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ABSTRACT

This paper analyzes several important features of trapped-particle instabilities. For trapped-electron modes, the complete two-dimensional (2D) spatial structure, including the effects of magnetic shear, is numerically calculated within the framework of a differential formulation for long radial wavelength modes. Growth rates obtained for representative cases correlate reasonably well with the usual one-dimensional (1D) estimates of shear stabilization. However, the spatial structure of the mode differs markedly; e.g., it typically extends over several mode-rational surfaces. At the shorter wavelengths, where the maximum growth rates of the modes typically occur, it is necessary to introduce an integral equation formulation for calculating the radial dependence. Growth rates from this 2D analysis are significantly smaller than 1D estimates, and the poloidal mode structure exhibits a pronounced localization at the magnetic field minimum. Specific collisional mechanisms affecting the linear stability of these modes are also studied. Collisional scattering of low energy electrons can reduce the nonadiabatic trapped-electron response, and collisional broadening can strongly modify the resonant response of the untrapped electrons. The saturation of the usual form of the dissipative trapped-ion instability by mode coupling is studied analytically and numerically. The "solitary wave" or "multi-mode" equilibria of LaQuey, et al. are found to be unstable and inaccessible. However, their "two-mode" equilibria can be both stable and accessible and can lead to transport levels well below the Kadomtsev-Pogutse estimates, provided wave dispersion is small and the parameters considered are not far from marginal stability. In addition to the standard trapped-ion modes, which are associated with the electron diamagnetic drift branch, it is found that ion diamagnetic modes of this type can also be generated. This new branch is destabilized by both resonant and nonresonant interactions with ions which have average unfavorable magnetic drifts. If the ion temperature gradient parameter, $\eta_i \equiv d\,\ell n\,T_i/d\,\ell n\,n$, is either large ($\eta_i \gtrsim 2/3$) or negative ($\eta_i < 0$), growth rates from the ion branch are dominant over the electron branch.

Trapped-Electron Mode

The first type of trapped-particle instability predicted to be encountered in tokamak devices is the dissipative trapped-electron mode. Here we investigate two important aspects of this instability, namely, (1) its two-dimensional spatial structure and the effect of magnetic shear on modes extending over several mode-rational surfaces; and (2) the influence of collisional scattering of low energy electrons and of collisional broadening of the untrapped resonant electron response.

In order to properly identify the trapped-electron mode and assess its effects, it is necessary to determine its spatial structure and to obtain a realistic estimate of the stabilizing influence of magnetic shear. Previous investigations of these problems have generally been one-dimensional; i.e., either the radial structure of the mode is ignored while solving for the structure parallel to the magnetic field, or the parallel structure is ignored while solving for the radial structure in the vicinity of a single mode-rational surface [1]. The present analysis deals with the complete two-dimensional (2D) structure of the mode over its full width, which may extend over several rational surfaces. Recalling that this is basically an electrostatic instability and that the appropriate frequency range lies between the thermal ion transit frequency, $\overline{\omega}_{ti}$, and the thermal electron bounce frequency, $\overline{\omega}_{be}$, the ion and electron density responses are calculated using standard procedures [1]. We then use the quasineutrality condition to obtain the 2D, integro-differential equation for the fluctuating electrostatic potential, $\phi(r,\theta,\zeta,t)$, where r is the minor radius and θ and ζ are the poloidal and toroidal angles. Without loss of generality, the slow and fast θ-dependence can be separated by expressing ϕ as

$$\phi(r,\theta,\zeta,t) = \tilde{\phi}(\theta,r)\exp(-i\omega t+im^{o}\theta-i\ell\zeta) \tag{1}$$

