**DESIGN GUIDE
FOR
CALCULATING HYDRODYNAMIC MASS
PART I: CIRCULAR CYLINDRICAL STRUCTURES**

S.S. Chen and Ho Chung

BASE TECHNOLOGY

U of C-AUA-USERDA

**COMPONENTS TECHNOLOGY DIVISION
ARGONNE NATIONAL LABORATORY
Argonne, Illinois**

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Part I: Circular Cylindrical Structures

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NOMENCLATURE

<u>Symbol</u>	<u>Description</u>	<u>Units</u>
c_m	Hydrodynamic mass coefficient	dimensionless
c_v	Viscous damping coefficient	kg/m-sec
D	Diameter	m
E	Young's modulus	Pa
f_c	Coupled natural frequencies of cylinders	Hz
f_e	Natural frequencies in vacuo	Hz
F_i	Hydrodynamic forces per unit length or per unit area	N/m or N/m ²
G	Gap between cylinders	m
h	Thickness	m
H	Parameter given by Eq. (4)	dimensionless
I	Moment of inertia	m ⁴
I_0, I_1	Modified Bessel functions of the first kind	dimensionless
K_0, K_1	Modified Bessel functions of the second kind	dimensionless
k	Number of bodies in a structural system	dimensionless
ℓ	Length of cylinder, or a half-wavelength of cylindrical shell	m
m	Mass of a body per unit length	kg/m
m_a	Hydrodynamic mass per unit length ($c_m \rho V$), or per unit area ($c_m \rho R_i$)	kg/m or kg/m ²
m_{ij}	Element of hydrodynamic mass matrix (per unit length) for cylinder	kg/m
$\bar{m}_{ij}$	Element of hydrodynamic mass matrix (per unit area) for cylindrical shell	kg/m ²
n	Circumferential wave number for shell	dimensionless
p	Distance between the centers of two cylinders ($R_1 + R_2 + G$)	m

NOMENCLATURE (Contd.)

<u>Symbol</u>	<u>Description</u>	<u>Units</u>
r	Ratio of cylinder radii (R_1/R_2)	dimensionless
R_1	Radius of cylinder	m
$\text{Re}(H)$, $\text{Im}(H)$	Real and imaginary parts of a complex quantity H	dimensionless
S	Dimensionless number ($\omega R_1^2/\nu$)	dimensionless
T	Temperature	degree
t	Time	sec
u	Displacement of a body	m
V	Volume of a body	m^3
α	Parameter given by Eqs. (4), (14), and (20)	dimensionless
α_1, α_2	Parameter given by Eq. (14)	dimensionless
α_s	Parameter given by Eq. (11)	dimensionless
β	Parameter given by Eq. (20)	dimensionless
ϵ	Eccentricity of two eccentric cylinders	dimensionless
μ_i	Principal values of hydrodynamic mass matrix	N/m
μ'_i	Principal values of hydrodynamic mass coefficient matrix	dimensionless
ν	Kinematic viscosity of fluid	m^2/sec
ν_{ij}	Element of hydrodynamic mass coefficient matrix	dimensionless
ρ	Fluid density	kg/m^3
ρ_t	Mass density of cylinder	kg/m^3
ω	Circular frequency of oscillation	sec^{-1}

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ABSTRACT

Many reactor and plant components contain, or are submerged in, a fluid. The fluid moving with a vibrating structure has an important effect on the dynamics of the structure, particularly on its natural frequencies. The effect of the fluid on natural frequencies can be accounted for using the hydrodynamic mass associated with the structure. This design guide provides formulas, graphs, and computer programs for calculating hydrodynamic mass. Part I of this design guide presents methods for calculating the hydrodynamic masses of circular cylindrical structures. The second part of the design guide, to be published in the future, will include structural components with other types of geometry, such as plates.

I. INTRODUCTION

When a structural component which is submerged in a fluid vibrates, the surrounding fluid must be displaced to accommodate these motions. As a result, fluid pressures are generated. In turn, the integrated effect of these pressures produces a hydrodynamic force acting on the structure.

The fluid moving with the structure has an important effect on the dynamics of the vibrating structure; particularly, its natural frequencies and damping characteristics. The influence on the natural frequency is associated with the hydrodynamic mass, while the damping effect is attributed to dissipative viscous loss and acoustic radiation.

Ideally, the hydrodynamic force should be calculated based on the three-dimensional Navier-Stokes equations. Obviously, this is difficult, even in simple cases. In many practical situations, the effects of fluid viscosity and compressibility on the hydrodynamic mass can be neglected. In these cases, incompressible potential flow theory can be used to calculate the hydrodynamic mass.

The objective of this design guide is to provide formulas and graphs for computing hydrodynamic mass. While it is recognized that fluid damping is important in structural response, analytical solutions including viscous effects are difficult to obtain. Therefore, fluid damping effect will be presented only for a few simple cases to illustrate the general characteristics.

Part I of this design guide presents methods for calculating the hydrodynamic masses of circular cylindrical structures. Part II will include structural components with other types of geometry, e.g., plates.

II. DEFINITION OF HYDRODYNAMIC MASS AND VISCOUS DAMPING

A structural component moving at a constant velocity in an infinite ideal fluid encounters no resistance. This phenomenon is commonly referred to as D'Alembert's paradox. In contrast, a body moving at a variable velocity, even in a condition of potential flow, experiences a resistance; the body behaves as though an added mass of fluid were rigidly attached to and moving with it. When the body is subjected to an excitation, not only must the mass of the body be accelerated, but also that of the added fluid mass. The total force required to accelerate the body is given by [Ref. 1]:

$$F = (m + m_a) \frac{\partial^2 u}{\partial t^2} + c_v \frac{\partial u}{\partial t} \quad (1)$$

Where m is the body mass, m_a is referred to as hydrodynamic mass (or added mass), and c_v is the viscous damping coefficient. Note that two force components are associated with the fluid motion:

$$(1) \quad m_a \frac{\partial^2 u}{\partial t^2}, \text{ in phase with the acceleration, arises}$$

because the fluid moves as the body oscillates.

$$(2) \quad c_v \frac{\partial u}{\partial t}, \text{ opposing the movement of the body, is}$$

attributed to fluid viscosity.

The hydrodynamic mass is proportional to the fluid density ρ and the body volume V , and is given by:

$$m_a = \rho V c_m, \quad (2)$$

where c_m is the hydrodynamic mass coefficient. The sum of the hydrodynamic mass and the body mass is called the virtual mass or the apparent mass.

The viscous damping coefficient depends on fluid viscosity, fluid density, vibration frequency, and others. If the fluid is inviscid, c_v is zero.

Equation (1) corresponds to a body moving in a particular direction.

For a body having three degrees of freedom both in translation and rotation, a complete description of the hydrodynamic mass requires a six-by-six matrix. The hydrodynamic masses for a body with N degrees of freedom are represented by $[m_{ij}]$ ($i, j = 1, 2, 3, \dots, N$). It can be shown that $[m_{ij}]$ is symmetric; thus for a body with N degrees of freedom, the number of terms needed to completely describe the hydrodynamic mass is $N(N + 1)/2$.

In a structural component consisting of multiple bodies, the situation is more complicated. For example, consider a structure containing k bodies, in which each body possesses N' degrees of freedom. The hydrodynamic masses again can be represented by a matrix $[m_{ij}]$ ($i, j = 1, 2, 3, \dots, N$; $N = N' \times k$). As in the case of a single body, $[m_{ij}]$ is symmetric.

Since $[m_{ij}]$ is symmetric, one can find the N eigenvalues of the hydrodynamic mass matrix $[m_{ij}]$. These eigenvalues are called effective hydrodynamic masses. In many practical applications, the effective hydrodynamic masses for multiple bodies play the same role as the hydrodynamic mass for a single body [Ref. 2].

III. HYDRODYNAMIC MASSES

A. A Single Cylinder in an Infinite Fluid

For a long, circular cylinder vibrating in an infinite, viscous fluid, in the direction perpendicular to its axis (Fig. 1), c_m and c_v are given by [Ref. 1]:

$$c_m = \text{Re}(H) ,$$

and

$$c_v = -\rho\pi R^2\omega \text{Im}(H) ,$$

where

$$H = 1 + \frac{4K_1(\alpha)}{\alpha K_0(\alpha)} ,$$

$$\alpha = \sqrt{iS} ,$$

$$S = \frac{\omega R^2}{\nu} .$$

When S is very large,

$$c_m = 1 + \frac{2\sqrt{2}}{\sqrt{S}} ,$$

and

$$-\text{Im}(H) = \frac{2\sqrt{2}}{\sqrt{S}} + \frac{2}{S} .$$

Furthermore, if the fluid is inviscid, H is equal to 1; thus

$$c_m = 1$$

and

$$c_v = 0 .$$

This is the familiar case in which the hydrodynamic mass is equal to the displaced mass of fluid.

Values of H are given in Fig. 1* as a function of S .

*Figures begin on page 40.

