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SOME HIGHLIGHTS OF THEORETICAL & COMPUTATIONAL PLASMA PHYSICS

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Some Highlights of Theoretical & Computational Plasma Physics*

Brendan McNamara

Lectures given to the Summer College on Physics & Contemporary Needs,
Nathiagali, Pakistan, 1977.

Introduction

These lectures are a personal selection of topics from plasma theory and computation. The objectives of the selection are to provide some insight into the basic models, to give some impression of the difficult nature of the equations, to show some of the present understanding of the nonlinear consequences of the equations, and to illustrate the remarks with computational studies. I have chosen to avoid giving the most trivial examples of what the models imply, since this is well documented in basic plasma texts, and instead try to jump quickly to the modern and very sophisticated 2 and 3 dimensional calculations. The physical situations envisaged are always closely related to the Tokamaks or Mirror Machines described in the first pair of lectures. The aim of this plasma theory is to describe equilibria, stability of the equilibria, and the transport of energy and particles out of a plasma. These three topics separate naturally according to the time and space scales of the phenomena involved and the plasma models reflect this.

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The Basic Magnetohydrodynamic Models

The gross characteristics of plasma equilibrium and dynamics on the fast timescales of thermal motion in a Tokamak are describable by the simple set of equations in Table I. The Tokamak equilibrium has zero fluid velocity on the fast timescales. I have only included the plasma resistivity as a non-ideal coefficient because it can lead to topological changes in the magnetic geometry by cutting and rejoining of field lines.

The primary effect of the magnetic field on plasma dynamics is to make the plasma flow anisotropic, it being much easier for plasma to flow along the field than across. In a mirror machine the pressure itself is anisotropic since a typical ion is only containable for one ion-ion scattering time before it escapes from the machine through the ends.

The equilibrium problem for a Tokamak or a mirror reduces to the equations of Table II. In the Tokamak case the axisymmetry can be used to reduce the equations to a single second order elliptic equations for the poloidal flux. The pressure is a function only of ψ . The equilibrium equation could also be derived by minimizing the total energy of the plasma-field system subject to the constraint that the poloidal and toroidal fluxes are also functions of ψ above. As described in the first lectures the ratio of these fluxes must be kept below the level at which the kink instability can arise.

The equation has two arbitrary functions which can be chosen (by a theorist!) to maximize the amount of plasma which can be contained in a Tokamak. Typical computational results are shown in Fig. 1 for $p(\psi)$ and $F(\psi)$ chosen as pieces of $\cos(\psi_{wall}-\psi)$, for increasing plasma pressure. The boundary condition was that $\psi=\psi_{wall}$ on the perfectly conducting wall

of the torus. There are a number of difficulties in reducing the flux equation, one of which is the possibility of bifurcation and several solutions being possible for the same parameters. The numerical scheme shown in Table II allows one to control the solution process. Typical Doublet equilibria are shown in Fig. 2, with doublet and droplet solutions. The elongations of the cross section of the Tokamak give a longer path in the poloidal direction for field lines. The plasma current and hence maximum plasma pressure allowed by the kink stability condition are raised to $\beta < (R/a) (L_{pol}/L_{tor})^2/q^2$, $q = L_{pol} B_{tor}/L_{tor} B_{pol}$. Note also that the plasma boundary in these calculations is set by the flux surface which touches the limiter.

A recent simple observation is that the plasma pressure can be raised dramatically whilst not violating the kink conditions if one starts from a stable equilibrium and holds the poloidal and toroidal fluxes constant as a constraint. This can be done in an experiment by powerful neutral injection into a stable discharge, raising the plasma pressure faster than the fluxes can diffuse. Such a flux conserving tokamak plasma is eventually unstable to more complicated MHD instabilities resulting from the combination of steep pressure gradients and poor field line curvature on the outside of the toroidal plasma. We will consider this topic later.

I have tried to show that even this very simple model of an axisymmetric equilibrium is very complicated. I would also emphasize the subject of the relative timescales on which the model is valid. Individual particle motions in such magnetic fields are highly convoluted and a large class of particle orbits are trapped on the large R side of the plasma by the mirror effect in the toroidal field, $B_p \sim 1/R$. Small helical fluctuations

in the plasma can resonate with the twist of the field lines and radically alter the small scale (across a few Larmor radii) character of the magnetic topology. The model is therefore valid only over the gross scale of the plasma and for times longer than a collision time but much shorter than the transport times, and it is not possible to relate a microscopic ion distribution function to the fluid model. Within these limits the model is good enough that General Atomic have coupled their plasma diagnostics directly to fast computations of the equilibria to analyze the nature of each shot in the Doublet II experiment!!

Mirror Equilibria

In describing a mirror confined plasma I want to bring out two main points which contrast strongly with toroidal equilibria:

- 1) The equilibrium can be described on all timescales and the complete connection between the microscopic distribution function and the pressure profiles can be made.
- 2) Minimum-B magnetic wells can confine plasma at arbitrarily high β , essentially by using the plasma diamagnetism to dig the well deeper.