where ℓ and m^{o} are the toroidal and poloidal mode numbers. Here $m^{o} = \ell q(r_o)$ with q being the safety factor and r_o designating the position of a reference mode-rational surface around which the mode is localized. The slow θ-variation is contained in $\tilde{\phi}(\theta,r)$, which must be periodic in θ and satisfy the condition $|(\partial\tilde{\phi}/\partial\theta)/\tilde{\phi}| << m^{o}$. Noting that the spacing of mode rational surfaces [i.e., the distance between the surfaces where $q = m/\ell$ and $q = (m\pm 1)/\ell$] is $\Delta r_s \simeq (\ell q')^{-1}$, it is convenient to express the radial distance from r_o in terms of $S(r) \equiv \ell[q(r)-q(r_o)] \simeq (r-r_o)/\Delta r_s$. Equation (1) thus becomes

$$\phi = [\tilde{\phi}(\theta,S)\exp(-iS\theta)]\exp\{-i\ell[\zeta-q(r)\theta]\}\exp(-i\omega t) \ . \tag{2}$$

Since Δr_s is generally much smaller than typical equilibrium scale lengths for realistic conditions with $rq'/q \sim 1$, the radial equilibrium gradients can be treated as constant to a good approximation. Hence, only the explicit S-dependence is considered.

For radial wavelengths longer than the ion gyroradius ($k_r\rho_i < 1$), k_r is treated as a differential operator; i.e. $k_r^2 = -(\Delta r_s)^{-2}\partial^2/\partial S^2$. The basic mode equation thus becomes

$$\left(A(\theta) \frac{\partial^2}{\partial S^2} + B\left(\frac{\partial}{\partial\theta} - iS\right)^2 + C + D(\theta) + E \frac{\partial^4}{\partial S^4} \right) \tilde{\phi}(\theta,S)$$

$$- \int_T d^3v\, F_e^M \left(\frac{\omega - \omega_{*e}[1 + \eta_e(E/T_e - 3/2)]}{\omega - \bar{\omega}_{De}(E/T_e) + i\nu_-(E/T_e)^{-3/2}} \right) \bar{\phi} + K\tilde{\phi} = 0 \qquad (3)$$

where A contains the first-order radial finite ion gyroradius terms, B includes the ion sound contribution $\propto (\bar{\omega}_{ti}/\omega)^2$, C is the sum of the electron adiabatic, ion hydromagnetic, and poloidal finite ion gyroradius terms, D includes the ion magnetic curvature drift terms, E contains the second-order corrections to A, and K represents the contribution from ion and untrapped electron Landau resonances. In the trapped-electron driving terms, $\omega_{*e} = k_\theta \rho_i v_i \tau / 2r_n > 0$, $\tau \equiv T_e/T_i$, $v_i = (2T_i/m_i)^{1/2}$, $r_n^{-1} \equiv -(d/dr)\,\ell n\, n$, $k_\theta \equiv m^0/r = \ell q(r_0)/r$, $\eta_e \equiv d\,\ell n\, T_e / d\,\ell n\, n$, $\bar{\omega}_{De}$ is the orbit-averaged electron magnetic drift frequency, $\nu_- \equiv \nu_e/\varepsilon$, $\varepsilon \equiv r/R$, and $\bar{\phi} \equiv \{\oint d\theta'\, \tilde{\phi}(S,\theta') \exp[iS(\theta-\theta')]/v_{\parallel}(\theta')\}/[\oint d\theta/v_{\parallel}(\theta)]$. This equation can be converted into a matrix equation by expanding $\tilde{\phi}(\theta,S)$ in complete sets of poloidal and radial basis functions, specifically,

$$\tilde{\phi}(\theta,S) = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \tilde{\phi}_{jn}\, g_j(\theta) h_n(S) \qquad (4)$$

where $g_j(\theta) = (2\pi)^{-1/2} \exp(ij\theta)$, $h_n(S) = M_n^{-1/2} H_n(\sigma^{1/2}S) \exp(-\sigma S^2/2)$, $M_n \equiv (\pi/\sigma)^{1/2} 2^n n!$, H_n denotes a Hermite polynomial, and σ is a parameter [with $\mathrm{Re}(\sigma) > 0$] which is adjusted to minimize the required number of radial basis functions. Poloidal periodicity is clearly satisfied, and the radial boundary condition, $\tilde{\phi} \to 0$ as $S \to \pm\infty$, is also satisfied for $\mathrm{Re}(\sigma) > 0$. The latter corresponds to "globally" localized modes which can spread over a number of mode-rotational surfaces. Substituting Eq. (4) into Eq. (3), multiplying by $g_{j'}^*(\theta) h_{n'}(S)$, and integrating over θ and S gives the basic matrix equation, which is solved by standard numerical procedures.

