Hi. 1282

LA-6882-MS
Informal Report

UC-32

Issued: July 1977

Maximum Entropy Eddington Factors

Gerald N. Minerbo

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87545

UNITED STATES
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 36

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161
Price: Printed Copy $3.50 Microfiche $3.00

MAXIMUM ENTROPY EDDINGTON FACTORS

by

Gerald N. Minerbo

ABSTRACT

A technique from statistical mechanics is applied t the problem of determining the most probable value of the Eddington tensor given the zeroth and first moment of the intensity. The result is applicable to two- and three-dimensional configurations and is intended for use in large radiation hydrodynamics calculations.

I. INTRODUCTION

As a technique for solving the equations of radiative transfer, the variable Eddington approximation[1,2] is more accurate than the diffusion approximation and much faster than transport calculations. It has received considerable attention [3-5] and is being used in an increasing number of applications.[6-11] The transfer equation for the intensity $I_\nu(\vec{\Omega})$ in a material in local thermodynamic equilibrium is[12]

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \vec{\Omega}\cdot\nabla\right] I_\nu(\vec{\Omega}) = \kappa_a' \left[B_\nu - I_\nu(\vec{\Omega})\right] + S_\nu(I) \quad , \qquad (1.1)$$

where B_ν is the Planck function, κ_a' is the absorption opacity (including the induced emission factor), and S_ν represents the scattering terms. By taking moments of this equation one obtains[12,13]

$$\begin{aligned} &\frac{\partial E}{\partial t} + \nabla\cdot\vec{F} = \kappa_a' \left[4\pi B - cE\right] \\ &\frac{1}{c}\frac{\partial}{\partial t}\vec{F} + c\nabla\cdot\vec{\vec{P}} = -\kappa_a' \vec{F} + \vec{S}_1 \quad , \end{aligned} \qquad (1.2)$$

where the energy density E, flux $\vec{F}$, and pressure tensor $\overleftrightarrow{P}$ are defined as

$$E = \frac{1}{c}\int d^2\Omega\ I(\vec{\Omega}) \quad , \tag{1.3}$$

$$\vec{F} = \int d^2\ \Omega\ \vec{\Omega}\ I(\vec{\Omega}) \quad , \tag{1.4}$$

$$\overleftrightarrow{P} = \frac{1}{c}\int d^2\ \Omega\ \vec{\Omega}\ \vec{\Omega}\ I(\Omega) \quad , \tag{1.5}$$

and $\vec{S}_1$ is the first moment of the scattering term S (the zeroth moment vanishes under fairly general conditions). The Eddington factor is defined as

$$\overleftrightarrow{f} = \overleftrightarrow{P}/E. \tag{1.6}$$

An equation similar in complexity to the diffusion equation is obtained if $\overleftrightarrow{f}$ can be expressed in terms of $\vec{F}$ and E. The usual approach to the variable Eddington factor approximation is to postulate a simple model for the $\vec{\Omega}$ dependence of the intensity. In order to obtain closure with this approach it is also necessary to approximate the scattering term $\vec{S}_1$ so that it couples only to the three moments of the intensity in Eqs. (1.3)-(1.5) (see Ref. 2). A variety of models[1-4] have been proposed for one-dimensional systems (slab or spherical geometry). No satisfactory treatment has been found for two- or three-dimensional geometries.

Here we consider a probabilistic formulation of the problem: given a monochromatic ensemble of photons where the zeroth and first moments of $I(\vec{\Omega})$ are specified by Eqs. (1.3), (1.4), what is the most probable form of the distribution $I(\vec{\Omega})$? Problems of this type occur in statistical mechanics[14,15] and communication theory.[16,17] The procedure used is to maximize the entropy functional, which is proportional to the logarithm of the probability W of the distribution

$$H = k\ \ln W. \tag{1.7}$$

Here k is the Boltzmann constant. Since photons obey Bose-Einstein statistics, the appropriate form for H is[14]

$$H = k\int d^2\Omega \left\{ [1+n(\vec{\Omega})]\ln[1+\frac{1}{n(\vec{\Omega})}] + \ln\ n(\vec{\Omega})\right\} , \tag{1.8}$$

where the occupation number density n is related to the intensity by

$$I(\vec{\Omega}) = \frac{2h\nu^3}{c^2}\, n(\vec{\Omega}).$$

Since we are considering a monochromatic ensemble of photons, the frequency ν will be a constant in the present problem. In communication theory,[16] it is shown that the information content is the negative of the entropy of the distribution. Thus, by using the maximum entropy criterion, one avoids introducing information that is not available. This approach is conceptually superior to the use of an ad hoc model for the intensity. Also it is easily generalized to two- or three-dimensional geometries.

