increases in \( \eta \) have two main effects. The real part of the uppermost branch is shifted downward slightly. This reduces the width in density of the propagating region of the second branch. At the same time there is a marked increase in the imaginary part of \( k_z \) associated with the propagating regions having large real \( k_z \) values. This damping also increases with increasing \( k_z \). The imaginary part as well as the real part of the lowest of the three branches is essentially unaffected by the small changes in \( \eta \).
In Fig. 5(e) \( \eta \) is further increased slightly to 41.1, where the driving frequency is now just above the harmonic. The real part of the uppermost branch shifts upward considerably and in turn the second wave conversion region shifts to a very low value of density. Again, with the frequency close to the ion cyclotron harmonic, the branches with large real \( k_x \) values are strongly damped. As \( \eta \) is increased slightly more to 41.3 and 41.5 in Figs. 5(e) and 5(f), the real part of the uppermost branch shifts downward by small amounts and the damping of the large real \( k_x \) branches decreases as the frequency shifts away from the cyclotron harmonic.

In this series of figures in which \( \eta \) is varied from 40.5 to 41.5 and all other parameters are fixed, the upper branches display a high degree of variation with \( \eta \). However, in the detail of the figures, the results from the case \( \eta = 40.5 \) are essentially indistinguishable from the case \( \eta = 41.5 \). In this sense the results are somewhat periodic in the parameter \( \eta \) with a period of unity. As \( \eta \) is increased beyond 41.5, similar cycles would be repeated.

It is important to note that ion cyclotron harmonic damping, damping which is strong only when the driving frequency is close to a harmonic of the ion cyclotron frequency, is significantly present only on the branches having relatively large real \( k_x \) values. Therefore, in order for the damping directly by the ions to take place, the original long wavelength oscillation must undergo transitions, brought about by the plasma inhomogeneity, which causes its \( x \)-directed wavelength to decrease considerably. That this type of damping does not take place in the long wavelength region can be explained through an examination of the dispersion relation, Eq. (3). The imaginary part which enters near an ion cyclotron harmonic originates from the term

\[
\frac{\omega}{k_x v_{ti}} Z \left( \frac{\omega - \kappa \omega_{ci}}{k_x v_{ti}} \right)
\]

Evaluating for the appropriate harmonic number \( n_2 \) which is 41 for Figs. 5–7. When the pure real argument of the \( Z \) function is sufficiently small, the function takes on an appreciable imaginary part. This term is independent of \( k_x \) and of density, yet ion cyclotron harmonic damping enters from this term only for large real \( k_x \) values. This is explained by the term

\[
\exp(-\lambda_1^e(\lambda_1^e))
\]

which multiplies the previous one as it is brought into the summation. \( \lambda_1 \) and therefore \( \alpha_x \) must be sufficiently large for this term to be significant.

The parameters of Fig. 7 are identical to those of Fig. 5(a) except for a decrease in ion temperature. The basic result of this decrease is an upward shift, relative to the \( n_e \) scale, of the upper branches and the lower wave conversion region is shifted to a higher density, closer to the lower hybrid resonance density. This behavior is similar to that under the two-fluid theory. The electron Landau damping, evident on the lowest branch in the largest \( k_x \) set, is unaffected, relative to the \( n_e \) scale, by the change in ion temperature.

Figure 8 is an example where \( \eta = 20.5 \), a value which differs significantly from that of the other figures, yet the same qualitative behavior is evident. In addition, in the course of this work, many other cases were studied in the lower hybrid frequency range for a wide range of parameters. Similar qualitative behavior was also displayed indicating that the figures presented here should be useful in describing the basic finite temperature effects of importance in the linear theory of the lower hybrid wave conversion problem.

It should be noted in the figures that the region in which \( \lambda_1 = 1 \) on the ordinate, i.e., where the local \( x \)-directed wavelength approaches the average ion Larmor radius, is not consistently related to any characteristic features of the curves. As \( \eta \) is varied from approximately 41, as in Figs. 5–7, to 20.5 as in Fig. 8, the relative location of the \( \lambda_1 = 1 \) region varies. Also the \( \lambda_1 = 1 \) region does not mark the onset of significant divergence between the results of two-fluid theory and the results of the kinetic theory. However, as in the two-fluid theory, the range \( \alpha_x = 1 \) does consistently mark a characteristic of these curves. When the real part of the propagating region of the second branch is extended by a straight line to zero density, this line intersects the zero density axis around \( \alpha = 1 \).

In each of the individual figures presented, the effects of the density inhomogeneity have been emphasized while
the dc magnetic field is held constant. Since the large real \( k_x \) branches of the curves are so sensitive to small changes in the dc magnetic field, the inhomogeneity of this field must be considered simultaneously with that of density in determining the spatial behavior of the lower hybrid oscillation after it has transformed to relatively short wavelengths. No attempt has been made to produce figures under some specified profile in which both the density and dc magnetic field are varied. They would display upper branches which are highly structured and highly dependent upon the specific profile considered. However, it can be concluded from the figures presented that the rf oscillation launched at the outside of the plasma will, primarily as a result of the inhomogeneity in density, decrease in wavelength. When the local wavelength is sufficiently small, the oscillation will damp locally about regions where the driving frequency equals a harmonic of the local value of the ion cyclotron frequency.

The waves represented by the large real \( k_x \) values of these curves propagate in a direction which is very nearly perpendicular to the dc magnetic field, i.e.,

\[
\tan^{-1}\left(\frac{k_y}{k_x}\right) = \frac{\pi}{2}.
\]

When the angle of propagation is sufficiently close to 90°, these waves have the character of ion Bernstein modes modified by the presence of small but finite value of \( k_x \). The behavior of the real parts of the portions of the upper branches of the figures where \( k_x/k_y \) is sufficiently large can be described in terms of the relatively familiar \( \omega \) vs \( k_y \) diagrams of the ion Bernstein modes for exact 90° \((k_x=0)\) propagation. However, the damping of these branches cannot be so described since perpendicular propagation implies the absence of collisionless damping. Figure 9, which is not to scale, indicates some general properties of these modes between two ion cyclotron harmonics above the lower hybrid frequency. The leading edges of the curves for various density values lie along a common envelope, while the trailing edges are asymptotic to the horizontal cyclotron harmonic frequency line lying just below the curves. The point of zero group velocity shifts to higher density values and to higher \( k_x \) values as \( \eta = (\omega_0/\omega_i) \) is increased. There is a stopband \((\omega \text{ complex})\) for frequencies between this zero group velocity point and the next higher cyclotron harmonic. The zero group velocity point corresponds closely to the upper wave conversion point on the \( k_x \) vs density plots. This explains the behavior of the \( k_x \) vs density branches that as the frequency approaches the cyclotron harmonic from above, the second conversion point shifts to lower densities. Also for a fixed density, as the frequency approaches the harmonic from above, the two upper real \( k_x \) vs density branches become more widely separated vertically.

IV. THE STRAIGHT LINE ION ORBIT TREATMENT

It is evident from these figures for the lower hybrid frequency range that on the upper branches of the \( k_x \) density curves the local wavelengths transverse to the dc magnetic field are small compared with the average ion Larmor radius \((\lambda_i \gg 1)\). Also, the frequency is much greater than the ion cyclotron frequency. Under these circumstances the ions may in some sense appear to the oscillation as having straight line zero order orbits, or equivalently as experiencing no dc magnetic field. Since this has previously been the basis of an approximation in describing lower hybrid waves, it is of interest to compare the results of Sec. III directly with those obtained from the dispersion relation derived by treating the ions explicitly in the straight line orbit approximation. This dispersion relation is given as

\[
D = k_x^2 + k_y^2 + \frac{\omega^2}{v_{fe}^2} \left[ 1 + \frac{\omega}{k_x v_{te}} \frac{Z}{k_x v_{te}} \right] \left[ 1 + \frac{\omega}{k_x v_{te}} \frac{Z}{k_x v_{te}} \right] = 0,
\]

This expression is simply the sum of the electron term from the dispersion relation previously considered, Eq. (3), the ion term from the electrostatic dispersion relation for an unmagnetized plasma, and the \( k_x^2 + k_y^2 \) term which is common to both the magnetized and unmagnetized plasma dispersion relations.

Figure 10 illustrates results for a typical case from this straight line ion orbit treatment. There is no third uppermost propagating branch. The lowest branch, its real part and its associated electron Landau damping, is essentially identical to that in the previous treatment. After undergoing wave conversion onto the second branch the wave has a phase velocity which ap-

![FIG. 9. General \( \omega \) vs \( k_x \) \((k_x=0)\) dependence, between two ion cyclotron harmonics above the lower hybrid frequency, for ion Bernstein modes. Not to scale.](image)

![FIG. 10. \( \sigma_x \) vs \( n_x^2 \) from Vlasov theory with straight line zero order ion orbits. \( \eta = 40.5, T_e/T_i = 2.17 \) (\( T_e = 2.6 \text{ keV}, T_i = 1.2 \text{ keV}\), deuterium ions.](image)
proaches, as density decreases, the ion thermal velocity. In so doing this branch becomes strongly damped by the process of direct ion Landau damping and this damping is not sensitive to the proximity of the driving frequency to an ion cyclotron harmonic. The predicted damping length becomes less than what would be an ion Larmor radius. In marked contrast to the results of Sec. III, there is no longer the high degree of structure about ion cyclotron harmonics.

Although these results differ significantly from the results of the analysis of the exact form of the dispersion relation, there may be more merit than what is evident in this straight line zero order ion orbit treatment in the problem of describing the evolution of the lower hybrid oscillation. In the previous section the zero order ion orbits are treated as exact helices. Perhaps there are cases in which deviations, which may exist, from exactly helical zero order orbits are significant. Derivation and analysis of the dispersion relation for a general type of zero order orbit would quite likely be impossible, except for the helical and straight line orbit cases. The two cases presented mark two extremes which may help to give some indication of the results for intermediate cases.

V. DISCUSSION

The results presented here have been obtained through numerical analysis of the exact forms of the expressions for the dispersion relations derived within various theories involving various approximations. They point out the general features, within each theory, of the finite temperature effects on driven waves in the lower hybrid frequency range in an inhomogeneous plasma. In this study the limitations of the two-fluid theory compared with the Vlasov theory are evident. This analysis should be of value in the selection of a theory and approximations upon which to base specific lower hybrid wave calculations. Also the numerical results should prove useful in serving as a reference for the comparison of the results of approximations which are inevitably required in more general analytical, rather than numerical, treatments of lower hybrid wave problems.

ACKNOWLEDGMENTS

This work was supported in part by National Science Foundation Grant GK-37979X1 while the author was at the Massachusetts Institute of Technology, and in part by the U. S. Energy Research and Development Administration under Contract AT (04-3)-767 at the California Institute of Technology.

Transient response from a small antenna in an anisotropic plasma

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(Received 19 November 1975)

The linear response of a cold anisotropic plasma to a point source under impulse excitation displays a frequency spectrum having maxima at the two frequencies determined by the resonance cone condition and at the upper hybrid frequency. A laboratory demonstration is reported.

This note demonstrates, through theory and experiment, that the linear response of a cold, homogeneous, anisotropic plasma to a small antenna driven by impulse excitation is directly related to the extensively studied resonance cone phenomenon. This follows from the fact that for small signal conditions the antenna-plasma configuration constitutes a linear system. Theoretical papers by Felsen and by Chen and Yen point out that the frequency components contained in the impulse response depend upon angle of observation with respect to the magnetic field and upon time, and that these frequencies may be determined from saddle points related to the cold plasma dispersion relation. The latter work contains a detailed numerical study of the location of these saddle points, and the curves therein designated as asymptotes, which correspond to long-time behavior, are, in fact, the two \( \omega \) vs \( \theta \) resonance cone branches. The theory in this report will be within the electrostatic approximation and therefore the analysis can only be valid for time...
scales which are long with respect to the time delay for propagation to the point of observation at the velocity $c$. The experiment represents the first reported laboratory observation of the transient response from a small antenna in an anisotropic plasma.

In this analysis the source term which will represent the impulse-driven point antenna within the plasma will be an external charge density $\rho(r, t) = q \delta(r) \delta(t)$, where $\delta$ indicates the Dirac delta function. A cold, homogeneous, collision-free plasma with an applied dc magnetic field along the $z$ direction is considered. Under the electrostatic approximation $E = -\nabla \phi$, and the cold plasma dielectric tensor $K$ governs the dynamics of the plasma. The elements of $K$ are expressed in terms of the Laplace transform variable $s$ rather than in terms of $\omega$. The system is initially at rest. Poisson’s equation then yields

$$\phi(r, t) = \mathcal{L}^{-1} \mathcal{L}^{-1} \left[ \phi(k, s) \right] = \mathcal{L}^{-1} \mathcal{L}^{-1} \left( \frac{q}{k \cdot K(s) \cdot k} \right)$$

(1)

where $k$ is the Fourier transform variable and $\mathcal{L}^{-1}$ and $\mathcal{L}^{-1}$ indicate inverse Laplace and Fourier transforms, respectively.

The difference between this impulse response problem and the resonance cone problem (as it is treated in Ref. 1 for the cold plasma case) lies solely in the time dependence of the source terms, $\exp(-i\omega t)$ for the resonance cone as opposed to $\delta(t)$ for the present case. The coefficient of $\exp(-i\omega t)$ in (3) of Ref. 1 may be regarded as the Laplace transform of the impulse response. Therefore,

$$\phi(r, t) = \mathcal{L}^{-1} \left( \frac{q}{4\pi \varepsilon_0} \left( (K^2 s K) (s^2 + \omega_0^2) \right)^{1/2} \right),$$

(2)

where $\rho$ is the coordinate measuring the distance to the $z$ axis, and

$$K_i = 1 + \frac{\omega_i^2}{s^2 + \omega_0^2}$$

where $\omega_i$ and $\omega_0$ are, respectively, the electron plasma and cyclotron frequencies. Ion terms are neglected which implies that the time scales being studied are short compared with the inverse of typical ion frequencies, the ion cyclotron and lower hybrid, for example.