To check the 2D code against previous 1D analytic calculations of the radial structure [1], the poloidal dependence as well as terms D, E, and K are suppressed in Eq. (3), and a typical very long wavelength mode ($k_\theta \rho_i \simeq 0.04$) is analyzed. In the same spirit, the radial structure and terms A, B, D, E, and K are suppressed to compare with previous analytic calculations of the radially-local poloidal structure [1]. The comparisons indicate good agreement in both cases. We then proceed to study the proper 2D functional dependence with all terms kept in Eq. (1). Growth rates calculated for $k_\theta \rho_i \simeq 0.2$ and $rq'/q = 1$ are found to be quite close to the results from the 1D radial analysis [2]. This seems to indicate that the 1D radial estimate of shear stabilization may be more accurate than expected. However, it should be emphasized that the actual mode structure, as shown in Fig. 1(a), is very different from the simple 1D result. Hence, the basis of comparison with experimentally measured mode structure, as well as the usual assumptions regarding the radial wave spectrum introduced in nonlinear theory, could be significantly altered.

The most serious limitation of the analysis described is the constraint, $k_r\rho_i < 1$, which is required to ensure the validity of the differential formulation for calculating the radial dependence. Typically, this condition is found to be violated for $k_\theta\rho_i \gtrsim 0.3$. Since the radially local results indicate that the maximum growth rates lie in the range $k_\theta\rho_i \gtrsim 1$, a reformulation of the analysis of the radial dependence is necessary. The term which primarily governs the radial structure comes from the finite ion gyroradius factor in the perturbed ion density response and can be expressed as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_r \exp[ik_r(r-r_o)]J_o^2[(v_\perp/\Omega)(k_\theta^2+k_r^2)^{1/2}] \int_{-\infty}^{\infty} dr \exp[-ik_r(r-r_o)]\tilde{\phi}(0,r) \ .$$

If $k_r\rho_i < 1$, the Bessel function term, J_O^2, can be expanded in the usual manner, and the integrals here are trivially performed to yield the familiar differential form. For the general case of arbitrary k_r, we can still carry out the r-integration analytically, but the k_r-integration must be calculated numerically. However, the basic structure of the matrix equation remains unchanged. Hence, the numerical procedures used for solving the long radial wavelength (differential) problem can be employed for the integral formulation as well. The influence of ion collisions and various corrections to lowest-order effects have also been incorporated in the analysis. After confirming that the new 2D code accurately reproduces the results from the previous 2D treatment of long wavelength modes, a case with $k_\theta\rho_i \simeq 1.4$ is studied. Here we find that the growth rate is roughly a factor of two below the 1D radial estimate [2], and that the poloidal structure, as shown on Fig. 1(b), exhibits a strong localization at the magnetic field minimum (where the trapped particles are most strongly concentrated). This indicates that the usual assumption of a flute-like poloidal structure is quite inaccurate for short wavelength modes. As $k_\theta\rho_i$ is further increased, the trend toward smaller growth rates (compared to 1D results) and poloidal localization is found to persist. This is likely related to the fact that the trapped-electron driving term in Eq. (3) is reduced. Specifically, since $S \propto k_\theta(r-r_o)$, the phase, $iS(\theta-\theta')$, in $\bar{\phi}$ can become quite large at short wavelengths.

