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IN SCATTERING POTENTIALS AND APPLIED FIELDS *

M. B. Fogel, S. E. Trullinger[†], A. R. Bishop, and J. A. Krumhansl

Laboratory of Atomic and Solid State Physics

and

Materials Science Center

Cornell University

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Classical Particle-Like Behavior of Sine-Gordon Solitons
in Scattering Potentials and Applied Fields *

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Abstract

We show that classical Sine-Gordon solitons maintain their integrity to a high degree in the presence of external perturbations. Two examples, of particular importance in condensed matter, are described in detail: (i) a model impurity is found to bind low-velocity solitons but merely phase-shift those with high-velocities, (ii) external static driving terms with damping accelerate the soliton to a terminal velocity. The importance of a translation mode is emphasized and it is concluded that the soliton behaves as a classical particle in all essential respects.

Non-linear fields¹, particularly those exhibiting solitary-wave solutions, are now of interest in many physical problems. Free field solutions have been investigated extensively, particularly for quantum fields²; but in condensed matter we require knowledge of the behavior and integrity of solitons in the presence of "impurities" or applied fields. Kink solutions, or domain walls, occur in magnetic³ and ferrodistortive⁴ materials, and in many Landau-Ginzburg expansion contexts⁵.

In this Letter we describe results of a general perturbation method for examining these kinds of problems. The technique is applicable to non-linear equations possessing translational invariance. For definiteness we consider the case of a Sine-Gordon (SG) soliton and examine the effect on its motion of (i) a weak model impurity potential, and (ii) a constant external driving term and a viscous term in the SG equation of motion.

For (i) we find that high velocity solitons pass through the impurity region with only a phase-shift while those with low velocity become trapped and oscillate. In case (ii) we find transient perturbations which decay rapidly for a large damping constant and leave a soliton moving with a terminal velocity determined by the relative magnitude of damping and forcing constants.

In all cases we conclude that the soliton retains a localized shape and that its dynamics are essentially those of a classical Newtonian particle.

The pure SG equation¹ is a non-linear wave equation of Lorentz covariant form:

$$\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} + \omega_0^2 \sin \psi = 0 \quad . \quad (1)$$

Here c_0 and ω_0 are a velocity and frequency characteristic of the particular physical context¹⁻⁵. The various solutions of (1) are known exactly¹⁻⁶.

For instance

$$\psi_{\pm}^v(x,t) = 4 \tan^{-1} \exp\left(\pm \frac{\omega_0}{c_0} \gamma(x-vt)\right) \quad , \quad (2)$$

with $\gamma \equiv (1-v^2/c_0^2)^{-\frac{1}{2}}$, are soliton (+) and antisoliton (-) solutions travelling with velocity v ($|v| < c_0$). Linearized perturbations $\phi(x,t)$ about $\psi_{\pm}^v$ have been studied extensively, e.g. to establish linear stability^{1,7}. They may be written as $\phi(x,t) = f(x)e^{-i\omega t}$, where $f(x)$ satisfies a 1-D Schrödinger-like eigenvalue equation:

$$-c_0^2 \frac{d^2 f}{dx^2} + \omega_0^2 (1 - 2 \operatorname{sech}^2 \frac{\omega_0}{c_0} x) f = \omega^2 f \quad . \quad (3)$$

The assumption $v=0$ made here is completely general since a Lorentz transformation to the soliton rest frame is always possible. Eqn. (3) has exactly⁷ one "bound" eigenstate with $\omega_0^2 = 0$ and

$$f_b(x) = \frac{2\omega_0}{c_0} \operatorname{sech} \frac{\omega_0}{c_0} x \quad . \quad (4)$$

The remaining eigenfunctions form a continuum with $\omega_k^2 = c_0^2 k^2 + \omega_0^2$ and

$$f_k(x) = (2\pi)^{-\frac{1}{2}} \frac{c_0}{\omega_k} e^{ikx} (k + i \frac{\omega_0}{c_0} \tanh \frac{\omega_0}{c_0} x) \quad . \quad (5)$$

These continuum solutions resemble those of the linearized version of (1) (Klein-Gordon eqn.), except for a perturbation localized around the soliton and an asymptotic phase shift^{7,8}. The zero frequency bound state (4) is of paramount importance to our discussion and may be viewed as a Goldstone mode because the soliton breaks the continuous translational symmetry. Since $f_b(x) = \partial \psi_+^0 / \partial x$, $\psi(x) = \psi_+^0(x) + a f_b(x)$ corresponds (in the linear order considered) to a soliton translated by a distance $(-a)$. Thus $f_b(x)$ may be termed the "translation mode", and allows us to describe the soliton motion⁹. The continuum solutions correspond to small perturbations of the soliton shape.

