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Location Errors in Angle-Measuring and Distance-Measuring Systems

Eugene A. Aronson

MASTER

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AND DISTANCE-MEASURING SYSTEMS

Eugene A. Aronson
Applied Mathematics Division 2642
Sandia Laboratories, Albuquerque, NM 87115

ABSTRACT

Methods are described for the location of an object by using angle-measuring systems, distance-measuring systems, or a mix of the two. With reasonable assumptions on the statistics of the data errors in these systems, an analysis is presented which yields the statistics of the errors in object location, a data weighting scheme is shown which minimizes the variance of the location errors, and, as an aid to designers of such systems, a computer program is described which computes and plots location error contours for desired configurations.

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LOCATION ERRORS IN ANGLE-MEASURING AND DISTANCE-MEASURING SYSTEMS

Introduction

In this report, we deal with two systems commonly used to estimate the position of an object. These systems utilize measurements made from a number of stations whose locations are known. Angle-measuring systems (AME) measure the azimuth and elevation of the line-of-sight from station to object. Distance-measuring systems (DME) measure the distance from station to object. In addition, we consider a third location system which mixes data from the other two.

The results of this study are fourfold:

1. Techniques are described for the reduction of AME, DME, and mixed data.
2. An analysis is presented which computes the statistics of the object position errors due to errors in the various measurements.
3. A data weighting scheme is given which minimizes the variance of the position errors under a reasonably general description of the measurement errors.
4. A computer program "TONER" is described which computes and plots error contours for given object location schemes. This program can aid the test designer to estimate position errors and to design optimal configurations for his various tests.

Notation

Before proceeding into the main body of the report, it is convenient to define some of the notation. We shall use lower-case symbols for scalars and upper-case symbols for matrices and (column) vectors. With deference to long established custom, we immediately violate these rules and use S for azimuth and H for elevation measurement.* To maintain a modicum of consistency in the violation, we shall use R to denote the (scalar) measurement of distance.

*The symbols S and H for azimuth and elevation have been used at Sandia over the life of the Laboratories. These symbols are the first letters of the German words for azimuth and elevation. The earliest AME devices used extensively at Sandia were Askania ciné-theodolites taken from Germany after WWII.

Let the general three-space position vector be

$$P \triangleq \{x, y, z\} \quad (1)$$

The true position of the object to be located is

$$P_o \triangleq \{x_o, y_o, z_o\} \quad (2)$$

The location of the n-th station is

$$P_n^s \triangleq \{x_n^s, y_n^s, z_n^s\} \quad (3)$$

We assume that P_n^s is known exactly for all n.

The measurements are

$$\begin{aligned} S_n &- \text{azimuth measurement for n-th station,} \\ H_n &- \text{elevation measurement for n-th station,} \\ R_n &- \text{distance measurement for n-th station,} \end{aligned} \quad (4)$$

It is also convenient to define the distances

$$\begin{aligned} x_n &= x - x_n^s \\ y_n &= y - y_n^s \\ z_n &= z - z_n^s \\ d_n &= \left(x_n^2 + y_n^2 \right)^{1/2} \\ r_n &= \left(d_n^2 + z_n^2 \right)^{1/2} \end{aligned} \quad (5)$$

We use the right-handed Cartesian coordinate system established at Sandia Laboratories' Tonopah Test Range (TTR), where the +y-axis points to the North, the +x-axis to the East, and the +z-axis points to the zenith. The AME devices measure azimuth from the +y-axis positive to towards the +x-axis. However, in order to maintain consistency with the literature, we take azimuth to be measured from the +x-axis positive toward the +y-axis. Elevation is measured from the horizontal xy-plane positive toward the +z-axis.

Data Reduction Procedures

Let there be L stations measuring angles and M stations measuring range (distance). Some of the L stations may be the identical stations as some of the M stations. We assume that if a station measures either azimuth or elevation it measures both. In order to estimate the three-space position of an object, data from at least two AME stations, or three DME stations, or one of each type is required. In general, there are data from more than the minimum required number of stations and the position is computed by the least-squares solution of an overdetermined system of equations.

Associated with each measurement is a function involving the object position, the station location, and the measurement. The least-squares solution is that value of P which minimizes the weighted sum of the squares of these functions. Let $N = 2L + M$ and let

$$F \triangleq \{f_1, f_2, \dots, f_N\} \quad (6)$$

be the vector of these functions. Also, let W denote a positive-definite N by N symmetric weighting matrix. We wish to find P which minimizes the scalar

$$v = F^T W F, \quad (7)$$

where F^T denotes F transpose. If W is a scalar multiple of the identity matrix, the system Equation (7) is called unweighted and W can be ignored. We carry W in the analysis for generality whether Equation (7) is weighted or not.

Two different sets of functions have been used with the AME data. The Davis method^(1, 2) deals with the measured angles directly. In this case the first $2L$ elements of F are

$$\begin{aligned} f_n &= \arctan(y_n/x_n) - S_n, \\ f_{L+n} &= \arctan(z_n/d_n) - H_n, \end{aligned} \quad (8)$$

for $n = 1, 2, \dots, L$. The Schutzberger method^(2, 3) reasons that each azimuth and elevation measurement defines a line-of-sight from station to object and the object is best located at that point where the sum of the squared distances from object to each line-of-sight is minimized. This method has the useful property that if the elements of W are constants, the solution of Equation (7) is found by solving a linear system of equations. In this formulation the first $2L$ elements of F are

$$f_n = -x_n \sin S_n + y_n \cos S_n \quad (9)$$

$$f_{L+n} = -\sin H_n (x_n \cos S_n + y_n \sin S_n) + z_n \cos H_n$$

$n = 1, 2, \dots, L$. The original Schutzberger formulation^(2,3) minimized the sum of squares of three function for each AME station. This formulation was inconvenient for our purposes, and thus the two function formulation was derived. The derivation is in Appendix A.