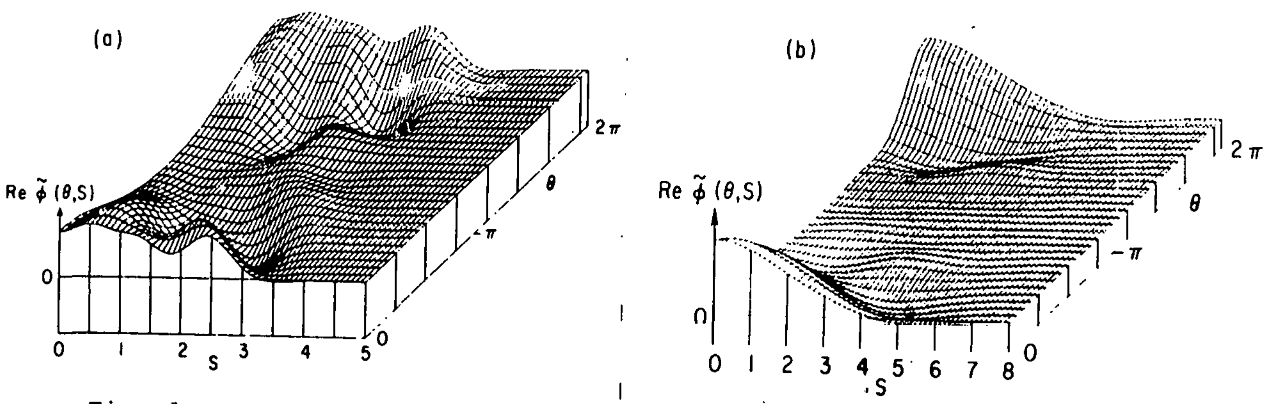

Fig. 1. Real part of the perturbed electrostatic potential eigenfunction $\tilde{\phi}(\theta,S)$ for $k_\theta\rho_i = 0.2$ [Fig. 1(a)] and $k_\theta\rho_i = 1.4$ [Fig. 1(b)]. Note that the poloidal angle θ is zero at the outside of the torus, and that the radial variable S is $S = (r-r_o)/\Delta r_s$, where r_o is the radius of the mode-rational surface around which the mode is centered, and Δr_s is the spacing of mode-rational surfaces. Only positive values of S are shown since the eigenfunction is symmetric in (Sθ). (PPPL 762223)

Two basic collisional mechanisms, which affect the linear stability of trapped-electron modes, are the collisional scattering of low energy electrons and the collisional broadening of the resonant response of the untrapped electrons. To a good approximation the trapped electrons satisfy the conditions, $v_{\|} < \epsilon^{1/2} v$ and $v > \nu_*^{1/4} v_e$, where the collisionality parameter, $\nu_* \equiv \nu_{-}/\bar{\omega}_{be}$, is equal to unity at the transition from the plateau to the banana regime. The latter constraint, which is ignored in most calculations, can lead to a significant reduction in the nonadiabatic trapped-electron response.

For the untrapped electrons, the nonadiabatic response is modified by the collisional broadening of the resonant interaction. In studying this effect we follow the familiar 1D radial eigenmode analysis [1] but include electron magnetic curvature drifts [2] and untrapped electron Landau resonances [3]. The radial dependence of the latter effect is treated perturbatively [3] and leads to an additional term in the usual eigenvalue equation. For the lowest (most unstable) radial eigenmode, this term is proportional to $\int_0^\infty dx \exp(-ux^2/2)\tilde{n}_{UT}$, where x is the radial variable, $u \equiv (\Omega_i/\omega)|k_\theta/L_s|$, $L_s^{-1} \equiv (\epsilon/q) d\,\ell n\, q/dr$, and $\tilde{n}_{UT}$ is the nonadiabatic untrapped electron density response. To include the resonance broadening effect in $\tilde{n}_{UT}$, we use an approximate Lorentz collision operator, $\nu_e(v) v^2 \partial^2/\partial v_{\|}^2$, in the drift kinetic equation determining the perturbed distribution function for the untrapped electrons. This is solved by a Fourier transform method which leads to the result

$$\tilde{n}_{UT} \propto \int_{UT} d^3v \, \frac{\omega-\omega_*^T}{|k_{\|}|} \exp(-v^2/v_e^2) \int_0^\infty dp \exp\left[-ip\left(v_{\|} - \frac{\omega}{|k_{\|}|}\right) - \frac{\nu_e p^3 v^2}{3|k_{\|}|}\right], \quad (5)$$

