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SELECTED TOPICS FROM CONTINUUM MECHANICS

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SELECTED TOPICS FROM CONTINUUM MECHANICS

PREFACE

The purpose of this paper is to provide a simple, nonrigorous introduction to some of the basic concepts and vocabulary of continuum mechanics. Some fundamental equations and "physical facts" are presented without significant derivation. The treatment is not at all comprehensive, but a number of references are provided that will serve as a starting point for study in greater depth.

I. INTRODUCTION

Continuum mechanics is taken to mean the mechanical behavior of continuous materials. The basic principles of the field are simply the conservation laws for mass, momentum, and energy. Despite its simplicity, the resulting set of equations is nonlinear, and analytic solutions are available for only the most elementary problems. In most practical situations, one finds the solutions with a computer, using finite-difference equations.

The properties of the continuous material itself are important. These are described by the equations of state (EOS) of the medium, and they provide relationships between thermodynamic variables. The classic example is the EOS of an ideal gas:

$$PV = nkT$$

Pressure is simply the force per unit area within the gas, and is the same in all directions. The ideal gas represents a form of EOS that is very simple. Of course, more complicated forms for gases can be developed (e.g., van der Waals). Most liquids are characterized by an EOS that is quite complicated compared to the ideal-gas equation. Solids have the ability to hold a shape, and this results in an even more complicated EOS.

The differential equations for inviscid fluid flow, for an observer at rest relative to the fluid, are as follows:

Conservation of Mass

$$\left(\frac{\partial}{\partial t} + \vec{U} \cdot \nabla \right) \rho = -\rho \nabla \cdot \vec{U}$$

Continuity equation

Conservation of Momentum

$$\rho \left(\frac{\partial}{\partial t} + \vec{U} \cdot \nabla \right) \vec{U} = -\nabla P$$

Equation of motion

Conservation of Energy

$$\rho \left(\frac{\partial}{\partial t} + \vec{U} \cdot \nabla \right) \epsilon = -P \nabla \cdot \vec{U}$$

Equation of State

$$P = P(\rho, \epsilon)$$

II. ELASTIC SOLIDS

We will examine all three states of matter (but not plasmas), beginning with solids. The standard simple example used to describe a solid is the Hooke's law EOS appropriate for elastic solids. Most springs behave approximately according to Hooke's law, in which the amount by which the spring extends is proportional to the force pulling on the spring. A cylindrical bar of length ℓ and cross-sectional area A will experience a change in length $\Delta \ell$ when a given force F is applied to it in the direction of the length of the bar:

$$\frac{F}{A} \propto \frac{\Delta \ell}{\ell}$$

In the case of an ideal gas, we stated that pressure is the same in all directions; hence it is normal to any surface within the gas. For the case of the cylindrical bar, the force is of value F only for a surface normal to the direction of the bar. However, we could take a solid sphere and immerse it in a fluid, and thereby apply a pressure uniformly around it. It would be compressed uniformly according to Hooke's law:

$$P \propto \frac{\Delta R}{R},$$

with the value of F/A being the same normal to all positions on the surface of the sphere. The point is that the presence of Hooke's law is not what really identifies a solid. We could in fact conceptually have a Hooke's law gas such that the following equation might be valid over a limited range of pressure, volume, and temperature:

$$P \propto \frac{\Delta V}{V} \left(\text{or } dP \propto \frac{dV}{V} \right)$$

Of course, in all practical cases, P exhibits a strong temperature dependence. The thing that makes solids unique is their ability to sustain shearing forces when static. In the example of the cylindrical bar, a force was applied along the axis of the bar, and a corresponding change in length in the bar was observed. There will also be a change in radius of the bar as a result of the shear stresses set up in the bar:

$$\frac{\Delta R}{R} = -\sigma \frac{\Delta l}{l},$$

where σ is called Poisson's ratio. Writing the equation for the change in length of the cylindrical bar, we have the following:

$$\frac{F}{A} = Y \frac{\Delta l}{l},$$

where Y is called Young's modulus.

