MASTER

ON A CLASS OF NON-ASSOCIATIVE FUNCTIONS OF A
SINGLE POSITIVE REAL VARIABLE

M. Creutz and R. M. Sternheimer

Brookhaven National Laboratory, Upton, NY 11973

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On a Class of Non-Associative Functions of a
Single Positive Real Variable

M. Creutz and R. M. Sternheimer

Brookhaven National Laboratory, Upton, NY 11973

In a previous paper, one of us (R.M.S.) has defined a function $F_{n,\alpha}(x) = \prod_{j=1}^n [g_j(x)]_{\alpha}$, which is a generalization of the function $f(x) = x^{x^{\dots}}$ with an infinite number of x 's. The value of $F_{n,\alpha}(x)$ depends on the bracketing order which is denoted by α (non-associativity), and on the number n of functions $g_j(x)$ included in the powering operation. The purpose of the present paper is to study the limit of $F_{n,\alpha}(x)$ as $n \rightarrow \infty$ for a number of choices of $g_j(x)$ which have been proposed in the earlier paper, in particular: $g_j^{(A)}(x) = x/j$, $g_j^{(B)}(x) = x/j^2$, $g_j^{(D)}(x) = x/2^j$, and $g_j^{(F)}(x) = x/j^{\frac{1}{2}}$. The bracketing order α considered in the main part of this paper is the bracketing (order of parentheses) denoted by a in the previous work. In an Appendix to the present paper, one of us (R.M.S.) has discussed several interesting properties of the functions x^y and $E(x,y) \equiv x^y + y^x$.

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In Ref. 1 [Eq. (56)], the following function has been proposed:

$$F_{n,\alpha}(x) = \prod_{j=1}^n [g_j(x)]_{\alpha} \equiv g_1(x)^{g_2(x) \cdot g_3(x) \cdots g_n(x)}, \quad (1)$$

where n denotes the number of functions $g_j(x)$ included in the powering operation, and α denotes the bracketing order, i.e., the order in which the parentheses and brackets are to be inserted on the right hand side of Eq. (1). Throughout the present paper, we will restrict ourselves primarily to the bracketing which has been denoted by a in Ref. 1, and of which typical examples are given in Eqs. (2), (18), and (62)-(69) of Ref. 1. To reiterate this definition, for the case $n=4$, the bracketing a of Eq. (1) corresponds to:

$$F_{4,2}(x) = \prod_{j=1}^4 [g_j(x)]_a \equiv g_1(x) \left\{ g_2(x) \left[g_3(x)^{g_4(x)} \right] \right\}. \quad (2)$$

For the special case $g_j(x) \equiv x$ (independent of j), $\lim_{n \rightarrow \infty} F_{n,a}(x)$ which will be denoted by $F_a(x)$, reduces to the function $f(x)$:

$$f(x) = x^{x^{x^{\cdots x}}}, \quad (3)$$

which constituted the start of the investigation of Ref. 1. This function (in the limit of an infinite number of x 's) is finite in the range for x from $x=0$ to $x = e^{\frac{1}{e}} \cong 1.445$, and takes on values ranging from $f(0) = 0$ to $f(e^{\frac{1}{e}}) = e = 2.718$. Graphs of the function $f(x)$ are given in Figs. 1 and 2 of Ref. 1. In addition, values of $f_2(x)$, $f_{3,\alpha}(x)$, and $f_{4,\alpha}(x)$ (for $n = 2, 3, 4$, respectively) for various bracketing orders α and values of x ranging from $x=2$ to $x=6$ have been presented in Table 1

of Ref. 1.

We now wish to consider the choices $g_j^{(A)}(x)$, $g_j^{(B)}(x)$, $g_j^{(D)}(x)$ proposed by one of us (R.M.S.) in Ref. 1. We have:

$$g_j^{(A)}(x) \equiv x/j, \quad (4)$$

$$g_j^{(B)}(x) \equiv x/j^2, \quad (5)$$

$$g_j^{(D)}(x) \equiv x/2^j. \quad (6)$$

We will also consider the additional choice

$$g_j^{(F)}(x) \equiv x/j^{0.5}. \quad (7)$$

The fact that the choices $g_j^{(A)}$, $g_j^{(B)}$, and $g_j^{(D)}$ lead to a convergent value of $F_\alpha(x)$ as $n \rightarrow \infty$ in Eq. (1) has been demonstrated in Ref. 1 by considering the values of $F_a(x)$ when x is an integer such that the right hand side of Eqs. (4)-(6) equals 1 for a particular value of j denoted by j_1 . This follows because 1 to any power equals 1, and therefore the functions $g_j(x)$ with $j > j_1$ can be disregarded in Eq. (1). This establishes the fact that the functions $F_a(x)$ corresponding to the g 's of Eqs. (4)-(6) are convergent and finite at least for the values of x for which $g_j(x)$ is an integer for a suitable value of j . However, when the right-hand side of Eqs. (4)-(7) is not an integer, we have the phenomenon of "dual convergence" which will be described below.

One of us (M.C.) has been particularly interested in the properties of these functions in the region $0 < x < 1$. He therefore programmed the operations involved in obtaining $F_{n,\alpha}(x)$ for the Hewlett-Packard

HP 9830A computer of the Theory Group of the Physics Department. In this fashion, he obtained several interesting results for the region $0 < x < 1$. Some of these results are shown directly by Figs. 1, 1a, and 1b, which pertain to

$$F_a^{(A)}(x) = \sum_{j=1}^{\infty} [x/j]_a = x^{x/2} x^{x/3} x^{x/4} \dots \quad (8)$$

It is seen that while for $x > 1$, $F_a^{(A)}(x)$ is a rapidly increasing function of x [with values given by Eqs. (62)-(66) of Ref. 1], for $x < 1$, the curve of $F_a^{(A)}(x)$ has two branches, one of which goes to $F_a^{(A)}(0) = 1$ and the other goes to $F_a^{(A)}(0) = 0$. It can be easily seen from the calculations that for $n = \text{even}$, the values approach the upper branch as $n \rightarrow \infty$, whereas for $n = \text{odd}$, the values approach the lower branch as $n \rightarrow \infty$. This behavior is also exhibited in the results of Table I, which give the values of $F_{n,a}^{(A)}(x)$ for various odd n and even n values for the three x values: $x = 0.10$, $x = 0.25$, and $x = 0.50$. The two limits at $x=0$ follow essentially from the following argument. Suppose that we start with a value $x = \epsilon$, which is finite but small, e.g., $\epsilon = 0.1$. We find in calculating $F_{n,a}^{(A)}(x) = F_{n,a}^{(A)}(\epsilon)$:

$$\left(\frac{\epsilon}{n-1}\right)^{\frac{\epsilon}{n}} \sim \left(\frac{\epsilon}{n-1}\right)^0 = 1 \quad (9)$$

$$\left(\frac{\epsilon}{n-2}\right)^1 = \left(\frac{\epsilon}{n-2}\right) \sim 0 \quad (10)$$

$$\left(\frac{\epsilon}{n-3}\right)^{\frac{\epsilon}{n-2}} \sim \left(\frac{\epsilon}{n-3}\right)^0 = 1, \quad (11)$$

which shows immediately the alternation of values close to 1 [for an even

number of $g(x)$'s] with values close to 0 [for an odd number of $g(x)$'s]. In connection with the latter statement, we note that for ϵ itself [i.e., a single $g_j(x) = x$] we have $\epsilon \rightarrow 0$, an example of an odd number of $g(x)$'s.

The convergence of the even and odd sequences $F_{2n,a}(x)$ and $F_{2n+1,a}(x)$ follows because they are monotonic in n and bounded. However, these two sequences converge in general to two different values as $n \rightarrow \infty$ (except at the crossing points of the two curves of $F_a(x)$ vs x , as will be discussed below). We will refer to this phenomenon as "dual convergence" of the functions involved.