Substitution of these into (2) results in

$$\phi(r, t) = \frac{q}{4\pi \varepsilon_0 r} \mathcal{L}^{-1} \left( s(s^2 + \omega_0^2) \right)$$

\[ \times \left[ (s^2 + \omega_0^2) \right] \left[ (s^2 + \omega_{ab}^2) \right] \left[ (s^2 + \omega_{ah}^2) \right]^{-1/2}, \]

(3)

where $r = |r|$; $\theta = \tan^{-1} p/x$; $\omega_{ab} = (\omega_i^2 + \omega_0^2)^{1/2}$, the upper-hybrid frequency; and $\omega_{ah}^2(\theta)$ and $\omega_{ah}^2(\theta)$ are the two positive roots for $\omega_0^2$ in the resonance cone angle formula, (5) of Ref. 1,

$$\sin^2 \theta = \frac{\omega_0^2 (\omega_i^2 + \omega_0^2 - \omega_{ah}^2)}{\omega_{ah}^2 \omega_0^2}.$$  

(4)

From the fact that

$$\mathcal{L}^{-1}[(s^2 + \alpha^2)^{1/2}] = J_0(\alpha t) U_1(t),$$

where $U_1(t)$ is the unit step function, it follows that

$$\phi(r, \theta, t) = \frac{q}{4\pi \varepsilon_0 r} \frac{d}{dt} \left( \frac{d^2}{dt^2} + \omega_0^2 \right)$$

\[ \times \left[ J_0(\omega_i t) U_1(t) \ast J_0(\omega_0 t) U_1(t) \ast J_0(\omega_{ab} t) U_1(t) \right], \]

(5)

where the asterisk denotes convolution. The time-domain description of the transient waveform will not be carried any further herein. The significant point is that $\phi(r, \theta, t)$ is dominated by three frequencies $\omega_i$, $\omega_0$, and $\omega_{ab}$, the singularities of the Laplace transform of $\phi$.

An experiment was performed with the afterglow of an rf-generated, argon plasma produced within a cylindrical glass vacuum chamber of 16-cm diam. A pair of solenoids mounted in a Helmholtz coil arrangement produced the dc magnetic field. Since the afterglow of the plasma decays with a time constant of about 3 msec and the impulse response measurements are
made within a 200-nsec interval, for the purposes of this experiment the pulsed plasma may be considered steady in time.

The transmitting antenna for the impulse measurement consists of a 1.8-mm o.d., semi-rigid, copper conductor, Teflon dielectric, 50-Ω coaxial cable with a 3.6-mm length of center conductor exposed to the plasma. It is directed at right angles to the magnetic field. A second antenna identical and parallel to the first is used for probing of the local plasma response.

The experiment consists of three measurements: (1) Recording of waveforms of the local plasma response to the input pulse as measured by means of the receiving probe. (2) Frequency spectrum analysis of these transient waveforms. This was performed by electronically recording the waveform and playing it back periodically into a spectrum analyzer. (3) cw measurement of the two resonance cone branches.\(^1\)

The second trace of Fig. 1 shows the vacuum response of the system. The coupling between the antennas is capacitive and as a result the received signal to the 50-Ω transmission line is in effect the time derivative of the incident pulse. Next in Fig. 1 are waveforms taken in the presence of plasma. The frequency spectrum of one of these waveforms is shown in Fig. 2. As predicted by theory, there are three frequencies which dominate the response, and these are identified as the lower and upper resonance cone branches (LB and UB), and the upper hybrid. The frequencies of the spectrum peaks for the plasma response waveforms of Fig. 1 are plotted in Fig. 3 along with the measured points of the cw resonance cone branches. The theoretical curves for these branches are determined by (4) using the value for \(\omega_c = 2\pi f_c\) as determined from a direct magnetic field measurement and for \(\omega_h\) as determined from the measured upper branch resonance cone frequency for \(\theta = 90^\circ\). The electron temperature of this plasma was determined from a Langmuir probe measurement to be approximately 0.2 eV.

As Fig. 3 indicates, the observed frequencies of the transient response are in general agreement with those of the resonance cone points, while both are in turn in general agreement with the theoretical curves.

This work was supported by the U. S. Energy Research and Development Administration under Contract AT(04-3)-787.

To appear in Physics of Fluids 18, May 1976

Eigenvalue Formulation for Linear Modes in
the Homogeneous Multiple-Fluid Plasma

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This work was supported by
the U.S. Energy Research and Development Administration
under Contract AT(04-3)-767.

California Institute of Technology
December 1975
Eigenvalue Formulation for Linear Modes in the Homogeneous Multiple-Fluid Plasma

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ABSTRACT

The linearized equations of the multiple-fluid model, including an isotropic pressure law, of a plasma are cast into a concise form directly suited for the calculation of the full set of characteristic modes of the system. A fully-ionized, homogeneous, drift-free, magnetized plasma having an arbitrary number of finite-temperature components is considered. For oscillations of the form $\exp[i(k \cdot r - \omega t)]$ the three separate problems of determining $\omega$ vs $(k, \theta)$, $k$ vs $(\omega, \theta)$, and $k_x$ vs $(\omega, k_z)$ are treated by formulating the set of algebraic equations into matrix eigenvalue problems for which the eigenvalues are $\omega^2$, $k^2$, and $k_x^2$, respectively. For each of the three cases, an example of the results of numerical evaluation of the eigenvalues is presented for an electron, two-ion species plasma. A method is described which may be used for calculating, directly within the eigensystem formulation, the group velocity of the characteristic modes.
I. INTRODUCTION

The dynamics of a magnetized plasma of $N$ components, or particle species, each having a finite temperature, are in many cases accurately described by the multiple-fluid model of the plasma along with an isotropic pressure law. While the model of an isotropic pressure law contains significant limitations such as the failure to describe the kinetic phenomena of collisionless damping and the existence of cyclotron harmonic waves, it has the feature of being considerably simpler than the kinetic theory. This often makes the approximate model deserving of attention before the kinetic theory is analyzed in the study of finite-temperature effects in plasmas. A common application of this multiple-fluid model is found in the study of the linear theory of waves in a homogeneous plasma, in which case the dispersion relation, derived from the Fourier transforms of the linearized form of the fluid equations and of Maxwell's equations, plays a fundamental role. The form of these linearized equations may appear simple. However, the process of deriving from them in their exact form the dispersion relation for a plasma of two or more finite-temperature components can readily become a most prolonged exercise in algebra.

This report describes a method of dealing with this problem in a general and systematic manner which is applicable to a fully-ionized, drift-free plasma having an arbitrary number of components. The report consists predominately of a concise mathematical treatment of the equations of this model; the question of the validity of the model will not be addressed. The approach to be taken will be to cast the system of linearized equations into the form of an algebraic matrix eigenvalue problem. The eigenvalue will represent the dependent variable of the dispersion relation for the system and the elements of the eigenvectors will represent selected...
dynamic variables of the system. The polynomial form of the dispersion relation, which is in fact the characteristic polynomial of the eigensystem matrix, will not be directly considered. With the eigensystem formulation it is possible to directly determine the solutions to the dispersion relation by calculating the eigenvalues, and the associated eigenvectors will contain information concerning the nature of the dynamic variables, i.e., polarizations, etc. for each mode of oscillation. The means of obtaining these solutions will generally be limited to numerical methods. Conveniently, there now exists highly developed, modern mathematical software¹ for solving numerical eigensystem problems², and this is readily available at many computer installations. An important additional advantage of this eigensystem formulation is, as will be shown in this paper, that the form and order alone of the related matrices imply certain general properties of the modes. These properties are generally not apparent without this formulation.

In this paper, all equations will be treated within a rectangular coordinate system in which the uniform equilibrium magnetic field is directed along the z direction. Space and time dependence of the dynamic variables is to be of the form \( \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \) where \( \mathbf{k} = k_x \hat{\mathbf{x}} + k_z \hat{\mathbf{z}} \) \((k_x = k \sin \theta, k_z = k \cos \theta)\) and \( k_y = 0 \). Three types of problems which commonly arise under these conditions will be analyzed:

1) A real value of \( k \), or equivalently of \( k \) and \( \theta \), is imposed in the plasma and \( \omega \) is to be determined for the various modes of oscillation predicted by the theory. This situation occurs in initial value problems, for example.

2) Real values for \( \omega \) and \( \theta \) are imposed, and \( k = (k_x^2 + k_z^2)^{1/2} = |\mathbf{k}| \) is to be determined. This problem is in a sense the inverse of 1), and
it arises in the calculation of index of refraction surfaces for waves in the anisotropic plasma. 3) Real values of \( \omega \) and \( k_z \) are imposed and solutions for \( k_x \) are to be found. This situation applies to the analysis of waves driven externally by a fixed frequency antenna source having an associated \( k_z \). In a manner similar to the treatment of this problem, the case of imposed real values of \( \omega \) and \( k_x \), with \( k_z \) to be determined may be analyzed, although it will not be done herein.

The first of the above three problems has been previously analyzed by means of an eigensystem formulation of the equations of the multiple-fluid model by Georges and Sakuda\(^3\) for a two-component (electron, single-ion species) plasma. Two very significant means for simplifying the formulation were overlooked in that work. In this paper the two simplifying procedures, one of which takes advantage of the specific form of the eigensystem matrices resulting from the fluid model, are described and applied to problem 1), \( \omega(k,0) \). In addition, the other two problems, 2) and 3), are similarly analyzed. In all three cases a plasma of an arbitrary number of components is considered. Whereas the formulation of reference 3 results in an eigenvalue problem involving a 14x14 real symmetric matrix, in this paper it will be shown that, for the identical problem, only a 6x6 real symmetric matrix is necessary.

An outline of the remainder of this paper is as follows. In Sec. II the precise form of the equations to be analyzed describing the dynamics of the plasma is presented. The basic definitions concerning the algebraic
eigenvalue problem are discussed in Sec. III. Also, basic matrix operations are described which can simplify a matrix eigenvalue problem of a specific form which will be found common to all three systems of equations in the three problems to be analyzed. In Sec. IV, V, and VI the eigenvalue formulations are derived for the three problems, respectively, $\omega(k, \theta)$, $k(\omega, \theta)$, and $k_x(\omega, k_z)$. Representative plots of numerical results from the three cases are given in Sec. VII where examples of waves in a three-component (electron, two-ion species) plasma are presented. These are not intended to serve as a comprehensive analysis of plasma waves under the multiple-fluid theory, because the variety of such waves is so great and their individual nature may be highly parameter dependent. Rather, the plots are intended merely to illustrate the basic nature of the results which can be readily obtained through this eigensystem formulation. Only plots of eigenvalues are presented; the eigenvectors and group velocities are not presented, although it should be recognized that they are an important part of the total description of the linear modes. The group velocities of the linear modes may be calculated directly within the eigensystem formulation through application of the results derived in the appendix.
II. BASIC EQUATIONS TO BE ANALYZED

In this analysis the equations governing the dynamics of each species of the plasma are the continuity and momentum fluid equations along with an isotropic pressure law. No collision terms will be included. The continuity and momentum equations may be written, respectively, as follows:

\[ \frac{\partial n_s}{\partial t} + \nabla \cdot n_s \mathbf{v}_s = 0 \]  

\[ n_s m_s \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla n_s = n_s q_s [\mathbf{E} + \mathbf{v}_s \times \mathbf{B}] - \nabla P_s. \]  

In the above the subscript \( s \) indicates particle species; \( m_s \) and \( q_s \) are the mass and charge of each particle species. The dynamic variables \( n_s, \mathbf{v}_s, \) and \( P_s \) are the particle density, velocity, and pressure of the fluid representing each species; \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields.

Maxwell's equations will be required.

\[ \nabla \times \mathbf{H} = \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]  

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]  

\[ \nabla \cdot \mathbf{B} = 0 \]

where \( \mathbf{B} = \mu_0 \mathbf{H}, \mathbf{j}, \) and \( \rho \), the current and charge densities, are defined in terms of the plasma variables as
Although Eqs. (5) and (6) generally follow from the continuity equation (1) and from (3) and (4) of Maxwell's equations, the explicit form of (5) and of (6) will be useful later in this paper.

In this analysis all equations are treated in their linearized, or small-signal, form in which the zero-order variables satisfy the equilibrium \((a/\partial t = 0)\) form of the equations. A homogeneous drift-free plasma is considered implying a uniform zero-order magnetic field; zero-order charge neutrality and the absence of a zero-order electric field are assumed. Under these conditions the general equilibrium solution is

\[
\vec{B}_0(\vec{r}) = B_0 \hat{z} \\
p_{s0}(\vec{r}) = p_{s0} \\
\vec{V}_{s0}(\vec{r}) = 0 \\
\vec{E}_0(\vec{r}) = 0 \\
\sum_s q_s n_{s0} = 0
\]

The subscript 0 indicates a zero-order variable, while 1 will indicate a first-order or small-signal variable. The first-order densities and pressures, \(n_{s1}\) and \(p_{s1}\), are related by the standard linearized isotropic pressure law approximation

\[
p_{s1} = \gamma_s kT_s n_{s1}
\]

in which \(\gamma_s\) is a constant of order unity and in this paper it will otherwise be left as an arbitrary parameter. \(k\) is Boltzmann's constant.
and $T_s$ is the species temperature which is also constant.