B. A Cylinder Vibrating in a Fluid Annulus

For transverse vibration, c_m and c_v are defined as in Eqs. (3), except H is given as follows [Ref. 1]:

$$\begin{aligned}
 H = & \{2\alpha^2[I_0(\alpha)K_0(\beta) - I_0(\beta)K_0(\alpha)] - 4\alpha[I_1(\alpha)K_0(\beta) + I_0(\beta)K_1(\alpha)] \\
 & + 4\alpha r[I_0(\alpha)K_1(\beta) + I_1(\beta)K_0(\alpha)] - 8r[I_1(\alpha)K_1(\beta) - I_1(\beta)K_1(\alpha)]\} \\
 & \div \{\alpha^2(1-r^2)[I_0(\alpha)K_0(\beta) - I_0(\beta)K_0(\alpha)] + 2\alpha r[I_0(\alpha)K_1(\beta) \\
 & - I_1(\beta)K_0(\beta) + I_1(\beta)K_0(\alpha) - I_0(\beta)K_1(\beta)] + 2\alpha r^2[I_0(\beta)K_1(\alpha) \\
 & - I_0(\alpha)K_1(\alpha) + I_1(\alpha)K_0(\beta) - I_1(\alpha)K_0(\alpha)]\} - 1,
 \end{aligned} \quad (7)$$

where

$$\alpha = \sqrt{1S},$$

$$\beta = \frac{1}{r} \sqrt{1S},$$

$$r = R_1/R_2,$$

and

$$S = \frac{\omega R_1^2}{v}.$$

When α and β are large,

$$H = \frac{[\alpha^2(1+r^2) - 8r]\sinh(\beta-\alpha) + 2\alpha(2-r+r^2)\cosh(\beta-\alpha) - 2r^2\sqrt{\alpha\beta} - 2\alpha r\sqrt{r}}{\alpha^2(1-r^2)\sinh(\beta-\alpha) - 2\alpha r(1+r)\cosh(\beta-\alpha) + 2r^2\sqrt{\alpha\beta} + 2\alpha r\sqrt{r}}. \quad (8)$$

Values of $\text{Re}(h)$ and $\text{Im}(H)$ are presented in Figs. 2 and 3.

C. A Circular Cylinder Near a Wall

The hydrodynamic mass for a circular cylinder near a parallel wall is independent of the direction of motion in an ideal fluid; c_m is given by Ref. [3]

$$c_m = 1 + 4 \sinh^2 \alpha \sum_{\ell=1}^{\infty} \ell \frac{e^{-3\ell\alpha}}{\sinh(\ell\alpha)} \quad (9)$$

where

$$\alpha = \ln \left(\frac{R+G + \sqrt{(R+G)^2 - R^2}}{R} \right).$$

Values of c_m are presented in Fig. 4.

D. A Circular Cylindrical Shell Containing Fluid

The hydrodynamic mass of a circular cylindrical shell depends on the shell vibrational modes. The closed-form solution is difficult to obtain, except for a shell with simply supported end conditions. For a simply supported shell containing ideal fluid [Ref. 4]:

$$m_a = \rho R c_m, \quad (10)$$

where m_a is the hydrodynamic mass per unit area, and c_m is given by

$$c_m = \frac{I_n(\alpha_s)}{\alpha_s \frac{d}{d\alpha_s} [I_n(\alpha_s)]} \quad (11)$$

$$\alpha_s = \frac{2\pi R}{\ell}.$$

Note that $I_n(\alpha_s)$ is the modified Bessel function of the first kind.

c_m depends on the circumferential wave number n and the axial wave number α_s . Values of c_m are presented in Fig. 5 as functions of α_s for different n .

E. Two Parallel Circular Cylinders

The hydrodynamic mass matrices are as follows [Ref. 5]:
for motion in the x direction (in-plane motion),

$$[m_{ij}] = \rho \pi \begin{bmatrix} R_1^2 v_{11} & - \left(\frac{R_1 + R_2}{2} \right)^2 v_{12} \\ - \left(\frac{R_1 + R_2}{2} \right)^2 v_{12} & R_2^2 v_{22} \end{bmatrix}; \quad (12)$$

and for motion in the y direction,

$$[m_{ij}] = \rho \pi \begin{bmatrix} R_1^2 v_{11} & \left(\frac{R_1 + R_2}{2} \right)^2 v_{12} \\ \left(\frac{R_1 + R_2}{2} \right)^2 v_{12} & R_2^2 v_{22} \end{bmatrix}. \quad (13)$$

The motions in the two directions are uncoupled. v_{11} , v_{12} , and v_{22} are

given by

$$\begin{aligned}
 \alpha_{11} &= 1 + \frac{p^4 - 2p^2(R_1^2 + R_2^2) + (R_2^2 - R_1^2)^2}{p^2 R_1^2} \sum_{\ell=1}^{\infty} \ell \frac{\exp[-\ell(\alpha + \alpha_1)]}{\sinh(\ell\alpha)} \\
 v_{22} &= 1 + \frac{p^4 - 2p^2(R_1^2 + R_2^2) + (R_2^2 - R_1^2)^2}{p^2 R_2^2} \sum_{\ell=1}^{\infty} \ell \frac{\exp[-\ell(\alpha + \alpha_2)]}{\sinh(\ell\alpha)} \\
 v_{12} &= \left[\frac{2R_1 R_2}{p(R_1 + R_2)} \right]^2 \left\{ 1 + \frac{p^4 - 2p^2(R_1^2 + R_2^2) + (R_2^2 - R_1^2)^2}{R_1^2 R_2^2} \sum_{\ell=1}^{\infty} \ell \coth(\ell\alpha) \exp(-2\ell\alpha) \right\} \\
 \alpha &= \ln \left\{ \frac{p^2 - R_1^2 - R_2^2}{2R_1 R_2} + \left[\left(\frac{p^2 - R_1^2 - R_2^2}{2R_1 R_2} \right)^2 - 1 \right]^{1/2} \right\} \\
 \alpha_1 &= 2 \ln \left\{ \frac{p^2 + R_1^2 - R_2^2}{2pR_1} + \left[\left(\frac{p^2 + R_1^2 - R_2^2}{2pR_1} \right)^2 - 1 \right]^{1/2} \right\} \\
 \alpha_2 &= 2 \ln \left\{ \frac{p^2 - R_1^2 + R_2^2}{2pR_2} + \left[\left(\frac{p^2 - R_1^2 + R_2^2}{2pR_2} \right)^2 - 1 \right]^{1/2} \right\}
 \end{aligned} \tag{14}$$

where $p = R_1 + R_2 + G$.

Values of v_{11} and v_{12} are presented in Figs. 6 and 7; v_{22} can be obtained from Fig. 6 by exchanging the subscripts 1 and 2. For example, consider two cylinders: $R_1 = 1.0$, $R_2 = 2.0$, and $G = 1.0$; these give $G/R_1 = 1$ and $R_2/R_1 = 2.0$. From Fig. 6, $v_{11} = 1.058$. Then let $R_1 = 2.0$, and $R_2 = 1.0$; these give $G/R_1 = 0.5$ and $R_2/R_1 = 0.5$. Again, from Fig. 6, $v_{11} = 1.036$. Therefore, using the original subscripts of $R_1 = 1.0$ and $R_2 = 2.0$, $v_{11} = 1.058$ and $v_{22} = 1.036$.

F. Two Eccentric Cylinders

The hydrodynamic masses for two eccentric cylinders are independent of the direction of motion; the hydrodynamic mass matrix is given by [Ref. 6]

$$[m_{ij}] = \rho \pi \begin{bmatrix} R_1^2 v_{11} & \left(\frac{R_1 + R_2}{2}\right)^2 v_{12} \\ \left(\frac{R_1 + R_2}{2}\right)^2 v_{12} & R_2^2 v_{22} \end{bmatrix}. \quad (15)$$

Values of v_{11} , v_{12} , and v_{22} are presented in Figs. 8, 9, and 10 as functions of the eccentricity $\epsilon (= \frac{G}{R_2 - R_1})$.

G. A Group of Cylinders

Consider a group of k parallel cylinders submerged in an infinite ideal fluid (Fig. 11). Each cylinder can move in the x and y directions. Let u_i and u_{k+i} designate the displacement components of cylinder i in the x and y directions, respectively. The hydrodynamic forces acting on cylinder i are given by F_i in the x direction and F_{k+i} in the y direction. F_i is given by [Refs. 2 and 6]

$$F_i = \sum_{j=1}^{2k} m_{ij} \frac{\partial^2 u_j}{\partial t^2}, \quad i = 1, 2, 3, \dots, 2k \quad (16)$$

where m_{ij} is the element of the hydrodynamic mass matrix. Equation (16) also may be written

$$F_i = \sum_{j=1}^{2k} \left(\frac{R_i + R_j}{2}\right)^2 \rho \pi v_{ij} \frac{\partial^2 u_j}{\partial t^2}, \quad (17)$$

where R_i is the radius of cylinder i , and v_{ij} is the element of the hydrodynamic mass coefficient matrix. When all cylinders are of the same radius, m_{ij} is equal to v_{ij} multiplied by the fluid mass displaced by the cylinder.

Since m_{ij} is symmetric [Refs. 2 and 6], for a group of k cylinders, it possesses $2k$ principal values, designated by μ_i ($i=1$ to $2k$). The principal values of the matrix $[m_{ij}]$ for a group of cylinders play the same role as the hydrodynamic mass for a solitary cylinder; those principal values are called the effective hydrodynamic masses.

Many cylinders are arranged in a hexagonal pattern as shown in Fig. 12. The hydrodynamic mass coefficient v_{ij} , and the effective hydrodynamic mass coefficient μ_i' , which is the eigenvalue of $[v_{ij}]$, are computed for three cases of seven, nineteen, and thirty-seven cylinders arranged in hexagonal arrays.

Figure 13 shows the variation of v_{ij} with p/D where p is tube pitch and D is tube diameter. Note that v_{ij} is positive and approaches unity as p/D becomes very large, while v_{ij} for $i \neq j$ may be negative and approaches zero as the pitch-to-diameter ratio (p/D) becomes infinite.

Figure 14 shows the upper and lower bounds of μ_i' as functions of p/D . For a group of k cylinders, there are $2k$ effective hydrodynamic coefficients distributed between the upper and lower bounds. All μ_i' approach unity as p/D is increased, while the upper bound increases and lower bound decreases as p/D is reduced.