where p is the transform variable associated with $v_{\|}$, $k_{\|} = k_\theta x/L_s$, and $\omega_*^T \equiv \omega_{*e}\{1 + \eta_e[(v/v_e)^2 - 3/2]\}$. For $|\omega + i(\nu_e/3)p^2v^2|(L_s|u^{1/2}|/k_\theta v_e) << 1$, we can analytically perform the x, p, and $v_{\|}$ integrations in the nonadiabatic untrapped electron contribution to the usual eigenvalue equation [2]. The influence of this term for $\nu_* \lesssim 1$ is generally found to be weak. It should also be noted that even if $\nu_* > 1$, the collisional broadening tends to strongly <u>reduce</u> the Landau damping for $\nu_e/\omega \gtrsim 3$. However, nonresonant collisional effects from the nonadiabatic untrapped electrons may be significant in this regime.

<u>Trapped-Ion Mode</u>

Operating conditions in present toroidal systems, such as PLT and T-10, as well as those in future larger tokamaks are expected to reach the high temperature regime where the trapped-ion instability is predicted to appear. Here we present (1) a comprehensive analysis of the nonlinear saturation of these dangerous modes by mode coupling; and (2) a calculation showing that a new branch of trapped-ion modes, which rotate in the ion-, rather than the usual electron-diamagnetic direction, can also appear.

In studying the saturation of the usual dissipative trapped-ion instability, we consider the two-dimensional Kadomtsev-Pogutse fluid equations modified by essential kinetic effects [4]. The basic mechanism considered is the process whereby energy in long wavelength unstable modes is nonlinearly coupled via $\underline{E} \times \underline{B}$ convection to short wavelength modes stabilized by ion Landau damping. The fundamental nonlinear equation for the potential, $\Phi \equiv e\phi/T$, in the absence of kinetic modifications can be expressed as

$$\frac{\partial\phi}{\partial t} + v_* \frac{\partial\Phi}{\partial y} + \frac{v_*^2}{\nu_-}\frac{\partial^2\phi}{\partial y^2} + \nu_+\phi + \frac{v_*}{\varepsilon^{1/2}}\frac{\partial\phi^2}{\partial y} + \frac{2r_n v_*^2}{\varepsilon^{1/2}\nu_-}\left(\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}\right) = 0 \quad (6)$$

where $\Phi << 1$, $T_e = T_i \equiv T$, $v_* \equiv \omega_o/k_y = (\varepsilon^{1/2}/2)(cT/eB)r_n^{-1}$, $r_n^{-1} \equiv -(d/dx)\ln n_o$, $\nu_+ \equiv \nu_i/\varepsilon$, and the slab coordinates, $x = r$ and $y = r(\theta-\zeta/q)$ are employed. Since $\omega/\nu_- << 1$, it is relevant to analyze the radially-local, one-dimensional problem which results when the last term in Eq. (6) is ignored. As noted in previous work [4], linear kinetic theory indicates that Landau damping by both trapped and circulating ions provides an energy sink at short wavelengths for sufficiently weak temperature gradients. Saturation results when the wave steepening due to the $\partial\Phi^2/\partial y$ term in Eq. (6) moves energy from the unstable long wavelength modes to the damped modes at short wavelength. Ion Landau damping is added to Eq. (6) in a perturbative fashion, and the resultant expression is then transformed to the drift frame moving with speed v_* to yield

$$\frac{\partial\psi}{\partial\tau} + \frac{\partial^2\psi}{\partial\xi^2} + \alpha\frac{\partial^4\psi}{\partial\xi^4} + \nu\psi + \frac{\partial\psi^2}{\partial\xi} = 0 \quad (7)$$

where $\xi \equiv (y-v_*t)/r$, $\psi \equiv (\nu_-/\omega_o)(\Phi/\varepsilon^{1/2})$, $\nu \equiv \nu_+\nu_-/\omega_o^2$, $\tau \equiv \omega_o^2 t/\nu_-$, $\alpha = A'(1-1.5\eta_i)(\nu_-/\bar{\omega}_{bi})(\omega_o/\bar{\omega}_{bi})^2$, $A' \sim 40$, $\eta_i \equiv d\ln T_i/d\ln n_o$, and $\bar{\omega}_{bi}$ is the thermal ion bounce frequency.