A more dramatic demonstration of this dual convergence emerges from Fig. 2 which shows the function $F_a^{(B)}(x)$ which is obtained with $g_j^{(B)}(x) = x/j^2$ [see Eq. (58) of Ref. 1, and Eq. (5) of the present paper]. In this case the upper curve for $x < 1$ was calculated by taking:

$$F_{20,a}^{(B)}(x) = \sum_{j=1}^{20} [(x/j^2)]_a. \quad (12)$$

This curve approaches $F_{20,a}(0) = 1$ at $x=0$, as expected (n =even). On the other hand, the lower curve for $x < 1$ was calculated from:

$$F_{21,a}^{(B)}(x) = \sum_{j=1}^{21} [(x/j^2)]_a. \quad (13)$$

This curve approaches $F_{21,a}(0) = 0$ at $x=0$ as is also expected, since n = odd. However, in contrast to the case of Fig. 1 for $F_a^{(A)}(x)$, the two curves at $x=1$ intersect making a finite (and appreciable) angle. Beyond $x=1$, $F_a^{(B)}(x)$ is again double-valued, from $x=1$ to $x=4$, for which we have the unique value $F_a^{(B)}(4) = 4$, as already derived in Eq. (67)

of Ref. 1. Beyond $x=4$, the two curves diverge again, and do not coincide again until $x=9$, where we have the unique value $F_a^{(B)}(9) = 9^{9/4} = 140.3$, as calculated in Eq. (68) of Ref. 1. Thus the only points for which $F_a^{(B)}(x)$ is single-valued are those for which $x = j_1^2$, with $j_1 = \text{integer}$. These are precisely the points on the x axis, where the number $1 = x/j_1^2$ occurs in the sequence of $g_j(x)$ values of Eq. (1). By the same token, the points $x = j_1^2$ constitute crossing-points of $F_a^{(B)}(x)$, and $F_a^{(B)}(x)$ is double-valued at all other values of x . However, it may be remarked that the two different values of $F_a^{(B)}(x)$ differ by at most $\sim 25\%$ for $x > 1$, with the differences being of course smaller as we approach the crossing-points $x = j_1^2$. Nevertheless, the double-valuedness is a very interesting mathematical property of the function $F_a^{(B)}(x)$. This property is not restricted uniquely to $F_a^{(B)}(x)$, as will be shown in the following discussion.

Thus we consider the function $F_a^{(D)}(x)$, which involves the function $g_j^{(D)}(x) = x/2^j$ of Eq. (6) [see also Eq. (70) of Ref. 1]. Actually we will consider the function

$$F_{7,a}^{(D)}(x) = \prod_{j=1}^7 \left[\frac{x}{2^{j-1}} \right]_a, \quad (14)$$

and

$$F_{8,a}^{(D)}(x) = \prod_{j=1}^8 \left[\frac{x}{2^{j-1}} \right]_a, \quad (15)$$

so that we obtain the first unique value $F_{7,a}^{(D)}(1) = F_{8,a}^{(D)}(1) = 1$, instead of the first unique value at $x=2$. We note that the choice of Eqs. (14) and (15) corresponds to the function:

$$F_{n,a}^{(D)}(x) = x^{\frac{x}{2} \cdot \frac{x}{4} \cdot \frac{x}{8} \cdots \frac{x}{2^{n-1}}} \quad (16)$$

The behavior of $F_{n,a}^{(D)}(x)$ is shown in Figs. 3 and 3a, for $n=7$ (odd) and $n=8$ (even). We now note that in analogy to $F_a^{(B)}(x)$, the function $F_{n,a}^{(D)}(x)$ is double-valued not only for $x < 1$, but also for x between 1 and 2, and for $2 < x < 4$, etc. Thus all of the points at which $F_{n,a}^{(D)}(x)$ has a single value, i.e., $x = 1, 2, 4, 8, \dots$ are also crossing points of the limiting functions. We may remark that the differences between the values of the two branches of $F_{n,a}^{(D)}(x)$ are small (a few percent) at points between the crossing points for $x > 1$. Nevertheless, we are here dealing with an interesting function from the mathematical point of view.

Referring again to the convergence of $F_a^{(A)}(x)$, $F_a^{(B)}(x)$ and $F_a^{(D)}(x)$, and leaving aside the question of the double-valuedness of these functions in varying regions of x , we may note that both $g_a^{(A)}(x)$ and $g_a^{(B)}(x)$ are of the form:

$$g_j(x) = x j^\beta \quad (17)$$

with $\beta = -1$ for the function $g_a^{(A)}(x)$, and $\beta = -2$ for $g_a^{(B)}(x)$. It can be rather easily shown that for $\beta > 0$ and $x > 1$, the function $F_{n,a}(x)$ of Eq. (1) diverges as $n \rightarrow \infty$, since for $\beta > 0$, j^β becomes arbitrarily large with increasing j . On the other hand, for $\beta = 0$, we obtain the choice $g_j(x) = x$ and hence the function $f(x)$ of Ref. 1, which converges for $0 < x < 1.445$.

If we now take β negative, it is easily shown that $F_{n,a}(x)$ of Eq. (1)

exhibits a dual convergence to finite values for $n \rightarrow \infty$ for both $x < 1$ and $x > 1$. Two examples have been discussed above, namely $g_j^{(A)}(x)$ and $g_j^{(B)}(x)$ with $\beta = -1$ and $\beta = -2$, respectively. However, even for $\beta = -0.1$, we obtain a convergence, albeit to a very large number. As an example, for $x=2$, we find

$$F_a(x) = 2^{\frac{2}{0.1}} \cdot 3^{\frac{2}{0.1}} \cdot 4^{\frac{2}{0.1}} \cdot \dots \cdot 1024^{\frac{2}{0.1}} \quad (18)$$

since $2^{10} = 1024$, and $2^{(10)(0.1)} = 2$. Thus in this case, we need 1024 quantities on the right hand side, but the result converges, although to a very large "meganumber" in the terminology of Ref. 1. (A meganumber is defined as a quantity which exceeds $10^{10^{10}}$.)

Figure 4 shows the behavior of $F_a^{(F)}(x)$ pertaining to the function $g_j^{(F)}(x) = x/j^{0.5}$ of Eq. (7). In this case, we have $\beta = -0.5$ [cf. Eq. (17)]. The two functions $F_{n,a}^{(F)}(x)$ for $n = \text{even}$ and $n = \text{odd}$ appear to converge to the same value as $n \rightarrow \infty$ for $x > 1$. Nevertheless, we believe that these functions are distinct, although they differ by very small amounts for $x > 1$. This phenomenon has been observed for the two functions $F_{n,a}^{(A)}(x)$ pertaining to $g_j = x/j$, as described in footnote 2.

Before proceeding to a partial explanation of the double-valuedness which generally exists for both $x < 1$ and $x > 1$, we wish to present Fig. 5 which shows the functions $f_n(x)$ of Ref. 1 for $n = 1, 2, 3, 4, 5$, and 6, as well as the limiting case $n \rightarrow \infty$, which has been denoted simply by $f(x)$ in Ref. 1 [see Eqs. (2)-(10) and Figs. 1 and 2 of Ref. 1]. Figure 5 shows that the functions $f_n(x)$ approach $f_n(0) = 1$ for $n = \text{even}$ and $f_n(0) = 0$ for $n = \text{odd}$, as long as n is finite. However, in the limit

$n \rightarrow \infty$, which is obtained by solving the equation

$$f = x^{\frac{f}{3}}, \quad (19)$$

which was originally proposed by H. Snyder, the value of $f(0)$ is 0, and there is no double-valuedness in the limit $n \rightarrow \infty$. It may be noted that the curve of f vs x approaches $x = 0$ with infinite slope. The very small resulting value of x for $f = 1/10$, namely $x = 10^{-10}$, has already been derived in Ref. 1 [see Eq. (10)].

Concerning the dual convergence of $F_{n,a}(x)$, we have noted the following correlation. We consider the five functions: $f(x)$, $F_a^{(F)}(x)$, $F_a^{(A)}(x)$, $F_a^{(D)}(x)$, $F_a^{(B)}(x)$ in the order of rapidity of decrease of $g_j(x)$ with increasing j . It turns out that this is the same order as the order of increasing double-valuedness of the functions $F_a(x)$ in the two regions of x , namely $x < 1$ and $x > 1$. Thus we can write down the successive terms $g_j(x)$ for these five functions, and correlate the decrease of $g_j(x)$ with increasing j with the extent of the double-valuedness.

For $f(x)$, we have:

$g(x) = x, x, x, x$: no decrease; $f(x)$ is single-valued for $0 < x < e^{\frac{1}{e}}$.

For $g_j^{(F)}(x) = \frac{x}{\sqrt{j}} = \frac{x}{1}, \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{3}}, \frac{x}{2}, \frac{x}{\sqrt{5}}$: slow decrease, and the resulting $F_a^{(F)}(x)$ is double-valued for $0 < x < 1$, but is approximately single-valued for $x > 1$.

For $g_j^{(A)}(x) = \frac{x}{j} = \frac{x}{1}, \frac{x}{2}, \frac{x}{3}, \frac{x}{4}$: somewhat faster decrease, and the resulting $F_a^{(A)}(x)$ is double-valued for $0 < x < 1$, but is approximately single-valued for $x > 1$, although there is a slight difference between $F_{n,a}^{(A)}(1.25)$ for $n = \text{even}$ and $n = \text{odd}$, and also for $F_{n,a}^{(A)}(1.50)$, as described in footnote 2.