So far, no mention has been made of the shear modulus. The presence of a nonzero shear modulus is what distinguishes solids, and it turns out that both Y and σ can be related to the shear modulus and some other as-yet-undefined quantity. First we will define the shear modulus. Consider a block of material having a side of unit square dimensions. The base of the block is held fixed, and the application of a force S (per unit length normal to the diagram below) shears the block as shown in Fig. 1. Assuming that the value of θ is small, the base width and height are the same before and after application of the force S . Thus the area of the side of the block is also constant.

Fig. 1. A block in shear

The force S (the shearing force) is not normal to the area, as the force F was in the case of the cylindrical bar; instead, S is tangential to the area. The shear modulus (μ) is then defined by the following equation:

$$\frac{S}{A} = \mu \theta$$

Let us return to the case of a spherical solid immersed in a fluid, in order to define a quantity called the bulk modulus. The change in radius was uniform over the sphere when the fluid pressure was increased. The volume is proportional to R^3 , but if $\Delta R/R$ is small, then

$$\frac{\Delta V}{V} \propto \frac{(R + \Delta R)^3 - R^3}{R^3} \cong \frac{3\Delta R}{R}$$

But we had

$$P \propto \frac{\Delta R}{R}$$

Let us rewrite it as

$$P = -K \frac{\Delta V}{V}$$

where K is called the bulk modulus.

The sphere did not change shape, but its compression did change. Conversely, the block to which we applied a shearing force did not experience compression, but it did change its shape.

It turns out that the pair of terms Y and σ can be expressed in terms of μ and K (and vice versa):

$$Y = \frac{9K\mu}{3K + \mu} \quad \text{and} \quad \sigma = \frac{3K - 2\mu}{2(3K + \mu)},$$

or

$$K = \frac{Y}{3(1 - 2\sigma)} \quad \text{and} \quad \mu = \frac{Y}{2(1 + \sigma)}$$

There are only two independent parameters for this simple description we have drawn for solids. One such pair of parameters, K and μ , can be considered as characterizing the change of density of a solid and the change of shape of a solid, respectively. Implicit in the above model for solids was the assumption that the material properties are the same in all directions. This assumption of isotropic behavior is certainly not appropriate on the scale of crystalline samples, but may be reasonably accurate for large macroscopic samples composed of many crystal grains randomly oriented. We will now briefly examine the more general description of anisotropic materials.

We begin by considering an infinitesimally small cube within a material. Let the strain in the x direction be represented by ϵ_{xx} and be equal to $\Delta l/l$ in the x direction. Likewise define ϵ_{yy} and ϵ_{zz} . Define ϵ_{xy} as Δl in the

in the x direction divided by l in the y direction, and similarly for ϵ_{yz} and ϵ_{zx} . Then for the face of the cube perpendicular to the x direction, we can have a stress normal to the face (S_{xx}) and a stress tangential to the face (S_{yz}), as shown in Fig. 2. The first is called a normal stress, and the second a shear stress.

Fig. 2. Normal (S_{xx}) and shear (S_{yz}) stresses on an infinitesimally small cube within a material.

For elastic behavior, but not assuming isotropy, we write Hooke's law as follows:

$$S_{ij} = \sum_{kl} C_{ijkl} \epsilon_{kl}.$$

Again, this says that stress is proportional to strain.

S_{ij} and ϵ_{kl} are both second-rank tensors, and C_{ijkl} , which contains the physical properties, is a fourth-rank tensor called the tensor of elasticity. Apparently there are $3^4 = 81$ different values for C_{ijkl} . However, one can show by symmetry arguments that many are identical and that only 36 unique values can exist. Only 2 unique values remain when the assumption of isotropic behavior is made. The equation for stress is then as follows:

$$S_{ij} = 2\mu\epsilon_{ij} + \lambda\left(\sum_k \epsilon_{kk} \delta_{ij}\right).$$

Here μ is the shear modulus we defined earlier, and λ is called the Lamé parameter of the first kind. We can express λ in terms of μ and K as follows:

$$\lambda = K - \frac{2}{3}\mu.$$

In hydrodynamics, accelerations are caused by pressure gradients. Likewise, stress gradients give rise to accelerations in elastic media. Let χ be the displacement of an infinitesimal element from the position it would occupy if there were no stress gradients. Then the equation of motion is

$$\rho \frac{\partial^2 \chi}{\partial t^2} = (\lambda + \mu) \nabla(\nabla \cdot \chi) + \mu \nabla^2 \chi$$

By taking the divergence and then the curl of the above equation, we find two wave equations. One wave equation is for compressional waves with the oscillations in the same direction as the wave is traveling. The other is for shear waves whose oscillations are perpendicular to the direction of the wave propagation.