II. MAXIMUM ENTROPY SOLUTION

To describe $\vec{\Omega}$ we use polar coordinates θ, ϕ ,

$$\vec{\Omega} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \quad , \tag{2.1}$$

with the polar axis along the direction of $\vec{F}$, so that

$$\vec{F}\cdot\vec{\Omega} = F\cos\theta, \quad \text{where } F = |\vec{F}|. \tag{2.2}$$

The constraint equations (1.3), (1.4) can thus be expressed as

$$E = \frac{2h\nu^3}{c^3}\int_{-1}^{1} du \int_0^{2\pi} d\phi\ n(u,\phi) \tag{2.3}$$

and

$$F = \frac{2h\nu^3}{c^2}\int_{-1}^{1} du \int_0^{2\pi} d\phi\ u\, n(u,\phi) \tag{2.4}$$

where $u = \cos\theta$. Following a standard procedure from the calculus of variations we introduce Lagrange multipliers η, λ for Eqs. (2.3), (2.4), respectively, and extremize the expression

$$H' = \int du \int d\phi \left\{ [1+n(u,\phi)]\ell n[1+1/n(u,\phi)] + \ell n\ n(u,\phi)\right\} + \eta\int du\int d\phi\ n(u,\phi) + \lambda\int du\int d\phi\ \ u\, n(u,\phi). \tag{2.5}$$

Variation with respect to n yields

$$\ell n\ [1+1/n(u,\phi)] + \eta + \lambda u = 0 \tag{2.6}$$

or

$$n(u,\phi) = \frac{1}{e^{-\eta-\lambda u} - 1} \quad . \tag{2.7}$$

The two constants η and λ must be determined by imposing conditions (2.3), (2.4).

The fact that n is independent of ϕ simplifies the expressions for the moments. If we define

$$m_\ell = 2\pi \int_{-1}^{1} du \; u^\ell \; n(u,\phi), \quad \ell = 0, 1, \ldots \quad , \tag{2.8}$$

then Eqs. (1.3) - (1.5) become

$$\begin{aligned} E &= \frac{2h\nu^3}{c^3} \; m_o \quad , \\ \vec{F} &= \frac{2h\nu^3}{c^2} \; m_1 (\vec{F}/F) \quad , \\ \vec{\vec{P}} &= \frac{h\nu^3}{c^3} \left[(m_o - m_2)\vec{\vec{1}} + (3m_2 - m_o)(\vec{F}\vec{F}/F^2) \right] \quad , \end{aligned} \tag{2.9}$$

and the Eddington factor

$$\vec{\vec{f}} = \frac{1}{2} \left(1 - \frac{m_2}{m_o}\right) \vec{\vec{1}} + \frac{1}{2} \left(3 \frac{m_2}{m_o} - 1\right)(\vec{F}\vec{F}/F^2). \tag{2.10}$$

We will consider explicitly only the case where the occupation numbers are small

$$n << 1 \tag{2.11}$$

or equivalently

$$E << 4\pi \; h\nu^3/c^3. \tag{2.12}$$

This case is the interesting one physically; in radiation hydrodynamics problems, when Eq. (2.11) is violated, the intensity is usually close to isotropic and the Eddington approximation holds. With this assumption Eq. (2.7) becomes

$$n(u,\phi) = Ce^{\lambda u} \quad , \qquad \text{where } C = e^{\eta} \,. \tag{2.13}$$