From Figs. 13 and 14, it is seen that as the number of cylinders increases, the absolute values of v_{ij} increase. The results for cylinders 1 and 2 obtained from 19 cylinders and 37 cylinders do not differ significantly. The implication is that the coupling between a cylinder and other cylinders which are not immediately surrounding it can be neglected. For example, in the hexagonal arrangement, only the effects of the six cylinders surrounding the central cylinder have to be considered if the motion of the central cylinder is of interest.

The principal values of the hydrodynamic mass coefficient matrix for a seven-cylinder bundle and nine-cylinder array are presented in Figs. 15 and 16. Note that the principal values are not distinct; there are five repeated roots in each case. This is due to the symmetrical arrangement. If the cylinders are arbitrarily arranged or the cylinders are of different sizes, the principal values will be distinct.

If all cylinders in a group are identical, the natural frequencies of the coupled modes can be easily obtained. For example, if m is the mass and f_e the natural frequency of the cylinders in vacuo, the coupled natural frequencies f_c are given by

$$f_c = \left(\frac{m}{m + \rho \pi R^2 \mu_1'} \right)^{1/2} f_e \quad (18)$$

The computer program for calculating hydrodynamic mass coefficients and principal values is given in the Appendix. This program can be used for a group of cylinders with different radius in an infinite fluid or in a circular container.

H. Two Coaxial Cylindrical Shells with a Fluid-Filled Gap

Two coaxial shells are shown in Fig. 17. The hydrodynamic forces acting on the two shells per unit area are F_1 and F_2 , these are given by

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{21} & \bar{m}_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \end{Bmatrix} \quad (19)$$

The hydrodynamic mass matrix $[\bar{m}_{ij}]$ per unit shell surface area is given by [Ref. 7]

$$[\bar{m}_{ij}] = \rho \begin{bmatrix} R_1 v_{11} & -R_1 v_{12} \\ -R_2 v_{21} & R_2 v_{22} \end{bmatrix}, \quad (20)$$

$$\begin{aligned}
v_{11} &= \frac{\beta}{\Delta} \left[-K'_n(\beta) I_n(\alpha) + I'_n(\beta) K_n(\alpha) \right] \\
v_{12} &= \frac{\alpha}{\Delta} \left[I'_n(\alpha) K_n(\beta) - K'_n(\alpha) I_n(\beta) \right] \\
v_{21} &= \frac{\beta}{\Delta} \left[-K'_n(\beta) I_n(\beta) + I'_n(\beta) K_n(\beta) \right] \\
v_{22} &= \frac{\alpha}{\beta} \left[I'_n(\alpha) K_n(\beta) - K'_n(\alpha) I_n(\beta) \right] \\
\Delta &= \alpha\beta \left[I'_n(\alpha) K'_n(\beta) - I'_n(\beta) K'_n(\alpha) \right] \\
\alpha &= \frac{2\pi R_1}{\ell}, \quad \beta = \frac{2\pi R_2}{\ell},
\end{aligned}
\tag{20}$$

(Contd.)

where the prime denotes differentiation with respect to the argument of the modified Bessel functions I_n and K_n , and n is the circumferential wave number.

When α and β are very small, it can be shown that

$$\begin{aligned}
v_{11} = v_{22} &= \frac{1}{n} \left[\frac{(R_2/R_1)^{2n+1}}{(R_2/R_1)^{2n-1} - 1} \right], \\
v_{12} = v_{21} &= \frac{2}{n} \left[\frac{(R_2/R_1)^n}{(R_2/R_1)^{2n} - 1} \right].
\end{aligned}
\tag{21}$$

Values of v_{11} , v_{12} , and v_{22} are presented in Figs. 18 to 32 for various values of R_2/R_1 , n , α , and r ($= R_2/R_1$).

IV. EXAMPLES OF APPLICATION

A. Natural Frequency of a Tube in Water

Consider the case of a single tube containing, and submerged in, water. The tube is fixed at both ends. The dimensions and material properties are given in Table 1. The problem is to find the lowest natural frequency and the fluid viscous damping coefficient.

The following steps are required to find the lowest natural frequency:

- (1) Tube mass per unit length (m).

$$\begin{aligned} m &= \rho_t \pi (R_o^2 - R_i^2) \\ &= 0.358 \text{ kg/m} \end{aligned}$$

- (2) Fluid mass per unit length inside the tube (m_1).

$$\begin{aligned} m_1 &= \rho \pi R_i^2 \\ &= 0.082 \text{ kg/m} \end{aligned}$$

- (3) Hydrodynamic mass per unit length associated with the fluid outside the tube (m_2).

$$\begin{aligned} m_2 &= c_m \rho \pi R_o^2 \\ &= 0.127 \text{ kg/m (assuming } c_m = 1.0) \end{aligned}$$

- (4) Total mass per unit length (M).

$$\begin{aligned} M &= m + m_1 + m_2 \\ &= 0.567 \text{ kg/m} \end{aligned}$$

- (5) Moment of inertia (I).

$$\begin{aligned} I &= \frac{\pi}{4} (R_o^4 - R_i^4) \\ &= 7.46 \times 10^{-10} \text{ m}^4 \end{aligned}$$

- (6) The lowest natural frequency.

$$f = \frac{1}{2\pi} \left(\frac{4.73}{l} \right)^2 \sqrt{\frac{EI}{M}} = 97.72 \text{ Hz}$$

Table 1. Data for Sample Problems A and C

Tube

Outside Diameter ($2 R_o$)	1.27 cm (0.5 in.)
Inside Diameter ($2 R_i$)	1.02 cm (0.4 in.)
Length (l)	76.20 cm (30 in.)
Material: Stainless Steel	
Density (ρ_t)	7965 kg/m ³ (74.53 x 10 ⁻⁵ lb-sec ² /in. ⁴)
Young's Modulus (E)	1.93 x 10 ¹¹ Pa (28 x 10 ⁶ lb/in. ²)

Coolant

Water

Density (ρ)	1000 kg/m ³ (9.35 x 10 ⁻⁵ lb-sec ² /in. ⁴)
Kinematic Viscosity (ν) (at T = 20°C)	1.01 x 10 ⁻⁶ m ² /sec (10.9 x 10 ⁻⁶ ft ² /sec)

- (7) Calculation of S.

$$S = \frac{2\pi f R_o^2}{v} = 24513$$

- (8) From Fig. 1, $c_m = 1.02$. The improved value of hydrodynamic mass is

$$m_2 = c_m \rho \pi R_o^2 = 0.129 \text{ kg/m}$$

- (9) The new total mass per unit length (M).

$$\begin{aligned} M &= m + m_1 + m_2 \\ &= 0.569 \text{ kg/m} \end{aligned}$$

- (10) The lowest natural frequency is now improved as

$$f = \frac{1}{2\pi} \left(\frac{4.73}{l} \right)^2 \sqrt{\frac{EI}{M}} = 97.55 \text{ Hz}$$

- (11) From Fig. 1, for $S = 24513$, $-\text{Im}(H) = 0.018$.

$$\begin{aligned} c_v &= -\rho \pi R_o^2 \omega \text{Im}(H) \\ &= 1.40 \text{ kg/m-sec.} \end{aligned}$$

If more accurate results are desirable, steps (7) to (10) can be repeated. However, as expected for large values of S, the effect of viscosity on added mass coefficient and, hence, frequency is small.

B. A Circular Cylindrical Shell Containing Fluid

Consider a circular cylindrical shell with radius = 1 m, length = 2 m, and thickness = 0.01 m. The shell density is 7965 kg/m^3 , and the shell is simply supported at both ends. The problem is to find the frequency ratios for the shell containing water, and the empty shell.

Let f_c be the frequency for the shell containing water, and f_e for the empty shell. The following steps can be used to find the ratio of f_c/f_e :

- (1) Shell mass per unit area (m).

$$m = \rho_t h = 79.65 \text{ kg/m}^2$$

- (2) Axial wave number (α_s).

$$\alpha_s = \frac{2\pi R}{l} = 1.57$$

where λ is the lowest wavelength, i.e., 4 m.

(3) Hydrodynamic mass coefficient (c_m).

From Fig. 5, for

$$n = 0, \quad c_m = 1.04$$

$$n = 1, \quad c_m = 0.64$$

$$n = 2, \quad c_m = 0.42$$

$$n = 3, \quad c_m = 0.30$$

$$n = 4, \quad c_m = 0.23$$

$$n = 5, \quad c_m = 0.19$$

$$n = 6, \quad c_m = 0.16$$

$$n = 7, \quad c_m = 0.14$$

$$n = 8, \quad c_m = 0.12$$

$$n = 9, \quad c_m = 0.11$$

$$n = 10, \quad c_m = 0.10.$$

(4) Frequency ratio (f_c/f_e).

$$f_c/f_e = \left(\frac{m}{m + \rho R c_m} \right)^{1/2} \quad (22)$$

Substituting m and c_m into Eq. (22) gives, for

$$n = 0, \quad f_c/f_e = 0.267$$

$$n = 1, \quad f_c/f_e = 0.333$$

$$n = 2, \quad f_c/f_e = 0.399$$

$$n = 3, \quad f_c/f_e = 0.458$$

$$n = 4, \quad f_c/f_e = 0.507$$

$$n = 5, \quad f_c/f_e = 0.543$$

$$n = 6, \quad f_c/f_e = 0.577$$

$$n = 7, \quad f_c/f_e = 0.602$$

$$n = 8, \quad f_c/f_e = 0.632$$

$$n = 9, \quad f_c/f_e = 0.648$$

$$n = 10, \quad f_c/f_e = 0.666.$$

C. A Group of Four Cylinders

Consider a group of four cylinders as shown in Fig. 33. The cylinder dimensions and material properties are the same as that of Example A. Here, the cylinders are assumed to be empty. The pitch is 1.91 cm. The fluid is assumed to be inviscid. The coupled natural frequencies can be obtained as follows:

- (1) Natural frequency of the cylinder in air.