The wave velocity of the compressional wave is given by C_{comp} and the wave velocity of the shear wave is given by C_{shear} , as follows:

$$C_{\text{comp}} = \sqrt{(\lambda + 2\mu)/\rho}$$

$$C_{\text{shear}} = \sqrt{\mu/\rho}$$

Note: $C_{\text{comp}} > C_{\text{shear}}$.

III. MATERIAL FAILURE

One of the advantages of the general notation of tensor analysis is that it helps sort out those mathematical quantities which are invariant. That is, tensor notation allows us to formulate quantities whose values are independent of the coordinate system used for the calculations. The average of the normal stresses is one invariant:

$$\bar{S} = \frac{1}{3} (S_{xx} + S_{yy} + S_{zz})$$

Note that this is equivalent to pressure in a fluid. To find another useful invariant, first define the deviatoric stresses (S_i):

$$\begin{aligned} S_x &= S_{xx} - \bar{S} \\ S_y &= S_{yy} - \bar{S} \\ S_z &= S_{zz} - \bar{S} \end{aligned}$$

Then the following invariant can be determined:

$$J_2 = \frac{1}{2} \left(S_x^2 + S_y^2 + S_z^2 \right) + S_{yz}^2 + S_{zx}^2 + S_{xy}^2$$

It turns out that the specific energy stored in a material is given by the following equation

$$W = \frac{J_2}{2\mu} + \frac{\bar{S}}{2K}$$

where W is the specific energy density. The first term corresponds to the energy stored by distorting the material (shear energy), and the second term corresponds to the energy stored by compressing the material.

A term called the octahedral shear stress can be defined as follows:

$$\tau_{oc} = (2J_2/3)^{1/2}$$

We can best interpret the meaning of τ_{oc} if we are working in the principal coordinate system, that system in which only normal stresses exist on the face of an infinitesimally small cube. There is one and only one set of such coordinate axes for any isotropic elastic material experiencing shear--that is, for which the value of J_2 is nonzero. In the principal coordinate system, τ_{oc} is the shear stress one would obtain across the plane whose normal has direction cosines $l = m = n = 1/\sqrt{3}$; and is thus inclined equally to the principal axes.

The presence of a nonzero τ_{oc} characterizes failure of solids. If a solid is placed under compressive loads only, it will not fail. However, at a given compressive load (pressure or $\bar{S}$), there can exist a critical value of τ_{oc} (or J_2) that a material cannot exceed without failing. Some materials, when they reach the critical value of τ_{oc} , flow (or yield) so that they maintain τ_{oc} at the critical value; such failure is said to be ductile. In other materials, when the critical value of τ_{oc} is exceeded, all the stresses may be catastrophically relieved, with propagating cracks and a breaking up of the material; such failure is said to be brittle.

The above description of material failure is very much simplified. Most materials do not behave in a perfectly elastic manner right up to the moment of failure. Also, the value of τ_{oc} at failure is not a function of $\bar{S}$ (or pressure) only, but also of a third invariant, J_3 :

$$J_3 = S_x S_y S_z$$

We will not discuss J_3 further here, but further description can be found in White.*

In many engineering calculations involving real materials, the invariant-quantity approach to stress analysis is glossed over. For example, suppose you are designing a spherical pressure vessel, in which the wall thickness is small compared to the radius of the vessel. Then the tensile stress in the wall of the vessel is given by the formula

$$\text{stress} = \frac{PR}{2t} ,$$

where P is the pressure of the gas inside the vessel, R is the radius, and t is the wall thickness. The stress acts normal to the radius vector of the vessel. In fact, there is a small stress in the direction of the radius vector, but to a first approximation the failure of the vessel is independent of it; so failure is considered as being dependent only on the maximum tensile stress. Thus, if the stress in the above formula exceeds the maximum allowable tensile stress, you conclude that the vessel will fail.