Within a rectangular coordinate system and under the

$\exp[i(k_x x + k_z z - \omega t)]$ ($k_y = 0$) dependence of the small-signal variables,

the dynamic equations are as follows:

Continuity equation:

$$\omega n_s = k_x n_s v_{x,s} + k_z n_s v_{z,s} \quad (7)$$

Momentum equation, $x$, $y$, and $z$ components:

$$\omega v_{x,s} = \frac{k_x y_s k T_s}{m_s n_s} n_s + \frac{iB_0 q_s}{m_s} v_{y,s} + \frac{i q_s}{m_s} E_x \quad (8a)$$

$$\omega v_{y,s} = -\frac{i q_s B_0}{m_s} v_{x,s} + \frac{i q_s}{m_s} E_y \quad (8b)$$

$$\omega v_{z,s} = \frac{k_z y_s k T_s}{m_s n_s} n_s + \frac{i q_s}{m_s} E_z \quad (8c)$$

Maxwell's $\nabla \times \vec{H}$ equation, $x$, $y$, and $z$ components:

$$\omega E_x = \frac{-i}{\varepsilon_0} \sum_{s} n_s q_s v_{x,s} + \frac{k_z}{\varepsilon_0} H_y \quad (9a)$$

$$\omega E_y = \frac{-i}{\varepsilon_0} \sum_{s} n_s q_s v_{y,s} - \frac{k_z}{\varepsilon_0} H_x + \frac{k_x}{\varepsilon_0} H_z \quad (9b)$$

$$\omega E_z = \frac{-i}{\varepsilon_0} \sum_{s} n_s q_s v_{z,s} - \frac{k_x}{\varepsilon_0} H_y \quad (9c)$$
Maxwell's \( \nabla \times \vec{E} \) equation, \( x, y, \) and \( z \) components:

\[
\omega H_x = -\frac{k_z}{\mu_0} E_y \\
\omega H_y = \frac{k_z}{\mu_0} E_x - \frac{k_x}{\mu_0} E_z \\
\omega H_z = \frac{k_x}{\mu_0} E_y 
\]

\( 10a \)

\( 10b \)

\( 10c \)

\( \nabla \cdot \vec{B} = 0: \)

\[
k_x H_x + k_z H_z = 0.
\]

\( 11 \)

Poisson's equation:

\[
\text{i} k_x E_x + \text{i} k_z E_z = \frac{1}{\epsilon_0} \sum_{s} q_s n_s l.
\]

\( 12 \)

The zero-order variables \( \vec{v}_{so} \) and \( \vec{E}_o \) are zero in this analysis and therefore do not appear in the equations being considered. Also, in this paper the zero-order magnetic field will always be expressed in terms of \( \vec{B}_o \) rather than \( \vec{H}_o \). Therefore, whenever any of the variables \( \vec{v}_s, \vec{E}, \) and \( \vec{H} \) appear lacking the appropriate subscript in this paper, they are to be understood as representing first-order variables.

Equations (7) through (12) are sufficient to completely describe the dynamics of the homogeneous plasma. Mathematically they constitute a linear system of algebraic equations. In Secs. IV, V, and VI of this paper this linear system will be represented in three different ways. Each will be in the form of a matrix eigenvalue problem and each will be applicable to one of three basic problems, \( \omega(k, \theta), k(\omega, \theta), \) and \( k_x(\omega, k_z) \), as discussed earlier.
III. GENERAL FORM OF THE MATRIX EIGENVALUE PROBLEM

The algebraic matrix eigenvalue problem is expressed

\[ A \hat{\Phi} = \lambda \hat{\Phi} \]  

(13)

where \( A \) is a square matrix of finite dimension \( n \) and its elements are defined in terms of the independent variables of the problem. \( \hat{\Phi} \) is a column vector of dimension \( n \), and \( \lambda \) is a scalar. Each of these three quantities may be complex, although in this paper it will be sufficient to deal with only real \( A \). Generally, \( A \) is specified and a set of \( n \) eigenvalues \( \lambda \) and a corresponding set of \( n \) eigenvectors \( \hat{\Phi} \) which satisfy this matrix equation are to be determined. In this paper the eigenvalue \( \lambda \) will represent the variable \( \omega \) (or a simple function thereof) when \( k \) and \( \theta \), or \( k_x \) and \( k_z \), are independent variables; the variable \( k \) (or a simple function thereof) when \( \omega \) and \( \theta \) are independent variables; and \( k_x \) (or a simple function thereof) when \( \omega \) and \( k_z \) are independent variables. The eigenvector \( \hat{\Phi} \) will represent in each case a selected set of dynamic variables. The method of determining directly within the matrix eigenvalue formalism the components of the group velocity vector \( \partial \omega / \partial k \) is discussed in the appendix.

In each of three cases to be studied it is possible to select variables for the elements of the vector \( \hat{\Phi} \) in such a way that the corresponding matrix \( A \) is pure real and is of the form

\[ A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \]  

(14)
where B and C are square matrices of dimension $n/2$. $n$, the dimension of the matrix $A$, will always be an even number. If $\hat{Y}$ is partitioned into two parts each of dimension $n/2$,

$$
\hat{Y} = \begin{bmatrix}
\hat{Y}_1 \\
\hat{Y}_2
\end{bmatrix},
$$

then the eigenvalue problem is written

$$
\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\hat{Y}_1 \\
\hat{Y}_2
\end{bmatrix} = \lambda
\begin{bmatrix}
\hat{Y}_1 \\
\hat{Y}_2
\end{bmatrix}.
$$

(15)

For any eigenvalue $\lambda$ and eigenvector $\hat{Y}$ of a matrix of this form, direct substitution indicates that $-\lambda$ is also an eigenvalue with an identical eigenvector, except $\hat{Y}_1$ is replaced by $-\hat{Y}_1$.

Now, pre-multiply both sides of Eq. (13) by $A$,

$$
A^2 \hat{Y} = \lambda A \hat{Y} = \lambda^2 \hat{Y}.
$$

This indicates that if $\lambda$ is an eigenvalue of $A$, then $\lambda^2$ is an eigenvalue of $A^2 = AA$. They share common eigenvectors. Applying this operation to Eq. (15),

$$
\begin{bmatrix}
B & C & 0 \\
0 & C & B
\end{bmatrix}
\begin{bmatrix}
\hat{Y}_1 \\
\hat{Y}_2
\end{bmatrix} = \lambda^2
\begin{bmatrix}
\hat{Y}_1 \\
\hat{Y}_2
\end{bmatrix}.
$$

(16)
This may be separated into two individual eigenvalue problems of order \( n/2 \),

\[
B C \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \lambda^2 \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \quad (17a)
\]

\[
C B \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \lambda^2 \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}. \quad (17b)
\]

Next it will be shown that the characteristic polynomial of \( A \) is identical to within a minus sign to the characteristic polynomial of \( B C \) and of \( C B \) when each is written as a polynomial in \( \lambda \). This will imply that the set of eigenvalues of \( B C \) is identical to that of \( C B \). Also the set of eigenvalues of \( A \) is given by \( \pm (\lambda^2)^{1/2} \) where \( \lambda^2 \) is the set of eigenvalues of \( B C \) or of \( C B \). In any case non-distinct \( \lambda^2 \) are counted to the multiplicity to which they appear. The characteristic polynomial of \( A \) is defined

\[
p(\lambda) \equiv \det \begin{bmatrix} -\lambda & I \\ B & C \end{bmatrix}.
\]

Row operations which do not change the value of the determinant may be performed upon this matrix.

\[
p(\lambda) = \det \begin{bmatrix} -\lambda & I + \frac{1}{\lambda} & B & C \\ 0 & C & -\lambda \end{bmatrix}.
\]

The determinant is then expanded in terms of the submatrices,

\[
p(\lambda) = \det \{( -\lambda I)(\frac{1}{\lambda} I + B C - \lambda I)\} = \det(\lambda^2 I - B C)
\]

\[
= (-1)^{n/2} \det(B C - \lambda^2 I),
\]
and the last expression in the characteristic polynomial of $BC$ except for the factor $(-1)^{n/2}$. A similar result can be derived for the matrix $CB$. The eigenvectors $\hat{V}_1$ and $\hat{V}_2$ are related, as can be verified by direct substitution into (15), by

$$\hat{V}_2 = \frac{1}{\lambda} C \hat{V}_1$$

(18a)

$$\hat{V}_1 = \frac{1}{\lambda} B \hat{V}_2$$

(18b)

The problem of eigenvectors associated with non-distinct eigenvalues will be discussed in Secs. IV, V, and VI, which deal with specific matrices.

It has been shown in this section that an eigenvalue problem in which the matrix $A$ is in the form of (14) can be replaced by a single matrix eigenvalue problem of half the dimension. The next three sections of this paper apply this result to three representations of the set of equations of Sec. II.

IV. $\omega(\vec{k})$

In this case $\omega$ is the dependent variable and will therefore be the matrix eigenvalue, while $k_x$ and $k_z$ are real independent variables of the matrix $A$. For this case Eqs. (7), (8a-c), (9a-c), and (10a-c) are sufficient to formulate an eigenvalue problem. These equations have been written in a manner which allows them to be put directly in that form. That is, the scalar $\omega$ has been isolated to one side of each of the equations and multiplies one of each of the dynamic variables appearing in
the equations. Each of these variables may be made the individual elements of a vector \( \vec{Y} \), and one side of these equations then becomes the term \( \omega \vec{Y} \). For an \( N \)-component plasma there are a total of \( 4N+6 \) equations and as many elements making up the vector \( \vec{Y} \). The right-hand side of (7) through (10) may be written as \( A \vec{Y} \) where \( A \) is a square matrix of dimension \( n = 4N+6 \), and \( A \) consists only of pure real and pure imaginary elements. Through a simple change of variables of some of the elements of \( \vec{Y} \) consisting of multiplying some of them by pure real or imaginary factors, the resulting matrix \( A \) can be made pure real and symmetric. This property of \( A \) implies that its eigenvalues \( \omega \) are pure real.

The procedure outlined in the preceding paragraph has been carried out in Ref. 3 for a two-component (\( N = 2 \)) plasma. The generalization to an \( N \)-component plasma is straightforward. Also, numerical results from the formalism are presented therein describing various modes of oscillation for several sets of plasma parameters. While the derived 14x14 real symmetric matrix in Ref. 3 may lend itself directly to numerical eigensystem analysis, it demands an excessive and possibly a prohibitive amount of numerical computation. Also, two of the 14 eigenvalues of the matrix have no physical significance whatsoever. These two eigenvalues are a pair of zeros. Their corresponding eigenvectors lie in the null space of \( A \), i.e., \( A \vec{Y} = \vec{0} \). The elements of these eigenvectors do not satisfy \( \nabla \cdot \vec{B} = 0 \) or Poisson's equation. This problem arises because, for \( \omega = 0 \), these two conditions do not necessarily follow from (7), (9), and (10). This problem can be avoided by using the explicit form of Poisson's equation, (12), and of \( \nabla \cdot \mu_0 \vec{H} = 0 \), (11), to eliminate from (7) through (10) the two variables \( H_x \) and \( E_x \) when \( k_x \)
is nonzero, or \( H_z \) and \( E_z \) when \( k_z \) is nonzero. Here, it will be assumed that \( k_x \) is finite and \( E_x \) and \( H_x \) will be eliminated by means of the following substitutions:

From Poisson's equation

\[
E_x = -\frac{k_z}{k_x} E_z - \frac{1}{k_x \varepsilon_0} \sum_s q_s n_s
\]

(19)

and from \( \nabla \cdot \mathbf{u}_0 \hat{n} = 0 \)

\[
H_x = -\frac{k_z}{k_x} H_z.
\]

(20)

Equations (9a) and (10a) are then excluded from the resulting system of equations. These operations decrease by two the dimension of the eigenvalue problem to \( n = 4N + 4 \).