Cylinder mass per unit length, $m = 0.358 \text{ kg/m}$

Moment of inertia, $I = 7.46 \times 10^{-10} \text{ m}^4$

Young's Modulus, $E = 1.93 \times 10^{11} \text{ Pa}$

Cylinder length, $l = 0.762 \text{ m}$

Natural frequency in air,

$$f_e = \frac{1}{2\pi} \left(\frac{4.73^2}{l^2} \right) \sqrt{\frac{EI}{m}}$$

$$= 122.98 \text{ Hz}$$

- (2) Hydrodynamic mass coefficients and principal values of hydrodynamic mass coefficient matrix.

For the cylinder arrangement given in Fig. 33, the outputs from the computer program AMASS are given in Table 2.

The eigenvalues μ_i' are as follows:

$$\mu_1' = 0.5768$$

$$\mu_2' = 0.7201$$

$$\mu_3' = 0.9091$$

$$\mu_4' = 0.9091$$

$$\mu_5' = 1.1345$$

$$\mu_6' = 1.1345$$

$$\mu_7' = 1.4076$$

$$\mu_8' = 1.8001$$

Table 2. Outputs from AMASS for the Case of Four Cylinders

Number of Cylinders = 4

Number of Terms Used = 10

This is not the case of confined cylinders.

Cylinder	R	X	Y
1	0.63500	0.0	0.0
2	0.63500	1.91000	0.0
3	0.63500	0.0	-1.91000
4	0.63500	1.91000	-1.91000

Hydrodynamic Mass Coefficients

0.10740E 01	-0.24208E 00	0.23559E 00	-0.52165E-01	-0.10733E-01	-0.31151E-01	0.31151E-01	0.12324E 00
-0.24208E 00	0.10740E 01	-0.52165E-01	0.23559E 00	0.31151E-01	0.10733E-01	-0.12324E 00	-0.31151E-01
0.23559E00	-0.52165E-01	0.10740E 01	-0.24208E 00	-0.31151E-01	-0.12324E 00	0.10733E-01	0.31151E-01
-0.52165E-01	0.23559E 00	-0.24208E 00	0.10740E 01	0.12324E 00	0.31151E-01	-0.31151E-01	-0.10733E-01
-0.10733E-01	0.31151E-01	-0.31151E-01	0.12324E 00	0.10740E 01	0.23559E 00	-0.24208E 00	-0.52165E-01
-0.31151E-01	0.10733E-01	-0.12324E 00	0.31151E-01	0.23559E 00	0.10740E 01	-0.52165E-01	-0.24208E 00
0.31151E-01	-0.12324E 00	0.10733E-01	-0.31151E-01	-0.24208E 00	-0.52165E-01	0.10740E 01	0.23559E 00
0.12324E 00	-0.31151E-01	0.31151E-01	-0.10733E-01	-0.52165E-01	-0.24208E 00	0.23559E 00	0.10740E 01

Principal Mass Coefficients

0.23258D 00
0.29038D 00
0.36658D 00
0.36658D 00
0.45746D 00
0.45746D 00
0.56756D 00
0.72584D 00

* Here, the principal values of the matrix $\left(\frac{R_i + R_j}{2} \right)^2 v_{ij}$ are calculated. For this particular case of $R_i = R_j = R = 0.635$, the nondimensional values of μ'_i in Eq. (18) may be obtained by dividing these values by $R^2 = 0.635^2$.

Eigenvectors of Hydrodynamic Mass Matrix

0.35355D 00	0.35355D 00	0.68994D-01	0.49522D 00	0.49656D 00	-0.58511D-01	-0.35355D 00	0.35355D 00
0.35355D 00	0.35355D 00	-0.49042D 00	0.97407D-01	0.29808D-01	0.49911D 00	0.35355D 00	-0.35355D 00
-0.35355D 00	-0.35355D 00	-0.49042D 00	0.97407D-01	0.29808D-01	0.49911D 00	-0.35355D 00	0.35355D 00
-0.35355D 00	-0.35355D 00	0.68994D-01	0.49522D 00	0.49656D 00	-0.58510D-01	0.35355D 00	-0.35355D 00
0.35355D 00	-0.35355D 00	-0.97407D-01	-0.49042D 00	0.49911D 00	-0.29809D-01	-0.35355D 00	-0.35355D 00
-0.35355D 00	0.35355D 00	-0.49522D 00	0.68994D-01	-0.58511D-01	-0.49657D 00	-0.35355D 00	-0.35355D 00
0.35355D 00	-0.35355D 00	-0.49522D 00	0.68994D-01	-0.58511D-01	-0.49656D 00	0.35355D 00	0.35355D 00
-0.35355D 00	0.35355D 00	-0.97407D-01	-0.49042D 00	0.49911D 00	-0.29808D-01	0.35355D 00	0.35355D 00

(3) Natural frequencies of coupled modes (f_c).

f_c is given by Eq. (18); i.e.,

$$f_c = \left(\frac{0.358}{0.358 + 7965\pi(0.00635^2 \mu_1')} \right)^{1/2} \times 122.98 \quad (23)$$

Substituting the values of μ_1' into Eq. (23) gives the following frequencies of coupled modes:

49.90 Hz

55.18 Hz

60.03 Hz

65.16 Hz

70.66 Hz

75.90 Hz

V. CONCLUDING REMARKS

The methods for calculating hydrodynamic mass presented in this design guide are primarily based on potential flow theory, except Cases A and B, in which linear viscous flow theory is used. Linear theory yields sufficiently accurate results, provided the vibration amplitude is small. In many practical situations, the cylinder displacement is much smaller than the cylinder diameter; in these cases linear theory is applicable. As the cylinder displacement becomes large, nonlinear effects of the fluid field become important. In fact, it has been shown that the hydrodynamic mass is a function of the vibration amplitude [Ref. 8], and that the damping is a function of the first and second powers of the cylinder velocity [Ref. 9].

The results presented for cylinders are based on two-dimensional flow theory; i.e., the axial motion of the fluid is neglected. Two-dimensional theory tends to overestimate the hydrodynamic mass. However, this is justified if the wavelength is much larger than the cylinder diameter. Many practical situations satisfy this requirement. However, when the wavelength is comparable with the diameter, such as a cylindrical shell, the three-dimensional effects of the flow become important.

The hydrodynamic masses of cylinders are independent of modes of vibration for large wavelengths. In the case of shell vibration, since the wavelength is comparable with the diameter, the hydrodynamic mass becomes dependent on the modes of vibration. The hydrodynamic masses presented in this design guide are for simply supported end conditions. However, the information presented can be applied to estimate the hydrodynamic masses for other types of end conditions. For example, for a cantilevered shell of length ℓ and radius R , the hydrodynamic masses can be approximated from the data for simply-supported condition by assuming

$$\alpha = \frac{\pi R}{2\ell} .$$

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APPENDIX

```

*****
*
*          *** SUBROUTINE  AMASS ***
*
*  HYDRODYNAMIC MASS COEFFICIENTS OF  A GROUP OF
*  CIRCULAR CYLINDER IN A FLUID
*
*****

```

THIS PROGRAM CALCULATES HYDRODYNAMIC MASS COEFFICIENTS OF A GROUP OF CIRCULAR CYLINDERS IMMERSED IN A FLUID CONTAINED IN A CYLINDER OR IN AN INFINITE FLUID.

ON INPUT:

NNN: NUMBER OF TERMS USED (IN GENERAL, NNN=10 WILL GIVE RESULTS WITH SUFFICIENT ACCURACY).

KKI: TOTAL NUMBER OF CYLINDERS INCLUDING THE OUTER CYLINDER.

ICLK: IF ICLK=1, GENERAL CASE, ONE INNER CYLINDER IS CONCENTRIC WITH THE OUTER CYLINDER (IK=0).

IF ICLK=2, NO INNER CYLINDER IS CONCENTRIC WITH THE OUTER CYLINDER (IK=1).

IF ICLK=3, A GROUP OF CYLINDERS IN AN INFINITE FLUID (IK=2).

RA(I),XA(I),YA(I): RADIUS AND COORDINATES OF CYLINDER I.

* INPUT SEQUENCE OF CYLINDER GEOMETRIES *

OUTER CYLINDER, CONCENTRIC INNER CYLINDER & OTHER CYLINDERS.

ON OUTPUT:

GAM(I,J): HYDRODYNAMIC MASS COEFFICIENT MATRIX.

PADD(I): PRINCIPAL VALUES OF HYDRODYNAMIC MASS MATRIX DIVIDED BY FLUID DENSITY AND BY $\Phi=3.1416$ (SHOULD BE MULTIPLIED BY FLUID DENSITY AND BY Φ TO YIELD THE EFFECTIVE HYDRODYNAMIC MASS PER UNIT LENGTH OF CYLINDERS).

ZZ(I,J): EIGENVECTORS OF THE HYDRODYNAMIC MASS MATRIX.

** DIMENSION SPECIFICATIONS **

(KKI)	RA,XA,YA
(KKK)	R,X,Y
(KKI*2)	PADD,D
(KKI,KKI)	RAIJ,PHYA,ALP,BET,SIG,TAU
(KKK,KKK)	RIJ,PHY
(KKI*2,KKI*2)	GAM,AM,ZZ
(KKI,NNN,KKI)	AL,BE,SI,TA,H
(KKK,NNN,KKK,NNN)	AA,BB,CC,DD
(KKI*NNN*2,KKI*NNN*2)	E
(KKI*NNN*2,KKI)	F

WHERE KKK=KKI+IK.