For $g_j^{(D)}(x) = \frac{x}{j^{j-1}} = \frac{x}{1}, \frac{x}{2}, \frac{x}{4}, \frac{x}{8}$: appreciably faster decrease, and the resulting $F_a^{(D)}(x)$ is double-valued both for $0 < x < 1$ and for $x > 1$, except at the points $x = 2^k$, where k is an integer. This behavior is shown clearly in Figs. 3 and 3a.

Finally for $g_j^{(B)}(x) = \frac{x}{j^2} = \frac{x}{1}, \frac{x}{4}, \frac{x}{9}, \frac{x}{16}$: the initial decrease of $g_j(x)$ with increasing j is fastest, and correspondingly the double-valuedness of $F_a^{(B)}(x)$ for small x is the most pronounced of any of the five functions, as shown by a comparison of Figs. 2 and 2a with the other figures of this paper.

In connection with the definition of the parameter β through Eq. (17), we note that the case $\beta = 0$ is unique, since the corresponding function F which has been denoted by $f(x)$ in Ref. 1 is convergent in the limited range from $x = 0$ to $x = e^{\frac{1}{e}} \approx 1.445$. The case $\beta > 0$ will be discussed below. For $\beta < 0$, the region of dual convergence of $F(x)$ extends over the entire positive x axis, as has been discussed extensively above [see in particular Eq. (18) for the case $\beta = -0.1$]. Thus the function $f(x)$ obtained by taking $g_j(x) = x$ independent of j occupies a unique position in the spectrum of functions $g_j(x) = xj^\beta$ defined by Eq. (17), particularly since the functions $f_{n,a}(x)$ converge to the same value for both $n = \text{even}$ and $n = \text{odd}$, as $n \rightarrow \infty$.

We believe that all functions $F_{n,\alpha}(x)$ except $f_n(x)$ are dual-convergent both for $x < 1$ and for $x > 1$, except at the points where $g_j(x) = 1$ for an appropriate j . However, in some cases, such as $g_j(x) = xj^{-0.5}$, the double-valuedness for $x > 1$ does not appear within the accuracy of the Hewlett-Packard calculations (a few parts in 10^9) (see Fig. 4). The fact that $f_n(x)$ is single-valued has been checked for $x = 0.1$. Thus for $n \sim 200$, the values of $f_n(x)$ oscillate between

0.399,012,975 and 0.399,012,982 for $n = \text{even}$ and $n = \text{odd}$, and the difference ($= 7 \times 10^{-9}$) decreases by $\sim 1 \times 10^{-9}$ as n is increased by 2 units beyond $n \sim 200$.

In addition to the five functions shown in Figs. 1-5, as discussed above, we have also evaluated two additional functions $F_{n,a}(x)$ which are based on the functions $g_j^{(E)}(x) = x/n_0^j$ [Eq. (71) of Ref. 1] and $g_j^{(C)}(x) = x^j$ [Eq. (59) of Ref. 1], respectively. These functions, which are denoted by $F_a^{(E)}(x)$ (for $n_0 = 3$) and $F_a^{(C)}(x)$, respectively, are shown in Figs. 6 and 7 of the present paper.

For $F_{n,a}^{(E)}(x)$, we have actually used

$$F_{n,a}^{(E)}(x) = \sum_{j=1}^n \left[\frac{x}{3^{j-1}} \right]_a \quad (20)$$

in analogy to Eqs. (14) and (15) for $n_0 = 2$. It can be easily seen from Fig. 6 that for this case, the difference between $F_{10,a}^{(E)}(x)$ and $F_{11,a}^{(E)}(x)$ is even greater than for the case $g_j(x) = x/j^2$ of Figs. 2 and 2a. This can be directly correlated with the steepness of the decrease of $g_j(x)$ with increasing x , as discussed above. Thus the successive functions $g_j(x)$ are given by:

$$g_j^{(E)}(x) = x, \frac{x}{3}, \frac{x}{9}, \frac{x}{27}, \frac{x}{81} \cdot \quad (21)$$

The first three functions are comparable with the corresponding functions $g_j^{(B)}(x)$, but the fourth function, namely $x/27$ is appreciably smaller than the corresponding $g_4^{(B)}(x) = x/16$, and a similar result holds for the fifth function [$x/81$ as compared to $x/25$ for $g_5^{(B)}(x)$].

Considering now Fig. 7, corresponding to $g_j^{(C)}(x) = x^j$, we have

dual convergence for $x \leq 1$. However, for $x > 1$, the function $F_{n,a}^{(C)}(x)$ diverges to infinity. The fact that there is no sharp break in the curve at $x = 1$ is due to the finite values of n considered in Fig. 7 ($n = 10$ and $n = 11$). Actually even for $n = 10$ and 11 , $F_{n,a}^{(C)}(x)$ becomes larger than 10^{100} at $x = 1.07$. An interesting feature of the functions $F_{10,a}^{(C)}(x)$ and $F_{11,a}^{(C)}(x)$ is that they coincide for all practical purposes from $x \sim 0.8$ to $x = 1.0$ (see Fig. 7).

In Eq. (17), the case of positive β will now be discussed. First we will consider the value $\beta = 1$, so that we have

$$F_{n,a}^{(G)}(x) = x^{2x} 3x^{4x} \cdots nx^{nx} \quad (22)$$

It was noted by one of us (M.C.) that if $x = 1/m$ where m is an integer, the powering operation of Eq. (22) converges to a non-trivial value (see below), since we have $mx = 1$, so that additional powers, i.e., $(m+1)x, (m+2)x, \dots$ do not change the result. In particular for $m = 1, m = 2, m = 3, m = 4$, we obtain

$$F_{1,a}^{(G)}(1) = 1 = F_a^{(G)}(1) \quad , \quad (23)$$

$$F_{2,a}^{(G)}(0.5) = 0.5^1 = 0.5 = F_a^{(G)}(0.5) \quad , \quad (24)$$

$$F_{3,a}^{(G)}(1/3) = (1/3)^{(2/3)} = 0.4808 = F_a^{(G)}(1/3), \quad (25)$$

$$F_{4,a}^{(G)}(1/4) = 1/4^{(2/4)} = 0.4384 = F_a^{(G)}(1/4), \quad (26)$$

where $F_a^{(G)}(x)$ indicates the function $F_{n,a}^{(G)}(x)$ obtained by taking $n \rightarrow \infty$.

In addition, it is easy to show that $F_a^{(G)}(x)$ (for $n \rightarrow \infty$) converges throughout the interval $0 \leq x \leq 1$, although the function has an

(infinite) number of discontinuities occurring at the values $x_m \equiv 1/m$.

(1) Firstly, for the interval $1/2 < x < 1$, it is easy to show that $F_a^{(G)}(x) = 0$. Thus taking as an example, $x = 0.7$, we find

$$F_{n,a}^{(G)}(0.7) = 0.7^{1.4^{2.1^{2.8^{\dots^{0.7n}}}}} \quad (27)$$

Now the quantity in the exponent of 0.7, namely

$$\text{exp} = 1.4^{2.1^{2.8^{\dots^{0.7n}}}} \quad (28)$$

becomes infinite as $n \rightarrow \infty$, and $0.7^\infty = 0$. The same argument can be used for any value in the region $0.5 < x < 1.0$.

(2) For the interval $1/3 < x < 1/2$, we find $F_a^{(G)}(x) = 1$. In order to prove this result, we consider $x = 0.4$, as an example:

$$F_{n,a}^{(G)}(0.4) = \left[0.4^{0.8^{1.2^{1.6^{\dots^{0.4n}}}}} \right]_a \quad (29)$$

where the bracketing order a is implied in the right-hand side. As before, the quantity

$$\text{Exp} = \left[1.2^{1.6^{\dots^{0.4n}}} \right]_a \quad (30)$$

becomes infinite as $n \rightarrow \infty$. Upon inserting this result in Eq. (29), we obtain $0.8^{\text{Exp}} = 0$, and hence the result

$$F_a^{(G)}(0.4) = 0.4^0 = 1 \quad (31)$$

Obviously, the same argument applies for any value of x in the interval $1/3 < x < 1/2$.