The M functions associated with the distance data are

$$f_n = r_n - R_n \quad (10)$$

$n = 2L + 1, \dots, N$.

Note that without weighting, the Davis AME functions cannot be mixed with DME functions since we would be attempting to minimize the sum of angles-squared plus distances-squared. However, unweighted Schutzberger AME data is compatible with DME data since both systems minimize distance-squared.

The minimization of v is achieved in the usual manner. We set $dv/dx = 0$, $dv/dy = 0$, and $dv/dz = 0$, and then solve the three simultaneous so-called "normal" equations for the three unknowns x , y , and z . Define the N by 3 matrix A as

$$A \triangleq dF/dP = \begin{pmatrix} df_1/dx & df_1/dy & df_1/dz \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ df_N/dx & df_N/dy & df_N/dz \end{pmatrix} \quad (11)$$

Then the three normal equations are

$$A^T W F = 0 \quad (12)$$

Except for the Schutzberger AME case with constant weights, these equations are non-linear and are usually solved by iteration. Let P^q be the q -th estimate of the solution. Expanding F in a Taylor's series about $P = P^q$ and retaining only up to linear terms yields the iteration

$$P^{q+1} = P^q - B^q A^q{}^T W^q F^q \quad (13)$$

where

$$A^q \triangleq A|_{P=P^q}, W^q \triangleq W|_{P=P^q}, F^q = F|_{P=P^q}, \quad (14)$$

$$B^q = (A^q{}^T W^q A^q)^*, \quad (15)$$

and * denotes inverse. If the data errors are "small", the initial estimate of P is "good", and the system is "well-conditioned", convergence of the iteration is rapid. These conditions are usually met in practice. The iteration can be terminated after a specified number of iterations or when the change in iterated position becomes less than some specified distance.

Measurements with "small" data errors is a matter of measurement equipment design and environment. "Good" initial estimates of P are almost always met in practice. The condition of the system has to do with station geometry and object position. A system is considered ill-conditioned if small data errors cause large errors in object-estimated position. This phenomenon occurs when $A^T W A$ becomes nearly singular and the elements of B subsequently become large.

For reference, we give the elements of the A matrix for all cases. Let $\tilde{A}_n$ be the n-th row of A. For the Davis method

$$\tilde{A}_n = (-y_n/d_n^2, x_n/d_n^2, 0), \quad (16)$$

$$\tilde{A}_{L+n} = (-x_n z_n / d_n r_n^2, -y_n z_n / d_n r_n^2, d_n / r_n^2),$$

$n = 1, 2, \dots, L$. For the Schutzberger scheme

$$\tilde{A}_n = (-\sin S_n, \cos S_n, 0), \quad (17)$$

$$\tilde{A}_{L+n} = (-\sin H_n \cos S_n, -\sin H_n \sin S_n, \cos H_n),$$

$n = 1, 2, \dots, L$. For DME data

$$\tilde{A}_n = (x_n/r_n, y_n/r_n, z_n/r_n), \quad (18)$$

$n = 2L + 1, \dots, N$.

Position Error Analysis

In order to arrive at a tractable representation of position error statistics, we assume that the data errors are "small" and that the system is not ill-conditioned. In effect this means that the normal equations, Equation (12), can be suitably approximated by a linear system.

Let G be the N -dimensional vector of data

$$\begin{aligned} G &\triangleq \{g_1, g_2, \dots, g_N\} \\ &= \{S_1, \dots, S_L, H_1, \dots, H_L, R_{2L+1}, \dots, R_N\} \end{aligned} \quad (19)$$

If G_0 is the exact (errorless) data, then the data error vector is

$$G_e = G - G_0 \quad (20)$$

The three-dimensional position error vector is

$$P_e = P - P_0 \quad (21)$$

where P is the solution of (12).

Define the N by N matrix C as

$$C \triangleq dF/dG ; \quad (22)$$

that is, the i, j element of C is df_i/dg_j . Expanding F in a Taylor's series about $P = P_0$ and $G = G_0$ and retaining only up to linear terms gives the approximation

$$F \doteq F_0 + A_0 P_e + C_0 G_e \quad (23)$$

where F_0 , A_0 , C_0 , and W_0 are respectively F , A , C , and W evaluated at $P = P_0$ and $G = G_0$. Since $F_0 = 0$, the linearized normal equations yield

$$P_e = - \left(A_0^T W_0 A_0 \right)^* A_0^T W_0 C_0 G_e = - B_0 A_0^T W_0 C_0 G_e \quad (24)$$

Linear terms in the expansion of W have been ignored since they contribute second-order terms to the normal equations. We note that Equation (24) states that P_e is a linear combination of the G_e . Thus, if the elements of G_e are jointly-gaussian, the elements of P_e are also jointly-gaussian. ⁽⁴⁾

We also assume that the mean of the data errors is zero; i.e.,

$$\overline{G_e} = 0 \quad (25)$$

The overbar will be used to denote expected value. This is not a very restrictive assumption since if systematic bias errors do occur in the data they usually can be estimated and then removed in the data reduction procedure. From Equation (25) we get

$$\overline{P_e} = 0 \quad (26)$$

It is now desired to evaluate the covariance matrix of the position errors. From Equation (24) this 3 by 3 matrix is

$$\begin{aligned} P_{\sigma} &= \overline{P_e P_e^T} = B_o A_o^T W_o C_o G_e \left(B_o A_o^T W_o C_o G_e \right)^T \\ &= B_o A_o^T W_o C_o \overline{G_e G_e^T} C_o^T W_o^T A_o B_o^T \\ &= B_o A_o^T W_o C_o G_o C_o^T W_o^T A_o B_o^T \end{aligned} \quad (27)$$

In Equations (27), we note that B and W are symmetric and define G_o as the covariance matrix of the data errors. If G_e is gaussian, then all the statistics of P_e are given by Equation (26) and (27).