IV. GASES AND SHOCK WAVES

For an ideal gas the macroscopic form of the EOS is as follows:

$$P = (\gamma - 1)E\rho ,$$

where γ is the ideal gas constant, and E is specific energy. For a monatomic gas $\gamma = 5/3$; for a diatomic gas $\gamma = 7/5$. As the number of degrees of freedom increases, γ approaches a value of 1. The lower the value of γ , the more compressible the gas. (An ideal-gas EOS would not be a good choice for describing liquids or solids, but if such a choice were made, the value of γ would be quite large for the low-pressure regime; an idealized incompressible fluid would be equivalent to $\gamma = \infty$.) In practical problems it may be useful to use an ideal gas whose γ value is a slowly varying function of energy and/or density. This represents a sort of perturbation approach. However, some gas descriptions are so complicated that they may require a high-order polynomial or a table of pressure values given as a function of energy and density.

*References are listed alphabetically by author on p. 23.

The most important characteristics of gases are that they are readily compressible, and that they expand readily to fill any given volume. Even materials which are solids at room temperature and atmospheric pressure are best described by a gas EOS when at high temperatures and pressures. Such conditions are usually attained in the presence of strong shocks. In this section, we will be interested in examining the physics of shock waves, using the EOS for an inviscid ideal gas in our examples. (Gases can exhibit viscosity in the same way that liquids can, but a discussion of viscosity will be postponed to the section on liquids.)

Like viscous processes, shock processes are accompanied by a change in entropy. Thus they are irreversible thermodynamic processes. For an ideal gas, we know that the following simple equation applies to reversible adiabatic processes:

$$P\rho^{-\gamma} = \text{constant}$$

Shock processes are approximately adiabatic (that is, virtually no heat transfer takes place), but they are not reversible and thus are not isentropic. We will examine a planar shock wave as portrayed schematically in Fig. 3.

E stands for specific energy, ρ for density, and P for pressure; the subscript i indicates the initial state of the gas in the undisturbed region, and the subscript f indicates the final state of the gas in the shocked region. In addition, let $\dot{\xi}$ be the shock speed relative to the undisturbed material, and let U_i and U_f be the material velocities of the two regions relative to an arbitrary reference frame.

Fig. 3. Schematic of a planar shock wave in a gas.

For an inviscid ideal gas, the shock is represented by a step function, which means both a physical and a mathematical discontinuity. Real gases possess viscosity, with the result that the discontinuity is removed. However, since the distance over which the thermodynamic variables change is on the order of the interatomic distance, the approximation of a step function is appropriate for most practical considerations.

By requiring that mass be conserved across the shock front, we obtain the following equation:

$$\rho_i U_i = \rho_f U_f$$

By also requiring that momentum be conserved, we obtain

$$\rho_i U_i^2 + P_i = \rho_f U_f^2 + P_f$$

Finally, by requiring that energy be conserved across the shock front, we obtain

$$\frac{1}{2} \rho_i U_i^3 + E_i \rho_i U_i + P_i U_i = \frac{1}{2} \rho_f U_f^3 + E_f \rho_f U_f + P_f U_f$$

By manipulating and combining terms, and rewriting the equations in a different way, we can obtain the Rankine-Hugoniot equations, as follows:

$$(\rho_i U_i)^2 = (\rho_f U_f)^2 = \frac{P_f - P_i}{(1/\rho_i) - (1/\rho_f)}$$

$$(U_i - U_f)^2 = (P_f - P_i) \left(\frac{1}{\rho_i} - \frac{1}{\rho_f} \right)$$

$$(E_f - E_i) = \frac{P_i + P_f}{2} \left(\frac{1}{\rho_i} - \frac{1}{\rho_f} \right)$$

These equations relate the change in a variable across the shock front to the changes in other variables across the shock front. Thus they are a convenient set of equations to work with.

The strength of a shock can be described in a number of ways. We will use the ratio of the pressure in the shocked region to the pressure in the undisturbed region. As this ratio approaches 1, the process approaches a reversible isentropic process. The larger the pressure ratio, the greater the change in entropy.