(3) For the interval $1/4 < x < 1/3$, we obtain $F_a^{(G)}(x) = x$. This can be easily proven, using the same technique as before. We consider the representative value $x = 0.3$, for which we can write:

$$F_{n,a}^{(G)}(0.3) = \left[0.3^{0.6^{0.9^{1.2^{\dots^{0.3n}}}}} \right]_a, \quad (32)$$

The part of the exponent to the right of 1.2 gives $1.2^N \rightarrow \infty$ as $n \rightarrow \infty$. Proceeding now to 0.9, we find $0.9^\infty = 0$, and further $0.6^0 = 1$. Thus Eq. (32) reduces to

$$F_a^{(G)}(0.3) = 0.3^1 = 0.3, \quad (33)$$

which proves that $F_a^{(G)}(x) = x$ in the region $1/4 < x < 1/3$.

(4) Finally for the interval $1/5 < x < 1/4$, we consider a typical value, namely $x = 0.22$. This gives:

$$F_{n,a}^{(G)}(0.22) = \left[0.22^{0.44^{0.66^{0.88^{1.10^{\dots^{0.22n}}}}} \right]_a. \quad (34)$$

The power to the right of 0.88 gives $0.88^\infty = 0$ in the limit $n \rightarrow \infty$.

From this result, we obtain $0.66^0 = 1$, so that $F_a^{(G)}(0.22)$ reduces to:

$$F_a^{(G)} = 0.22^{0.44} = 0.5137, \quad (35)$$

and $F_a^{(G)} = x^{2x}$ in the interval $1/5 < x < 1/4$.

(5) It is easily shown by induction that for x in the interval

$1/m < x < 1/(m-1)$, $F_a^{(G)}$ is given by:

$$F_a^{(G)} [1/m < x < 1/(m-1)] = \sum_{j=1}^{m-3} [xj]_a . \quad (36)$$

Equation (36) leads to a discontinuity at each value of $x = 1/m$.

Thus we find, for ϵ very small (and positive):

$$F_a^{(G)} \left(x = \frac{1}{m} - \epsilon \right) = \sum_{j=1}^{m-2} \left[\frac{j}{m} \right]_a , \quad (37)$$

$$F_a^{(G)} \left(x = \frac{1}{m} + \epsilon \right) = \sum_{j=1}^{m-3} \left[\frac{j}{m} \right]_a . \quad (38)$$

The value of $F_a^{(G)}$ at $x =$ exactly $1/m$, which we have calculated above in Eqs. (23)-(26), is intermediate between the results (37) and (38), and is given by:

$$F_a^{(G)} (x = 1/m) = \sum_{j=1}^{m-1} \left[\frac{j}{m} \right]_a . \quad (39)$$

The set of segments defined by Eq. (36) for the function $F_a^{(G)}$ have been calculated using the Hewlett-Packard calculator, and are shown in Fig. 8, together with the individual values (singularities) obtained from Eq. (39). These latter values are shown as separate points between the successive line or curved segments derived from Eq. (36). In particular, we note the values of Eqs. (23)-(26), and similar values at $x = 1/5$, $x = 1/6$, and $x = 1/7$. These points which are intermediate between the line segments lie on a fairly smooth curve, which has however not been drawn. This hypothetical curve is continued by and large by the curve which is valid for $x < 0.1$, where the consecutive line segments

are essentially continuous, i.e., without the sharp breaks which are encountered for $x > 1/6$, and which become progressively greater as we approach $x = 0.5$. (Although we have not mentioned it specifically, it should be obvious that for all values of x greater than 1, the function $F_{n,a}^{(G)}(x)$ diverges to ∞ as $n \rightarrow \infty$.)

Returning to the discontinuities observed for $x < 1$, which persist visibly down to $x = 0.1$, and in principle down to $x = 0$, we believe that these discontinuities and the resulting alternation between larger and smaller values of $F_a^{(G)}(x)$ are both caused by a phenomenon which is closely related to the dual convergence of the functions $F_a^{(A)}(x)$, $F_a^{(B)}(x)$ and $F_a^{(F)}(x)$ having $\beta < 0$, which has been previously pointed out. This correspondence is especially likely because the dual convergence is most marked, i.e., the differences between the $n = \text{even}$ and $n = \text{odd}$ cases are particularly large in the region $0 < x < 1$, which is also the region being investigated for $F_{n,a}^{(G)}(x)$. In fact, referring to Eq. (36), the regions where $F_a^{(G)}$ is large correspond to Ξ functions for which the upper limit $m-3$ is even, whereas those regions for which $F_a^{(G)}$ is relatively small correspond to Ξ functions where $m-3$ is odd, in complete analogy to the similar situation in the cases of Figs. 1-7, where the function $F_{n,a}$ for $n = \text{even}$ has the large values for $x < 1$, with $F_{n,a}(0) = 1$ and where $F_{n,a}$ for $n = \text{odd}$ goes to zero at $x = 0$. As an example of a relatively small $F_a^{(G)}(x)$ function, we consider the segment extending from $x = 1/4$ to $x = 1/3$, which gives $m=4$ in Eq. (36), so that $m-3 = 1$ is odd. It may also be noted that the curve of $F_a^{(G)}(x)$ for $x \lesssim 0.1$, which has been calculated only down to $x \cong 0.006$, appears to approach the value $F_a^{(G)}(x=0) = 0$ with infinite slope, in similarity to the behavior of $f_{n,a}(x)$ for $n = \infty$, as shown in Fig. 5 (see also

Fig. 2 of Ref. 1).

As an additional example of Eq. (17) with $\beta > 0$, we have also calculated the case $\beta = 0.5$, for which we have

$$g_j^{(G')}(x) \equiv xj^{0.5}. \quad (40)$$

For this function $g_j^{(G')}(x)$, the discontinuities of $F_a^{(G')}(x)$ occur at the points $x = 1/m^{\frac{1}{2}}$ where $m = \text{integer}$. The resulting values of $F_a^{(G')}(x)$ are shown in Fig. 9. We again have the segments $F_a^{(G')} = 1$ for $1/2^{\frac{1}{2}} < x < 1$, $F_a^{(G')} = 0$ for $1/3^{\frac{1}{2}} < x < 1/2^{\frac{1}{2}}$, and $F_a^{(G')} = x$ for $1/4^{\frac{1}{2}} < x < 1/3^{\frac{1}{2}}$.

For $x < 1/3^{\frac{1}{2}}$, $F_a^{(G')}(x)$ is given by [cf. Eq. (36)]:

$$F_a^{(G')} \left[1/m^{\frac{1}{2}} < x < 1/(m-1)^{\frac{1}{2}} \right] = \sum_{j=1}^{m-3} [xj^{\frac{1}{2}}]_a. \quad (41)$$

As an example, for x between $1/5^{\frac{1}{2}}$ and $1/2$, Eq. (41) gives

$$F_a^{(G')}(x) = x^{(2^{\frac{1}{2}}x)}. \quad (41a)$$

We have the same alternation between relatively large values of $F_a^{(G')}(x)$ for $m-3 = \text{even}$ and relatively small values for $m-3 = \text{odd}$, as for the function $F_a^{(G)}(x)$ of Fig. 8. However, the limited convergence of the two types of segments is more rapid than for $F_a^{(G)}(x)$, i.e., the breaks between consecutive segments decrease faster as x is decreased below $x \sim 0.5$. In fact, for $x \lesssim 0.4$, the two types of segments merge into a practically continuous curve, which if continued in the region $x > 0.4$ would pass through the singular values which are marked as isolated points in Fig. 9. In analogy to Eq. (39), these isolated points are given by the following equation:

$$F_a^{(G')}(x = 1/m^{\frac{1}{2}}) = \sum_{j=1}^{m-1} \left[\frac{j^{\frac{1}{2}}}{m^{\frac{1}{2}}} \right]_a \quad (42)$$

We note that because of the nearly complete coalescence of the segments of $F_a^{(G')}(x)$ with $m = \text{odd}$ and $m = \text{even}$, for the region $0.28 < x \lesssim 0.40$, we have calculated only the "isolated points" [which are intermediate between the segments given by Eq. (41)] for the region $0.14 < x < 0.28$. Although the pertinent calculations have not been carried out, we strongly believe that for $x \lesssim 0.1$, the curve of $F_a^{(G')}(x)$ would bend downward toward $F_a^{(G')}(0) = 0$, in the same manner as seems to be the case for $F_a^{(G)}(x)$ in Fig. 8.

In this connection, we note that if $F_a^{(G)}(x)$ behaves like $f(x)$ (see Fig. 5) near $x=0$, we would expect the same to be true for $F_a^{(G')}(x)$, particularly since $f(x)$ corresponds to Eq. (17) with $\beta=0$, and the case $\beta = 1/2$ of Fig. 9 is expected to be intermediate between the cases $\beta=1$ and $\beta=0$.