The diagonal elements of the 3 by 3 matrix P_o give the variance of the position errors in the x, y, and z directions, respectively. The off-diagonal elements give the error cross-correlations. Similarly, the diagonal elements of the N by N covariance matrix G_o give the variance of each data error and the off-diagonal elements give the cross-correlations. Given G_o , a weighting scheme, and the station locations, P_o can be evaluated at any position of the object to be located--if B_o exists. If $A_o^T W_o A_o$ is ill-conditioned, small elements of G_e cause large elements of P_o and the system is ill-conditioned.

For both the Davis and Schutzberger methods, including DME data, the C_o matrix is diagonal. Let x_n^o , y_n^o , z_n^o , d_n^o , and r_n^o be x_n , y_n , z_n , d_n , and r_n , respectively, evaluated at $P = P_o$. Then, if c_n^o is the diagonal element of the n-th row (column) of C_o ,

$$c_n^o = -1 \quad (28)$$

for $n = 1, 2, \dots, N$ in the Davis scheme and also for $n = 2L - L, \dots, N$ in both schemes. The first $2L$ elements in the Schutzberger method are

$$\begin{aligned} c_n^o &= -d_n^o \\ c_{L+n}^o &= -r_n^o \end{aligned} \quad (29)$$

for $n = 1, \dots, L$.

Because of the way data errors are characterized, care must be taken in the formulation of G_σ . The elements of G_σ associated with elevation and range data are the usual variance and covariance estimates of the angle and distance errors themselves. However, the azimuth errors are defined as angle projections into the azimuth plane--the plane of zero elevation. If $\sigma_{S_n}^2$ is the given (projected) error variance of azimuth for the n -th station, then the n -th diagonal element of G_σ is $\sigma_{S_n}^2 (r_n^o/d_n^o)^2$. If some data errors are correlated, then the off-diagonal elements associated with S_n have the forms $\sigma_{S_n H_m} r_n^o/d_n^o$, $\sigma_{S_n R_m} r_n^o/d_n^o$, and $\sigma_{S_n S_m} r_n^o r_m^o/d_n^o d_m^o$. Typically, the data errors are uncorrelated and G_σ is diagonal.

If the data errors are small and the system is well-conditioned, the linear approximations used herein give excellent estimates of the error variances. If the system is ill-conditioned, the error estimates are not as accurate, but they clearly indicate that small data errors will cause large position errors. Most test designers are not interested in great accuracy in error estimates if the errors are overly large, but they are concerned with discovering when a system is ill-conditioned and finding what actions will remedy the situation. It may be possible to derive a higher-order error theory and thereby improve the accuracy of large error estimates, but we doubt that the effort would be worthwhile.

For reference we present here the elements of A_o . If $\tilde{A}_n^o$ is the n th row of A_o , we have

$$\begin{aligned} \tilde{A}_n^o &= \left(-y_n/d_n^2, x_n/d_n^2, 0 \right) \\ \tilde{A}_{L+n}^o &= \left(-x_n z_n/d_n r_n^2, -y_n z_n/d_n r_n^2, d_n/r_n^2 \right) \end{aligned} \quad (30)$$

$n = 1, \dots, L$, for Davis;

$$\begin{aligned} \tilde{A}_n^u &= \left(-y_n/d_n, x_n/d_n, 0 \right) \\ \tilde{A}_{L+n}^o &= \left(-x_n z_n/d_n r_n, -y_n z_n/d_n r_n, d_n/r_n \right) \end{aligned} \quad (31)$$

$n = 1, \dots, L$, for Schutzberger; and

$$\tilde{A}_n^0 = (x_n/r_n, y_n/r_n, z_n/r_n) \quad , \quad (32)$$

$n = 2L + 1, \dots, N$, for both methods. All the terms in Equations (30), (31), and (32) are evaluated at $P = P_0$.

Maximum-Likelihood, Minimum-Variance Weighting

We now address the problem of choosing the weighting matrix W . A very judicious choice of W can be made if the data errors have certain statistical properties, usually met in practice. The reasonability of $\bar{G}_e = 0$ has been established. It is also reasonable to assume that the probability density functions of the elements of G_e are unimodal and reasonably symmetric about their means. This description suggests that the data errors may be modeled as jointly-gaussian random variables with zero means.

Under the linear assumption that the object position errors are linear combinations of the data errors, Equation (24), it can be shown that the position errors will have minimum variance if W is taken as the inverse of the data errors covariance matrix.⁽⁵⁾ Thus, we choose

$$W_0 = (C_0 G_0 C_0^T)^{-1} \quad , \quad (33)$$

where we note that if G_e is gaussian, $C_0 G_e$ is also gaussian. With Equation (33), Equation (27) reduces to

$$P_0 = R_0 \quad . \quad (34)$$

Although Equations (33) and (34) properly yield the minimum variance estimates only if G_e is gaussian, there is no reason to assume that the expression is not "robust"; i.e., the results are not very sensitive to the gaussian assumption. Unless evidence exists that the error probability densities are strongly multimodal or asymmetric, the weighting scheme presented here should give excellent results.