* THIS SUBROUTINE IS FOR NNN=10 AND KKI=4 *

SUBROUTINE AMASS(NNN,KKI,ICLK,RA,XA,YA,GAM,PADD,ZZ)

REAL*8 CNA,CMA,CNMA,CNA1,CMA1,CNMAA,CNA2,CMA2,CMMNA
 REAL*8 AGG,BGG,CGG,DGG,AGG1,AGG2

```

REAL*8 AA,BB,CC,DD,H,E,F,DETERM
REAL*8 AG1,BG1,CG1,DG1,AG11,BG11,CG11,DG11
REAL*8 RAM,ZZ,PADD,D

```

C

```

DIMENSION RA(4),XA(4),YA(4),R(6),X(6),Y(6)
DIMENSION RAIJ(4,4),PHYA(4,4),RIJ(6,6),PHY(6,6)
DIMENSION AA(6,10,6,10),BB(6,10,6,10)
DIMENSION CC(6,10,6,10),DD(6,10,6,10)
DIMENSION H(4,10,4),E(80,80),F(80,4)
DIMENSION AL(4,10,4),SI(4,10,4),TA(4,10,4),BE(4,10,4)
DIMENSION ALP(4,4),SIG(4,4),TAU(4,4),BET(4,4)
DIMENSION GAM(8,8),RAM(8,8),ZZ(8,8),PADD(8),D(8)

```

C

```
KKI2=KKI*2
```

```
C ::::: SET UP CONTROL VARIABLES :::::
```

```
GO TO (4001,4002,4003),ICLK
```

```
4001 IK=0
```

```
IKK=2
```

```
GO TO 4004
```

```
4002 IK=1
```

```
IKK=1
```

```
GO TO 4004
```

```
4003 IK=2
```

```
IKK=0
```

```
4004 KKK=KKI+IK
```

```
C ::::: COMPUTE RAIJ(I,J) AND PHYA(I,J) :::::
```

```
DO 201 I=1,KKI
```

```
DO 201 J=1,KKI
```

```
GO TO (4011,4012,4012),ICLK
```

```
4011 IF (I-2) 4013,4013,4012
```

```
4013 IF (J-2) 4014,4014,4012
```

```
4014 RAIJ(I,J)=0.0
```

```
PHYA(I,J)=0.0
```

```
GO TO 201
```

```
4012 IF (I-J) 4015,4014,4015
```

```
4015 ARG=(XA(J)-XA(I))**2+(YA(J)-YA(I))**2
```

```
RAIJ(I,J)=SQRT(ARG)
```

```
ARG1=YA(J)-YA(I)
```

```
ARG2=XA(J)-XA(I)
```

```
PHYA(I,J)=ATAN2(ARG1,ARG2)
```

```
201 CONTINUE
```

```
C ::::: CLEAR WORK SPACES :::::
```

```
DO 4005 I=1,KKK
```

```
R(I)=0.0
```

```
X(I)=0.0
```

```
Y(I)=0.0
```

```
DO 4005 J=1,KKK
```

```
RIJ(I,J)=0.0
```

```
PHY(I,J)=0.0
```

```
DO 4005 N=1,NNN
```

```
DO 4005 M=1,NNN
```

```
AA(I,N,J,M)=0.0
```

```
BB(I,N,J,M)=0.0
```

```
CC(I,N,J,M)=0.0
```

```
4005 DD(I,N,J,M)=0.0
```

```
C ::::: GO TO INDEX KKK FROM KKI :::::
```

```
GO TO (4031,40331,4033),ICLK
```

```
4031 DO 4032 I=1,KKK
```

```
X(I)=XA(I)
```

```

      Y(I)=YA(I)
      R(I)=RA(I)
      DO 4032 J=1, KKK
      RIJ(I,J)=RAIJ(I,J)
4032  PHY(I,J)=PHYA(I,J)
      GO TO 4040
40331 X(1)=XA(1)
      Y(1)=YA(1)
      R(1)=RA(1)
      RIJ(1,1)=RAIJ(1,1)
      PHY(1,1)=PHYA(1,1)
4033  DO 4036 I=3, KKK
      IIK=I-IK
      X(I)=XA(IIK)
      Y(I)=YA(IIK)
      R(I)=RA(IIK)
      GO TO (4034,4035), IK
4034  IM1=I-1
      RIJ(I,1)=RAIJ(IM1,1)
      RIJ(1,I)=RAIJ(1,IM1)
      PHY(I,1)=PHYA(IM1,1)
      PHY(1,I)=PHYA(1,IM1)
4035  DO 4036 J=3, KKK
      JJK=J-IK
      RIJ(I,J)=RAIJ(IIK,JJK)
4036  PHY(I,J)=PHYA(IIK,JJK)
4040  CONTINUE
C ::::: FORM AA, BB, CC, DD MATRICES :::::
      GO TO (4050,4050,5010), ICHK
4050  DO 5000 I=1, IKK
      DO 5000 N=1, NNN
      GO TO (8881,8882), I
8881  MMM=N
      GO TO 8883
8882  MMM=NNN
8883  CONTINUE
      DO 5000 J=1, KKK
      IF(ICHK.EQ.1) GO TO 4051
      IF(J.EQ.2) GO TO 5000
4051  CONTINUE
      DO 5000 M=1, MMM
      GO TO (4052,7770), I
4052  CONTINUE
      IF (J-2) 4053,4053,4100
4053  IF (N-M) 4061,4054,4061
4054  IF (I-J) 4058,4055,4058
4055  AA(I,N,J,M)=N
      BB(I,N,J,M)=N
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
4058  AA(I,N,J,M)=-N*(R(2)/R(1))**(N+1)
      BB(I,N,J,M)=-N*(R(2)/R(1))**(N+1)
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 5000
4061  AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0

```



```

DD(I,N,J,M)=0.0
GO TO 5000
4100 MM1=M-1
    NMM=N-M
    ARG11=(M-1)*PHY(1,J)
    AGG1=R(J)*R(J)/(R(I)*RIJ(1,J))
    DO 4101 MC=1,M
4101 AGG1=AGG1*RIJ(1,J)/R(J)
    CNA1=1.0
    DC 4102 MC=1,N
    CNA1=CNA1*MC
4102 AGG1=AGG1*R(J)/R(I)
    CMA1=1.0
    IF (M-1) 4103,4105,4103
4103 DO 4104 MC=1,MM1
4104 CMA1=CMA1*MC
4105 CNMMA=1.0
    IF (NMM) 4106,4108,4106
4106 DO 4107 MC=1,NMM
4107 CNMMA=CNMMA*MC
4108 AA(I,N,J,NMM+1)=-AGG1*CNA1*COS(ARG11)/(CMA1*CNMMA)
    BB(I,N,J,NMM+1)=AA(I,N,J,NMM+1)
    CC(I,N,J,NMM+1)=AGG1*CNA1*SIN(ARG11)/(CMA1*CNMMA)
    DD(I,N,J,NMM+1)=-CC(I,N,J,NMM+1)
    GO TO 5000
7770 CONTINUE
    IF(J.GT.2) GO TO 7774
    IF(N.NE.M) GO TO 7772
    IF(I.EQ.J) GO TO 7771
    IF(N.EQ.1) GO TO 7773
    AA(I,N,J,M)=N*(R(2)/R(1))**(N-1)
    BB(I,N,J,M)=N*(R(2)/R(1))**(N-1)
    CC(I,N,J,M)=0.0
    DD(I,N,J,M)=0.0
    GO TO 5000
7771 AA(I,N,J,M)=-N
    BB(I,N,J,M)=-N
    CC(I,N,J,M)=0.0
    DD(I,N,J,M)=0.0
    GO TO 5000
7772 AA(I,N,J,M)=0.0
    BB(I,N,J,M)=0.0
    CC(I,N,J,M)=0.0
    DD(I,N,J,M)=0.0
    GO TO 5000
7773 AA(I,N,J,M)=1.0
    BB(I,N,J,M)=1.0
    CC(I,N,J,M)=0.0
    DD(I,N,J,M)=0.0
    GO TO 5000
7774 IN1=N-1
    IM1=M-1
    INM1=N+M-1
    ARG1=(N+M)*PHY(I,J)
    AGG=R(I)/RIJ(I,J)*R(J)/R(I)
    DO 15051 MC=1,M
15051 AGG=AGG*(-1.)*R(J)/R(I)
    CNA=1.0
    IF (IN1) 15052,15054,15052

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```

15052 DO 15053 MC=1,IN1
15053 CNA=CNA*MC
15054 CMA=1.0
      IF (IM1) 15055,15057,15055
15055 DO 15056 MC=1,IM1
15056 CMA=CMA*MC
15057 CNMA=1.0
      DO 15058 MC=1,INM1
15058 CNMA=CNMA*MC/(RIJ(I,J)/R(I))
      AA(I,N,J,M)=AGG*CNMA/CNA/CMA*COS(ARG1)
      BB(I,N,J,M)=-AA(I,N,J,M)
      CC(I,N,J,M)=AGG*CNMA/CNA/CMA*SIN(ARG1)
      DD(I,N,J,M)=CC(I,N,J,M)
5000 CONTINUE
      IF (KKK-2) 206,206,5010
5010 CONTINUE
      DO 205 I=3,KKK
      DO 205 N=1,NNN
      DO 205 J=1,KKK
      GO TO (5011,5012,5013),ICHK
5012 IF (J-2) 5011,205,5011
5013 IF (J-2) 205,205,5011
5011 CONTINUE
      DO 205 M=1,NNN
      IF (I-J) 5017,5014,5017
5014 IF (N-M) 5016,5015,5016
5015 AA(I,N,J,M)=-N
      BB(I,N,J,M)=-N
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 205
5016 AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 205
5017 IN1=N-1
      IM1=M-1
      INM1=N+M-1
      MMN=M-N
      IF (J-1) 5050,5018,5050
5018 IF (M-N) 5019,5020,5020
5019 AA(I,N,J,M)=0.0
      BB(I,N,J,M)=0.0
      CC(I,N,J,M)=0.0
      DD(I,N,J,M)=0.0
      GO TO 205
5020 ARG12=(M-N)*PHY(1,I)
      AGG2=R(1)/RIJ(1,I)
      CMA2=1.0
      DO 5021 MC=1,M
      AGG2=AGG2*RIJ(1,I)/R(1)
5021 CMA2=CMA2*MC
      CNA2=1.0
      IF (N-1) 5022,5024,5022
5022 DO 5023 MC=1,IN1
      AGG2=AGG2*R(I)/RIJ(1,I)
5023 CNA2=CNA2*MC
5024 CMMNA=1.0