This form could also have been obtained directly by assuming Boltzmann statistics for the photons. With this form for n it is easy to evaluate the quantities

$$m_o = (4\pi C/\lambda)\sinh\lambda \quad ,$$

$$m_1 = (4\pi C/\lambda^2)(\lambda\cosh\lambda - \sinh\lambda) \quad , \tag{2.14}$$

and

$$m_2 = m_o - (2/\lambda)m_1 .$$

We will use the abbreviations

$$R_1 = m_1/m_o = F/cE$$

and

$$R_2 = m_2/m_o = (\vec{F}\cdot\vec{\vec{f}}\cdot\vec{F})/F^2 . \tag{2.15}$$

From Eq. (2.14) one obtains

$$R_1 = \coth\lambda - 1/\lambda \tag{2.16}$$

and

$$R_2 = 1 - 2R_1/\lambda \,. \tag{2.17}$$

To obtain R_2 from R_1, the transcendental equation (2.16) must be solved for λ and R_2 then computed from (2.17). Only nonnegative values of λ are of interest since $R_1 \geq 0$ from Eq. (2.15). The $\lambda = 0$ limit corresponds to an isotropic intensity

$$R_1 \underset{\lambda\to 0}{\sim} \frac{1}{3}\lambda - \frac{1}{45}\lambda^3$$

$$R_2 \underset{\lambda\to 0}{\sim} \frac{1}{3} + \frac{2}{5}R_1^2 \quad . \tag{2.18}$$

Streaming is obtained in the large λ limit,

$$R_1 \underset{\lambda\to\infty}{\sim} 1 - 1/\lambda$$
$$R_2 \underset{\lambda\to\infty}{\sim} 1 - 2R_1 + 2R_1^2 \, . \tag{2.19}$$

A plot of R_2 vs R_1 is shown in Fig. 1 (solid curve). The following rational approximation to this function was obtained by a computer fit to data generated from Eqs. (2.16), (2.17).

$$R_2 \simeq \frac{1}{3} + \frac{0.01932R_1 + 0.2694R_1^2}{1 - 0.5953R_1 + 0.02625R_1^2} \, . \tag{2.20}$$

The absolute error produced by this approximation is less than 0.004 on the interval $0 \leq R_1 \leq 1$.

III. LINEAR APPROXIMATION

In this section we consider a linear approximation to the exponential solution in Eq. (2.13),

$$n(u) = Ce^{\lambda u} \simeq C(1+\lambda u) \, . \tag{3.1}$$

Since n is nonnegative, we use the form

$$n(u) = \max\,\{0,\ A(u-b)\} \, , \tag{3.2}$$

where A and b are constants. The first three moments in Eq. (2.8) are easily computed

$$m_0 = \begin{cases} \pi A(1-b)^2, & -1 \leq b \leq 1 \\ -4\pi Ab \quad , & b < -1 \end{cases}$$

$$m_1 = \begin{cases} \frac{\pi}{3} A(1-b)^2(2+b) \, , & -1 \leq b \leq 1 \\ \frac{4\pi}{3} A \quad , & b < -1 \end{cases} \tag{3.3}$$

$$m_2 = \begin{cases} \frac{\pi}{6} A(1-b)^2(3+2b+b^2), & -1 \leq b \leq 1 \\ -\frac{4\pi}{3} Ab, & b < -1 \, . \end{cases}$$

The relation between R_2 and R_1 is found to be

$$R_2 = \frac{1}{3} , \qquad 0 \leq R_1 \leq \frac{1}{3}$$
$$R_2 = \frac{1}{2} - R_1 + \frac{3}{2} R_1^2 , \qquad \frac{1}{3} \leq R_1 \leq 1 . \tag{3.4}$$

This function is shown as the dashed curve in Fig. 1. Compared to the rational approximation in Eq. (2.20), Eq. (3.4) is a rather crude approximation, but this approach is attractive because of its simplicity and should be adequate in many applications. The use of the linear approximation (3.2) also simplifies the calculation of the scattering term $\vec{S}_1$ in Eq. (1.2).

IV. CONCLUSIONS

It may be objected that the prescription in Sec. II has limited applicability since an intensity with the angular dependence in Eq. (2.7) or Eq. (2.13) is not often encountered in radiation transfer problems. It is important to interpret Eq. (2.13) not as a model for a specific system but as representative of an ensemble of systems. Relative to an ensemble of systems, one can state that the prescription in Sec. II for computing $\vec{f}$ will be correct more often than any other prescription which uses the local values of $\vec{F}$ and E as the only input information.