For the analysis of the resulting equations, the \( 4N+4 \) elements of \( \hat{\mathbf{\gamma}} \) will be defined in form and in order as follows:

\[
\hat{\mathbf{\gamma}}^t = [i \mu_0^{1/2} H_z \varepsilon_0^{1/2} E_z \{ \gamma_s \mathbf{m}_s \}^{1/2} \mathbf{v}_{y,s} \{ i \mathbf{\gamma}_s \mathbf{k}_T \}^{1/2} \mathbf{n}_s]
\]

\[\begin{align*}
\{i \mu_0^{1/2} H_y \varepsilon_0^{1/2} E_y \{ i \gamma_s \mathbf{m}_s \}^{1/2} \mathbf{v}_{x,s} \{ i \mathbf{\gamma}_s \mathbf{k}_T \}^{1/2} \mathbf{n}_s\}
\end{align*}\]

(21)

where the inner brackets indicate a set of \( \mathbf{N} \) elements which consist of the enclosed term for each value of the species index \( s \). These elements of \( \hat{\mathbf{\gamma}} \) will be represented symbolically

\[
\hat{\mathbf{\gamma}} = [i H_z \quad E_z \quad \{ \mathbf{v}_{y,s} \} \quad \{ i N_s \}]
\]

\[\begin{align*}
H_y \quad i E_y \quad \{ i \mathbf{v}_{x,s} \} \quad \{ i \mathbf{v}_{z,s} \}
\end{align*}\]

(22)
All of the elements of $\hat{\mathbf{Y}}$ have been defined to have the same units, energy volume density to the one-half power, and therefore the elements of $A$ must all have the same units as $w$. With this definition of $\hat{\mathbf{Y}}$ the resulting eigenvalue problem is written

$$A \hat{\mathbf{Y}} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \omega \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \omega \hat{\mathbf{Y}}$$

(23)

where

$$A = \begin{bmatrix} 0 & kc \sin \theta \\ -kc \sin \theta & 0 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \hat{\theta} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & \varepsilon_s \Pi_s \\ -\varepsilon_s \Omega_s \delta_{st} & 0 \end{bmatrix}$$

(24)
\[ \mathbf{C} = \begin{bmatrix} 0 & \frac{kc}{\sin \theta} & \hat{0} \\ \frac{kc}{\sin \theta} & 0 & \epsilon_t \Pi_t \\ \epsilon_t \Pi_t & 0 & \epsilon_t \Pi_t \end{bmatrix} \]

\[ \mathbf{P} = \begin{bmatrix} \hat{0} \\ \frac{\epsilon_s \Pi_s}{\tan \theta} & -\epsilon_s \Omega \delta_{st} \\ -\epsilon_s \Pi_s & 0 \end{bmatrix} \]

\[ \mathbf{C} = \begin{bmatrix} \hat{0} \\ -\epsilon_s \Pi_s & 0 \\ 0 & k_c \cos \theta \delta_{st} \end{bmatrix} \]

(25)

\[ \hat{\mathbf{Y}}_1 \] and \[ \hat{\mathbf{Y}}_2 \] consist of the first and second halves, respectively, of \( \hat{\mathbf{Y}} \) as defined in (22).
In the submatrices into which \( B \) and \( C \) are partitioned, the subscript \( s \) indicates column position, the subscript \( t \) indicates row position, and these indices run from 1 to \( N \), the number of particle species. Also,

\[
\delta_{st} = \begin{cases} 
1 & , s = t \\
0 & , s \neq t 
\end{cases} \quad \text{Kronecker delta function}
\]

\[
\epsilon_s = \text{sgn} \left( q_s \right) \quad \text{sign of the species' charge}
\]

\[
\Omega_s = \left( \frac{n_{so} q_s^2}{m_s e_o} \right)^{1/2} \quad \text{plasma (radian) frequency}
\]

\[
\Omega_s = \left( \frac{q_s B}{m_s} \right) \quad \text{cyclotron (radian) frequency}
\]

\[
\sqrt[3]{\gamma_s kT_s} \quad \text{thermal velocity}
\]

\[
c = \frac{1}{\left( \mu_0 e_o \right)^{1/2}} \quad \text{velocity of light in vacuum}
\]

It should be noted that the matrix \( A \) which follows from the substitutions of (19) and (20) is real and reduced in dimension by two over that which would be obtained without these substitutions; but at the same time it is no longer symmetric since

\[
B^t \neq C
\]

Now, through the simplification outlined earlier for problems of the form
of (15) two separate eigenvalue problems each of order $2N$ follow

$$\omega^2 \vec{Y}_1 = B C \vec{Y}_1$$

$$\omega^2 \vec{Y}_2 = C B \vec{Y}_2$$

The results of the multiplication of matrices are

$$B C = \begin{bmatrix}
  \begin{array}{cc}
    k^2 c^2 & 0 \\
    0 & k^2 c^2 + \sum_s \Pi_s^2 \\
  \end{array}
  & \begin{array}{cc}
    kc \epsilon_t \Pi_t \sin \theta & 0 \\
    0 & kc \epsilon_t \Pi_t (c_t^2 - 1) \cos \theta \\
  \end{array}
  \\
  \begin{array}{cc}
    \frac{k c c_s \Pi_t}{\sin \theta} & -\frac{\Pi_s \Omega_s}{\tan \theta} \\
    -\frac{\Pi_s \Omega_s}{\tan \theta} & \Pi_s \Omega_s \epsilon_t \Pi_t \\
  \end{array}
  & \begin{array}{cc}
    \frac{\Omega_s \epsilon_t \Pi_t}{kc_t \sin \theta} & -kc_s \epsilon_s \Omega_s \sin \theta \delta_{st} \\
    -kc_s \epsilon_s \Omega_s \sin \theta \delta_{st} & k^2 c_s^2 \delta_{st} + \epsilon_s \epsilon_t \Pi_t \Pi_s c_t \\
  \end{array}
\end{bmatrix}$$

(26)
\[
C_B = \begin{bmatrix}
\kappa c^2 & 0 & \kappa c \varepsilon_t \Pi_t \cos \theta & \kappa c \varepsilon_t \Pi_t \sin \theta \\
0 & \kappa c^2 + \sum_s \Pi_s^2 & -\Pi_t \Omega_t & 0 \\
-\kappa c \varepsilon_s \Pi_s \cos \theta & -\Pi_s \Omega_s & (\Omega_s^2 + \kappa c^2 \sin^2 \theta) \delta_{st} + \varepsilon_s \varepsilon_t \Pi_s \Pi_t & \kappa c \varepsilon_s \Pi_s \cos \theta \sin \theta \delta_{st} \\
\kappa c \varepsilon_s \Pi_s \sin \theta & 0 & \kappa c \varepsilon_t \Pi_t \cos \theta \sin \theta \delta_{st} & \kappa c \varepsilon_t \Pi_t \cos \theta \sin \theta \delta_{st}
\end{bmatrix}
\]
It should be noted that $C B$ is a real symmetric matrix, while $B C$ is real but not symmetric. Therefore, for purposes of general numerical computation of the eigenvalues and eigenvectors of the dynamic system, the matrix $C B$ would be preferred over $B C$ since there is available eigensystem mathematical software specialized to real symmetric matrices. Its use would be more efficient over methods suited for more general matrices. The symmetric property of $C B$ implies that to any multiple or non-distinct eigenvalues of this matrix there are associated as many linearly independent eigenvectors. The symmetric property of $C B$ also implies that its eigenvalues, the set of $\omega^2$, are pure real. In addition they must be positive since, as explained earlier, the formalism of Ref. 3 implies that all values of $\omega$ for that system are real, and except for the pair of zero eigenvalues which have been eliminated in the present formalism, the same solutions for $\omega$ must result in the two formulations. That its eigenvalues are positive implies that this symmetric matrix $C B$ is positive definite, although it is not apparently a simple matter to show that this follows from the specific form of $C B$.

V. $k(\omega, \theta)$

Here, $k = |\vec{k}|$ is the dependent variable, and $\omega$ and $\theta$ are independent variables which will be restricted to real values. For this problem, $k(\omega, \theta)$, the selection of equations to be included in the matrix eigenvalue problem is less direct than for that of the previous section, $\omega(k, \theta)$. In Eqs. (7) through (12) $k_x$ and $k_z$ are replaced by $k \sin \theta$ and $k \cos \theta$. The difficulty in selecting
equations for the eigenvalue problem arises because in Eqs. (7), (9b), and (10b) following this substitution, the dependent variable $k$ appears as a coefficient of two dynamic variables rather than always of a single one as in the previous case. 

The derivation of a suitable set of equations is outlined here.

From (10b) and (12) obtain

$$kE_x = \omega_0 \cos \theta H_y - i \sin \theta \frac{1}{\varepsilon_0} \sum q_s \n_s l$$  

(28a)

From (10c):

$$kE_y = \frac{\omega_0}{\sin \theta} H_z$$  

(28b)

From (8c), (12), (8a), (8b), (7), (33a), and (33b):

$$k v_{xs} = \frac{i q_s}{d_s m_s} \cos \theta \omega_0 H_y - \frac{1}{d_s} \frac{\varepsilon s \Omega_s}{\omega} \cos \theta \frac{q_s}{m_s} \frac{\cos^2 \theta}{\sin \theta} \mu_0 H_z + \frac{1}{d_s} \frac{\omega \sin \theta}{n_s n_0} \n_s l$$  

(28c)

where

$$d_s = 1 - \cos^2 \theta \frac{\Omega_s^2}{\omega^2}$$

From (9a):

$$k H_y = \frac{\omega \varepsilon_0}{\cos \theta} E_x + \frac{i}{\cos \theta} \sum \n_s q_s v_{xs}$$  

(28d)

From (9b), (11), and (8b):

$$k (\sin \theta)^{-1} H_z = \omega (1 - \sum \frac{\Omega^2_s}{\omega^2}) \varepsilon_0 E_y + \varepsilon_0 B_0 \sum \frac{\Omega^2_s}{\omega} v_{xs}$$  

(28e)

From (8a) and (8b):
\[
kn_{s1} = \frac{-i}{\sin \theta} \frac{q_s}{m_s} \frac{n_{so}}{c_s^2} E_x + \frac{1}{\sin \theta} \frac{q_s^2 B_0}{m_s^2} \frac{n_{so}}{c_s^2 \omega} E_y + \frac{1}{\sin \theta} \frac{n_{so}}{c_s^2 \omega} (\omega^2 - \Omega_s^2) v_{x,s} \tag{28f}
\]

These may now be written directly in the form of (15). The elements of \( \mathbf{Y} \) and \( \mathbf{Y}^t \) will be defined as follows, each having the units of energy volume density to the one-half power:

\[
\mathbf{Y}^t_1 = \left[ i \varepsilon_0^{1/2} E_x \varepsilon_0^{1/2} E_y \{ (n_s n_{so})^{1/2} v_{x,s} \} \right] = \left[ i E_x E_y \{ V_{x,s} \} \right] \tag{29}
\]

\[
\mathbf{Y}^t_2 = \left[ i \mu_0^{1/2} H_y (\sin \theta)^{-1} \mu_0^{1/2} H_z \{ \frac{Y_{s T}^{K}}{n_{so}} n_s \} \right] = \left[ i H_y (\sin \theta^{-1}) H_z \{ M_s \} \right].
\]

Since \((\sin \theta)^{-1}\) appears as a coefficient of \(H_z\) in the definition of \(\mathbf{Y}^t_2\), it appears that the formalism may fail for \(\theta = 90^\circ\). However, (11) indicates that

\[(\sin \theta)^{-1} \mu_0 H_z = -(\cos \theta)^{-1} \mu_0 H_x\]

and either side of this equation may be taken interchangeably as the definition of the element of \(\mathbf{Y}^t_2\). Note that \(\mathbf{Y}^t_1\) and \(\mathbf{Y}^t_2\) are of dimension \(N+2\), in contrast to the \(\omega(k,\theta)\) problem of the previous section where they were of dimension \(2N+2\). Since \(\omega\) may be treated as real constant, the index of refraction \(n = kc/\omega\) will be used in place of \(k\) as is commonly done in the literature. Since \(n\) is dimensionless, then all of the elements of \(\mathbf{A}\) are also. The resulting system is of
the form

\[
\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\vec{y}_1 \\
\vec{y}_2
\end{bmatrix}
= n
\begin{bmatrix}
\vec{y}_1 \\
\vec{y}_2
\end{bmatrix}
\]

and according to (16) and (17)

\[
B \cos \theta \vec{y}_1 = n^2 \vec{y}_1
\]
\[
C \sin \theta \vec{y}_1 = n^2 \vec{y}_1
\]

\[
B =
\begin{bmatrix}
\cos \theta & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\sin \theta \frac{\epsilon t \Omega t}{c} \\
\frac{c}{c_t}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos \theta \frac{\epsilon_s \Omega s}{\omega} \\
\frac{-\cos^2 \theta}{d_s} \frac{\Pi s_s}{\omega^2} \\
\frac{\sin \theta \frac{c}{c_s}}{d_s}
\end{bmatrix}
\]

(30)
\[ C = \begin{bmatrix}
\frac{1}{\cos \theta} & 0 & -\frac{1}{\cos \theta} \frac{\epsilon t \Omega t}{\omega} \\
0 & 1 - \sum s \frac{\Pi_s^2}{\omega^2} & \frac{\Pi t \Omega t}{\omega^2} \\
-\frac{1}{\sin \theta} \frac{\epsilon_s \Pi_s}{\omega} c_s & \frac{1}{\sin \theta} \frac{\Pi_s \Omega_s}{\omega^2} c_s & \frac{1}{\sin \theta} \frac{c_s}{c_s} (1 - \frac{\Omega_s^2}{\omega^2}) \delta_{st}
\end{bmatrix},
\]

(31)
\[
C_B = \begin{bmatrix}
1 - \sum_s \frac{\pi_s^2}{\omega^2} \frac{1}{d_s} & \cos \theta \times \\
& \sum_s \frac{\pi_s^2 e_s \Omega_s}{\omega^3} \frac{1}{d_s} - \sin \theta \cos \theta \frac{\varepsilon_t \Pi_t \Omega_t^2}{d_t} \frac{1}{\omega^3}
\end{bmatrix}
\]
As in the case treated in Sec. IV, C B is real and symmetric, while B C is real but not symmetric. However, in the present case C B is not in general positive definite, as evidenced by negative real solutions for $k^2$ to be shown in the examples in Sec. VI. As in the previous section, the symmetric property of C B implies that to any non-distinct eigenvalues which occur, there are as many linearly independent eigenvectors.