LaQuey, et al. [4] obtained two types of steady-state analytic solutions to Eq. (7); namely, "two-mode" equilibria and "multi-mode" or "solitary wave" equilibria. The question of the stability of the multi-mode equilibria to linear perturbations can be cast in the form of an eigenvalue problem. Using a Nyquist analysis we find that such equilibria are always unstable. Equation (7) has also been numerically integrated in time with the initial conditions being random noise in one case and the actual multi-mode equilibria in another case. In both situations the inaccessibility of such equilibria is clearly demonstrated, and the conclusions of the Nyquist analysis are confirmed.

The two-mode equilibria, on the other hand, are found to be both stable and accessible for a certain range of α. Considering the case where only a linearly unstable mode, ℓ, and its stable harmonic, 2ℓ, are significantly excited, one can obtain a steady-state solution to Eq. (7) which is stationary in the drift frame and has the form

$$\psi(\xi) = \sum_{\ell=1}^{\infty} a_\ell \sin(\ell\xi) \quad (8)$$

with $a_\ell = \pm\ell(\gamma_\ell|\gamma_{2\ell}|)^{1/2}$, $a_{2\ell} = -\ell\gamma_\ell$, and $\gamma_\ell = \ell^2(1-\alpha\ell^2) - \nu$ for $1/4 < \alpha\ell^2 < 1$. Linear perturbations, $\delta a_m \exp(-i\omega t)$, on the basic mode-coupling equation must satisfy

$$-i\omega\delta a_m = \gamma_m\delta a_m - m\sum_{\ell=-\infty}^{\infty} a_\ell\delta a_{m-\ell}\,. \quad (9)$$

Since the higher harmonics are heavily damped, it is adequate to just consider the perturbed modes m, $\ell \pm m$, $2\ell \pm m$, (ℓ being the fundamental mode number) in solving this matrix equation. The results of this analytic procedure indicate that with $\nu = 0$, two-mode equilibria are stable to linear perturbations for $0.6 \lesssim \alpha\ell^2 \lesssim 0.7$. Direct numerical integration of Eq. (7) with random noise initial conditions confirms this conclusion and also demonstrates the accessibility of this class of equilibria.

Ion collisional effects are important in determining the linear stability threshold for the trapped-ion modes and also introduce an energy sink at long wavelengths. However, their nonlinear influence is relatively weak because the dominant nonlinear wave steepening behavior (described earlier) transfers energy to short wavelengths. Wave dispersion, on the other hand, can be an important nonlinear effect in that it can hinder mode coupling and force the saturated amplitudes to significantly higher levels. Kinetic effects, such as finite ion banana-width excursions, give rise to dispersion and can be modeled by adding the term, $\delta\partial^3\psi/\partial\xi^3$, to the left side of Eq. (7) with $\delta \equiv (\nu_-/\omega_o)(\rho_i^2 q^2/\varepsilon r^2)$. This leads to equilibria with a finite group velocity in the drift frame and with a structure similar to the previously described two-mode equilibria. The group velocity, u, is introduced in Eq. (7) by replacing $\partial/\partial\tau$ with $-u\partial/\partial\xi$, and is found to scale as $u \propto \delta/\alpha$. The corresponding increase in energy content is $\Delta\langle(\partial\psi/\partial\xi)^2\rangle_\xi \propto \delta^2/\alpha$ with $\langle\ \rangle_\xi$ denoting averaging over ξ. Hence, the influence of dispersive effects on the saturation of the trapped-ion instability is negligible only if $\delta^2/\alpha < 1$.