If we start with a gas (or a liquid or a solid) at standard conditions, we can apply to it shock waves of greater and greater strengths. The initial pressure, density, and energy density are known. The final pressure is deter-

mined by the strength of the shock applied and can usually be controlled. We then can usually measure the resulting compression (ρ_f/ρ_i) resulting from the shock. From the third Rankine-Hugoniot equation we can calculate the energy density of the shocked material, and thus the relation between P , ρ , and E is determined. If we repeat the process for a continuous range of shock strengths, we can map out a region of a material's EOS. The locus of EOS end points traced out by shocks of varying strengths is called a Hugoniot curve. Note that such a curve is not a thermodynamic path along which the thermodynamic variables of the material trace their history in time. The Hugoniot curve is also called the Hugoniot adiabat, even though the curve is not a reversible adiabatic path. Figure 4 shows such a Hugoniot curve schematically.

A shock raises the pressure from initial pressure P_0 to final pressure P_s along the dashed line. The dashed curve (Hugoniot adiabat) is the locus of all possible values of P_s . This is an idealized-shock Hugoniot, without polymorphic transitions or elastic limit. The unloading curve, shown solid from (P_s, ρ_s) to (P_0, ρ_f) , is a true adiabatic curve (a succession of equilibrium states).

Fig. 4. Hugoniot curve.

Let us examine the case of an infinitely strong shock. By this we mean $P_f \gg P_i$ and $E_f \gg E_i$. The third Rankine-Hugoniot equation then becomes

$$E_f = \frac{P_f}{2} \left(\frac{1}{\rho_i} - \frac{1}{\rho_f} \right)$$

The equation of state for an ideal (gamma-law) gas is

$$P = (\gamma - 1)E\rho$$

This we can rewrite as follows:

$$E_f = \frac{P_f}{(\gamma - 1)\rho_f}$$

Equating this value for E_f with the one obtained for an ideal gas, we obtain (after simplifying terms)

$$\frac{\rho_f}{\rho_i} = \frac{\gamma + 1}{\gamma - 1}$$

This tells us that there are limits to the compressibility of an ideal gas by a single shock, even though the pressure and energy density may not be limited. An unlimited compressibility would result only for $\gamma = 1$ (and hence $P = 0$), and this result would obtain only for a vacuum.

We can use the Rankine-Hugoniot relations to carry the analysis still further for the case in which the infinitely strong shock is reflected off a perfectly rigid wall. In that case we obtain

$$\rho_r = \frac{\gamma}{\gamma - 1} \rho_f$$

or

$$\rho_r = \frac{\gamma}{\gamma - 1} \frac{\gamma + 1}{\gamma - 1} \rho_i$$

V. LIQUIDS

The EOS description for a liquid is much like that for a solid, with one notable exception: the value of the shear modulus is zero for a liquid. At low pressures a liquid may have a constant compressibility much like a solid:

$$dP = -K \frac{dV}{V}$$

However, as the pressures and/or temperatures become very high, the EOS will become more like that of a gas.

Between these two extremes, there have been many attempts to develop fairly general equations of state. One such example is the Grüneisen equation:

$$P = P_H + \gamma(\mu + 1)(E - E_H)$$

where P_H and E_H are the pressure and specific energy associated with the Hugoniot appropriate to the value of μ ($\mu = \frac{\rho}{\rho_0}$), and γ is the Grüneisen gamma,

$$\gamma = \frac{\gamma_0 + a\mu}{\mu + 1},$$

where γ_0 and a are constant. For a derivation, see Slater.

In order to develop a Grüneisen EOS for a material, it is obviously necessary to have Hugoniot values from shock-wave experiments. This approach represents an attempt to extend by theoretical means a limited EOS obtained by experimental means. Of course, there are attempts to develop an entire EOS theoretically from first principles or to determine one entirely from experimental results. The last may result in an equation which simply fits the data via a polynomial, e.g.,

$$\begin{aligned} P = & a_1\mu + a_2\mu^2 + a_3\mu^3 + \dots \\ & + E(b_1\mu + b_2\mu^2 + b_3\mu^3 + \dots) \\ & + E^2(c_1\mu + c_2\mu^2 + c_3\mu^3 + \dots) \end{aligned}$$

At low pressures and temperatures, liquids will be "stiff" and relatively incompressible, like an ideal gas with a large γ . Liquids also can exhibit viscous properties in the same manner as do gases, and like gases they can sustain compressional waves.