```

```

IF (MMN) 5025,5027,5025
5025 DO 5026 MC=1,MMN
5026 CMMNA=CMMNA*MC
5027 AA(I,N,J,M)=AGG2*CMA2*COS(ARG12)/(CNA2*CMMNA)
      BB(I,N,J,M)=AA(I,N,J,M)
      CC(I,N,J,M)=AGG2*CMA2*SIN(ARG12)/(CNA2*CMMNA)
      DD(I,N,J,M)=-CC(I,N,J,M)
      GO TO 205
5050 ARG1=(N+M)*PHY(I,J)
      AGG=R(I)/RIJ(I,J)*R(J)/R(I)
      DO 5051 MC=1,M
5051 AGG=AGG*(-1.)*R(J)/R(I)
      CNA=1.0
      IF (IN1) 5052,5054,5052
5052 DO 5053 MC=1,IN1
5053 CNA=CNA*MC
5054 CMA=1.0
      IF (IM1) 5055,5057,5055
5055 DO 5056 MC=1,IM1
5056 CMA=CMA*MC
5057 CNMA=1.0
      DO 5058 MC=1,INM1
5058 CNMA=CNMA*MC/(RIJ(I,J)/R(I))
      AA(I,N,J,M)=AGG*CNMA/CNA/CMA*COS(ARG1)
      BB(I,N,J,M)=-AA(I,N,J,M)
      CC(I,N,J,M)=AGG*CNMA/CNA/CMA*SIN(ARG1)
      DD(I,N,J,M)=CC(I,N,J,M)
205  CONTINUE
206  CONTINUE
      GO TO (5200,5100,5100),ICHK
5100 DO 50 N=1,NNN
      DO 50 M=1,NNN
      DO 50 I=3,KKK
      II1=I-1
      IIK=I-IK
      GO TO (11,12),IK
11  AA(II1,N,1,M)=AA(I,N,1,M)
      BB(II1,N,1,M)=BB(I,N,1,M)
      CC(II1,N,1,M)=CC(I,N,1,M)
      DD(II1,N,1,M)=DD(I,N,1,M)
12  DO 50 J=3,KKK
      GO TO (21,23),IK
21  IF (I-4) 22,23,23
22  JJ1=J-1
      AA(1,N,JJ1,M)=AA(1,N,J,M)
      BB(1,N,JJ1,M)=BB(1,N,J,M)
      CC(1,N,JJ1,M)=CC(1,N,J,M)
      DD(1,N,JJ1,M)=DD(1,N,J,M)
23  JJK=J-IK
      AA(IIK,N,JJK,M)=AA(I,N,J,M)
      BB(IIK,N,JJK,M)=BB(I,N,J,M)
      CC(IIK,N,JJK,M)=CC(I,N,J,M)
      DD(IIK,N,JJK,M)=DD(I,N,J,M)
50  CONTINUE
5200 KN=KKI*NNN
C ::::: PREPARE TO SOLVE SYSTEM OF EQUATIONS FOR X-AXIS :::::
      DO 230 I=1,KKI
      DO 230 N=1,NNN
      IN=(I-1)*NNN+N

```

```

      II=IN+KN
      DO 230 J=1,KKI
      DO 230 M=1,NNN
      JM=(J-1)*NNN+M
      JJ=JM+KN
      E(IN,JM)=AA(I,N,J,M)
      E(II,JJ)=BB(I,N,J,M)
      E(IN,JJ)=CC(I,N,J,M)
      E(II,JM)=DD(I,N,J,M)
230  CONTINUE
      DO 220 I=1,KKI
      DO 220 N=1,NNN
      DO 220 L=1,KKI
      IF (N-1) 62,60,62
60   IF (I-L) 62,61,62
61   H(I,N,L)=1.0
      GO TO 220
62   H(I,N,L)=0.0
220  CONTINUE
      DO 240 I=1,KKI
      DO 240 N=1,NNN
      IN=(I-1)*NNN+N
      II=IN+KN
      DO 240 L=1,KKI
      F(IN,L)=H(I,N,L)
240  F(II,L)=0.0
      KNKN=2*KN
C ::::: SOLVE SYSTEM OF EQUATIONS :::::
      CALL CROUT(E,KNKN,F,KKI,DETERM,80)
      DO 250 I=1,KKI
      DO 250 N=1,NNN
      IN=(I-1)*NNN+N
      II=IN+KN
      DO 250 L=1,KKI
      AL(I,N,L)=F(IN,L)
      TA(I,N,L)=F(II,L)
250  CONTINUE
C ::::: PREPARE TO SOLVE SYSTEM OF EQUATIONS FOR Y-AXIS :::::
      DO 270 I=1,KKI
      DO 270 N=1,NNN
      IN=(I-1)*NNN+N
      II=IN+KN
      DO 270 L=1,KKI
      F(IN,L)=0.0
270  F(II,L)=H(I,N,L)
      DO 231 I=1,KKI
      DO 231 N=1,NNN
      IN=(I-1)*NNN+N
      II=IN+KN
      DO 231 J=1,KKI
      DO 231 M=1,NNN
      JM=(J-1)*NNN+M
      JJ=JM+KN
      E(IN,JM)=AA(I,N,J,M)
      E(II,JJ)=BB(I,N,J,M)
      E(IN,JJ)=CC(I,N,J,M)
231  E(II,JM)=DD(I,N,J,M)
C ::::: SOLVE SYSTEM OF EQUATIONS :::::
      CALL CROUT(E,KNKN,F,KKI,DETERM,80)