VI. $k_x(\omega, k_z)$

In this section real values of $\omega$ and $k_z$, the independent variables, are to be imposed, and $k_x$, the dependent variable, is to be determined. Again, indices of refraction $n_x = \frac{k_x}{\omega}$ and $n_z = \frac{k_z}{\omega}$ will be used in place of the parameters $k_x$ and $k_z$, respectively. The equations to appear explicitly in the matrix eigenvalue problem are those of (7) through (10) which have $k_x$ as a coefficient, namely, (7), (8a), (9b,c), and (10b,c). There are $2N+4$ such equations in this case. The remaining of (7) through (10) are used to eliminate from these equations the dynamic variables which do not appear as elements of $\vec{Y}_1$ or $\vec{Y}_2$ which are defined as follows:

$$\vec{Y}_1 = \left[ \varepsilon_0 E_y, -i\mu_0 H_y \right] \left\{ (m_s n_s)^{1/2} v_{x,s} \right\} = [E_y, -iH_y, \{v_{x,s}\} ] \quad (34a)$$

$$\vec{Y}_2 = \left[ \varepsilon_0 H_z, i\varepsilon_0 E_z \right] \left\{ (\frac{\gamma_s^T n_s}{n_s})^{1/2} n_s \right\}$$

$$= [H_z, iE_z, \{n_s\}] \quad (34b)$$

Each of these vectors is of dimension $N+2$ as in the $k(\omega, \theta)$ problem of the previous section. Again, the elements of these vectors are each de-
fined as having the units of energy density to the one-half power. With $n_x$ as the eigenvalue and with $\vec{Y}_1$ and $\vec{Y}_2$ as defined in (34), the eigenvalue problem for this case is

$$
\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\vec{Y}_1 \\
\vec{Y}_2
\end{bmatrix} = n_x
\begin{bmatrix}
\vec{Y}_1 \\
\vec{Y}_2
\end{bmatrix}
$$

and according to (16) and (17),

$$
B C \vec{Y}_1 = n_x^2 \vec{Y}_1
$$

$$
C B \vec{Y}_2 = n_x^2 \vec{Y}_2
$$

The elements of $A$ all have the dimensions of $n_x$, which is actually dimensionless. The matrices $B$ and $C$ of dimension $N+2$ consist of only real elements and are defined.
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 - \sum \frac{\Pi_s^2}{s \omega}
\end{bmatrix}
\begin{bmatrix}
\vec{0} \\
- \varepsilon_t \frac{\Pi_t}{\omega} n_z
\end{bmatrix}
\]

\[B = \begin{bmatrix}
1 & 0 \\
0 & - \varepsilon_s \frac{\Pi_s}{\omega} \times n_z
\end{bmatrix}
\begin{bmatrix}
\frac{\Pi_s}{\omega} \\
\frac{\Pi_t}{\omega} \delta_{st}
\end{bmatrix}\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
- \sum \frac{\Pi_s^2}{s \omega} & 0 & 0 \\
0 & 1 - n_z^2 & 0
\end{bmatrix}
\begin{bmatrix}
\Pi_t \Omega_t \\
0 \\
\varepsilon_t \frac{\Pi_t}{\omega} n_z
\end{bmatrix}
\]

\[C = \begin{bmatrix}
\varepsilon_s \frac{\Pi_s}{\omega} \\
\frac{\Pi_s}{\omega} \times n_z \\
(1 - \frac{\Omega_s^2}{\omega^2}) \delta_{st} \varepsilon_s \varepsilon_t \frac{\Pi_s}{\omega} \frac{\Pi_t}{\omega}
\end{bmatrix}\]
The results of the matrix multiplications are

\[
B C = \begin{bmatrix}
1 - n_x^2 - \sum \frac{\pi_s^2}{\omega^n} & 0 & \frac{\Pi_t}{\omega} \Omega_t \\
-\sum \frac{\epsilon_s \omega_s}{\omega} \frac{\pi_s^2}{\omega^n} n_x & 1 - n_x^2 - \sum \frac{\pi_s^2}{\omega^n} & \frac{\Pi_t}{\omega} \Omega_t \\
\frac{\pi_s}{\omega_s} \Omega_s \frac{\epsilon_s^2}{c_s^2} (n_x^2 - 1) & \frac{\pi_s}{\omega_s} n_x \frac{\epsilon_s^2}{c_s^2} - 1 & \frac{\pi_s}{\omega_s} \Omega_s \delta_s t \\
\epsilon_s \frac{\pi_s}{\omega_s} \frac{\epsilon_s^2}{c_s^2} - n_x^2 (1 - \frac{\Omega_s^2}{\omega_s^2}) & \epsilon_s \frac{\pi_s}{\omega_s} \frac{\epsilon_s^2}{c_s^2} - n_x^2 & -\epsilon_s \frac{\pi_s}{\omega_s} \frac{\epsilon_s^2}{c_s^2} \frac{\Omega_s^2}{\omega_s^2}
\end{bmatrix}
\]  

\(38\)

and

\[
C B = (B C)^T
\]  

\(39\)

This condition follows from the fact that B and C are both symmetric matrices. Neither of the real matrices B C and C B is symmetric, and therefore \(n_x^2\) may, in general, be complex. However, since these matrices
are pure real, \( n_x^2 \), if complex, must appear in complex conjugate pairs. When non-distinct eigenvalues occur in this system it is, in general, difficult to determine the number of linearly independent eigenvectors associated with such eigenvalues. However, each mode described in this \( k_x(\omega, k_z) \) formalism is also contained, at least when \( k_x \) and \( k_z \) are real, in the \( \omega(k, \theta) \) and \( k(\omega, \theta) \) formalisms which are in terms of symmetric matrices and therefore which contain a complete set of linearly independent eigenvectors. Therefore, for real non-distinct eigenvalues in the \( k_x(\omega, k_z) \) formalism the number of associated linearly independent eigenvectors is equal to the multiplicity of the eigenvalue for that mode in the \( \omega^2 \) or \( k^2 \) formalisms.

VII. EXAMPLES OF MODES IN A THREE-COMPONENT PLASMA

Examples of numerical solutions of each of the three eigenvalue problems formulated are presented in this section. In each case a three-component plasma consisting of one electron and two positive ion species is considered. Waves in a three-component plasma are less commonly treated in the literature than are those in a two-component plasma. Also, some of the modes may not be physically accurate as a result of failure of isotropic pressure law approximation. However, it is not the intention of this paper to discuss in any detail the physical nature of these modes. Rather, it is merely to indicate that such modes which are predicted by the multiple-fluid theory may conveniently be studied numerically through the application of the derived matrix eigenvalue formulations. Although this section is restricted to a study of eigenvalues of the linear mode problem, it should be recognized that the corresponding eigenvectors and group velocity provide a further description of these modes.
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The second example, Fig. 2, applies to the problem $k(\omega, \theta)$ discussed in Sec. V. All plasma parameters and a chosen frequency expressed in terms of $\omega/\Omega_e$ are held constant, and $k$ is determined as a function of $\theta$ the angle of propagation. The plasma parameters taken are identical to those of Fig. 1. For the three-component plasma the eigensystem problem in this case is of fifth order. Thus there are five real branches for $k^2$, although some may be negative implying pure imaginary values of $k$. Each of the five branches represents a pair of positive and negative real or imaginary solutions. It is interesting to note that cold plasma theory predicts only two branches regardless of the number of component species. In Fig. 2 the wavenumber $k$ is normalized to the index of refraction $n = kc/\omega$. If these branches were presented versus angle on a polar plot rather than in rectangular coordinates, the result would be what is termed the index of refraction surfaces.

In Fig. 2 the lowest propagating branch is identified as the whistler wave, for angles well below $90^\circ$. This branch is not significantly affected by the finite plasma temperature. The upper two propagating branches may be termed acoustic in nature since the magnitudes of their phase velocities are of the order of those of the ion thermal velocities. According to the Vlasov theory, these modes may actually be strongly damped by means of collisionless absorption. It is noted that for $\theta = 30^\circ$, the angle taken for Fig. 1, the real solutions for $n (= kc/\omega)$ in Fig. 2 are in precise agreement with the corresponding $k$ values in Fig. 1 at the $\omega/\Omega_e$ value of Fig. 2. This must be the case since the results of both figures are derived from the identical set of equations. For the parameters of Fig. 2 at least two of
the five branches represent evanescent waves, i.e., waves characterized by imaginary values for their wavenumbers. However, there do exist parameter ranges in which all five branches are simultaneously real over a finite range of angle.

Results from the $k_x(\omega,k_z)$ problem are presented in Fig. 3. The wavenumbers $k_x$ and $k_z$ are normalized to indices of refraction $n_x$ and $n_z$. The magnetic field $B_0$, or rather the dimensionless parameter $\Omega_e/\omega$, is varied while all other independent variables, $n_z$, $\omega$, density, etc., are held constant. There are five branches for $n_x^2$, and these may in general assume complex conjugate values. In Fig. 3 the magnitude of $n_x$, $|n_x|$, is plotted on a log scale vs $\Omega_e/\omega$ on a linear scale. Each $n_x$ branch shown represents a pair of $n_x$ solutions which are negatives of each other. Complex $n_x$ branches represent in addition complex conjugate pairs.

The lowest propagating branch, which actually appears in two segments, is identified as the fast or compressional Alfvén wave. However, near the value $\Omega_e/\omega = 2.3 \times 10^3$ the left-hand segment makes a nearly vertical downward transition toward zero, while the right-hand segment makes a nearly vertical upward transition to an acoustic type of mode. The region of this upward transition is, in fact, that of the two-ion hybrid resonance, and this type of transition from one type of wave to another is termed a mode conversion. In the limit of zero temperature this acoustic mode would shift vertically upward and disappear, and a true 90° resonance, $n_x \rightarrow \infty$, would take place at the ion-ion hybrid resonance. The remaining branches for the particular parameter range are imaginary or complex and not of direct physical interest.
VIII. DISCUSSION

An objective of this work has been to present the equations of the multiple-fluid theory of linear waves in a homogeneous plasma in forms which are concise and relatively general in applicability. Yet, it is sometimes useful to apply to the theory certain approximations which cannot be readily incorporated into the specific eigensystem matrices presented in Secs. IV through VI. One such example is the electrostatic approximation in which the electric field is expressed as the gradient of a scalar potential, \( \mathbf{E} = -\nabla \phi \); Poisson's equation, Eq. (12), replaces the remainder of Maxwell's equations; and the first order magnetic field \( \mathbf{H}_1 \) may be neglected. For each of the three cases, \( \omega(k,\theta) \), \( k(\omega,\theta) \) and \( k_x(\omega,k_z) \), one may derive the corresponding eigensystem formulations by starting from the basic physical equations which explicitly include this electrostatic approximation. The orders of the resulting systems will be reduced over those of the systems derived in this paper.

Another case for which the matrices derived in this paper do not directly apply is that of the zero-temperature limit of one or more of the particle species. In this limit some of the matrix elements become infinite in the \( k(\omega,\theta) \) and \( k_x(\omega,k_z) \) formulations of this paper and at the same time some of the eigenvalues become infinite. Also, in the zero-temperature limit for at least two species simultaneously some of the eigenvalues in the \( \omega(k,\theta) \) problem become zero. For these cases of zero species temperature certain dynamic variables related to the cold species may be eliminated in the original equations through algebraic operations. The resulting set of equations may again be expressed as an eigenvalue formulation of reduced order. These operations are somewhat involved and
will not be presented, although the resulting orders of the three systems are given as follows: For $\omega(k,0)$ the order of the eigenvalue problem in $\omega^2$ is $2+2N$ when none or only a single species is cold; and $3+2N-N_C$ where $N_C$ is the number of cold species, when two or more species are cold. For $k(\omega,0)$ and $k_x(\omega,k_z)$ the orders of the eigenvalue problems in $k^2$ and $k_x^2$ are both $2+N - N_C$.

ACKNOWLEDGMENT

This work was supported by the U.S. Energy Research and Development Administration under Contract AT(04-3)-767.
APPENDIX

Calculation of the components of the group velocity vector
\[ \vec{V}_g = \frac{\partial \omega}{\partial k} \]
for each mode of oscillation may be made directly within the preceding eigensystem formulation of the linear system. The fundamental requirement for this calculation is a method of determining partial derivatives of the eigenvalue with respect to the independent variables of the problem. This applies for all of the three cases analyzed in this paper. The matrix analysis involved in determining these partial derivatives will be reviewed in this section.

Let \( \xi \) be an independent variable of the matrix \( M \) in the matrix eigenvalue problem, \( \lambda \) be the eigenvalue, and \( \vec{Y} \) be the eigenvector
\[ \lambda \vec{Y} = M(\xi) \vec{Y} \]

Differentiate this equation with respect to the independent variable, \( \xi \)
\[ \frac{\partial \lambda}{\partial \xi} \vec{Y} + \lambda \frac{\partial \vec{Y}}{\partial \xi} = \frac{\partial M}{\partial \xi} \vec{Y} + M \frac{\partial \vec{Y}}{\partial \xi} \]  \hspace{1cm} (A1)

Now, consider the eigenvalue problem for the matrix \( M^t \) which is the transpose of \( M \)
\[ \lambda_t \vec{Y}_t = M^t \vec{Y}_t \]  \hspace{1cm} (A2)
where \( \lambda_t \) represents an eigenvalue of \( M^t \), and \( \vec{Y}_t \) the corresponding eigenvector. It is a fact that a matrix and its transpose have the same eigenvalues, i.e., \( \lambda_t = \lambda \), but equality of the corresponding eigenvectors does not necessarily hold. Premultiply (A1) by \( \vec{Y}_t \) which is the transpose of \( \vec{Y}_t \)
\[ \begin{align*} \dot{\gamma}^t \frac{\partial \lambda}{\partial \xi} \dot{\gamma} + \lambda \dot{\gamma}^t \frac{\partial \gamma}{\partial \xi} &= \dot{\gamma}^t \frac{\partial M}{\partial \xi} \dot{\gamma} + \dot{\gamma}^t M \frac{\partial \gamma}{\partial \xi} \end{align*} \]

and substitute
\[ \lambda \dot{\gamma}^t = \dot{\gamma}^t M \]

which is obtained from the transpose of (A2) and from the fact that \( \lambda_t = \lambda \).