The essence of our technique is to recognize that $f_i(x)$ ($i=b,k$), being eigenfunctions of a self-adjoint operator, form a complete set. Their orthogonality and completeness relations are readily established⁷. We repeatedly used this complete set to expand linearized perturbations about various functions¹⁰.

(1) Impurity-Soliton interaction. Here we consider the Hamiltonian density

$$3(x) = A \left[\frac{1}{2} \left(\frac{\partial \psi}{\partial t} \right)^2 + \frac{1}{2} c_0^2 \left(\frac{\partial \psi}{\partial x} \right)^2 + \omega_0^2 (1 - \cos \psi) - \lambda \frac{\partial \psi}{\partial x} g(x) \right], \quad (6)$$

i.e. the usual SG density¹ plus a term representing the interaction¹¹ of ψ with the impurity potential $g(x)$. The constant A sets the energy scale, and for simplicity we take $g(x) = \theta(x - x_0) - \theta(x + x_0)$. This simple step-function form, chosen for analytic tractability, is not a limitation of our method. It is expected to exhibit all of the qualitative features of more realistic impurity potentials. In the following the coupling constant λ is assumed small.

We first form the equation of motion for ψ from (6):

$$\frac{\partial^2 \psi}{\partial \tau^2} - \frac{\partial^2 \psi}{\partial z^2} + \sin \psi = \alpha \left\{ \delta(z + z_0) - \delta(z - z_0) \right\}, \quad (7)$$

where $\tau \equiv \omega_0 t$, $z \equiv \omega_0 x / c_0$, $\alpha \equiv \lambda / (c_0 \omega_0)$. A soliton (or antisoliton) given by (2) and initially moving with velocity v will be modified if $\alpha \neq 0$.

Labelling this perturbation by $\phi(z, \tau)$, we assume $\phi \ll 1$ and using (7) form the equation governing it to linear order. Upon Lorentz transforming to the rest frame of the unperturbed soliton this becomes (with $\beta \equiv v/c_0$)

$$\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial z^2} + (1 - 2 \operatorname{sech}^2 z) \phi = \frac{\alpha}{\gamma} \left\{ \delta(z + \beta \tau + \frac{z_0}{\gamma}) - \delta(z + \beta \tau - \frac{z_0}{\gamma}) \right\}. \quad (8)$$

Eqn. (8) can be solved using the Fourier time transform

$$\phi(z, \bar{\omega}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} d\tau e^{i\bar{\omega}\tau} \phi(z, \tau) \quad (\bar{\omega} \equiv \omega/\omega_0) \quad (9)$$

and expanding $\phi(z, \bar{\omega})$ in the complete set (4) and (5):

$$\phi(z, \bar{\omega}) = \phi_b(\bar{\omega}) f_b(z) + \int_{-\infty}^{+\infty} d\kappa \phi(\kappa, \bar{\omega}) f_\kappa(z) \quad (\kappa \equiv k c_0 / \omega_0) \quad (10)$$

After projection and inverse Fourier transformation we find $\phi_b(\tau)$ and $\phi(\kappa, \tau)$.