A reconsideration of the situation for $g_j^{(F)}(x) = xj^{-0.5}$ of Eq. (7) (see also Fig. 4) led us to a better understanding of the double-valuedness of $F_{n,a}(x)$ when $g_j(x)$ depends on j . Thus we note that for $g_j^{(F)}(x)$, we have

$$g_j^{(F)}(x) = x/\sqrt{j} \quad (43)$$

This implies that for $x=1$, $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{4} = 2$, $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, etc., we have one of the $g_j^{(F)}(x) = 1$, namely $j = x^2$. Since we know that at these points, $F_{n,a}^{(F)}(x)$ is single-valued, the regions of double-valuedness become progressively smaller as x is increased, and moreover they are substantially smaller than for $g_j^{(A)}(x) = x/j$ and

$g_j^{(B)}(x) = x/j^2$, for example. Thus for $g_j^{(B)}(x)$, the points at which $F_{n,a}^{(B)}(x)$ is single-valued are at $x=1, 4, 9, 25, 36$, etc. where, in fact, the intervals of double-valuedness increase with increasing x . Of course, the density of points where $F_{n,a}(x)$ is single-valued decreases directly in proportion to the steepness of the decrease of $g_j(x)$ with increasing x . As the regions of double-valuedness are made smaller in going from $g_j^{(A)}$ and $g_j^{(B)}$ to $g_j^{(F)}$, we can also expect the differences between the two functions $F_{n,a}(x)$ for $n = \text{even}$ (and large) and $n = \text{odd}$ (and large) to decrease, as is indeed observed (see Fig. 4). This tendency becomes even more marked if we consider

$$g_j^{(G)}(x) = x/j^{\frac{1}{4}}, \quad (44)$$

in which case the points of single-valuedness are $x=1, 2^{\frac{1}{4}} = 1.189, 3^{\frac{1}{4}} = 1.316, 4^{\frac{1}{4}} = 1.414, \dots, 13^{\frac{1}{4}} = 1.8988, 14^{\frac{1}{4}} = 1.9343, 15^{\frac{1}{4}} = 1.9680, 16^{\frac{1}{4}} = 2.000$. Thus these points are very closely spaced throughout the region $x=1$ to $x=2$ (and even more so from $x=2$ to $x=3$, etc.) Thus we expect the double-valuedness of $F_{n,a}^{(G)}(x)$ corresponding to $g_j^{(G)}(x)$ of Eq. (44) to be minimal, and non-existent for all practical purposes.

We may note that for the choice

$$g_j^{(H)}(x) = xj^{-0.10}, \quad (45)$$

the number of "crossing points" of the two functions $F_{n,a}^{(H)}(x)$ for $n = \text{even}$ and $n = \text{odd}$ is $N_c = 2^{10} - 1 = 1023$ between $x=1$ and $x=2$, and it is:

$$N_c = 3^{10} - 2^{10} = 59,049 - 1,024 = 58,025 \quad (46)$$

between $x=2$ and $x=3$, and similarly

$$N_c = 4^{10} - 3^{10} = 1,048,576 - 59,049 = 989,527 \quad (47)$$

between $x=3$ and $x=4$, and so on.

Thus we can safely extrapolate to the case:

$$g_j^{(I)}(x) = xj^{-\epsilon}, \quad (48)$$

where ϵ is arbitrarily small, to find an infinite number of crossings in any finite interval of x for the choice $g_j(x) = x$ which gives rise to the function $f(x)$ of Ref. 1. This argument provides an alternative proof that $f(x)$ is a single-valued function of x in the region in which $f(x)$ is convergent, i.e., from $x=0$ to $x = e^{\frac{1}{e}} \cong 1.445$, as was already observed in Ref. 1. On the other hand for $g_j(x) = xj^\beta$ with $\beta < 0$ the limiting function $F_{n,a}(x)$ is in principle double-valued, except at a large number N_c of "crossing points", which increases rapidly as $|\beta|$ is made smaller, as has been discussed above. The same double-valuedness is expected and observed for the functions $g_j^{(A)} = x/j$, $g_j^{(B)}(x) = x/j^2$, $g_j^{(D)}(x) = x/2^{j-1}$, and $g_j^{(E)}(x) = x/3^{j-1}$, where the density of crossing points is rather small, as has also been discussed.

The preceding considerations were restricted to the bracketing order denoted by a in Ref. 1, of which an example is given by Eq. (2) of the present paper. We now wish to consider the opposite bracketing order, which has been denoted by μ in Eq. (45) of Ref. 1. An example of the bracketing order μ for the case of four functions $g_j(x)$ is given by:

$$F_{4,\mu}(x) = \prod_{j=1}^4 [g_j(x)]_\mu = \left[\left(g_1(x)^{g_2(x)} \right)^{g_3(x)} \right]^{g_4(x)} \quad (49)$$

It can be easily seen that $F_{4,\mu}(x)$ can be rewritten as follows:

$$F_{4,\mu}(x) = g_1(x)^{g_2(x)g_3(x)g_4(x)}, \quad (50)$$

and for arbitrary n , we find:

$$F_{n,\mu}(x) = g_1(x)^{\prod_{j=2}^n g_j(x)}, \quad (51)$$

where $\prod$ denotes as usual the product. For the special case of $g_j(x) = x$, we thus obtain:

$$f_{n,\mu}(x) = x^{(x^{n-1})}, \quad (52)$$

as can be easily verified from the results of Table I of Ref. 1. As an example $f_{4,e}(5) = f_{4,\mu}(5) = 5^{(5^3)} = 5^{125}$.

We now note that if $g_j(x)$ decreases to zero as $j \rightarrow \infty$, e.g., $g_j^{(A)}(x) = x/j$, the function $F_{n,\mu}^{(A)}(x)$ will at first increase with n , and then attain a maximum value for $g_n^{(A)} = x/n = 1$, i.e., at $x = n$. For still larger values of n , $F_{n,\mu}(x)$ will asymptotically decrease to $[g_1(x)]^0 = 1$.

Thus for the function $g_j^{(A)}(x) = x/j$, we find [cf. Eqs. (62)-(66) of Ref. 1],

$$F_{1,\mu}^{(A)}(1) = 1, \quad (53)$$

$$F_{2,\mu}^{(A)}(2) = 2 = \max_n F_{n,\mu}^{(A)}(2) , \quad (54)$$

$$F_{3,\mu}^{(A)}(3) = 3^{3/2} = 5.196 = \max_n F_{n,\mu}^{(A)}(3) , \quad (55)$$

$$F_{4,\mu}^{(A)}(4) = 4^{\left(\frac{4}{2}\right)\left(\frac{4}{3}\right)} = 4^{\frac{8}{3}} = 40.317 = \max_n F_{n,\mu}^{(A)}(4) , \quad (56)$$

$$F_{5,\mu}^{(A)}(5) = 5^{\left(\frac{5}{2}\right)\left(\frac{5}{3}\right)\left(\frac{5}{4}\right)} = 5^{\frac{125}{24}} = 4,369.9 = \max_n F_{n,\mu}^{(A)}(5) , \quad (57)$$

where the symbol $\max_n$ indicates the maximum of $F_{n,\mu}^{(A)}(x)$ as a function of n .

In a similar fashion, the function $F_{n,\mu}^{(B)}(x)$ pertaining to $g_j^{(B)}(x) = x/j^2$ attains its maximum value for $x = n^2$. These maximum values are as follows [see Eqs. (67)-(69) of Ref. 1]:

$$F_{2,\mu}^{(B)}(4) = 4^{4/4} = 4 = \max_n F_{n,\mu}^{(B)}(4) , \quad (58)$$

$$F_{3,\mu}^{(B)}(9) = 9^{9/4} = 140.30 = \max_n F_{n,\mu}^{(B)}(9) , \quad (59)$$

$$F_{4,\mu}^{(B)}(16) = 16^{(16/4)(16/9)} = 16^{64/9} = 3.6528 \times 10^8 = \max_n F_{n,\mu}^{(B)}(16) , \quad (60)$$

$$\begin{aligned} F_{5,\mu}^{(B)}(25) &= 25^{(25/4)(25/9)(25/16)} = 25^{15625/576} = 8.3474 \times 10^{37} \\ &= \max_n F_{n,\mu}^{(B)}(25) . \end{aligned} \quad (61)$$

It is apparent from the plots of Figs. 1-7, that already beyond $x = 1$, and certainly for $x > 2$, the functions $F_{n,a}(x)$ increase with x much faster than exponentially. This is, of course, a result of the definition of $F_{n,a}(x)$ in Eq. (1), which shows that a large (and sometimes infinite) number of powers $g_j(x)$ is involved. This feature is also shown by the results of Table I of Ref. 1 which exhibit the rapid increase with x of all of the functions $f_{n,\alpha}(x)$, and particularly the functions $f_{3,a}(x)$

and $f_{4,a}(x)$, with the bracketing order of type a which has been considered throughout most of this paper [see Eq. (2)].