A description of flowing compressible fluids is extremely difficult and really requires the use of finite-difference methods on large computers. However, liquids can frequently be considered as incompressible. Let $\vec{v}$ denote the fluid velocity. Then define $\vec{\Omega}$ to be the vorticity as follows:

$$\vec{\Omega} = \nabla \times \vec{v}.$$

(If $\vec{\Omega} = 0$, the fluid flow is said to be irrotational.) Let ϕ be the specific potential energy derived from a conservative field. Then the equation of motion is as follows:

$$\rho \left\{ \frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2} \nabla v^2 \right\} = -\nabla P - \rho \nabla \phi.$$

Assuming that flow is steady ($\partial \vec{v} / \partial t = 0$), Bernoulli's theorem can be obtained from the above, by taking the dot product of $\vec{v}$ with the remaining terms:

$$\frac{P}{\rho} + \frac{1}{2} v^2 + \phi = \text{constant}.$$

Within a fluid, different values of the constant trace out streamlines. Rewriting the equation of motion to include viscous forces results in the following for incompressible flow:

$$\rho \left\{ \frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2} \nabla v^2 \right\} = -\nabla P - \rho \nabla \phi + \eta \nabla^2 \vec{v} .$$

The last term includes the effect of viscous friction, which permits the flowing fluid to maintain a shear stress in the presence of velocity gradients. Here η is called the shear viscosity coefficient. These stresses are calculated in a way fairly similar to that used for shear stresses in elastic solids:

$$S_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) .$$

It's as if one were to consider thin sheets of fluid sliding over each other and accounting for their frictional forces through the shear viscosity coefficient.

By taking the curl of the equation of motion, we obtain the following:

$$\frac{\partial \vec{\Omega}}{\partial t} + \nabla \times (\vec{\Omega} \times \vec{v}) = \frac{\eta}{\rho} \nabla^2 \vec{\Omega} .$$

First, if we neglect the middle term in the above equation, we are left with a diffusion equation. Thus we expect the vorticity to diffuse and spread out. Second, we note that an initially irrotational field could never become rotational. However, we can introduce vorticity by working on the boundary conditions. An example would be stirring a glass of water with a spoon. If $\eta = 0$, the water would simply slide around the spoon without any effect. In the general study of flows, the Reynolds number is a useful index to determine what type of flow to expect (laminar, turbulent, etc). It is defined as follows:

$$R = \frac{\rho}{\eta} v D ,$$

where D is the characteristic dimension around which the flow is taking place.

VI. INSTABILITIES

In fluid mechanics there are many types of instabilities. In plasma physics a widely known problem is the "two-stream" instability, and in accelerator beam design one is frequently concerned with the "hose" instability. These instabilities and others are concerned with the breaking up of fluid flow patterns. The tendency of the breakup is to convert fluid flow patterns from a more orderly pattern to a more turbulent one. In this section we will discuss two types of instabilities which occur when two fluids have a common planar interface (e.g., the interface between the air and the sea).

The first of these instabilities is the so-called Rayleigh-Taylor instability or simply the Taylor instability. Taylor instability is the instability of the surface between two fluids when the heavier material is being accelerated by the lighter material. It is traditionally illustrated by the instability of the surface between air and water when a beaker of water is turned upside down.