Recalling that the radial transport is primarily driven by the $\underline{E} \times \underline{B}$ drift and assuming that wave dispersion is small, the diffusion coefficient can be expressed as

$$D = \frac{\varepsilon^{3/2}}{\nu_-}\left(\frac{\omega_o}{\nu_-}\right)^2\left(\frac{cT}{reB}\right)^2\left\langle\left(\frac{\partial\psi}{\partial\xi}\right)^2\right\rangle_\xi . \tag{10}$$

We use Eq. (8) to find that $\langle(\partial\psi/\partial\xi)^2\rangle_\xi \propto \alpha^{-2}$. Comparison with the Kadomtsev-Pogutse estimate then yields (with r_n and R in cm)

$$D/D_{K-P} \approx 2\times10^7 r_n^4(\varepsilon/qR)^6(1-1.5\eta_i)^{-2}B(50\text{ kG})^2 n(10^{14}\text{cm}^{-3})^{-4}T(\text{keV})^7 . \tag{11}$$

This indicates that saturation via one-dimensional mode-coupling leads to transport levels significantly more optimistic than those given by Kadomtsev and Pogutse for parameters just above the instability threshold. For conditions well above threshold (i.e. at higher temperatures or lower densities), D/D_{K-P} rapidly becomes very large. However, our basic model ceases to be valid in such regimes. In fact, it should be emphasized that in order to satisfy the various validity conditions for the basic model (i.e., $\omega_o \ll \omega_{bi}$, $\omega_o \ll \nu_-$, $\nu_+ < \omega_o^2/2\alpha\nu_-$, $\delta < \alpha^{1/2}$), we find that for typical PLT parameters our analysis is only meaningful for situations close to marginal stability.

As a final point we note that the two-dimensional nonlinear mode structure is currently being investigated. We find that the inclusion of radial perturbations via the last term in Eq. (6) can have a destabilizing influence on the one-dimensional equilibria given by Eq. (8), provided $|k_x r_n|(\omega/\nu_-)$ is sufficiently large. Since all k_x modes will be nonlinearly generated, this result emphasizes the importance of studying radial features such as magnetic shear and the stabilizing effects associated with ion banana-width dispersion (together with ion collisions). Inclusion of such effects provides an energy sink at short radial wavelengths, thus allowing the possible formation of two-dimensional saturated states.

In addition to the usual dissipative trapped-ion instability, which is basically an electron diamagnetic drift mode, we have found that ion diamagnetic modes of this type can also be generated. This new branch can be driven unstable by both resonant and nonresonant interactions with ions which have average unfavorable magnetic drifts (i.e. $\bar{\omega}_{Di} < 0$). Unlike the familiar trapped-ion modes, which can be stabilized in tokamak systems with flat or reversed gradient profiles [2], these modes can persist as residual instabilities. Moreover, for sufficently large ion temperature gradients ($\eta_i > 2/3$), the ion branch is usually dominant even for normal density profiles. To arrive at these conclusions, we analyze the local form of the Kadomtsev-Pogutse trapped-ion mode dispersion relation,

$$\frac{1+\tau^{-1}}{(2\varepsilon)^{1/2}} = \left\langle \frac{\Omega + (\varepsilon_T\eta)_i^{-1} + (\varepsilon_T)_i^{-1}(\bar{E} - 3/2)}{\Omega + \bar{E} + i\nu_+(\bar{E})^{-3/2}} \right\rangle + \left\langle \frac{\Omega/\tau - (\varepsilon_T\eta)_e^{-1} - (\varepsilon_T)_e^{-1}(\bar{E} - 3/2)}{\Omega - \tau\bar{E} + i\nu_-(\bar{E})^{-3/2}} \right\rangle \quad (12)$$