The result is
\[ \dot{\gamma}^t \frac{\partial \lambda}{\partial \xi} \dot{\gamma} = \frac{\partial \lambda}{\partial \xi} \dot{\gamma}^t \dot{\gamma} = \dot{\gamma}^t \frac{\partial M}{\partial \xi} \dot{\gamma} \]

and finally
\[ \frac{\partial \lambda}{\partial \xi} = \frac{\dot{\gamma}^t \frac{\partial M}{\partial \xi} \dot{\gamma}}{\dot{\gamma}^t \dot{\gamma}}. \quad (A3) \]

Therefore, calculation of the derivative of the eigenvalue requires the derivative of \( M \), and the corresponding eigenvectors of \( M \) and of \( M^t \).

Through application of (A3) the calculation of the group velocity vector is straightforward. When the eigenvalue is not simply \( \omega \) or \( k \) but rather a function of either one of these, or a function of both of these, as in case 2) where \( n^2 = \left( \frac{k}{\omega} \right)^2 \) is the eigenvalue, then this result is to be applied along with the appropriate rules for partial differentiation.
REFERENCES


Figure Captions

Fig. 1. $\omega$ vs $k$ at a fixed angle $\theta$ for an electron, two-ion species plasma.

Fig. 2. Index of refraction $n$ vs $\theta$ at a fixed frequency $\omega$ for an electron, two-ion species plasma.

Fig. 3. $n_x$ vs magnetic field $\left(\frac{\Omega e}{\omega}\right)$ at fixed $\omega$ and $n_z$ for an electron, two-ion species plasma.
Figure 1

\[ \frac{\omega}{\Omega_e} \]

\[ \frac{\Pi_e}{\Omega_e} = 0.3 \]
\[ \theta = 30^\circ \]

- \( \frac{c_e}{c} = 0.070 \) Ion1 - Hydrogen
- \( \frac{c_I}{c} = 0.00080 \) Ion2 - Deuterium
- \( \frac{c_2}{c} = 0.00052 \)
- \( \frac{n_1}{n_2} = \frac{2}{3} \)

\[ k \lambda_{De} \]

Figure 1
\[ \frac{\Pi_e}{\Omega_e} = 0.3 \quad \frac{\Omega_e}{\omega} = 1.25 \]
\[ \frac{c_e}{c} = 0.070 \quad \frac{n_1}{n_2} = \frac{2}{3} \]
\[ \frac{c_1}{c} = 0.00080 \]
\[ \frac{c_2}{c} = 0.00052 \]

Figure 2
Figure 3

\[ \frac{\Pi_e}{\omega} = 600 \quad \frac{c_e}{c} = 0.070 \quad \text{Ion}_1 - \text{Hydrogen} \]

\[ n_Z = 5 \quad \frac{c_1}{c} = 0.00080 \quad \text{Ion}_2 - \text{Deuterium} \]

\[ \frac{c_2}{c} = 0.00052 \quad \frac{n_1}{n_2} = \frac{2}{3} \]

- Real
- Complex
- Imag.

\[ \frac{\Omega_e}{\omega} \]

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Submitted to Physics of Fluids, May 1976

RESONANCE CONES IN A WARM MAGNETIZED BOUNDED PLASMA

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Research supported by the
U.S. Energy Research and Development Administration
under
Contract AT(04-3)-767

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February, 1976
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ABSTRACT

The resonance cone structure in a warm bounded slab magnetoplasma has been obtained in the electrostatic approximation for small electron and ion temperatures and a large magnetic field. It is found that the conducting boundaries act as perfect mirrors as in the cold plasma case--the cones reflect off the boundaries without any distortion of shape. The thermal effects produce the shifting of the maximum of the potential and the appearance of an interference structure inside each cone as in unbounded plasma theory. The amount of the shift is a uniformly increasing function of the axial distance from the source of each cone, while the spatial frequency is a uniformly decreasing function of that distance. Because of the spreading there is significant interference between nearby cones, so the total potential is more complicated than in the unbounded case. The solution is first obtained for a homogeneous plasma and studied, and then extended to the inhomogeneous plasma. The effects of Landau and collisional damping are included.
I. Introduction

The theory of resonance cones produced by a driven source in a magnetized plasma has been a subject of research in the area of wave propagation in plasmas in the past few years. In the special case of a point source simple cold plasma theory predicts that for certain ranges of the driving frequency the fields produced become singular along a double-cone surface emanating from the source with axis parallel to the magnetic field, hence the name resonance cone. Resonance cones were first discussed by Kuehl, and were experimentally measured by Fisher and Gould, who also investigated thermal effects on the cones. More recently, Burrell has studied the low frequency branch of the cones for a warm plasma, and Gould has shown the connection between the cold plasma cones in a bounded plasma and guided wave modes. Briggs and Parker have discussed how the resonance cones excited by a gap source are important in energy transport into the lower hybrid layer, while Bellan and Porkolab and Kuehl have studied the mode conversion of the cones near the lower hybrid layer.

The purpose of this paper is to look at the form of the electrostatic field from the resonance cone point of view in a warm bounded plasma with conducting boundaries, for small electron and ion temperatures. Landau and collisional damping are included. As discussed by Fisher and Gould there are three ranges of the driving frequency for which resonance cones exist, referred to as the three branches of the resonance cone. The range of plasma parameters assumed here is such that $\omega_{ci} < \omega_{pi} < \omega < \omega_{pe} < \omega_{ce}$ where $\omega_{pe,i}$ are the electron and
ion plasma frequencies, \( \omega_{ci} \) the electron and ion cyclotron frequencies, and \( \omega \) the driving frequency of the source. Hence it is only the middle branch of the cones (\( \omega_{LH} < \omega < \omega_{pe} \) where \( \omega_{LH} \) is the lower hybrid frequency) that is of concern here.

II. Homogeneous Plasma

The magnetic field is taken to be along the z-axis and the boundaries as conducting planes parallel to this field at \( x = 0 \) and \( x = a \). A gap source is assumed at \( x = z = 0 \) which extends to infinity in both directions along the y-axis. This enables us to treat the problem purely in two dimensions, i.e., in the x-z plane (see Fig. 1). The time dependence of the source and all fields produced by it is taken to be \( e^{-i\omega t} \). We will use the electrostatic approximation \( \hat{E} = -\nabla \phi \) and calculate the potential produced by a gap source for a warm plasma. The electrostatic approximation assumes \( \omega^2 |k^2|/c^2 >> |K_{ij}| \) for all \( i \) and \( j \) where \( k \) is the wavenumber and \( K_{ij} \) any component of the dielectric tensor. This approximation is best in the zones closest to the source.

We assume a very strong magnetic field \( (\omega_{ci}^2 << \omega_{pe}^2) \) and small electron temperature \( (v_e << \omega/k_z, \text{ where } v_e = \sqrt{2kT_e/\mu_e} \text{ is the electron thermal velocity and } k_z \text{ is the wavenumber in the z-direction.}) \) Also, \( \omega^2 << \omega^2 \) is assumed. Our approximations ensure that the electron Larmour radius \( r_{ci} \) is much smaller than any other characteristic length of interest \( (r_{ci} << \lambda_{De} << \omega, \text{ etc., where } \lambda_{De} \text{ is the Debye length}) \) and hence does not significantly influence the boundary conditions. We
assume an ion temperature $T_i \ll T_e$ so that $v_i^2 \ll v_e^2$. The ion energies are then such that almost all of them are repelled by the sheaths at the boundaries in their Larmour orbit hence the boundary conditions are not significantly affected by finite Larmour radius effects. This assures that we may use normal boundary conditions.

The form of the dielectric tensor is, to first order in the thermal terms,

$$
\bar{K} = \begin{bmatrix}
K_\perp & K_{xy} & 0 \\
-K_{xy} & K_\perp & 0 \\
0 & 0 & K_\parallel
\end{bmatrix}
$$

where $K_\perp = 1 - \frac{\omega_{pi}^2}{\omega(\omega + iv)} + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{3}{2} k_x^2 \left[ \frac{v_i^2 \omega_{pi}^2}{\omega^4} + \frac{v_e^2 \omega_{pe}^2}{4 \omega_{ce}^4} \right]$ and $K_\parallel = 1 - \frac{\omega_p^2}{\omega(\omega + iv)} - \frac{3}{2} \frac{k_v^2 v_e^2 \omega_{pe}^2}{\omega_{ce}^4} + 2i \frac{\omega_{pi}^2}{\omega_{ce}^2} - \frac{2 v}{k_{\parallel}^2}$, where $\omega_p = \omega_{pe} + \omega_{pi}$ and $v$ is a phenomenological collision frequency. All of the other ion thermal terms were neglected because they are much smaller than corresponding electron thermal terms. The other off-diagonal terms are negligible compared to the on-diagonal terms by our strong magnetic field assumption, hence were set to zero. It is assumed that $v \ll \omega$ and $\omega_{pi} \ll \omega$ so that the appearance of $v$ in the thermal terms in $K_\perp$ and $K_\parallel$ gives rise to second order terms, which may be neglected. We will assume $1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega_{ce}^2} \gg \frac{3}{2} k_x^2 \left[ \frac{v_i^2 \omega_{pi}^2}{\omega^4} + \frac{v_e^2 \omega_{pe}^2}{4 \omega_{ce}^4} \right]$, so that we may take from the electrostatic dispersion relation

$$
k_x = \left( \frac{K_\parallel}{K_\perp} \right)^{1/2} \frac{\omega_p^2 - \omega^2}{\omega_{pi}^2 + \omega_{pe}^2} \frac{1}{\omega_{pi}^2 + \omega_{pe}^2} \frac{1}{\omega_{ce}^2} \frac{1}{\omega_{ce}^2} \frac{k_z}{k_z}
$$

and substitute this into the thermal terms of $K_\perp$. (Thermal and damping contributions to $k_x$ are neglected because they give rise to second order terms in $K_\parallel$.)
assumption is valid for $\omega \gtrsim 1.5 \omega_{\text{LH}}$ where the lower hybrid frequency is $\omega_{\text{LH}} = \omega_{\text{pi}}/(1 + \omega_{\text{pe}}^2/\omega_{\text{ce}}^2)^{1/2}$, and for this form of $k_x$ it is seen that the thermal terms in $K_{\perp}$ become negligible compared to the thermal terms in $K_{\parallel}$ for $\omega \gtrsim 3\omega_{\text{LH}}$.

The cold plasma resonance cone angle is given by

$$\tan \theta_c = -\frac{K_{\perp}}{K_{\parallel}}$$

The thermal terms in $K_{\perp}$ become negligible compared to the thermal terms in $K_{\parallel}$ for $\omega \gtrsim 3\omega_{\text{LH}}$.

When thermal effects are included the resonance angle for the $k_z$ component of the outgoing wave is shifted to

$$\tan \theta_c = \frac{\omega - \omega_{\text{LH}}}{\omega_{\text{p}} - \omega} \left\{1 - \frac{3}{2} \frac{k_z^2 v_{\text{e}}^2}{\omega^2} \left[1 + \frac{\omega_{\text{pe}}^2}{\omega_{\text{p}}^2} \right] + \frac{(\omega_{\text{p}}^2 - \omega^2)}{(\omega_{\text{p}}^2 - \omega_{\text{LH}}^2)^2} \frac{v_{\text{e}}^2}{4} \right\}$$

where the small imaginary parts of $K_{\perp}$ and $K_{\parallel}$ have been neglected. This breaks down at sufficiently large $k_z$, but the resonance angle for each component is shifted to a smaller angle than the cold plasma cone angle.

Poisson's equation, after being Fourier transformed in the $z$-direction, takes the form

$$K_{\perp} \frac{\partial^2 \phi(x,k_z)}{\partial x^2} - K_{\parallel} k_z^2 \phi(x,k_z) = 0$$

where $\phi(x,z) = \int_{-\infty}^{\infty} \tilde{\phi}(x,k_z) e^{ik_zz} \frac{dk_z}{2\pi}$ is the electrostatic potential.