Equations (33) and (34) are valid for correlated data errors. However, we shall develop the minimum-variance weighting results further only for the very common and reasonable assumption that the data errors are uncorrelated. Thus, G_0 is assumed diagonal, and hence W_0 is diagonal. If w_n^0 is the diagonal element of the n -th row of W_0 , the Davis method gives

$$w_n^o = d_n^o / r_n^o \sigma_{S_n}^2, \quad (35)$$

$$w_{L+n}^o = 1/\sigma_{H_n}^2,$$

$n = 1, \dots, L$; and the Schutzberger scheme gives

$$w_n^o = 1/r_n^o \sigma_{S_n}^2 \quad (36)$$

$$w_{n+L}^o = 1/r_n^o \sigma_{H_n}^2,$$

$n = 1, \dots, L$. The last M elements for both methods are

$$w_n^o = 1/\sigma_{R_n}^2, \quad (37)$$

$N = 2L + 1, \dots, N$. The variances $\sigma_{S_n}^2$, $\sigma_{H_n}^2$, and $\sigma_{R_n}^2$ are the variances of the (projected) azimuth, elevation, and range data errors, respectively, for the n -th station.

Computation of $A_o^T W_o A_o$ for minimum-variance weighting shows the important result that P_o becomes identical for the Davis and Schutzberger methods, with range data added or not. Let p_{ij} be the element of the i -th row and j -th column of the 3 by 3 symmetric covariance matrix P_o . For both schemes we get

$$\begin{aligned}
P_{11} &= \sum_{n=1}^L \left(y_n^2 / \sigma_{S_n}^2 + x_n^2 z_n^2 / r_n^2 \sigma_{H_n}^2 \right) / d_n^2 r_n^2 + \sum_{n=2L+1}^N x_n^2 / r_n^2 \sigma_{R_n}^2 \\
P_{12} &= \sum_{n=1}^L \left(z_n^2 / r_n^2 \sigma_{H_n}^2 - 1 / \sigma_{S_n}^2 \right) x_n y_n / d_n^2 r_n^2 + \sum_{n=2L+1}^N x_n y_n / r_n^2 \sigma_{R_n}^2 \\
P_{13} &= \sum_{n=1}^L - x_n z_n / r_n^4 \sigma_{H_n}^2 + \sum_{n=2L+1}^N x_n z_n / r_n^2 \sigma_{R_n}^2 \\
P_{21} &= P_{12} \\
P_{22} &= \sum_{n=1}^L \left(x_n^2 / \sigma_{S_n}^2 + y_n^2 z_n^2 / r_n^2 \sigma_{H_n}^2 \right) / d_n^2 r_n^2 + \sum_{n=2L+1}^N y_n^2 / r_n^2 \sigma_{R_n}^2 \\
P_{23} &= \sum_{n=1}^L - y_n z_n / r_n^4 \sigma_{H_n}^2 + \sum_{n=2L+1}^N y_n z_n / r_n^2 \sigma_{R_n}^2 \\
P_{31} &= P_{13} \\
P_{32} &= P_{23} \\
P_{33} &= \sum_{n=1}^L d_n^2 / r_n^4 \sigma_{H_n}^2 + \sum_{n=2L+1}^N z_n^2 / r_n^2 \sigma_{R_n}^2
\end{aligned} \tag{38}$$

where, of course, all of the functions in Equations (38) are evaluated at $P = P_0$.

It is evident that the optimum weights depend on the true position of the object--a quantity that is not known. However, the results should not be too sensitive to small errors in the weights since no worse than second-order effects can ensue. The weights can be suitably

approximated by the location iterates. Indeed, if the initial guess of position is close to the true position, little error is induced by using this estimate to compute the weights and holding them fixed until the next location is to be found.

The weights used herein depend on the statistics of the data errors--not on the errors themselves. It may be possible to devise a scheme which generates improved iterated weights by estimating the actual data errors. Such a method would likely use some combination of the a priori data error statistics and the iterated residuals--the values of F^q . Also, object trajectory constraint information may be incorporated in a method. We do not attempt to create such a method here, but point out that the minimum-variance weights would likely be the initial estimates for such a technique.

Position estimates using the weighting scheme here have the property of "maximum likelihood." That is, the method gives a "best"; i.e., most likely, solution for each data set in addition to the solutions of minimum variance over the ensemble of data sets. Hence, any improvement in position estimation by adjusting weights according to each data set is possible only if there is an indication that the a priori assumed gaussian distribution is not valid for that set of data. Such a situation usually arises when the data contains obvious spurious values. The simplest action in that case is merely to eliminate those values.

Program TONER

The computer program TONER (Tonopah Errors) computes and plots the contours of position error statistics for specified station configurations and data error statistics. Although designed with Sandia Laboratories' Tonopah Test Range in mind, the program can be used with any system which uses right-handed Cartesian coordinates and angle and/or range data. The "small" data error and linear assumptions are used.

Position error statistics are computed in either specified horizontal planes--parallel to the xy-plane--or vertical planes--perpendicular to the xy-plane. The errors are in yards. For each desired plane two plots are given. One plot gives the standard deviation contours of the errors in the z (vertical) direction. This statistic is the square root of the element in the third row and third column of P_σ . The other plot gives contours of the Circular Error Probable, at the 50% level, in the xy (horizontal) direction. A discussion of CEP is in Appendix B. The value of CEP depends on the 2 by 2 upper left elements of P_σ . For each plot the median error for that plot is computed.