Consider a stable liquid with a standing wave. In Fig. 5, the problem is shown in two dimensions; the liquid extends for a unit length in the direction normal to the page. We will treat this in an approximate manner to illustrate the nature of the equations. The amplitude is represented by a . Then the rate of change of amplitude is $\dot{a}$. To an order of magnitude, $\dot{a}$ is the velocity of the material which flows from the trough of the wave to the crest. In considering the motion of the fluid from the trough to the neighboring crest as indicated in the diagram, it is clear that we should not assume that the whole body of fluid moves, but we should consider a depth of fluid of the order of magnitude of one wavelength -- certainly not more. The volume of fluid involved in the motion is thus of the order of magnitude of $(\lambda/2\pi)^2$, and the order of magnitude of the kinetic energy is:

$$KE \approx \rho \left(\frac{\lambda}{2\pi} \right)^2 \dot{a}^2$$

Fig. 5. Profile of the liquid surface.

The potential energy can be represented as the weight of displaced fluid multiplied by the distance it is displaced. To a degree of approximation similar to that used above, $(\lambda/2\pi)a$ is the volume of fluid moved. Its center of gravity is displaced by a distance about equal to a . Therefore the potential energy is

$$PE \approx \rho g \left(\frac{\lambda}{2\pi} \right) a^2 ,$$

where ρ is the density and g is the acceleration due to gravity. Now we see that:

$$\text{mass} = \rho \left(\frac{\lambda}{2\pi} \right)^2 ,$$

$$\text{restoring force constant} = \rho g \frac{\lambda}{2\pi} ,$$

and therefore the equation of motion of the liquid is

$$\rho \left(\frac{\lambda}{2\pi} \right)^2 \ddot{a} = -\rho g \left(\frac{\lambda}{2\pi} \right) a ,$$

$$\ddot{a} = - \frac{ga}{\lambda} 2\pi .$$

The solution for this is

$$a = a_0 \cos \sqrt{\frac{2\pi g}{\lambda}} t , \quad \text{i.e.,} \quad \omega = \sqrt{\frac{2\pi g}{\lambda}} .$$

This is the case of gravity waves in liquids and corresponds to the stable surface, or to the beaker when it is right side up.

The unstable case is the one in which the acceleration is in the opposite direction--when the beaker is turned upside down. In this case the acceleration (or gravity in the example we have taken) no longer acts as a restoring force but acts to accentuate any existing displacements. The equation now becomes

$$\ddot{a} = \frac{2\pi}{\lambda} g a \quad ,$$

where the sign of the "restoring" force is reversed.

The solution for this is

$$a = a_0 \exp \left(\sqrt{\frac{2\pi g}{\lambda}} t \right) \quad .$$

The amplitude of the disturbance will thus increase exponentially with time--which constitutes the Taylor instability. It can also be seen that the exponential coefficient of the time is inversely proportional to $\lambda^{1/2}$; short wavelengths will increase in amplitude faster than long wavelengths.

Note that the above analysis has an important limitation: it is limited to small perturbations. After the exponential growth of the surface amplitude has proceeded for a significant time, the approximations made in the derivation no longer apply.

The treatment of Taylor instability has also been extended to include compressible materials (see Richtmyer). Experimentally, an instability has also been observed as a shock passes normal to a material interface regardless of the relative densities at the interface (see Meshkov).

Again let the accelerating force be normal to the interface and let both fluids have a velocity component parallel to the plane of the interface. Let ρ be the density of the lower fluid and ρ' the density of upper fluid. U is the velocity component of the lower fluid parallel to the interface, and U' is the one for the upper fluid. Then an instability known as the Helmholtz instability can occur. The most common example of this instability is the ocean waves created as the wind passes over the water of the sea. Again let a be the amplitude of a sinusoidal disturbance at the interface. The derivation is omitted here, but it can be found in Lamb. The result is

$$a = a_0 \exp \left[i \left(\sigma t - \frac{2\pi x}{\lambda} \right) \right] \quad ,$$

where

$$\sigma = \frac{2\pi}{\lambda} \left\{ \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g\lambda}{2\pi} \frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'}{(\rho + \rho')^2} (U - U')^2} \right\}$$

Note that there are imaginary components of σ that would mean instabilities if

$$(U - U')^2 > \frac{g\lambda}{2\pi} \frac{\rho^2 - \rho'^2}{\rho\rho'}$$

Thus sufficiently small λ 's are always unstable unless $U - U' = 0$. Also, if $g = 0$ or if $\rho = \rho'$, the instability will exist for all λ 's provided $U \neq U'$. It should be noted that effects such as viscosity and surface tension have not been included in the above analysis. Both would tend to damp the growth of any perturbation.*