Even for the step-function form of $g(z)$ these are cumbersome expressions, and here we simply summarize their physical interpretation:

(a) Since f_b is the translation mode, $\phi_b(\tau)$ describes the "center of mass" motion of the soliton in the presence of the impurity. We find that as the impurity approaches $z=0$ (soliton center) in the rest frame the soliton begins to be affected, and when the leading edge passes $z=0$ the soliton acquires a velocity $\beta^* = \alpha/(8\beta)$ in the negative direction (for $\alpha > 0$) which is retained until the trailing edge passes $z=0$. Thus for $\alpha > 0$ (< 0) the soliton slows down (speeds up) in the impurity region. (Precisely the opposite effects occur for the antisoliton.) The soliton acquires an asymptotic phase shift $\delta = \frac{1}{4}\alpha z_0 \beta^{-2}$ in the lab frame. (b) The continuum contributions in (10) consist of two pieces (both proportional to α): one localized in the impurity region and persisting for all time; the other occurring only while the soliton is near the impurity and describing a slight distortion of the soliton waveform. We have illustrated some of this behavior in Fig. 1.

Our results show that ϕ diverges $\sim \beta^{-2}$, as $\beta \rightarrow 0$, so that the linear perturbation theory is invalid for low velocities even for $\alpha \ll 1$. This suggests that low velocity solitons may become trapped or repelled by the impurity. We have examined this possibility by considering a soliton at rest a distance

$\xi c_0/\omega_0$ from the impurity center ($z=0$), and computing from (6) the change $\Delta V(\xi)$ in the static soliton energy due to the impurity:

$$\Delta V(\xi) = 4A\omega_0 c_0 \alpha \tan^{-1}(\sinh z_0 / \cosh \xi) . \quad (11)$$

Note that $\Delta V(\xi=0)$ is a minimum (maximum) for $\alpha < 0 (> 0)$, with opposite conditions for the antisoliton. We have also found the classical condition for binding ($\alpha < 0$) or reflecting ($\alpha > 0$) a soliton by comparing the initial soliton kinetic energy with $|\Delta V(0)|$. Trapping or reflection occurs for $\beta^2 < \beta_c^2 = 1 - \left\{1 + \frac{1}{2}|\alpha| \tan^{-1}(\sinh z_0)\right\}^{-2}$.

If the soliton behaved as a Newtonian particle (of mass $M = 8A\omega_0 c_0$), it would execute oscillatory motion about $\xi=0$ (for $\alpha < 0, \beta^2 < \beta_c^2$) described by $M \partial^2 \xi / \partial \tau^2 = -\partial(\Delta V) / \partial \xi$. We studied this in two regimes - where the soliton width is (a) small and (b) large compared with the impurity width.

In case (a) we sought harmonic oscillations (at least for small β) determined by expanding $\Delta V(\xi)$ to quadratic order in ξ , i.e. $\xi(\tau) = \xi_0 \sin(\Omega\tau)$ with $\Omega^2 = \frac{1}{2}|\alpha| \tanh z_0 / \cosh z_0$. In case (b) we anticipated that the finite soliton extent would result in distinctly anharmonic oscillations. Indeed expanding $\Delta V(\xi)$ to quartic order and assuming Newtonian motion led to a nonlinear oscillator equation with Jacobi elliptic functions as oscillatory solutions⁴, which have the expected character.

We studied the validity of the Newtonian ansatz by considering $\psi = \psi_+^0(z - \xi(\tau)) + \phi(z, \tau)$. The first term describes a soliton oscillating about $z = 0$ and ϕ is the difference between this ansatz and the exact solution ψ to (7). Anticipating small ϕ we again linearized and expanded in the basis functions (4) and (5). For both the limits (a) and (b) above the amplitudes $\phi_b(\tau), \phi(x, \tau)$ were found to be small, as required for consistency.

(ii) External Fields and Damping. Consider the equation of

motion

$$\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} + \omega_0^2 \sin \psi = E - \eta \frac{\partial \psi}{\partial t}, \quad (12)$$

where $\eta \partial \psi / \partial t$ is a viscous term and E is a constant driving force.

For small E , we proceed in the same spirit as for (i), assuming a

solution to (12) of the form $\psi(z, \tau) = \psi_{\pm}^v(z - \beta \tau) + \phi(z, \tau)$, linearizing in ϕ

and transforming to a co-ordinate frame moving with velocity v . The perturbation

ϕ satisfies

$$\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial z^2} + (1 - 2 \operatorname{sech}^2 z) \phi + \gamma \Gamma \frac{\partial \phi}{\partial \tau} - \beta \gamma \Gamma \frac{\partial \phi}{\partial z} = \chi + 2\beta \gamma \Gamma \operatorname{sech} z, \quad (13)$$

with $\Gamma \equiv \eta / \omega_0$, $\chi \equiv E / \omega_0^2$. We again expand $\phi(z, \tau)$ in the complete set (4)

and (5) and project out the amplitudes $\phi_b(\tau)$ and $\phi(\alpha, \tau)$ (c.f. (10)).