The program computes position errors with given configurations by using either the (unweighted) Davis, (unweighted) Schutzberger, or minimum-variance weighted methods. Recall that with this weighting, the errors for the other two schemes are the same. If the

Davis method is desired, all requested range measurements are not considered since they are not compatible with the angle data.

There are three types of cards used to request the program. Also, any card that is blank in columns 1-8 inclusive is used as a terminator card. The cards are:

HV Card

<u>Col.</u>	<u>Format</u>	<u>Use</u>
1	A1	Use "H" to plot in a horizontal plane. Use "V" to plot in a vertical plane.
2	A1	Use "D" for Davis method, "S" for Schutzberger method, and "W" for weighted method.
6-8	Not used	
9-16	F8.0	If H, first altitude of horizontal plane in feet. If V, bearing angle in degrees (see below).
17-24	F8.0	If H, next altitude. If V, not used.
.	.	.
.	.	.
.	.	.
72-80	F8.0	If H, last altitude. If V, not used.

Plotting in the H-mode is terminated at the first altitude which is less than or equal to zero--after the first altitude given. Thus, up to nine horizontal planes may be specified in one HV card, but only one vertical plane.

If in V-mode, the vertical plane is assumed to pass through the coordinate origin. The bearing angle α is measured from +y toward +x. A point in the plane has the coordinates range, altitude (r, z). Range is measured as the distance from the origin to the projection of the point into the xy-plane, along the bearing angle direction. For example, if $0 \leq \alpha < 90^\circ$, points which project into the (+x, +y) quadrant have positive range, points projecting into the (-x, -y) quadrant have negative range. If $180^\circ \leq \alpha < 270^\circ$, the signs of range are reversed. Hence the system coordinates of a point are

$$x = r \sin \alpha$$

$$y = r \cos \alpha \quad (39)$$

The altitude coordinate is the usual z direction.

Grid Card

Let X and Y be, respectively, the horizontal and vertical directions of the contour plots themselves. In the H-mode, X is the x-direction and Y is the y-direction. In the V-mode, X is the r-direction and Y is the z-direction.

<u>Col.</u>	<u>Format</u>	<u>Use</u>
1-8	F8,0	H - x value of lower-left corner of plot (feet). V - r value of lower-left corner of plot (feet).
9-16	F8,0	H - x spacing between plotted grid points (feet). V - r spacing between plotted grid points (feet).
18-19	I2	N_x - number of <u>spaces</u> in X direction. $5 \leq N_x \leq 50$.
25-32	F8,0	H - y value of lower-left corner of plot (feet). V - z value of lower-left corner of plot (feet).
33-40	F8,0	H - y spacing between plotted grid points (feet). V - z spacing between plotted grid points (feet).
42-43	I2	N_y - number of <u>spaces</u> in Y direction. $5 \leq N_y \leq 50$.

All spacing entries (columns 9-17 and 33-40) must be positive.

Station Card

<u>Col.</u>	<u>Format</u>	<u>Use</u>
1-10	A10	Identification of station - cannot be blank or will be treated as terminator
11-20	F10,0	x coordinate of station.
21-30	F10,0	y coordinate of station.
31-40	F10,0	z coordinate of station.
41-50	F10,0	standard deviation of azimuth error (projected) for station-degrees.
51-60	F10,0	standard deviation of elevation error for station (degrees).
61-70	F10,0	standard deviation of range error for station (yards).

Up to nine stations is allowed for each plot. If either the azimuth or elevation errors is read as less than or equal to zero, it is assumed that the station does not measure angles. If the range error for a station is less than or equal to zero, that station does not measure range.

A set of sample plots are shown in Appendix C, along with a data card sample set. The program first reads an HV and grid card, in that order. Then the station cards are read, until a terminator card is read. The configuration is then computed and plotted. The next card is assumed to be the first card of a new station set--to be processed with the original HV and grid cards. If this is a new station set, it is processed, etc. If this next card is a terminator it is assumed that the following card is a new HV card, which must be followed by its associated grid card. If, however, a third terminator is read, the job is ended. If the card is indeed an HV card, processing continues as before.

The program runs on the CDC6600 with the FTN compiler. It uses the SCORS plotting package and produces HARDCOPY plots. The plots for each HV, Grid, and Station set have an identification code which is the hour, minute, second, month, and day at which the pass was processed--as given by HOROLOG. Error contours are plotted at 1/2-, 1-, 2-, 5-, 10-, 20-, 50-, and 100-yard errors. Two devices are used to assist in labeling the contours. The areas between adjacent pairs of contours have symbols plotted to help identify levels. The symbols are (.) between 1/2 and 1, (+) between 2 and 5, (x) between 10 and 20, and (*) between 50 and 100. In addition, the errors themselves are printed out for a grid of four equi-spaced points in both the X and Y directions. As an aid in keeping track of which station configurations go with which plots, a page appears with the plots giving the identification code and the associated station position and data error configuration.

Conclusions

Equations for position estimation using both the Davis and Schutzberger methods have been given. Also, it has been shown how range (distance) data may be combined with angle (azimuth and elevation) data in the Schutzberger scheme. Such combination is not meaningful in the (unweighted) Davis method.

For the typical situation where the angle and range data errors are "small" and the system is well-conditioned, the position errors may be represented as a linear combination of the data errors, Equation (24). Adding the fact that the data errors can be taken as random variables with zero mean leads to the result that the position errors are random variables with zero mean. The covariance matrix of the position was shown to be a linear combination of the covariance matrix of the data errors, Equation (27). With the reasonable assumption that the data errors are jointly-gaussian random variables, the position errors are also jointly-gaussian random variables and all their statistics can be found from their covariance matrix P_{σ} .