Referring to the functions $F_{n,\mu}^{(A)}(x)$ for which the maximum values have been given in Eqs. (53)-(57), it is of interest to calculate these functions for all significant n values, i.e., for all n values for which $F_{n,\mu}^{(A)}(x)$ differs appreciably from the asymptotic value, which is 1.0 for large n . The results of this calculation are given in Table II for $x = 2, 3, 4, 5$, and 6. It is seen that in all cases (i.e., for all x values considered), $F_{n,\mu}^{(A)}(x)$ has a sharp maximum which is attained at the two neighboring n values, namely $n = x-1$ and $n = x$. This property results, of course, from the definition of $F_{n,\mu}^{(A)}(x)$ according to Eq. (50), since we have $g_{n=x}(x) = x/x = 1$, so that the product in the exponent does not change as n is increased from $n = x-1$ to $n = x$. For $x \geq 3$, the increase to the maximum value as n is increased is rapid, and there is also an almost equally rapid decrease as n is made larger than x , until a value of $F_{n,\mu}^{(A)}(x)$ is approached which is not much larger than 1. Even in this region, the values of $F_{n,\mu}^{(A)}(x) - 1$ approach 0 rapidly.

Similar results were obtained for the functions $F_{n,\mu}^{(B)}(x)$ pertaining to $g_j^{(B)}(x) = x/j^2$. These results are listed in Table III for the following x values: $x = 2, 4, 6, 9, 16$, and 25. In this case, when x is a perfect square (i.e., for $x = 4, 9, 16, 25$), the values of $F_{n,\mu}^{(B)}(x)$ attain their maximum values both for $n = x^{\frac{1}{2}} - 1$ and $n = x^{\frac{1}{2}}$. In the same fashion as for $F_{n,\mu}^{(A)}(x)$, the reason is that when the highest $j = x^{\frac{1}{2}}$, we find: $x/j^2 = x/x = 1$, so that the product in the exponent of Eq. (51) is not changed as n is increased from $n = x^{\frac{1}{2}} - 1$ to $n = x^{\frac{1}{2}}$.

For a given value of $x = 6$, the maximum of $F_{n,\mu}^{(B)}(x)$ is consid-

erably smaller than that of $F_{n,\mu}^{(A)}(x)$, due to the more rapid falloff of $g_j^{(B)}(x) = x/j^2$ as compared to $g_j^{(A)}(x) = x/j$. In fact, the maxima of $F_{n,\mu}^{(B)}(x = 16)$ and $F_{n,\mu}^{(A)}(x = 6)$ are comparable ($\sim 3 \times 10^8$). On the other hand, the maximum of $F_{n,\mu}^{(B)}(x = 25)$ is enormously larger than that of $F_{n,\mu}^{(B)}(x = 16)$ (8.347×10^{37} vs 3.653×10^8), illustrating the extremely rapid increase of the functions $F_{n,\mu}(x)$ with increasing x , as discussed above.

Because of the rapid falloff of $g_j^{(B)}(x) = x/j^2$ with increasing j , the decrease of $F_{n,\mu}^{(B)}(x)$ on either side of the maximum, and especially on the side of large n , is extremely rapid. Thus for $x = 25$, in spite of the enormous value at the maximum for $n = 4$ and 5 , the values of $F_{n,\mu}^{(B)}(x)$ are already down to 1.0001 at $n = 15$. For large x , the functions $F_{n,\mu}^{(B)}(x) - 1$ are somewhat reminiscent of the corresponding δ function: $\delta(n - x^{\frac{1}{2}})$.

The corresponding values of $F_{n,\mu}^{(D)}(x)$ pertaining to the function $g_j^{(D)}(x) = x/2^{j-1}$ are listed in Table IV for the x values $x = 2, 4, 6, 8, 12$, and 16 . Because of the rapid fall-off of $g_j^{(D)}(x)$ with increasing j which is similar to the corresponding decrease of $g_j^{(B)}(x)$, the results of Table IV are similar to those of Table III, which have been discussed above. We may note that for the case $x = 16$ (last column), the values of $F_{n,\mu}^{(D)}(x)$ are symmetric about $n = 4$ and $n = 5$, in the range from $n = 1$ to $n = 8$. This property can be easily derived by writing down the explicit expressions for the various values of $F_{n,\mu}^{(D)}(16)$, as obtained from Eq. (51).

We wish to conclude this part of the paper with several general relations concerning the bracketing orders a and μ in connection with the powering operation denoted by Ξ .

Firstly, in connection with the bracketing order μ which has been considered in Eq. (45) of Ref. 1 and Eqs. (49)-(61) of the present paper, we note that if the product

$$\prod_{j=1}^n g_j(x)$$

has a non-zero limit as $n \rightarrow \infty$, to be denoted by $H(x)$, then $F_{n,\mu}(x)$ for $n \rightarrow \infty$ of Eq. (51) can be rewritten as follows:

$$F_{\mu}(x) = g_1(x) \prod_{j=2}^{\infty} g_j(x) = g_1(x)^{H(x)/g_1(x)} \quad (62)$$

Furthermore, if $H(x) \rightarrow \infty$ as $n \rightarrow \infty$, then for $g_1(x) > 0$, $F_{\mu}(x)$ has the following values:

$$\begin{aligned} F_{\mu}(x) &= 0 & \text{if } g_1(x) < 1, \\ F_{\mu}(x) &= 1 & \text{if } g_1(x) = 1, \\ F_{\mu}(x) &= \infty & \text{if } g_1(x) > 1. \end{aligned} \quad (63)$$

Of course, if $H(x) \rightarrow 0$ as $n \rightarrow \infty$, the limit of $F_{n,\mu}(x)$ as $n \rightarrow \infty$ is $[g_1(x)]^0 = 1$, as discussed above (see Tables II-IV).

We can also write down two recursion relations for the bracketing orders a and μ , which follow naturally from the definitions of Eqs. (2) and (45) of Ref. 1 [see also Eqs. (2) and (49) of the present paper]. We find:

(1) for the bracketing order a :

$$\prod_{j=m}^n [g_j(x)]_a = g_m(x) \left\{ \prod_{j=m+1}^n [g_j(x)]_a \right\}, \quad (64)$$

(2) for the bracketing order μ :

$$\prod_{j=1}^m [g_j(x)]_{\mu} = \left\{ \prod_{j=1}^{m-1} [g_j(x)]_{\mu} \right\}^{g_m(x)} . \quad (65)$$

The recursion relations Eqs. (64) and (65) define the two functions $F_{n,a}(x)$ and $F_{n,\mu}(x)$ completely, provided that we also specify:

$$\prod_{j=n}^n [g_j(x)]_a \equiv g_n(x) , \quad (66)$$

$$\prod_{j=1}^1 [g_j(x)]_{\mu} \equiv g_1(x) , \quad (67)$$

or, more generally:

$$\prod_{j=m}^m [g_m(x)]_{\alpha} \equiv g_m(x) , \quad (68)$$

regardless of the bracketing order α ($= a$ or μ).

APPENDIX A
by
R. M. Sternheimer

In connection with the preceding discussion of the powering operation represented by the symbol Ξ , for the bracketing orders denoted by α and μ , it seems of interest to mention some properties of the function x^y , where x and y are positive, and of the related function $E(x,y)$ defined by:

$$E(x,y) \equiv x^y + y^x \quad . \quad (A.1)$$

The present author became interested in the function $E(x,y)$ in 1962, when he noted that the number 100 can be written as:

$$100 = 2^6 + 6^2 = 64 + 36 = E(2,6). \quad (A.2)$$

It therefore became of interest to calculate $E(x,y)$ for other choices of x and y (= integers) in order to see whether other powers 10^n could be represented in this manner. Although the notes for these calculations are no longer available, it is believed that we showed that no power of 10 up to at least 10^7 (except for 10^2) can be expressed as a function $E(x,y)$. A cursory examination of a textbook on the theory of numbers by Hardy and Littlewood showed that the combination $E(x,y)$ does not seem to have been considered before, although we did not make a detailed search of the literature. The major concern in this connection has been with the combination $x^n + y^n$ (Fermat's theorem).