The solutions will be presented in detail elsewhere. Here we note only

that $\phi_b(\tau)$ satisfies¹²

$$\frac{\partial^2 \phi_b}{\partial \tau^2} + \gamma \Gamma \frac{\partial \phi_b}{\partial \tau} = \beta \gamma \Gamma + \frac{1}{4} \pi \chi. \quad (14)$$

Assuming $\partial \phi_b / \partial \tau (\tau=0) = 0$, i.e. a soliton initially at rest in the frame

moving with velocity v , and perturbations turned on at $\tau = 0$, we find from (14)

that $\phi_b(\tau) = (\beta + \frac{1}{4} \pi \chi \gamma^{-1} \Gamma^{-1}) \left\{ \tau - \gamma^{-1} \Gamma^{-1} (1 - e^{-\gamma \Gamma \tau}) \right\}$. Since $\phi_b(\tau)$ is the trans-

lation mode amplitude we see that the soliton achieves a terminal velocity (- z

direction) $\beta^T = \beta + \frac{1}{4} \pi \chi \gamma^{-1} \Gamma^{-1}$. If $\chi = 0$ then $\beta^T = \beta$ and the soliton comes

to rest in the lab frame¹³. Thus the soliton behaves precisely as a

Newtonian particle moving in a force field and viscous medium. Indeed (14) is

just Newton's second law for the position ($-\phi_b$) of such a particle. Except for some

initial transient time-dependence, the continuum contributions to $\phi(z, \tau)$,

describing modifications of the soliton shape, are localized about the soliton¹⁴

and time-independent.

In conclusion we have illustrated a simple perturbative method for examining the influence of external perturbations on soliton or solitary wave solutions of nonlinear wave equations. In translationally invariant cases we find that the solitons behave essentially as classical¹⁵ extended particles. This conclusion includes the " ϕ^4 " problem², although there the analog of (3) has an extra bound state to which external perturbations can couple. These results may be applied, for example, to Bloch walls in impure materials, "magnetic solitons"¹⁶ in ^3He , and "charged phase solitons"⁵ in 1-D charge density wave condensates.

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8. Note that their dispersion relation is unaffected by the soliton; the "potential" in Eq. (3) is "reflectionless" (see Ref. 7). The bound state corresponds to a binding of the $k = 0$ mode by the soliton.
9. The translation mode has also required special attention in field theory quantization procedures (see Ref. 2).
10. The inverse integral transform of a function $F(k)$ has to be replaced by its Cauchy principal value if $F(k)$ has a singularity on the real k axis.
11. Other interaction forms can be treated; this one was motivated by Ref. 5.
12. In fact there are coupling terms between ϕ_b and $\phi(x)$ which result in a "polarization" of the soliton by a cloud of linearized solutions. These have been omitted from Eq. (13) since they are of order $\beta \phi(x)$.
13. Even though $\phi_b(\tau)$ becomes large for $\tau \gg 1$, absorbing $\phi_b(\tau)$ into the argument of $\psi_{\pm}(z)$ is equivalent to the procedure of removing secular terms and makes the perturbation theory valid for large τ . See A. H. Nayfeh, Perturbation

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14. Except for a small uniform shift of the "equilibrium" value of ϕ .
15. It seems likely that a Schrödinger equation involving a wave function of ϕ_b would constitute a quantum generalization; this is an interpretation of recent attempts to quantize the translation mode in field theory (see Ref. 2).
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FIGURE CAPTION

Fig. 1. A representative soliton-impurity collision is shown in the initial rest frame of the soliton (a) before (b) during, and (c) after the interaction. The parameter values are $\alpha = 1.5$, $\beta = 0.9$, and $z_0 = 5$. The "center of mass" (CM) position of the soliton suffers a phase shift (see text). The vertical arrows indicate the boundaries of the "Lorentz-contracted" impurity region.