ACKNOWLEDGMENTS

The author wishes to thank George R. Spillman who provided valuable insight on this subject. James G. Sanderson computed the rational approximation in Eq. (2.20).

REFERENCES

1. G. C. Pomraning, An extension of the Eddington approximation, J.Q.S.R.T. 9, 407-422 (1969).
2. The Radiation Group, The VERA system of radiation-hydrodynamics codes, Systems Science and Software, La Jolla, CA, DASA-2258, Vols. I and II (1969).
3. L. H. Auer and D. Mihalas, On the use of variable Eddington factors in non-LTE stellar atmospheres computations, M.N.R.A.S. 149, 65-74 (1970).
4. L. H. Auer, The stellar atmospheres problem, J.Q.S.R.T. 11, 573-587 (1971).
5. C. M. Leung, Numerical solution of the radiative transfer equation in spherically symmetric dust shells, J.Q.S.R.T. 16, 559-574 (1976).
6. J. P. Apruzese, Radiative transfer in spherical circumstellar dust envelopes, Ap. J. 196, 753-760 (1975).

7. L. H. Auer and J. N. Heasly, An alternative formulation of the complete linearization method for the solution of non-LTE transfer problems, Ap. J. 205, 165-171 (1976).

8. C. G. Davis, Radiative transfer effects in Cepheid atmospheres, J.Q.S.R.T. 11, 647-653 (1971).

9. C. M. Leung, Radiation transport in dense interstellar dust clouds, I. Ap. J. 199, 340-360 (1975), II. Ap. J. 209, 75-93 (1976).

10. D. Mihalas, P. B. Kunasz and D. G. Hummer, Solution of the comoving-frame equation of transfer in spherically symmetric flows, Ap. J. 210, 419-433 (1976).

11. R. E. Tamm and R. D. Schwartz, Radiative transfer in circumstellar dust clouds, Ap. J. 204, 842-852 (1976).

12. G. C. Pomraning, The Equations of Radiation Hydrodynamics, Pergamon, New York, 1973, Chaps. II, III.

13. S. -H. Hsieh and E. A. Spiegel, The equations of photohydrodynamics, Ap. J. 207, 244-252 (1976).

14. D. ter Haar, Elements of Statistical Mechanics, Rinehart, New York, 1965, Sec. 4.3.

15. R. K. Pathria, Statistical Mechanics, Pergamon, New York, 1972, Chap. 6.

16. S. Goldman, Information Theory, Prentice-Hall, New York, 1953, Chap. 4.

17. E. T. Jaynes, Prior probabilities, Trans. IEEE, SSC4, 227-241 (1968).

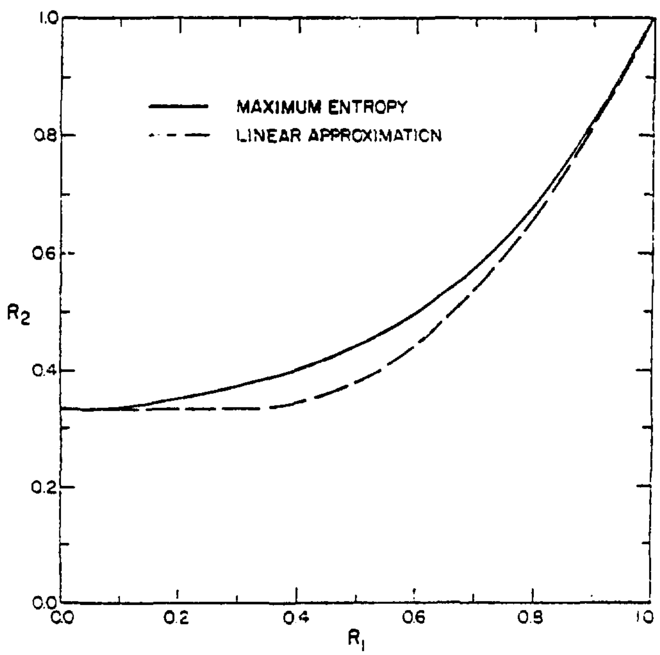

Fig. 1.
Functional relation between R_2 and R_1 for the maximum entropy solution of Sec. II (solid curve) and the linear approximation of Sec. III (dashed curve).