```

```

DO 280 I=1,KKI
DO 280 N=1,NNN
IN=(I-1)*NNN+N
II=IN+KN
DO 280 L=1,KKI
SI(I,N,L)=F(IN,L)
BE(I,N,L)=F(II,L)
280 CONTINUE
C ::::: CALCULATE ADDED MASS COEFFICIENTS :::::
GO TO (6001,6002,6003),ICHK
6001 I=1
R21=RA(2)*RA(2)/(RA(1)*RA(1))
DO 600 L=1,KKI
ALP(1,L)=AL(1,1,L)+AL(2,1,L)*R21
SIG(1,L)=SI(1,1,L)+SI(2,1,L)*R21
TAU(1,L)=TA(1,1,L)+TA(2,1,L)*R21
BET(1,L)=BE(1,1,L)+BE(2,1,L)*R21
IF (KKI.LE.2) GO TO 600
DO 6000 J=3,KKI
RJI2=RA(J)*RA(J)/(RA(I)*RA(I))
ALP(1,L)=ALP(1,L)+AL(J,1,L)*RJI2
SIG(1,L)=SIG(1,L)+SI(J,1,L)*RJI2
TAU(1,L)=TAU(1,L)+TA(J,1,L)*RJI2
6000 BET(1,L)=BET(1,L)+BE(J,1,L)*RJI2
600 CONTINUE
I=2
DO 7004 L=1,KKI
ALP(2,L)=AL(1,1,L)+AL(2,1,L)
SIG(2,L)=SI(1,1,L)+SI(2,1,L)
TAU(2,L)=TA(1,1,L)+TA(2,1,L)
BET(2,L)=BE(1,1,L)+BE(2,1,L)
IF (KKI.LE.2) GO TO 7004
DO 7003 J=3,KKI
AGG=0.0
BGG=0.0
CGG=0.0
DGG=0.0
RJIJ=RA(J)/RAIJ(2,J)
DO 7002 N=1,NNN
ARG=(N+1)*PHYA(2,J)
AG1=N*RJIJ*(COS(ARG)*AL(J,N,L)+TA(J,N,L)*SIN(ARG))
BG1=N*RJIJ*(COS(ARG)*SI(J,N,L)+BE(J,N,L)*SIN(ARG))
CG1=N*RJIJ*(SIN(ARG)*AL(J,N,L)-TA(J,N,L)*COS(ARG))
DG1=N*RJIJ*(SIN(ARG)*SI(J,N,L)-BE(J,N,L)*COS(ARG))
DO 7001 MC=1,N
AG1=-AG1*RJIJ
BG1=-BG1*RJIJ
CG1=-CG1*RJIJ
7001 DG1=-DG1*RJIJ
AGG=AGG+AG1
BGG=BGG+BG1
CGG=CGG+CG1
7002 DGG=DGG+DG1
ALP(2,L)=ALP(2,L)+AGG
SIG(2,L)=SIG(2,L)+BGG
TAU(2,L)=TAU(2,L)+CGG
7003 BET(2,L)=BET(2,L)+DGG
7004 CONTINUE
DO 605 L=1,KKI

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ALP(2,L)=-ALP(2,L)
SIG(2,L)=-SIG(2,L)
TAU(2,L)=-TAU(2,L)
605 BET(2,L)=-BET(2,L)
GO TO 6003
6002 DO 700 L=1,KKI
ALP(1,L)=AL(1,1,L)
SIG(1,L)=SI(1,1,L)
TAU(1,L)=TA(1,1,L)
BET(1,L)=BE(1,1,L)
DO 700 J=2,KKI
RJ12=RA(J)*RA(J)/(RA(1)*RA(1))
ALP(1,L)=ALP(1,L)+AL(J,1,L)*RJ12
SIG(1,L)=SIG(1,L)+SI(J,1,L)*RJ12
TAU(1,L)=TAU(1,L)+TA(J,1,L)*RJ12
700 BET(1,L)=BET(1,L)+BE(J,1,L)*RJ12
6003 ISS=IKK+1
IF(KKI-ISS) 899,6004,6004
6004 CONTINUE
DO 800 I=ISS,KKI
DO 800 L=1,KKI
ALP(I,L)=AL(I,1,L)
SIG(I,L)=SI(I,1,L)
TAU(I,L)=TA(I,1,L)
BET(I,L)=BE(I,1,L)
DO 790 J=1,KKI
AGG=0.0
BGG=0.0
CGG=0.0
DGG=0.0
IF (I-J) 801,770,801
801 GO TO (802,802,805),ICHK
802 IF (J-1) 805,803,805
803 DO 750 N=1,NNN
ARG1=(N-1)*PHYA(1,I)
R1IJ=RA(1)/RAIJ(1,I)
RIJ1=RAIJ(1,I)/RA(1)
AG11=N*R1IJ*(COS(ARG1)*AL(1,N,L)+TA(1,N,L)*SIN(ARG1))
BG11=N*R1IJ*(COS(ARG1)*SI(1,N,L)+BE(1,N,L)*SIN(ARG1))
CG11=N*R1IJ*(-SIN(ARG1)*AL(1,N,L)+TA(1,N,L)*COS(ARG1))
DG11=N*R1IJ*(-SIN(ARG1)*SI(1,N,L)+BE(1,N,L)*COS(ARG1))
DO 804 MC=1,N
AG11=AG11*RIJ1
BG11=BG11*RIJ1
CG11=CG11*RIJ1
804 DG11=DG11*RIJ1
AGG=AGG+AG11
BGG=BGG+BG11
CGG=CGG+CG11
750 DGG=DGG+DG11
GO TO 770
805 RJIJ=RA(J)/RAIJ(I,J)
DO 760 N=1,NNN
ARG=(N+1)*PHYA(I,J)
AG1=N*RJIJ*(COS(ARG)*AL(J,N,L)+TA(J,N,L)*SIN(ARG))
BG1=N*RJIJ*(COS(ARG)*SI(J,N,L)+BE(J,N,L)*SIN(ARG))
CG1=N*RJIJ*(SIN(ARG)*AL(J,N,L)-TA(J,N,L)*COS(ARG))
DG1=N*RJIJ*(SIN(ARG)*SI(J,N,L)-BE(J,N,L)*COS(ARG))
DO 806 MC=1,N

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      AG1=-AG1*RJIJ
      BG1=-BG1*RJIJ
      CG1=-CG1*RJIJ
806   DG1=-DG1*RJIJ
      AGG=AGG+AG1
      BGG=BGG+BG1
      CGG=CGG+CG1
760   DGG=DGG+DG1
770   CONTINUE
      ALP(I,L)=ALP(I,L)+AGG
      SIG(I,L)=SIG(I,L)+BGG
      TAU(I,L)=TAU(I,L)+CGG
790   BET(I,L)=BET(I,L)+DGG
800   CONTINUE
      DO 299 I=ISS,KKI
      DO 299 L=1,KKI
      ALP(I,L)=-ALP(I,L)
      SIG(I,L)=-SIG(I,L)
      TAU(I,L)=-TAU(I,L)
299   BET(I,L)=-BET(I,L)
899   CONTINUE
      DO 8800 I=1,KKI
      DO 8800 L=1,KKI
      ALP(I,L)=ALP(I,L)*RA(I)*RA(I)
      SIG(I,L)=SIG(I,L)*RA(I)*RA(I)
      TAU(I,L)=TAU(I,L)*RA(I)*RA(I)
8800  BET(I,L)=BET(I,L)*RA(I)*RA(I)
      DO 900 I=1,KKI
      DO 900 L=1,KKI
      RAM(I,L)=ALP(I,L)
      RAM(I,KKI+L)=SIG(I,L)
      RAM(KKI+I,L)=TAU(I,L)
900   RAM(KKI+I,KKI+L)=BET(I,L)
      DO 810 I=1,KKI
      DO 810 L=1,KKI
      RIL=4./((RA(I)+RA(L))*2)
      ALP(I,L)=ALP(I,L)*RIL
      SIG(I,L)=SIG(I,L)*RIL
      TAU(I,L)=TAU(I,L)*RIL
      BET(I,L)=BET(I,L)*RIL
      GAM(I,L)=ALP(I,L)
      GAM(I,KKI+L)=SIG(I,L)
      GAM(KKI+I,L)=TAU(I,L)
810   GAM(KKI+I,KKI+L)=BET(I,L)
      CALL TRED2(KKI2,KKI2,RAM,PADD,D,ZZ)
      CALL IMTQL2(KKI2,KKI2,PADD,D,ZZ,IER)
      RETURN
      END
C *****
C *
C *      *** SUBROUTINE CROUT ***
C *
C *****
C
C THIS PROGRAM SOLVES MATRIX EQUATION AX=B BY THE CROUT METHOD.
C
C ON INPUT:
C
C      A:      MATRIX.

```

C N: ORDER OF MATRIX A.
 C B: COLUMN MATRIX.
 C M: NUMBER OF COLUMN VECTOR B.
 C DETERM: RETURN CODE, IF ZERO A IS SINGULAR.
 C NMAX: ROW DIMENSION FOR MATRIX A AND COLUMN VECTOR B.

ON OUTPUT:

B: SOLUTION VECTOR X STORED.

```

-----
SUBROUTINE CROUT(A,N,B,M,DETERM,NMAX)
REAL*8 A,B,DETERM,DOTP,V,AMAX,TEMP
DIMENSION A(NMAX,N),B(NMAX,M)
DIMENSION V(80)
DETERM=1.0D0
DO 1000 K=1,N
  KP1=K+1
  KM1=K-1
  TEMP=0.0
  DO 20 I=K,N
    DO 2 L=1,K
      2 V(L)=A(I,L)
      A(I,K)=A(I,K)-DOTP(V,A(1,K),KM1)
      IF(DABS(A(I,K)).LT.TEMP) GO TO 20
      3 TEMP=DABS(A(I,K))
      IMAX=I
      20 CONTINUE
      AMAX=A(IMAX,K)
      IF(DETERM.EQ.0.0) RETURN
      IF(IMAX.EQ.K) GO TO 600
      DETERM=-DETERM
      DO 50 J=1,N
        TEMP=A(K,J)
        A(K,J)=A(IMAX,J)
        A(IMAX,J)=TEMP
      50 CONTINUE
      IF(M.LE.0) GO TO 600
      DO 400 J=1,M
        TEMP=B(K,J)
        B(K,J)=B(IMAX,J)
        B(IMAX,J)=TEMP
      400 CONTINUE
      600 DO 666 L=1,K
      666 V(L)=A(K,L)
      IF(K.EQ.N) GO TO 850
      DO 700 I=KP1,N
      700 A(I,K)=A(I,K)/AMAX
      DO 800 J=KP1,N
      800 A(K,J)=A(K,J)-DOTP(V,A(1,J),KM1)
      850 IF(M.LE.0) GO TO 1000
      DO 900 J=1,M
      900 B(K,J)=B(K,J)-DOTP(V,B(1,J),KM1)
      1000 CONTINUE
      IF(M.LE.0) RETURN
      DO 8000 I=1,N
      K=N+1-I
  
```



```

C *****
C *
C *
C *
C *
C *****

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```

C *****
C *
C *
C *
C *
C *****

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```

C
C   THIS SUBROUTINE REDUCES A REAL SYMMETRIC MATRIX TO A
C   SYMMETRIC TRIDIAGONAL MATRIX USING AND ACCUMULATING
C   ORTHOGONAL SIMILARITY TRANSFORMATIONS.

```

C
C NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C DIMENSION STATEMENT:

C
C A CONTAINS THE REAL SYMMETRIC INPUT MATRIX. ONLY THE
C LOWER TRIANGLE OF THE MATRIX NEED BE SUPPLIED.