In Eq. (A.1), we note that when $x=1$, the result is trivial. Thus

$$E(1,y) = 1^y + y^1 = y+1 \quad . \quad (A.3)$$

Therefore we will be concerned with cases where x and y are ≥ 2 , and we

will assume that $y \geq x$.

We can also define the related function:

$$E'(x,y) \equiv x^y - y^x \quad (A.4)$$

Of course, for $y = x$, we have $E(x,x) = 2x^x$, $E'(x,x) = 0$. Values of $E(x,y)$ and $E'(x,y)$ are listed in Table V. There does not seem to be an excessive number of entries $[E(x,y) \text{ or } E'(x,y)]$ which are prime numbers. Perhaps the only slightly "unusual" number is $E'(4,6) = 2800$, i.e., a multiple of 100 $= E(2,6)$.

We note the negative value of $E'(2,3)$:

$$E'(2,3) = 2^3 - 3^2 = 8 - 9 = -1 \quad (A.5)$$

We also note that $E'(2,4) = 0$:

$$E'(2,4) = 2^4 - 4^2 = 16 - 16 = 0 \quad (A.6)$$

In all other cases, if $y > x$, then $x^y > y^x$, and hence $E'(x,y) > 0$. In order to determine the conditions for which $x^y > y^x$ if $y > x$, we calculate the two derivatives:

$$\frac{\partial x^y}{\partial x} = yx^{y-1} \quad (A.7)$$

$$\frac{\partial x^y}{\partial y} = x^y \ln x \quad (A.8)$$

so that:

$$\frac{\partial x^y / \partial y}{\partial x^y / \partial x} = \frac{x}{y} \ln x . \quad (A.9)$$

Thus if we have a variation Δx of x and Δy of y , the total change of x^y is:

$$\Delta(x^y) = \frac{\partial x^y}{\partial x} \left[\Delta x + \frac{x}{y} \ln x \Delta y \right] . \quad (A.10)$$

If $x \approx y$, and we consider variations such that $\Delta x = -\Delta y$, we obtain:

$$E'(x,y) = x^y - y^x \approx x^x (y-x) (\ln x - 1) . \quad (A.11)$$

Equation (A.11) shows that for $\ln x > 1$, i.e., $x > e$, $E'(x,y)$ will be positive if $(y-x) > 0$. Conversely for $x < e$, $E'(x,y)$ will be negative if y exceeds x . If $x < e$, $y > e$, then the preceding argument based on derivatives will not be decisive, especially if the difference $y-x$ is rather large. This is essentially the reason why $E'(2,3)$ is negative, and $E'(2,4) = 0$. Table V shows that even for $x=2$, if y is sufficiently large, i.e., $y > 4$, then $E'(x,y) > 0$. And, of course, for $x = 3, 4, 5$, etc., we always have $E'(x,y) > 0$ for $y > x$.

The formula of Eq. (A.11) can be easily tested both for $x, y < e$ and for $x, y > e$. As an example: $2.5^{2.6} = 10.83$ is somewhat smaller than $2.6^{2.5} = 10.92$, and similarly $0.5^{0.6} = 0.6598$ is appreciably smaller than $0.6^{0.5} = 0.7745$. On the other hand, for $x, y > e$, we have, for example: $3.5^{3.6} = 90.95$ which is larger than $3.6^{3.5} = 88.54$. For comparison with Eq. (A.11) for the value

$$E'(0.5, 0.6) = 0.5^{0.6} - 0.6^{0.5} = 0.6598 - 0.7745 = -0.1147 , \quad (A.12)$$

we note that taking the average $\frac{1}{2}(x+y) = 0.55$, we find $0.55^{0.55} = 0.7198$,
and $\ln 0.55 = -0.5979$, giving:

$$E'(0.5, 0.6) = (0.7198)(0.1)(-1.5979) = -0.1150, \quad (\text{A.13})$$

in good agreement with the result -0.1147 of Eq. (A.12).

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Table I. Values of $F_{n,a}^{(A)}(x) = \sum_{j=1}^n [x/j]_a$ for $n = 1, 2, \dots, 12$, and limiting values for $n = \text{even}$ ($n=24$) and $n = \text{odd}$ ($n=25$), for $x = 0.1, 0.25$, and 0.50 .

n	$F_{n,a}^{(A)}(0.1)$	$F_{n,a}^{(A)}(0.25)$	$F_{n,a}^{(A)}(0.50)$
1	0.1000	0.2500	0.5000
2	0.8913	0.8409	0.8409
3	0.1245	0.3117	0.5769
4	0.8633	0.7916	0.7954
5	0.1317	0.3357	0.6056
6	0.8554	0.7717	0.7758
7	0.1337	0.3460	0.6192
8	0.8533	0.7633	0.7660
9	0.1342	0.3503	0.6263
10	0.8528	0.7598	0.7607
11	0.1343	0.3521	0.6301
12	0.8527	0.7584	0.7579
24	0.8527	0.7577	0.7549
25	0.1343	0.3531	0.6343

Table II. Values of $F_{n,\mu}^{(A)}(x)$ corresponding to the function $g_j^{(A)}(x) = x/j$ and the bracketing order μ given by Eq. (49). The functions $F_{n,\mu}^{(A)}(x)$ are listed for $x = 2, 3, 4, 5$, and 6 , and with the range of n values extending up to n_1 , where $F_{n_1,\mu}^{(A)}(x) < 1.0001$, i.e., very close to the asymptotic value of 1.0 .

n	$x = 2$	$x = 3$	$x = 4$	$x = 5$	$x = 6$
1	2	3	4	5	6
2	2	5.1962	16	55.902	216
3	1.5874	5.1962	40.317	817.29	46,656
4	1.2599	3.4416	40.317	4369.87	10,077,696
5	1.0968	2.0992	19.248	4369.87	253,532,421
6	1.0313	1.4489	7.1823	1080.75	253,532,421
7	1.0088	1.1722	3.0853	146.87	15,975,604
8	1.0022	1.0614	1.7565	22.612	252,693
9	1.0005	1.0201	1.2845	5.6547	3,996.95
10	1.0001	1.0060	1.1053	2.3780	144.89
11		1.0016	1.0371	1.4825	15.092
12		1.0004	1.0122	1.1783	3.8848
13		1.0001	1.0037	1.0651	1.8708
14			1.0011	1.0228	1.3079
15			1.0003	1.0075	1.1134
16			1.0001	1.0024	1.0411
17				1.0007	1.0143
18				1.0002	1.0047
19				1.0001	1.0015
20					1.0004
21					1.0001

Table III. Values of $F_{n,\mu}^{(B)}(x)$ corresponding to the function $g_j^{(B)}(x) = x/j^2$ and the bracketing μ given by Eq. (49). The functions $F_{n,\mu}^{(B)}(x)$ are listed for $x = 2, 4, 6, 9, 16$, and 25 , and with the range of n values extending up to n_1 , where $F_{n_1,\mu}^{(B)}(x) < 1.0001$, i.e., very close to the asymptotic value of 1.0 .

n	$x = 2$	$x = 4$	$x = 6$	$x = 9$	$x = 16$	$x = 25$
1	2	4	6	9	16	25
2	1.4142	4	14.697	140.30	65,536	5.459×10^8
3	1.0801	1.8517	6	140.30	365,284,284	1.861×10^{24}
4	1.0097	1.1665	1.9580	16.133	365,284,284	8.347×10^{37}
5	1.0008	1.0250	1.1750	2.7213	302,053	8.347×10^{37}
6	1.0000	1.0027	1.0272	1.2844	272,64	2.160×10^{26}
7		1.0002	1.0033	1.0470	6.2416	2.729×10^{13}
8		1.0000	1.0003	1.0065	1.5806	177,177
9			1.0000	1.0007	1.0946	41.675
10				1.0001	1.0146	2.5408
11				1.0000	1.0019	1.2125
12					1.0002	1.0340
13					1.0000	1.0050
14						1.0006
15						1.0001

Table IV. Values of $F_{n,\mu}^{(D)}(x)$ corresponding to the function $g_j^{(D)}(x) = x/2^{j-1}$ and the bracketing order μ given by Eq. (49). The functions $F_{n,\mu}^{(D)}(x)$ are listed for $x = 2, 4, 6, 8, 12$, and 16 , and with the range of n extending up to n_1 , where $F_{n_1,\mu}^{(D)}(x) < 1.0001$, i.e., very close to the asymptotic value of 1.0 .

n	x = 2	x = 4	x = 6	x = 8	x = 12	x = 16
1	2	4	6	8	12	16
2	2	16	216	4096	2,985,984	4.295×10^9
3	1.4142	16	3174.5	16,777,216	2.662×10^{19}	3.403×10^{38}
4	1.0905	4	422.92	16,777,216	1.374×10^{29}	1.158×10^{77}
5	1.0109	1.4142	9.6571	4096	7.135×10^{21}	1.158×10^{77}
6	1.0007	1.0443	1.5299	8	1.567×10^8	3.403×10^{38}
7	1.0000	1.0027	1.0407	1.2968	34.401	4.295×10^9
8		1.0001	1.0019	1.0164	1.3933	16
9		1.0000	1.0000	1.0005	1.0157	1.1892
10				1.0000	1.0004	1.0054
11					1.0000	1.0001

Table V. Values of $E(x,y) \equiv x^y + y^x$ and $E'(x,y) \equiv x^y - y^x$ for $x,y =$ integers ≥ 2 (see the discussion in Appendix A).