A weighting scheme has been described, Equation (33), which yields the minimum-variance estimate of the position errors. The weights depend on system geometry and the data error covariance matrix G_{σ} . For at least a diagonal G_{σ} , which means that the data errors are

uncorrelated, it has been shown that the position error statistics are identical for both the Davis and Schutzberger methods--with or without the addition of range data. These weights are not difficult to compute and we recommend that their use be studied for inclusion in the USXYZ program.

An easily used computer program TONER has been described as that which computes and plots contours of position errors for specified geometries and G_{σ} . This program can be used to examine the condition of various test configurations and to ascertain the effects of station locations and data error statistics on position error statistics. A sample of TONER input and output is shown in Appendix C. An analysis of position errors for various configurations at the Tonopah Test Range will appear in a subsequent document. ⁽⁶⁾

APPENDIX A DERIVATION OF THE TWO-FUNCTION SCHUTZBERGER METHOD

In this appendix, we derive the two-function formulation of the Schutzberger method. Let there be L stations measuring azimuth and elevation. The measured direction cosines of the line-of-sight from the n -th station to the object are

$$\begin{aligned}\lambda_n &= \cos S_n \cos H_n \\ \mu_n &= \sin S_n \cos H_n \\ \nu_n &= \sin H_n\end{aligned}\tag{A-1}$$

Also, define the auxiliary quantities

$$\begin{aligned}\xi_n &= \lambda_n x_n + \mu_n y_n + \nu_n z_n \\ \xi_n^S &= \lambda_n x_n^S + \mu_n y_n^S + \nu_n z_n^S\end{aligned}\tag{A-2}$$

In the original formulation we minimize

$$v = \sum_{n=1}^L (x_n - \lambda_n \xi_n)^2 + (y_n - \mu_n \xi_n)^2 + (z_n - \nu_n \xi_n)^2\tag{A-3}$$

The normal equations are

$$\begin{aligned}x \sum (1 - \lambda_n^2) - y \sum \lambda_n \mu_n - z \sum \lambda_n \nu_n &= \sum x_n^S - \lambda_n \xi_n^S \\ -x \sum \lambda_n \mu_n - y \sum (1 - \mu_n^2) - z \sum \mu_n \nu_n &= \sum y_n^S - \mu_n \xi_n^S \\ -x \sum \lambda_n \nu_n - y \sum \mu_n \nu_n - z \sum (1 - \nu_n^2) &= \sum z_n^S - \nu_n \xi_n^S\end{aligned}\tag{A-4}$$

All sums are 1 to L .

Another formulation can be shown as follows: For each station, translate the origin to the station and rotate the coordinates so that the measured line-of-sight is the x-axis in the new coordinates. Then the distance-squared of any point to this line is the sum of the squares of the new y and z coordinates of the point. The transformation is

$$\begin{pmatrix} x'_n \\ y'_n \\ z'_n \end{pmatrix} = \begin{pmatrix} \cos S_n \cos H_n & \sin S_n \cos H_n & \sin H_n \\ -\sin S_n & \cos S_n & 0 \\ -\cos S_n \sin H_n & -\sin S_n \sin H_n & \cos H_n \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \quad (A-5)$$

The quantity to be minimized is now

$$\begin{aligned} v &= \sum_{n=1}^L (y'_n)^2 + (z'_n)^2 \\ &= \sum_{n=1}^L (-x_n \sin S_n + y_n \cos S_n)^2 \\ &\quad + \left[-\sin H_n (x_n \cos S_n + y_n \sin S_n) + z_n \cos H_n \right]^2 \end{aligned} \quad (A-6)$$

Setting the derivatives of v with respect to x , y , and z to zero, it is not difficult to show that the normal equations associated with Equations (A-6) are identical to Equation (A-4).

APPENDIX B

COMPUTATION OF CEP

Let x and y be jointly gaussian random variables with variances σ_x^2 and σ_y^2 and covariance σ_{xy} . Without loss of generality we can assume $\bar{x} = \bar{y} = 0$. Rotating through the angle

$$\gamma = \arctan \left(\frac{2\sigma_{xy} / \sigma_x^2 - \sigma_y^2}{\sigma_x^2} \right) / 2$$

yields new jointly gaussian independent random variables x_1 and x_2 with zero means and variances⁽⁴⁾

$$\sigma_1^2 = \sigma_x^2 \cos^2 \gamma + \sigma_y^2 \sin^2 \gamma + 2 \sin \gamma \cos \gamma \sigma_{xy}$$

$$\sigma_2^2 = \sigma_x^2 \sin^2 \gamma + \sigma_y^2 \cos^2 \gamma - 2 \sin \gamma \cos \gamma \sigma_{xy}$$

The new quantities can be named so that

$$\sigma_1^2 \geq \sigma_2^2$$

is true.

The CEP for percentile p is the radius R of a circle centered at $(0,0)$ such that the volume of the joint probability density function of x_1 and y_1 inside the circle is $p/100$. Transforming to polar coordinates we are required to find R such that

$$\frac{1}{2\pi\sigma_1\sigma_2} \int_0^R dr \int_0^{2\pi} d\theta \exp \left\{ -\frac{1}{2} \left[\left(\frac{r \cos \theta}{\sigma_1} \right)^2 + \left(\frac{r \sin \theta}{\sigma_2} \right)^2 \right] \right\} = p/100$$

$$0 \leq p/100 < 1$$

The expression from Reference 7 is equivalent to

$$\frac{1}{c} \int_0^{R/\sigma_1} dr \exp \left[\frac{-r^2 (1+c^2)}{4c^2} \right] I_0 \left[\frac{r^2 (1-c^2)}{4c^2} \right] r = p/100 \quad (B-1)$$

where $c = \sigma_2/\sigma_1$ and I_0 is a Bessel function.