VII. DETONATION PROPAGATION

Many materials will undergo exothermic reactions and burn rapidly when subjected to well-defined conditions. A combustible mixture of gases can exhibit a flamefront propagating through it, and the velocity of such a flamefront is typically a few meters per second. The movement of the front is called a deflagration. However, under different conditions the release of energy is so rapid that the burning process results in a propagating front whose velocity is typically a few thousand meters per second. This latter wave propagation is called a detonation. In this discussion, we will assume all burn fronts to be one-dimensional and discontinuous, as we did with respect to shocks. This approximation is quite good for detonations, although not for deflagrations.

Consider Fig. 6, which shows two Hugoniot's for different materials that start out at the same original specific volume ($1/\rho$) but at different pressures. However, let's assume that these materials have Hugoniot's of very similar shapes! Remember that the Hugoniot's are not thermodynamic paths, but are the loci of points attainable by shocks of varying strengths.

* Our analysis of instabilities has been very qualitative and intuitive. Readers seeking a more mathematical treatment should consult Lamb.

Fig. 6. Two Hugoniot curves for different materials that start out at the same original specific volume but at different pressures.

The curves A and B are presented as having been obtained from two different materials whose densities happen to be the same when subjected to different pressures. However, another model could be proposed for curves A and B. Suppose we have an exothermic material which traces out the curve A when compressed by shocks of varying strength but does not undergo its exothermic reaction. Now suppose that infinitesimal amounts of a catalytic agent can be added, so that when the material experiences a shock (even a weak one) it will instantly release its energy. This energy release, if density is kept constant, would mean an instantly higher pressure. Curve B, then, is the Hugoniot curve for this exothermic material, which starts out at pressure P_1 and specific volume τ_1 and does undergo an instantaneous exothermic reaction. Now we can expand our graphical description by drawing Rayleigh lines, straight lines from (P_1, τ_1) to the Hugoniot.

All Hugoniot curves are concave upward. Thus any Rayleigh line will intersect a detonation Hugoniot in two places, with one exception: the Rayleigh line that is tangent to the Hugoniot curve (see Fig. 7). Such a detonation is called a Chapman-Jouguet (CJ) detonation. The pressure resulting from such a detonation is called the Chapman-Jouguet pressure (P_{CJ}).

The CJ detonation is the one found in nature as a stable propagating wave front. It can be shown (Courant and Friedrichs) that it minimizes both the entropy change and the velocity of the wavefront. If a detonation should be started away from the CJ condition, it will die out or else adjust itself so that it meets the CJ condition. A strong detonation ($P > P_{CJ}$) can be maintained only if a piston continually applies a driving pressure greater than P_{CJ} . Sustained weak detonations are not found in nature.

Fig. 7. Hugoniot function for a detonation.

The above statements can be derived by considerations involving the conservation equations across the wavefront. The analysis is quite similar to that performed in deriving the Rankine-Hugoniot equations across a shock front. Of course, the energy equation is modified to show the potential energy available in the unburned material, and this energy is converted into kinetic and internal energy in the burned material.

In a stable, self-sustaining CJ detonation, the propagating wave moves at the speed of sound relative to the burned material. Thus, relative to the unburned material, it moves at the speed of the burned material plus the sound speed of the burned material (Fig. 8).

Fig. 8. Velocity of the propagating wave in a CJ detonation.

By manipulating the conservation equations one can show that the following is true for detonations or deflagrations:

$$\frac{P_f - P_i}{\tau_f - \tau_i} = -\rho_f^2 U_f^2 = -\rho_i^2 U_i^2.$$

In other words, the quantity on the left side of the equation must be negative. This means that pressure and density must both increase or both decrease. The result is that certain regions are forbidden for detonations and/or deflagrations. Figure 9 shows generalized Hugoniot for both detonations and deflagrations. The forbidden regions are:

- 1) $P < P_0$ and $\tau < \tau_0$,
- 2) $P > P_0$ and $\tau > \tau_0$.

Fig. 9. Hugoniot curve for detonations and deflagrations.

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