The basis of mirror confinement is that individual particle (ion) motions in the magnetized plasma conserve three constants of motion: the energy, $\epsilon = \frac{1}{2} m v^2$, the magnetic moment, $\mu = \frac{1}{2} m v_{\perp}^2 / B$, and the longitudinal invariant or action integral along a field line between bounce points, $J = \oint v_{\parallel} dl$. The mirror effect can be seen by rewriting the energy as

$$\epsilon = \frac{1}{2} m v_{\parallel}^2 + \mu B$$

so that $v_{\parallel} = 0$ when $\mu B(L) = \epsilon$. Particles with small μ cannot satisfy this condition before the particle reaches the peak field, $B(L)$, and escape.

The ion distribution function in equilibrium is a function of the constants of motion, $f = f(\epsilon, \mu, J)$, and has no particles in the loss cone where

$\mu B < \epsilon$. The pressure, $\underline{P} = \int v: v f d^3V$ is therefore anisotropic,

$$\underline{P} = p_{\perp} \hat{b} + (p_{\perp} - p_{\parallel}) \underline{B} \cdot \underline{B}.$$

If we now consider the symmetries of the Baseball magnetic well, Table III, and expand the drift surfaces $\psi(r, \theta, z, v)$ defined by the invariant J about the axis of the well, it is easy to see that the plasma is composed of a set of concentric shells. The distribution function f can therefore be factored into a weight function ω , determining the amount of plasma on each shell, and a distribution $F(\epsilon, \mu)$ determined by the neutral injection, coulomb scattering and charge exchange of particles on each shell. The pressure tensor factorises likewise $\underline{P} = \omega(\psi) \hat{P}(\underline{B})$, and it remains to find ψ, B self consistently from the MHD equilibrium equations. We have therefore related f and $\underline{P}$ on the timescales of the particle invariants μ, J , on the fast MHD scales, on the scattering timescales of F , and the radial influx and diffusion times determining ω .

The property of omnigenity (drift surfaces independent of $v = \mu/\epsilon$) is illustrated by plotting drift surfaces in the center plane of a Baseball field for particles bouncing near the center of the machine ($vB(0)^{-1}$, solid) and out to the mirrors ($vB(L)^{-1}$), dotted, Fig. 3).

The MHD theory is conveniently reformulated in the classical form of magnetostatics of a medium with tensor magnetization $\underline{M}(\underline{B})$, as defined on Table IV. The terms in the magnetization vector can be identified with the form of the distribution function. This formulation reduces the MHD equations to Eqs. (2), (3), (4) for the functions Γ, ω, Ω which are to be found

by solving two magnetic differential equations and a Poisson equation. The system is nonlinear, being coupled by the definition of B , Eqs. (1), and must be solved iteratively in 3D, taking account of the symmetries S_1, S_2, S_3 . Fig. 4 shows a constant ω surface in a finite β plasma whilst Fig. 5, 5 show the well digging effect of the plasma on original vacuum $|B|$ contours.

The next question to be answered is, 'how much plasma can be put in such a magnetic well?' The well digging effect is adequately described by ignoring field curvature in the long thin approximation to pressure balance:

$$B_{vac}^2 = B^2 + 8\pi \hat{p}_\perp (B) \quad (1)$$

The ultimate limit on the plasma pressure is the mirror mode limit when the equilibrium equations change character from an elliptic to a hyperbolic system:

$$\frac{\partial H_v}{\partial B} \equiv \frac{\partial}{\partial B} (8\pi p_\perp + B^2) = 0. \quad (2)$$

Ion distribution functions have been computed by Fokker Planck diffusion codes and can be approximated by simple analytic fits:

$$F(\epsilon, \mu) = G(\epsilon) (\mu B_{mirror} - \epsilon)^n$$

$$p_\perp(B) = \frac{C(B_{mirror} - B)^{n+3/2}}{B^n}$$

and the equilibrium relation

$$p_\perp = -B^2 \frac{\partial}{\partial B} \left(\frac{p_\perp}{B} \right)$$

Analytic and computed curves of $\beta_i/R_V^2 \equiv 8\pi p_i/B_{\text{mirror}}^2$ vs. mirror ratio R are shown in Fig. 7 for the pressure balance and stability relations (1,2). Typical peak β 's for mirror machines are 0(80%) compared with 10% for toroidal devices. This leads to a higher fusion power density and smaller machine size for a given output.

Finally, one would like to be able to optimize a mirror design and, for example, minimize the stresses in the coils by finding the shallowest vacuum magnetic well for a required fusion power output. A fairly weak well is shown in Fig. 8, taken from an engineering design. As the plasma pressure is increased and magnetic flux pushed out of the well the magnetic field line curvature may be reversed before the mirror mode limit is reached. A typical field line is shown in Fig. 9 at zero β and at $\beta \sim 20\%$. It is now possible that a ballooning mode - an MHD embolism - could occur on such a fieldline. However, an elaborate 3D stability calculation is needed to find out precisely when this will happen.