Define the quantity $\alpha = (-K_{\parallel}/K_{\perp})^{1/2}$. Expand $\alpha$ for $\nu \ll \omega$ and for small imaginary parts of $K_{\parallel}$ and $K_{\perp}$ relative to their real parts. Then to first order in imaginary and thermal terms
\[ \alpha = (1 - \frac{\omega_{\text{LH}}}{\omega^2})^{-1/2} \left\{ \frac{\omega^2}{\omega^2} - 1 + \frac{3}{2} k_z^2 \frac{v_e^2}{\omega^4} + \frac{\omega^2}{\omega^2_{\text{LH}}} \left( \frac{v_i^2}{\omega^2} \right) \frac{1}{4} \left( \frac{p_{i1}}{\omega^2_{\text{LH}}} + \frac{v_e^2}{\omega^2_{\text{LH}}} \right) \right\}^{1/2} \]

\[ -i\sqrt{\pi} e^{-\frac{\omega^2}{k_z^2}} \frac{1}{2\omega^2} (\omega^2 - \omega^2_{\text{LH}}) \frac{\omega_{\text{LH}}}{\omega^2} \left\{ \frac{\omega^2}{\omega^2} - 1 \right\}^{1/2} \]

Here the assumptions \( \frac{3}{2} k_z^2 \frac{v_e^2}{\omega_{\text{pe}}^4} \ll \frac{\omega^2}{\omega^2} - 1 \) and \( \frac{3}{2} k_z^2 \frac{v_i^2}{\omega_{\text{LH}}^2} \) were utilized, and the thermal terms in the imaginary part of \( \alpha \) were neglected, since they are second order. Split \( \alpha \) into its real and imaginary parts: \( \alpha = \alpha_r + i\alpha_i \). Then upon further expansion of \( \alpha_r \) we obtain

\[ \alpha_r = D + E k_z^2 \]

where

\[ D = \left[ \frac{\omega^2}{\omega - \omega_{\text{LH}}} \right]^{1/2} \]

\[ E = \frac{3[\omega_{\text{pe}} v_e^2 + (\frac{\omega^2}{\omega - \omega_{\text{LH}}} \frac{v_i^2}{\omega_{\text{LH}}})]}{4\omega^2 (\omega^2_{\text{pe}} - \omega^2_{\text{LH}})^{1/2} (\omega^2_{\text{LH}})^{1/2}} \]

and

\[ \alpha_i = \frac{-\sqrt{\pi} e^{-\frac{\omega^2}{k_z^2}} \frac{1}{2\omega^2} (\omega^2 - \omega^2_{\text{LH}})^{1/2}}{(\omega^2_{\text{pe}} - \omega^2)^{1/2} (\omega^2_{\text{LH}})^{1/2}} \]

The assumption \( \frac{3}{2} k_z^2 \frac{v_e^2}{\omega_{\text{pe}}^4} \ll \frac{\omega^2}{\omega^2} - 1 \) breaks down as \( \omega \) approaches \( \omega_{\text{pe}} \), but since \( v_e \ll \omega/k_z \) has already been assumed, this should be okay
for \(2\omega^2 \leq \omega_{pe}^2\), i.e., for cold plasma cone angle \(\theta_c \leq \pi/4\). Also, the assumption \(\frac{3}{2} k^2 \left[ \frac{v_i^2 \omega_i^2}{2} + \frac{v_e^2 \omega_e^2}{4 \omega_{ce}} \right] < \frac{1 - \omega_{lh}^2}{\omega^2}\) breaks down as \(\omega\) approaches \\
\(\omega_{lh} = \frac{\omega_{pi}^2}{(1 + \omega_{pe}^2/\omega_{ce}^2)^{1/2}}\), but it is okay for \(\omega^2 \geq 3\omega_{lh}^2\), i.e., for cold plasma cone angle \(\theta_c \geq 2m_e/m_i\). Recall that the condition for the existence of (cold plasma) resonance cones is \(K \ll K < 0\), and this requires \\
\(\omega_{lh}^2 < \omega^2 \rho_{lh}^2\) in our case (the middle branch of the cones).

A solution to equation (2.3) which fits the boundary condition \(\tilde{\phi}(x=a) = 0\) and is causal (produces proper exponential decay away from the source for \(\alpha_i < 0\) is

\[
\tilde{\phi}(x,k_z) = \frac{-i\alpha k_z x}{1 - e^{-2i\alpha k_z a}} e^{i\alpha k_z(x - 2a)}.
\]

One may obtain the guided modes of the plasma by doing the inverse Fourier transform utilizing the poles in \(\tilde{\phi}(x,k_z)\), but to get the form of the resonance cones we must expand the denominator:

\[
[1 - e^{-2i\alpha k_z a}]^{-1} = \sum_{n=0}^{\infty} e^{-2i\alpha k_z n}.
\]

This converges since \(\alpha_i < 0\). Then

\[
\phi(x,z) = \sum_{n=0}^{\infty} \left\{ \int_{0}^{\infty} \tilde{\phi}(0,k_z)e^{-ik_z z} \left[ e^{-i\alpha k_z x} - e^{i\alpha k_z(x-2a)} \right] \frac{dk_z}{2\pi} \right\} e^{-2i\alpha k_z a} \frac{dk_z}{2\pi}.
\]

\[
\cdot e^{-i\alpha k_z(x-2a)} \frac{dk_z}{2\pi} \left\{ e^{-i\alpha k_z x} - e^{i\alpha k_z(x-2a)} \right\} e^{2i\alpha k_z a} \frac{dk_z}{2\pi}.
\]

(2.8)
It should be noted that our approximation $k_z^2 v_e^2 \ll \omega^2$ breaks down in the inverse transform above, since the integral extends to $|k_z| = \infty$. Thus the above form of the solution is valid only when the part of the integrals for $|k_z| \gg \omega/v_e$ makes a negligible contribution to the total integral.

For a point gap excitation the second boundary condition is of the form $\phi(x=0, z) = \delta(z)$. Then $\tilde{\phi}(0, k_z) = 1$ and

$$\phi(x, z) = -\sum_{n=0}^{\infty} \sum_{\delta=\pm} \sum_{\varepsilon=\pm} \epsilon f[\delta z + \varepsilon \alpha(a-x) + (2n+1) \alpha]$$

(2.9)

where $f(\xi) = \int e^{-ik_z \xi(z)} \frac{dk_z}{2\pi}$. Then this solution is consistent with the $k_z^2 v_e^2 \ll \omega^2$ assumption if the primary contribution to $f$ comes from $k_z$ satisfying this criterion. In the special case that we may evaluate $f$ by the saddle point method, the solution is valid if for each saddle point $k_0$ contributing to the integral, $|k_0|^2 \ll \omega^2/v_e^2$.

We want to evaluate the integral $f(\xi)$ for $\xi(k_z) = \delta z + \varepsilon \alpha(a-x) + (2n+1)\alpha$. Assume $\alpha = \alpha_r$ as a first approximation, since $\alpha_i \ll \alpha_r$ from previous assumptions. This will give us the undamped solution. We will include in the effects of $\alpha_i$ by a perturbation on the undamped solution to take the effect of Landau and collisional damping into account. We may thus write $\xi = \xi_0 + i\xi_1$, where $\xi_0$ and $\xi_1$ are the real and imaginary parts of $\xi$, and $\xi = \xi_0$ is the first approximation. Then

$$\xi_0(k_z) = \mu_n(\delta z, x, \varepsilon) + \nu_n(x, \varepsilon) k_z^2$$

(2.10)

where
\[ \nu_n(\delta z, x, \epsilon) = \delta z + [(2n+1+\epsilon)a - cx] D \]
\[ \nu_n(x, \epsilon) = [(2n+1+\epsilon) - \epsilon x] E \]

We have

\[ \phi(x, z) = - \sum_{n=0}^{\infty} \sum_{\epsilon = \pm} \sum_{\delta = \pm} e^{i(n_k z + \nu_n k_z^3) \frac{dz}{2\pi}}. \]  

(2.11)

From the Airy integral forms

\[ \text{Ai}(n) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \cos(\nu_n + \frac{\nu^3}{3}) \, dv \]
\[ \text{Gi}(n) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \sin(\nu_n + \frac{\nu^3}{3}) \, dv \]

we obtain

\[ \phi(x, z) = - \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{\epsilon = \pm} \sum_{\delta = \pm} \frac{\epsilon}{(3\nu_n)^{1/3}} F\left[\frac{\mu_n}{(3\nu_n)^{1/3}}\right] \]  

(2.12)

where \( F(\xi) = \text{Ai}(\xi) - i\text{Gi}(\xi) \). This is the form of the potential neglecting damping. The corresponding field components are

\[ E_z = - \frac{\partial \phi}{\partial z} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{\epsilon = \pm} \sum_{\delta = \pm} \left\{ \frac{\delta \epsilon}{(3\nu_n)^{2/3}} F\left[\frac{\mu_n}{(3\nu_n)^{1/3}}\right] \right\} \]

\[ E_x = - \frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{\epsilon = \pm} \sum_{\delta = \pm} \frac{1}{(3\nu_n)^{4/3}} \left\{ \frac{EF\left[\frac{\mu_n}{(3\nu_n)^{1/3}}\right]}{(3\nu_n)^{1/3}} \right\} \]

\[ + \frac{E_{\mu_n}}{(3\nu_n)^{1/3}} F\left[\frac{\mu_n}{(3\nu_n)^{1/3}}\right] \]  

(2.13)
Plots of $|\phi|$ for typical plasma parameters are shown in Figs. 3-4. The latter include Landau damping, which we will add in our analytical solution later.

There are several observations that can be made from the form of the potential in equation (2.12). First, as $T_e \to 0$, then $E \to 0$ so $v_n \to 0$ for all $n$. Then since as $v_n \to 0$

$$\frac{1}{\sqrt{\pi}} (3v_n)^{-1/3} \text{Ai}\left(\frac{\nu_n}{(3v_n)^{1/3}}\right) + \delta(\nu_n)$$

and we recover the cold bounded plasma resonance cones given by Gould.

In this limit it is easy to identify each term with a given $n, \varepsilon$ and $\delta$ with an individual member of the multiply-reflected resonance cones which in cold plasma theory is singular for $\nu_n(\delta z, x, \varepsilon) = 0$. The cones in $z > 0$ have $\delta = -$, and those in $z < 0$ have $\delta = +$. The cones reflected off of the $x = a$ boundary have $\varepsilon = +$, and the ones originating from or reflected off of the $x = 0$ boundary have $\varepsilon = -$ (see Fig. 2).

The effect of the temperature is the shifting of the maximum of the potential of a given cone (which is no longer singular) to a smaller angle and the appearance of an interference structure (secondary maxima) inside the cone as was first noted by Fisher and Gould. This is caused by interference between the cold plasma cone and a warm plasma wave, as can be
seen by taking the asymptotic limit of equation (2.12) (valid for very small $T_e$ away from the cold plasma cone line). For $\frac{\mu_n}{(3\nu_n)^{1/3}} >> 1$

$$\phi \sim - \sum_{n=0}^{\infty} \sum_{\epsilon = \pm} \sum_{\delta = \pm} \left\{ \frac{-i}{\mu_n} + \frac{1}{2\sqrt{\pi}(3\nu_n)^{1/4}} \exp\left[-\frac{2\mu_n^{3/2}}{3(3\nu_n)^{1/2}}\right] \right\} (2.15)$$

and for $\frac{\mu_n}{(3\nu_n)^{1/3}} << 1$

$$\phi \sim - \sum_{n=0}^{\infty} \sum_{\epsilon = \pm} \sum_{\delta = \pm} \left\{ \frac{-i}{\mu_n} + \frac{(3\nu_n)^{1/12}}{\sqrt{\pi}(-\mu_n)^{1/4}} \cos\left[-\frac{2\mu_n^{3/2}}{3(3\nu_n)^{1/2}} - \frac{\pi}{4} \right] \right\} (2.16)$$

This first term of the potential (the Gi term) may be identified with the cold plasma cone term and the second term (the Ai term) with the thermal wave term which produces the thermal interference structure. The width of the peaks and spatial oscillations spreads out in a uniform fashion away from the source, while the height of the peaks correspondingly decreases as $(3\nu_n)^{-1/3}$. If we let $\Delta z = z - z_c(x, \delta, \epsilon, n)$ be the distance from the cold plasma cone line for a given $x$ and $z$ and a given cone labeled by $n$, $\delta$ and $\epsilon$, then the argument of $F$ is $\frac{\mu_n}{(3\nu_n)^{1/3}} = \frac{-\delta \Delta z}{(3x_n(\epsilon)E)^{1/3}}$, where $x_n(\epsilon) = [(2n+1+\epsilon) a - \epsilon x]$. (The quantity $x_n(\epsilon)$ acts like a "total vertical distance" from the source for a given cone labeled by $n$ and $\epsilon$. ) Thus the width of the peaks and spatial oscillations goes like $T_e^{-1/3}$, and spreads out from the source as $x_n^{1/3}(\epsilon)$. The corresponding angular shift of the maximum of the potential of a given cone from that of the cold plasma cone goes like $\Delta \theta \sim \Delta z/x_n \sim x_n^{-2/3}$.

The solution for an unbounded plasma is obtained by letting $a \to \infty$. (This gives us the solution for a half-space plasma with a conductor at
Then only the \( n = 0, \epsilon = -1 \) terms survive. Thus the form of the potential for the unbounded case is the same as the \( n, \epsilon = -1 \) term of the potential for the bounded case for \( 2n < x < 2n+1 \), and the same as the negative of the \( n, \epsilon = +1 \) term of the bounded case potential for \( 2n+1 < x < 2n+2 \). Thus the individual terms in \( \phi(x, z) \) correspond to perfect repeated multiple reflection of the unbounded form of the potential with the conducting boundaries acting as mirrors. That is, the resonance cones reflect off the boundaries, changing sign upon each reflection, and each (reflected) cone with a given \( (n, \epsilon, \delta) \) has exactly the same shape (up to a \(-1\) factor) as a successive segment of the corresponding unbounded cone. But in the bounded case more than one term in the sum may contribute to the potential at a given point in space, hence there is interference between nearby reflected cones because of their nonlocality (spreading away from the source). Thus the effect of boundaries is to produce interference between the cone coming into a boundary and the one reflected off of that boundary. Indeed adjacent cones come together (interfere perfectly) at the boundary to give \( \phi = 0 \) there—the boundary condition produces the reflected cone—so interference is most important near the boundary. The higher \( T_e \) the greater the width of the region near the boundary in which interference of adjacent cones is important, and the greater the importance of interference. Also, the higher the driving frequency \( \omega \) the larger the cold plasma cone angle \( \theta_c \) and hence the greater the importance of interference.