(x,y)	$E(x,y)$	$E'(x,y)$
(2,2)	8	0
(2,3)	17	-1
(2,4)	32	0
(2,5)	57	7
(2,6)	100	28
(2,7)	177	79
(2,8)	320	192
(3,3)	54	0
(3,4)	145	17
(3,5)	368	118
(3,6)	945	513
(3,7)	2530	1844
(4,4)	512	0
(4,5)	1649	399
(4,6)	5392	2800
(4,7)	18,785	13,983
(5,5)	6250	0
(5,6)	23,401	7849
(6,6)	93,312	0

REFERENCES

1. R.M. Sternheimer, "On a Set of Non-Associative Functions of a Single Positive Real Variable", BNL Physics Department Informal Report PD-128 (June 1977) (BNL 23081).
2. A closer examination shows that for $x = 1.5$, the function $F_{n,a}^{(A)}(x) = \sum_{j=1}^n \left[\left(\frac{x}{j} \right) \right]_a$ oscillates between the value $F_{n,a}^{(A)}(1.5) = 1.39831$ for $n = \text{odd}$ and $F_{n,a}^{(A)}(1.5) = 1.39712$ for $n = \text{even}$. The difference $\Delta F = 0.00119$ is less than 0.1% of $F_{n,a}^{(A)}(1.5)$. For $x = 1.25$, the two values are $F_{n=\text{odd},a}^{(A)}(1.25) = 1.18292$ and $F_{n=\text{even},a}^{(A)}(1.25) = 1.18053$, giving a difference $\Delta F = 0.00239$ or $\sim 0.2\%$ of $F_{n,a}^{(A)}(1.25)$.
3. Hartland Snyder, private communication to R.M.S. (1960).

FIGURE CAPTIONS

Figs. 1, 1a and 1b: The functions $F_{n,a}^{(A)}$ pertaining to $g_j^{(A)}(x) = x/j$ for $n = \text{even}$ and $n = \text{odd}$. As in all subsequent cases, the two functions $F_{n,a}^{(A)}$ approach the values 1 and 0 at $x = 0$ for $n = \text{even}$ and $n = \text{odd}$, respectively. In Fig. 1b, the region near $x = 1$ has been enlarged in order to exhibit the slight thickening of the Hewlett-Packard trace for $x > 1$, corresponding to the limited double-valuedness of the function $F_{n,a}^{(A)}$ between $x = 1$ and $x = 2$, as described in footnote 2.

Figs. 2 and 2a. The functions $F_{n,a}^{(B)}(x)$ pertaining to $g_j^{(B)}(x) = x/j^2$ for $n = \text{even}$ and $n = \text{odd}$. Both figures show the marked double-valuedness of the functions $F_{n,a}^{(B)}(x)$, and Fig. 2a shows that the function $F_{n,a}^{(B)}(x)$ becomes single-valued again at $x = 4 (= j^2$ for $j = 2)$.

Figs. 3 and 3a. The functions $F_{n,a}^{(D)}(x)$ pertaining to $g_j^{(D)}(x) = x/2^{j-1}$ for $n = \text{even}$ and $n = \text{odd}$. The two figures taken together show that the functions are double-valued for $0 < x < 1$, $1 < x < 2$, and $2 < x < 4$, and are single valued at $x = 1 (= 2^0)$, $x = 2 (= 2^1)$ and $x = 4 (= 2^2)$.

Fig. 4 The functions $F_{n,a}^{(F)}(x)$ pertaining to $g_j^{(F)}(x) = xj^{-0.5}$ for $n = \text{even}$ and $n = \text{odd}$. These functions were evaluated at rather large intervals, $\Delta x = 0.1$. Although the two curves appear to coalesce for $x > 1$, and actually already for $x > 0.5$, the functions are believed to be double-valued for all x , except the points $x = \sqrt{j}$ where j is an integer, as discussed in connection with Eq. (43) of the text.

Fig. 5 The functions $f_{n,a}(x)$ as defined by Eq. (11) of Ref. 1 for $n = \text{even}$ ($n = 2, 4, 6$) and $n = \text{odd}$ ($n = 1, 3, 5$), and also the limiting function $f(x)$ for $n \rightarrow \infty$ [see Eqs. (1) and (2) of Ref. 1]. For this special case where all of the $g_j(x)$ are equal to x , the limit $f(x)$ is single-valued, i.e., it does not depend on whether $n = \text{even}$ or $n = \text{odd}$ as $n \rightarrow \infty$. This property has been checked by explicitly calculating $f_n(0.1)$ for n values up to $n = 200$; we find

$$f_n(0.1) = 0.399,012,98.$$

Fig. 6 The functions $F_{n,a}^{(E)}(x)$ pertaining to $g_j^{(E)}(x) = x/3^{j-1}$ for $n = \text{even}$ and $n = \text{odd}$. Except for $x = 1 (= 3^0)$ and $x = 3 (= 3^1)$, the functions are double-valued, and the difference between $F_{10,a}^{(E)}(x)$ and $F_{11,a}^{(E)}(x)$ is appreciable for $1 < x < 3$, corresponding to the small density of "crossing points" ($x = 1, 3, 9, 27, \dots$), as discussed in the text following Eq. (48).

Fig. 7 The functions $F_{n,a}^{(C)}(x)$ pertaining to $g_j^{(C)}(x) = x^j$ for $n = \text{even}$ and $n = \text{odd}$. The two curves appear to coalesce for $x > 0.8$. The portion of the curve beyond $x = 1.0$ is fictitious, i.e., it is the result of using finite n values ($n = 10$ and $n = 11$). For $n \rightarrow \infty$, it can be easily seen from the expression for $F_{n \rightarrow \infty, a}^{(C)}(x)$ [see Eq. (61) of Ref. 1] that this function becomes infinite as x is made larger than 1.0. Thus the correct continuation of $F_{n,a}^{(C)}(x)$ at $x = 1.0$ is the vertical line shown in Fig. 7.

Fig. 8 The function $F_a^{(G)}(x)$ pertaining to $g_j^{(G)}(x) \equiv x^j$. For $x < 1/3$, the various segments of this discontinuous function are given by Eq. (36). In addition to these segments, there exist the isolated points (singularities) at the positions of the discontinuities $x = 1/m$ ($m = \text{integer}$). The values of $F_a^{(G)}$ at these singularities

are given by Eq. (39) of the text.

Fig. 9 The function $F_a^{(G')}(x)$ pertaining to $g_j^{(G')}(x) \equiv xj^{\frac{1}{2}}$. For $x < 1/3^{\frac{1}{2}}$, the various segments of this discontinuous function are given by Eq. (41). The values of $F_a^{(G')}$ at the singularities (isolated points) are given by Eq. (42) of the text.

FIG. 1.

FIG. 1a.

FIG. 18.

FIG. 2.

$$F(x) = F_a^{(B)}(x)$$

FIG. 2a.

$$F(x) = F_a^{(D)}(x)$$

↑ 16

FIG. 3.

$$F(x) = F_a^{(n)}(x)$$

FIG. 3a.

FIG. 4

$f_n(x) = \frac{[x \cdot x \cdots x]_e}{n(x's.)}$
 $f_n(x) = y$
 $\uparrow 3 \uparrow$

FIG. 6.

$F_a^{(c)}(x)$

↑

FIG. 7

$\sum_{j=1}^{10} [x^j]_a \rightarrow$

$\sum_{j=1}^{11} [x^j]_a \leftarrow$

0

→ x

$$F_a = \lim_{j \rightarrow \infty} (x_j)_a$$

↑

FIG. 9.

$$F_c \quad f(x) = \sum_{j=1}^{\infty} [x f^j]_a$$