where $\langle A \rangle \equiv (2/\pi^{1/2}) \int_0^\infty d\bar{E}\, \bar{E}^{1/2} \exp(-\bar{E}) A$, $\bar{E} \equiv E/T_j$, $\tau \equiv T_e/T_i$, $\varepsilon_{Tj} \equiv r_{Tj}/R$, $r_{Tj}^{-1} \equiv -(d/dr)\, \ell n\, T_j$, $\eta_j \equiv (r_n/r_T)_j$, and $\Omega \equiv -\omega/\bar{\omega}_{Di}$. The Nyquist analysis of this equation indicates the existence of two unstable branches over a wide range of parameters. However, the actual numerical solutions for Ω indicate that only one branch is dominant. Specifically, if η_i is either large ($\eta_i > 2/3$) or negative ($\eta_i < 0$), the growth rates from the ion branch are usually larger than those from the electron branch and also persist at much higher collision frequencies. With regard to the flat ($\eta_i^{-1} = 0$) and reversed ($\eta_i < 0$) density gradient cases, we find that these ion diamagnetic "residual" modes generally have smaller growth rates than the standard type of trapped-ion modes (e.g., with $\eta_i = 1/2$), especially at lower collision frequencies.

The important features of the ion branch can be analytically derived from Eq. (12) in relevant asymptotic limits. In particular, we consider (i) $\nu_i = \nu_e = 0$ corresponding to the collisionless or interchange mode and (ii) $\nu_i = 0$ with $\nu_e = \infty$ corresponding to the lowest-order limit of the dissipative trapped-ion mode. In both cases we can express the energy integrals of Eq. (12) in the form of plasma dispersion functions (Z-functions). The usual collisionless trapped-ion mode can be recovered by assuming $|\bar{\omega}_D/\omega| << 1$ and expanding the equation to second order in this quantity. For large or negative η_i, it is necessary to carry the series to $|\bar{\omega}_D/\omega|^4$ before truncation. In accordance with the numerical solutions to Eq. (12), this algebraic equation yields two unstable roots with approximately equal growth rates and equal but opposite real frequencies. For the lowest-order limit of the dissipative trapped-ion mode, Eq. (12) can be expressed as

$$\xi = G_2 - \Lambda G_1 \quad (13)$$

where $\xi \equiv [(1 + \tau^{-1})/(2\varepsilon)^{1/2} - 1][\varepsilon_T/(1 - \varepsilon_T)]$, $\Lambda \equiv (3/2 - \eta_i^{-1})/(1 - \varepsilon_T)$, $G_1 \equiv 2[1 + yZ(y)]$, $G_2 \equiv [1 + 2y^2 + 2y^3 Z(y)]$, and $y \equiv -(-\Omega)^{1/2}$. Writing

$\Omega = \Omega_r + i\tilde{\gamma}$ and separating Eq. (13) into its real and imaginary parts, it is easily demonstrated that this equation yields no unstable solutions if $\Lambda \leq 0$ and/or $\xi \geq 1$. Since $\varepsilon_T << 1$, the first condition is equivalent to requiring $0 \leq \eta_i \leq 2/3$, and the latter condition is just $(2\varepsilon)^{1/2} \geq \varepsilon_T(1 + \tau^{-1})$. These conclusions are in complete agreement with the numerical solutions to Eq. (12) for all cases considered. Analytic estimates of growth rates for these dissipative modes can again be obtained by expanding the Z-functions for $|\varpi_D/\omega|' << 1$. For very weak density gradients, where $|\eta_i^{-1}|$ is of the same order or less than $|\varepsilon_T|$, we find $\Omega \simeq i[(3/\varepsilon_T)(\varepsilon/2)^{1/2}(1 + \tau^{-1})^{-1}]^{1/2}$ with ε, ε_T, $\varepsilon_n << 1$. This is a purely growing (nonresonant) type instability that is very similar in character to the familiar collisionless trapped-particle mode. For the more general case we find

$$\Omega \simeq -5/2 + (1/2C_1)[-C_2 + i(4C_1C_3 - C_2^2)^{1/2}] \qquad (14)$$

where $C_1 \equiv [1 + \tau^{-1} - (2\varepsilon)^{1/2}]/(2\varepsilon)^{1/2}$, $C_2 \equiv 3/2 - (\varepsilon_T\eta_i)^{-1}$, and $C_3 \equiv (\varepsilon_T)^{-1}(3/2 - \eta_i^{-1})$. Comparisons with numerical solutions to Eq. (12) indicate that this analytic estimate is reasonably accurate, especially in a qualitative sense.

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