Rather than do the numerical integration needed for each CEP computation, it is easy to evaluate (B-1) for some set of c and p . Note that $0 \leq c \leq 1$ and $0 \leq p \leq 100$. A cubic curve fit was found to yield three decimal accuracy for all values of c and p . We use the approximation

$$R(p) = \sigma_1 \left[d_0(p) + c d_1(p) + c^2 d_2(p) + c^3 d_3(p) \right]$$

The tabulated values of the $d_i(p)$ are

p	d_0	d_1	d_2	d_3
10	0.1145	0.5565	-0.3149	0.1029
20	0.2369	0.4685	0.0822	-0.1229
30	0.3713	0.3022	0.5038	-0.3389
40	0.5155	0.1427	0.8229	0.4768
50	0.6716	0.0125	0.0164	0.5281
60	0.8440	-0.0858	0.1034	-0.5114
70	1.0422	-0.1353	1.0442	-0.3996
80	1.2888	-0.1386	0.8498	-0.2028
90	1.6489	-0.0663	0.4348	0.1345

Interpolation for other values of p should yield reasonable accuracy.

In our situation x and y are the errors in the x and y directions, respectively. These errors are jointly gaussian random variables since they are linear combinations of the data errors, which are jointly gaussian random variables. The required elements of P_G are $p_{11} = \sigma_x^2$, $p_{12} = p_{21} = \sigma_{xy}$, and $p_{22} = \sigma_y^2$. The CEP value at the 50% level means that half of all computed positions, projected into the xy -plane, will be within CEP of the true position.

APPENDIX C SAMPLE RUN OF PROGRAM TONER

For illustration of the use of TONER, we present a sample run here. We desire to plot position error contours in a vertical plane with a certain station configuration. Next, we wish an error plot in the same vertical plane with a different station configuration and with different data error statistics. Both of these sets are to be computed by using the (unweighted) Schutzberger method. Finally, we wish to plot error contours in each of two horizontal planes by using the weighted method. This pass took 30 seconds of CPU time on the 6600. We list the input deck, the output listing, and show the plots in Figures C-1 through C-11.

Input Deck

VS	347.								
-90000.	2000.	50	5000.	1000.	25				
LA-24	-14170.		-37447.	5979.	.3	.5		3.	
ME-16	20602.		-436.	5456.				5.	
R-33	-7024.		804.	5375.	.4	.25		4.	
									BLANK
XXX	5000.		-7000.	6000.				1.	
LA-24A	-14170.		-37447.	5979.	.1	.1		1.	
R-33A	-7024.		804.	5375.	.15	.12		1.1	
ME-16A	20602.		-436.	5456.				.9	
									BLANK
									BLANK
HW	10000.	20000.							
-10000.	2000.	50	-90000.	2000.	50				
R-33A	-7024.		804.	5375.	.15	.12		1.1	
LA-24A	-14170.		-37447.	5979.	.1	.1		1.	
ME-16A	20602.		-436.	5456.				.9	
XXX	5000.		-7000.	6000.				1.	
									BLANK
									BLANK
									BLANK

Output Listing

TONOPAH ERROR DEVIATION
VS 347.

-90000. 2000. 50 5000. 1000. 25

ID=(1536440201)

STATION	X (FEET)	Y (FEET)	Z (FEET)	AER (DEG)	EER (DEG)	RER (YDS)
1 LA-24	-14170.	-37447.	5979.	.3	.5	3.
2 ME-16	20602.	-436.	5456.			5.
3 R-33	-7024.	804.	5375.	.4	.25	4.

ID=(1536440201)-S CEP IN YARDS

347 LINE MD= 44.72

ID=(1536440201)-S Z ERROR IN YARDS

347 LINE MD= 55.08

RANGE (KFEET) ALT

CEP (YDS) Z ERROR

-90.00	5.00	82.14	111.27
-90.00	13.33	83.43	111.29
-90.00	21.67	87.63	111.37
-90.00	30.00	94.30	111.52
-56.67	5.00	49.18	63.77
-56.67	13.33	51.15	63.70
-56.67	21.67	57.27	63.50
-56.67	30.00	66.31	63.24
-23.33	5.00	25.16	40.34
-23.33	13.33	27.58	38.72
-23.33	21.67	34.99	34.84
-23.33	30.00	45.91	31.08
10.00	5.00	21.18	71.17
10.00	13.33	27.44	69.35
10.00	21.67	40.00	65.19
10.00	30.00	52.66	61.25

ID=(1536590201)

STATION	X (FEET)	Y (FEET)	Z (FEET)	AER (DEG)	EER (DEG)	RER (YDS)
1 XXX	5000.	-7000.	6000.			1.
2 LA-24A	-14170.	-37447.	5979.	.1	.1	1.
3 R-33A	-7024.	804.	5375.	.15	.12	1.1
4 ME-16A	20602.	-436.	5456.			.9

ID=(1536590201)-S CEP IN YARDS

347 LINE MD= 15.48

ID=(1536590201)-S Z ERROR IN YARDS

347 LINE MD= 16.19

RANGE (KFEET) ALT

CEP (YDS) Z ERROR

-90.00	5.00	29.93	36.86
-90.00	13.33	30.30	36.86
-90.00	21.67	31.49	36.86
-90.00	30.00	33.41	36.86
-56.67	5.00	17.82	22.76
-56.67	13.33	18.43	22.74
-56.67	21.67	20.32	22.67
-56.67	30.00	23.12	22.57
-23.33	5.00	7.87	11.65
-23.33	13.33	9.26	11.09
-23.33	21.67	12.41	9.99
-23.33	30.00	16.21	9.22
10.00	5.00	6.51	14.58
10.00	13.33	8.46	14.07
10.00	21.67	12.40	12.96
10.00	30.00	16.61	12.07

The quantity MD is the median value of all errors computed for the plot.