D CONTAINS THE DIAGONAL ELEMENTS OF THE TRIDIAGONAL MATRIX;
E CONTAINS THE SUBDIAGONAL ELEMENTS OF THE TRIDIAGONAL
MATRIX IN ITS LAST N-1 POSITIONS. E(1) IS SET TO ZERO;

C Z CONTAINS THE ORTHOGONAL TRANSFORMATION MATRIX
 C PRODUCED IN THE REDUCTION;
 C
 C A AND Z MAY COINCIDE. IF DISTINCT, A IS UNALTERED.
 C
 C -----

C SUBROUTINE TRED2 (NM,N,A,D,E,Z)

C INTEGER I,J,K,L,N,II,NM,JP1
 C REAL*8 A (NM,N),D (N),E (N),Z (NM,N)
 C REAL*8 F,G,H,HH,SCALE
 C REAL*8 DSQRT,DABS,DSIGN

C DO 100 I=1,N
 C DO 100 J=1,I
 100 Z (I,J)=A (I,J)
 C IF (N.EQ.1) GO TO 320
 C ::::: FOR I=N STEP -1 UNTIL 2 DO -- :::::
 C DO 300 II=2,N
 C I=N+2-II
 C L=I-1
 C H=0.0D0
 C SCALE=0.0D0
 C IF (L.LT.2) GO TO 130
 C ::::: SCALE ROW (ALGOL TOL THEN NOT NEEDED) :::::
 C DO 120 K=1,L
 120 SCALE=SCALE+DABS (Z (I,K))
 C IF (SCALE.NE.0.0D0) GO TO 140
 130 E (I)=Z (I,L)
 C GO TO 290
 140 DO 150 K=1,L
 C Z (I,K)=Z (I,K)/SCALE
 C H=H+Z (I,K)*Z (I,K)
 150 CONTINUE
 C F=Z (I,L)
 C G=-DSIGN (DSQRT (H),F)
 C E (I)=SCALE*G
 C H=H-F*G
 C Z (I,L)=F-G
 C F=0.0D0
 C DO 240 J=1,L
 C Z (J,I)=Z (I,J)/H
 C G=0.0D0
 C ::::: FORM ELEMENT OF A*U :::::
 C DO 180 K=1,J
 180 G=G+Z (J,K)*Z (I,K)
 C JP1=J+1
 C IF (L.LT.JP1) GO TO 220
 C DO 200 K=JP1,L
 200 G=G+Z (K,J)*Z (I,K)
 C ::::: FORM ELEMENT OF P :::::
 220 E (J)=G/H
 C F=F+E (J)*Z (I,J)
 240 CONTINUE
 C HH=F/(H+H)
 C ::::: FORM REDUCED A :::::
 C DO 260 J=1,L
 C F=Z (I,J)

```

      G=E(J) -HH*F
      E(J)=G
      DO 260 K=1,J
      Z(J,K)=Z(J,K) -F*E(K) -G*Z(I,K)
260   CONTINUE
290   D(I)=H
300   CONTINUE
320   D(1)=0.0D0
      E(1)=0.0D0
C   ::::: ACCUMULATION OF TRANSFORMATION MATRICES :::::
      DO 500 I=1,N
      L=I-1
      IF(D(I).EQ.0.0D0) GO TO 380
      DO 360 J=1,L
      G=0.0D0
      DO 340 K=1,L
340   G=G+Z(I,K)*Z(K,J)
      DO 360 K=1,L
      Z(K,J)=Z(K,J) -G*Z(K,I)
360   CONTINUE
380   D(I)=Z(I,I)
      Z(I,I)=1.0D0
      IF(L.LT.1) GO TO 500
      DO 400 J=1,L
      Z(I,J)=0.0D0
      Z(J,I)=0.0D0
400   CONTINUE
500   CONTINUE
      RETURN
      END
C   *****
C   *
C   *          *** SUBROUTINE  IMTQL2 ***
C   *
C   *****
C
C   THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE IMTQL2,
C   NUM. MATH. 12, 377-383(1968) BY MARTIN AND WILKINSON,
C   AS MODIFIED IN NUM. MATH. 15, 450(1970) BY DUBRULLE.
C   HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 227-240(1971).
C
C   THIS SUBROUTINE FINDS THE EIGENVALUES AND EIGENVECTORS
C   OF A SYMMETRIC TRIDIAGONAL MATRIX BY THE IMPLICIT QL METHOD.
C   THE EIGENVECTORS OF A FULL SYMMETRIC MATRIX CAN ALSO
C   BE FOUND IF TRED2 HAS BEEN USED TO REDUCE THIS
C   FULL MATRIX TO TRIDIAGONAL FORM.
C
C   ON INPUT:
C
C       NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C       ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C       DIMENSION STATEMENT;
C
C       N IS THE ORDER OF THE MATRIX;
C
C       D CONTAINS THE DIAGONAL ELEMENTS OF THE INPUT MATRIX;
C
C       E CONTAINS THE SUBDIAGONAL ELEMENTS OF THE INPUT MATRIX
C       IN ITS LAST N-1 POSITIONS.  E(1) IS ARBITRARY;

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      C=1.0D0
      P=0.0D0
      MML=M-L
C   ::::: FOR I=M-1 STEP -1 UNTIL L DO -- :::::
      DO 200 II=1,MML
      I=M-II
      F=S*E(I)
      B=C*E(I)
      IF(DABS(F).LT.DABS(G)) GO TO 150
      C=G/F
      R=DSQRT(C*C+1.0D0)
      E(I+1)=F*R
      S=1.0D0/R
      C=C*S
      GO TO 160
150  S=F/G
      R=DSQRT(S*S+1.0D0)
      E(I+1)=G*R
      C=1.0D0/R
      S=S*C
160  G=D(I+1)-P
      R=(D(I)-G)*S+2.0D0*C*B
      P=S*R
      D(I+1)=G+P
      G=C*R-B
C   ::::: FORM VECTOR :::::
      DO 180 K=1,N
      F=Z(K,I+1)
      Z(K,I+1)=S*Z(K,I)+C*F
      Z(K,I)=C*Z(K,I)-S*F
180  CONTINUE
200  CONTINUE
      D(L)=D(L)-P
      E(L)=G
      E(M)=0.0D0
      GO TO 105
240  CONTINUE
C   ::::: ORDER EIGENVALUES AND EIGENVECTORS :::::
      DO 300 II=2,N
      I=II-1
      K=I
      P=D(I)
      DO 260 J=II,N
      IF(D(J).GE.P) GO TO 260
      K=J
      P=D(J)
260  CONTINUE
      IF(K.EQ.I) GO TO 300
      D(K)=D(I)
      D(I)=P
      DO 280 J=1,N
      P=Z(J,I)
      Z(J,I)=Z(J,K)
      Z(J,K)=P
280  CONTINUE
300  CONTINUE
      GO TO 1001
C   ::::: SET ERROR -- NO CONVERGENCE TO AN EIGENVALUE
C   AFTER 30 ITERATIONS :::::

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1000 IERR=L  
1001 RETURN  
END
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Fig. 1. $\text{Re}(H)$ and $\text{Im}(H)$ for a single cylinder in an infinite viscous fluid.

Fig. 2. Real values of H as a function of R_2/R_1 for a cylinder vibrating in a viscous fluid annulus.

Fig. 3. Imaginary values of H as a function of R_2/R_1 for a cylinder vibrating in a viscous fluid annulus.

Fig. 4. Hydrodynamic mass coefficient for a cylinder vibrating near a wall.

Fig. 5. Hydrodynamic mass coefficient for a circular cylindrical shell containing fluid.

Fig. 6. Hydrodynamic mass coefficient ν_{11} for two parallel cylinders.

Fig. 7. Hydrodynamic mass coefficient ν_{12} for two parallel cylinders.

Fig. 8. Hydrodynamic mass coefficient v_{11} for two eccentric cylinders.

Fig. 9. Hydrodynamic mass coefficient ν_{12} for two eccentric cylinders.

Fig. 10. Hydrodynamic mass coefficient v_{22} for two eccentric cylinders.

Fig. 11. A group of k circular cylinders vibrating in a liquid.

Fig. 12. Schematic of a tube bank arranged in a hexagonal pattern.

Fig. 13. Hydrodynamic mass coefficients as functions of the pitch-to-diameter ratio.

Fig. 14. Upper and lower bounds of the effective hydrodynamic mass coefficients as functions of the pitch-to-diameter ratio.

Fig. 15. Principal values of the hydrodynamic mass coefficient matrix for a seven-cylinder bundle.

Fig. 16. Principal values of the hydrodynamic mass coefficient matrix for a nine-cylinder array.

Fig. 17. Two coaxial shells with a fluid-filled gap.

Fig. 18. Hydrodynamic mass coefficient v_{11} for two coaxial shells for $r = 1.1$

Fig. 19. Hydrodynamic mass coefficient v_{11} for two coaxial shells for $r = 1.2$

Fig. 20. Hydrodynamic mass coefficient ν_{11} for two coaxial shells for $r = 1.3$

Fig. 21. Hydrodynamic mass coefficient v_{11} for two coaxial shells for $r = 1.4$

Fig. 22. Hydrodynamic mass coefficient ν_{11} for two coaxial shells for $r = 1.5$

Fig. 23. Hydrodynamic mass coefficient v_{22} for two coaxial shells for $r = 1.1$

Fig. 24. Hydrodynamic mass coefficient v_{22} for two coaxial shells for $r = 1.2$

Fig. 25. Hydrodynamic mass coefficient v_{22} for two coaxial shells for $r = 1.3$

Fig. 26. Hydrodynamic mass coefficient v_{22} for two coaxial shells for $r = 1.4$

Fig. 27. Hydrodynamic mass coefficient v_{22} for two coaxial shells for $r = 1.5$

Fig. 28. Hydrodynamic mass coefficient ν_{12} or ν_{21} for two coaxial shells for $r = 1.1$

Fig. 29. Hydrodynamic mass coefficient ν_{12} or ν_{21} for two coaxial shells for $r = 1.2$

Fig. 30. Hydrodynamic mass coefficient ν_{12} or ν_{21} for two coaxial shells for $r = 1.3$

Fig. 31. Hydrodynamic mass coefficient ν_{12} or ν_{21} for two coaxial shells for $r = 1.4$

Fig. 32. Hydrodynamic mass coefficient ν_{12} or ν_{21} for two coaxial shells for $r = 1.5$

Fig. 33. A Group of four cylinders.