We now include in Landau and collisional damping. Go back to the integral in equation (2.9). We have \( \xi(k_z) = \xi_0(k_z) + i \xi_1(k_z) \) where \( \xi_1(k_z) = \alpha_i[(2n+1+\epsilon)a - \epsilon x] \) was neglected before. The integral
\[ f(\xi_o) = \int_0^\infty e^{-ik_z \xi_0(k_z)} \frac{dk_z}{2\pi} \]  
may be solved by saddle point integration, 
and if we so solved it we would get an asymptotic form of the exact solution \( f(\xi_o) \approx \text{Ai}(\xi_o) - i\text{Gi}(\xi_o) \). Thus treating the \( \alpha_i \) term as a perturbation,

\[
f(\xi) = \int_0^\infty e^{-ik_z \xi(k_z)} \frac{dk_z}{2\pi} = e^{i \kappa_0 [(2n+1+\epsilon)a - \epsilon x] \alpha_i(k_0)} 
\cdot \int_0^\infty e^{-ik_z \xi(k_z)} \frac{dk_z}{2\pi} = e^{i \kappa_0 [(2n+1+\epsilon)a - \epsilon x] \alpha_i(k_0)} F(\xi_o)
\]

where \( \kappa_0 \) is the appropriate saddle point of \( f(\xi_o) \). Since \( k_z \xi_0(k_z) = \mu_n k_z + \nu_n k_z^3 \), then \( \kappa_0 = \pm\left[\frac{-\nu_n}{3\nu_n}\right]^{1/2} \). Since \( \alpha_i < 0 \) we must choose the + sign in the saddle point because the negative sign violates causality.

We thus obtain

\[
\phi(x,z) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} e^{\frac{-x}{(3\nu_n)^{1/3}}} \text{F}
\frac{\mu_n}{(3\nu_n)^{1/3}} e^{-\Gamma_n(\delta,\epsilon)} \]

where

\[
\Gamma_n(\delta,\epsilon) = \left[\frac{-\nu_n}{3\nu_n}\right]^{1/2} \left[(2n+1+\epsilon)a - \epsilon x\right] - \frac{3\nu_n \omega^2}{\mu_n \nu e} e^{3\nu_n \omega^2 / \mu_n \nu e} 
\cdot \left\{ \frac{\nu_n \omega^2}{\omega_p - \omega^2} \right\} e^{\frac{3\nu_n \omega^2}{\mu_n \nu e}} 
+ \frac{\nu}{2\omega} \left[ \frac{\omega_p^2}{\omega_p - \omega^2} - \frac{\omega^2}{\omega - \omega_{\text{LH}}} \right] \}
\]

This form of the damping is quantitatively correct only when the asymptotic form of \( F \) is valid, i.e., for argument \( [\mu_n/(3\nu_n)^{1/3}] \gg 1 \). However, for smaller arguments it should give a qualitative indications of the nature of the damping.
One interesting observation is that the Landau damping factor is exponentially decaying away from the source for \( u_n < 0 \) (inside the cone) but introduces a spatially-dependent phase for \( u_n > 0 \) (outside the cone). This phase does not influence the magnitude of the potential at points where only one cone contributes significantly to it, but at points where two or more cones are interfering this phase is important in determining the way the two contributions add, hence has an important influence on the potential.

For a given cone the damping exponent goes like

\[
|\Gamma_n(\epsilon,\delta)| \sim e^{-\delta z + x_n D/2} \frac{1}{3E \delta x_n} x_n \sim |\Delta z|^{1/2} x_n^{1/2}.
\]

This means, for example, that the parts of the thermal interference structure inside a given cone which are the farthest from the cold plasma cone line are the most highly damped and thus become negligible the fastest as one goes away from the source. Damping thus reduces the importance of interference between nearby cones. It is to be noted that the higher \( T_e \) is, the greater the Landau damping contribution to \( \Gamma_n \), but the less the collisional damping contribution.

In Figs. 3 and 4 the form of \( \phi \) is shown for \( 2\omega = \omega_{pe} \) and for two different temperatures. These graphs are for collision frequency \( \nu = 0 \). Only the \( z > 0 \) portion is shown, since \( \phi \) is completely symmetrical about \( z = 0 \). Near the source a peak shows up at \( z = 0 \) which is left over from the \( \delta \)-function at the source and which involves interference of the tails of the \((n=0, \epsilon=-1, \delta=-1)\) cone and the \((n=0, \epsilon=-1, \delta=+1)\) cone, but this rapidly goes away. The interference structure due to thermal effects with its primary maximum and secondary maxima clearly shows up.
Also appearing is the interference between the \((n=0, \varepsilon=-1, \delta=-1)\) cone with its frequency of thermally induced spatial oscillations and the \((n=0, \varepsilon=+1, \delta=-1)\) cone with its lower frequency of spatial oscillations just past the cold plasma cone line of the latter cone. Since the frequency of the spatial oscillations decreases away from the source of a cone, then as we go from \(x = 0\) to \(x = a\), the frequency of the oscillations of the \((n=0, \varepsilon=-1, \delta=-1)\) cone (originating from the gap source) decreases while that of the \((n=0, \varepsilon=+1, \delta=-1)\) cone (originating from the \(x = a\) boundary) increases, and the interference structure between the two cones changes accordingly. The nature of the interference between any two adjacent cones will be qualitatively the same as between these two cones. It is to be noted that near the \(x = 0\) boundary the interference of the \((n=1, \varepsilon=-1, \delta=-1)\) cone (originating from the \(x = 0\) boundary) interferes significantly with the other two cones at a value of \(z\) just past where the other two cones start interfering, so the interference structure in that region is rather complicated. Typical plasma parameters for which these graphs apply might be \(f = 1\) GHz and \(f_{pe} = 2\) GHz. Then Fig. 3 would correspond to \(T_e = 22.5\) eV, while Fig. 4 would correspond to \(T_e = 140\) eV, for a plasma of dimension \(a = 10\) cm.

III. Inhomogeneous Plasma

Having studied the solution for a homogeneous plasma we are interested in seeing how this is modified by the presence of weak inhomogeneities. Assume the inhomogeneities to be perpendicular to \(\mathbf{B}_0\) and in the \(x\)-direction, since that is the case of interest in most plasmas. Now \(\omega_{pe} = \omega_{pe}(x)\), \(\omega_{pi} = \omega_{pi}(x)\) and \(K_{\parallel}\) and \(K_{\perp}\) are functions of \(x\).
Poisson's equation is
\[ \frac{\partial}{\partial x} \left( K_L(x) \frac{\partial \phi(x,k_z)}{\partial x} \right) - k_z^2 K_R(x) \phi(x,k_z) = 0 \] (3.1)

Then in the WKB approximation solutions which fit the boundary condition at \( x = a \) are of the form
\[ \phi(x,k_z) = \tilde{\phi}(x=0,k_z) \left[ \alpha(0) K_L(0) \right]^{1/2} \frac{1}{\alpha(x) K_L(x)} \left\{ e^{-ik_z \int_0^x \alpha(x') dx'} - e^{-2ik_z \int_0^{2a-x} \alpha(x') dx'} \right\} - 2ik_z \int_0^a \alpha(x') dx' \]
(3.2)

Let \( \eta(x) = \int_0^x \alpha(x') dx' \). Then we may expand the denominator as before.

Letting \( G(x) = \int_0^x D(x') dx' \) and \( H(x) = \int_0^x E(x') dx' \) we straightforwardly obtain the solution for a point gap excitation:
\[ \phi(x,z) = \frac{-1}{\sqrt{\pi} L(\omega_p^2(x) - \omega_e^2(x))} \left[ \frac{(2n+1+\varepsilon)(\omega_H^2 - \omega_e^2)}{(2n+1)(\omega_p^2(x) - \omega_e^2 - \omega_H^2(x))} \right]^{1/2} \]
\[ \times \sum_{n=0}^{\infty} \sum_{\varepsilon, \delta = \pm} \frac{\varepsilon}{(3n)^{1/3}} \int F[M_n/(3N_n)^{1/3}] \]
(3.3)

where \( M_n(\delta z, x, \varepsilon) = \delta z + (2n+1+\varepsilon)G(a) - \varepsilon G(x) \) and \( N_n(\delta z, x, \varepsilon) = (2n+1+\varepsilon)H(a) - \varepsilon H(x) \). The cold plasma cone lines satisfy \( M_n = 0 \), so we see inhomogeneities cause a bending of the cold plasma cone lines and of the maxima of the warm plasma cones. They also cause a spatial modulation of the field amplitudes.

The WKB approximation assumes \( dk_x/dx < k_x^2 \) where \( k_x \) is the wave number in the \( x \)-direction. From the electrostatic dispersion relation,
\[ k_x = \left( -\frac{K_H}{K_L} \right)^{1/2} k_z = (\frac{\omega_p^2(x) - \omega^2}{\omega - \omega_{LH}(x)})^{1/2} k_z \]  

The criterion for the validity of the WKB approximation then becomes

\[ \frac{\omega^2 d(\omega_p^2)}{dx} \ll 2[(\omega_p^2 - \omega^2)^3(\omega^2 - \omega_{LH}^2)]^{1/2} k_z \]  \hspace{1cm} (3.4)

The WKB solution breaks down for strong inhomogeneities, for very small \( k_z \) components and as \( \omega > \omega_p(x) \) or \( \omega > \omega_{LH}(x) \). (By our assumptions \( \omega_{LH}(x) < \omega < \omega_p(x) \) for all \( x \).) These two frequencies are turning points: the plasma frequency cutoff and the low hybrid resonance, respectively. In the case \( \omega = \omega_{LH}(x) \) for some \( x \), i.e., for a lower hybrid layer present, \( K_{\perp}(x) \to 0 \) in our approximation and thermal terms become important in \( K_{\perp} \), and mode conversion of the resonance cones may take place.\(^6\) The WKB approximation breaks down near the plasma frequency because the perpendicular wavelength becomes as large as the order of the scale length of the density variation, but breaks down near the lower hybrid resonance because the perpendicular wavenumber becomes very large so rapidly as the lower hybrid layer is approached. It is to be noted that the WKB approximation breaks down for \( k_z \) near 0, so that since we integrate over \( k_z \) in this region when taking the inverse Fourier transform, our solution's validity depends upon the contribution to the integral from this region being negligible compared to the total integral.

The damping factor associated with each cone for the inhomogeneous case is just a generalization of that for the homogeneous case and is

\[ \Gamma_n(\delta, \epsilon) = \left[ \frac{M_n}{\gamma N_n} \right]^{1/2} \{(2n+1+\epsilon)\gamma_n(a) - \bar{\gamma} n_1(x)\} \]  \hspace{1cm} (3.5)
where

\[ n_1(x) = \int_0^x \alpha_1(x')dx' \mid_{k_z=k_o=\left[\frac{-M_n}{3N_n}\right]^{1/2}} \]

\[ = \left\{ \sqrt{\pi} \omega^2 e^{\frac{3N_n \omega^2}{M_n} v^2} \int_0^x \frac{dx'}{(\omega_p(x')-\omega^2)^{1/2}(\omega^2-\omega_p^2(x'))^{1/2}} \right\} \]

\[ + \frac{\nu}{2\omega} \int_0^x \left[ \frac{\omega_p^2(x') (\omega^2-\omega_p^2(x')) - \omega_p^2(x') \omega_{pi}(x') + \omega_p^2(x') \omega_{pi}(x')}{(\omega_p^2(x')-\omega^2)^{1/2} (\omega^2-\omega_p^2(x'))^{3/2}} \right] dx' \]

IV. Conclusion

The fields produced by a gap source in a warm magnetized plasma bounded by a conducting boundary have been calculated in the resonance cone picture for the frequency regime in which the middle branch of the cones exists. An important result was the observation of a pronounced interference structure between nearby cones—those coming into a boundary and those reflected off the boundary—for plasmas in which the damping is sufficiently small. The nature of the spreading out of the structure of the potential of each cone and its influence on the interference between nearby cones was studied. Landau and collisional damping were included in the solution and their effects studied. It was found that outside the cold plasma cone the severity of the damping increased with the distance from the cold plasma cone line, but inside the cold plasma cone a spatially dependent phase instead of exponential
decay was introduced into the cone term in the potential. This phase introduced important phase effects in the interference between nearby cones.

The introduction of inhomogeneities in the plasma was seen to cause the bending of the profile of peaks and valleys of the fields which is the well-known characteristic of the JWKB solution. One particular condition of interest under which this solution breaks down is when the local lower hybrid frequency becomes close to the driving frequency at some point in the plasma. At such points additional thermal ion plasma modes (ion Bernstein modes) become important, so that if the lower hybrid and driving frequencies become equal at some point (i.e., a resonance layer exists), then mode conversion may excite these ion Bernstein modes.6,7,9 This case will be the subject of a future paper.

ACKNOWLEDGMENTS

I would like to thank Dr. Roy Gould and Dr. Mario Simonutti for useful suggestions and discussions concerning this problem. This work was supported in part by the U. S. Energy Research and Development Administration.
REFERENCES

FIGURE CAPTIONS

Fig. 1. Plasma Geometry.

Fig. 2. Surfaces of singularity (dotted lines) of the cold plasma resonance cones, labelled by the cone which is singular there in cold plasma theory.

Fig. 3. Plots of $|\phi|$ versus $z$ for various values of $x$, for $\omega_{pe} = 2\omega$, $\omega_{pe} \ll \omega_{ce}$ and $v_e = 0.02 \omega$. The dotted lines show the positions of the cold plasma cone singularities as given in Fig. 2.

Fig. 4. Same as Fig. 3 except that $v_e = 0.05 \omega$. 