ID=(1519530218)

	STATION	X (FEET)	Y (FEET)	Z (FEET)
1	LA-24	-14170.	-37447.	5979.
2	ME-16	20602.	-436.	5456.
3	R-33	-7024.	804.	5375.

		AER (DEG)	EER (DEG)	RER (YDS)
1	LA-24	.3	.5	3.
2	ME-16			5.
3	R-33	.4	.25	4.

0.5-1.0	YDS=(.)	2.0-5.0	YDS=(+)
10-20	YDS=(X)	50-100	YDS=(*)

Figure C-1. First configuration--vertical plane

Figure C-2. CEP--first configuration, vertical plane

TONOPAH ERROR DEVIATION
HW 10000. 20000.

-10000. 2000. 50 -90000. 2000. 50

ID=(1537160201)

STATION	X (FEET)	Y (FEET)	Z (FEET)	AER (DEG)	EER (DEG)	RER (YDS)
1 R-33A	-7024.	804.	5375.	.15	.12	1.1
2 LA-24A	-14170.	-37447.	5979.	.1	.1	1.
3 ME-16A	20602.	-436.	5456.			.9
4 XXX	5000.	-7000.	6000.			1.

ID=(1537160201)-W CEP IN YARDS Z= 10000FT MD= 2.28

ID=(1537160201)-W Z ERROR IN YARDS Z= 10000FT MD= 23.42

X (KFEET)	Y	CEP (YDS)	Z ERROR
-10.00	-90.00	2.95	25.40
-10.00	-56.67	2.14	9.00
-10.00	-23.33	1.42	2.74
-10.00	10.00	1.21	3.61
23.33	-90.00	2.39	31.16
23.33	-56.67	1.89	19.26
23.33	-23.33	1.72	9.84
23.33	10.00	1.78	4.42
56.67	-90.00	2.60	41.08
56.67	-56.67	2.30	32.02
56.67	-23.33	2.47	24.83
56.67	10.00	3.22	18.02
90.00	-90.00	3.03	52.93
90.00	-56.67	2.83	45.80
90.00	-23.33	3.08	41.06
90.00	10.00	4.01	38.14

ID=(1537160201)-W CEP IN YARDS Z= 20000FT MD= 3.50

ID=(1537160201)-W Z ERROR IN YARDS Z= 20000FT MD= 12.48

X (KFEET)	Y	CEP (YDS)	Z ERROR
-10.00	-90.00	4.41	14.58
-10.00	-56.67	2.94	4.28
-10.00	-23.33	1.64	1.07
-10.00	10.00	1.53	1.99
23.33	-90.00	4.15	21.47
23.33	-56.67	3.07	10.85
23.33	-23.33	2.09	3.78
23.33	10.00	2.00	1.73
56.67	-90.00	4.46	21.00
56.67	-56.67	3.82	20.50
56.67	-23.33	3.42	11.98
56.67	10.00	3.52	7.11
90.00	-90.00	5.02	42.68
90.00	-56.67	4.77	33.10
90.00	-23.33	4.75	25.05
90.00	10.00	5.08	19.55

Figure C-3. Z-error--first configuration, vertical plane

ID=(1520090218)			
STATION	X (FEET)	Y (FEET)	Z (FEET)
1 XXX	5000.	-7000.	6000.
2 LA-24A	-14170.	-37447.	5979.
3 R-33A	-7024.	804.	5375.
4 ME-16A	20602.	-436.	5456.
	AER (DEG)	EER (DEG)	RER (YDS)
1 XXX			1.
2 LA-24A	.1	.1	1.
3 R-33A	.15	.12	1.1
4 ME-16A			.9

0.5-1.0 YDS=(.)	2.0-5.0 YDS=(+)
10-20 YDS=(X)	50-100 YDS=(*)

Figure C-4. Second configuration--vertical plane

Figure C-5. CEP--second configuration, vertical plane

Figure C-6. Z-error--second configuration, vertical plane

ID=(1520230218)

	STATION	X (FEET)	Y (FEET)	Z (FEET)
1	R-33A	-7024.	804.	5375.
2	LA-24A	-14170.	-37447.	5979.
3	ME-16A	20602.	-436.	5456.
4	XXX	5000.	-7000.	6000.

		AER (DEG)	EER (DEG)	RER (YDS)
1	R-33A	.15	.12	1.1
2	LA-24A	.1	.1	1.
3	ME-16A			.9
4	XXX			1.

0.5-1.0 YDS=(.) 2.0-5.0 YDS=(+)
 10-20 YDS=(X) 50-100 YDS=(*)

Figure C-7. Configuration--horizontal planes

Figure 8. CEP--first horizontal plane

Figure 9. Z-error--first horizontal plane

Figure 10. CEP--second horizontal plane

Figure 11, Z-error--second horizontal plane

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