Summary of Equilibrium Studies

The basic form of Tokamak and Mirror equilibria is easily described in terms of pressure profiles $p(\psi)$ or $\hat{p}(B)$ but the optimization of plasma pressure and coil configuration leads to elaborate free boundary computations. Elementary stability criteria can identify stable equilibria but in the optimization process more complicated modes of instability are possible.

Energy Principles for Plasma Stability

Minimum-B magnetic wells can provide plasma confinement stable to all low frequency modes of oscillation of the plasma. This powerful statement is easily demonstrated by considering the energy of an individual particle in the well:

$$\epsilon = \frac{1}{2} m v^2 = \frac{1}{2} m v_{\perp}^2 + \mu B$$

Any low frequency ($\omega < \Omega_c$) fluctuation which preserves the adiabatic invariant μ has to give energy, $\mu \Delta B$, to the particle to move it outwards in a magnetic well from the bounce surface ($v_{\parallel} = 0$) at B to $B + \Delta B$. Provided the energy distribution function is monotonically decreasing, $\partial f / \partial \epsilon | \mu < 0$, there is no free energy available to feed such a fluctuation and the plasma is stable. Of course, a mirror plasma has the loss cone in the ion distribution function and is susceptible to high frequency ($\omega \geq \Omega_c$) instabilities which destroy μ . This will be discussed later.

In the case of the Tokamak, or in a high β plasma in a mirror machine when B can be altered by a fluctuation, the story is more complicated. The stability condition is expressed by the famous MHD energy principle: an equilibrium is stable if the potential energy (ΔH , Slide 3) of the plasma-field system increases for any small perturbation of the equilibrium. The form of the MHD energy principle, as given, identifies the various types of driving term: pressure gradient, field curvature, and so on. In a minimum-B mirror machine in which $\underline{p} = \underline{p}(B)$ all but the first two terms vanish identically. These terms are positive definite in ξ provided the coefficients $H_{\perp}/B$, $\partial H_{\perp} / \partial B$ are positive. These conditions are necessary and sufficient for stability in this simple case.

In the Tokamak, there is no alternative but to minimize ΔW over all possible perturbations, ξ . (Slide 4). It is desirable to be able to identify the growth rates ω and mode structures of the actual instabilities and so it is preferable to extremise the Lagrangian. The Galerkin procedure is used: the perturbation is expanded in a suitable set of eigenfunctions as $\xi = \sum a_u \bar{\phi}_u$ and the problem is one of finding the eigenvalues ω and eigenfunctions $\bar{a}$ of the reduced problem. The important steps in making such computations trustworthy and accurate are to choose a suitable coordinate system and an appropriate physical decomposition of the possible modes ξ . The coordinate system (Slide 4) is best defined by the magnetic surfaces ψ , a polar angle θ related to the rotational transform of the field lines on each surface, and the toroidal angle ϕ . The first three eigenfunctions, $\cos \theta$, are shown in Fig. 10 and are seen to vary most rapidly on the inside of the torus. This is appropriate for modes with large amplitude on the bad curvature side of the plasma and small amplitude inside the torus. The physical decomposition of ξ (Slide 4) splits the perturbations into component amplitudes π , ζ , δ representing sound, shear Alfvén, and compressible fast Alfvén modes.

Since the equilibrium is independent of φ the modes ξ can be Fourier analysed in this coordinate and a distinct eigenproblem solved for each mode number, n . The techniques used for actually solving these problems are devious adaptations of well known methods. I am only trying to give an impression of the scope of these calculations and Fig. 11 shows a typical ballooning mode, ξ , in the TFTR tokamak at a $\beta \sim 5\%$. It is now a matter of ingenuity in parametrizing the equilibrium profiles $p(\psi)$, $f(\psi)$, and searching the parameter space with such stability codes to find the maximum practical value of β in TFTR.

The actual experiments are run so as to investigate the limits of the capabilities of the machines. This means that these instabilities are generated in practice and one would like to understand the nonlinear consequences of these modes as well as the conditions under which they can be avoided. I would like to emphasize that the stability calculations described are done in real Tokamak geometry with real coil systems and so on, a very considerable achievement. I will now illustrate the qualitative nonlinear behavior of external and internal kink modes.

Magnetic Bubbles

The development of an external kink mode is shown in the 'cartoon approximation' in Fig. 12. These modes are not strongly affected by toroidal curvature and the nonlinear phenomena can be understood most easily by straightening the toroidal plasma column into a cylinder. The cylinder, length $2\pi R$, carries a uniform current J , in a uniform axial field B_ϕ . The poloidal field B_θ from the current J twists the field lines by an angle giving a safety factor $q = aB_\phi / RB_\theta$. In this case there is no shear and all the field lines twist through the same angle. When the safety factor drops to 1, as J is increased, the field lines are closed in one length of the cylinder - a (1, 1) state which is unstable to a surface mode which can carry field lines inside the plasma cylinder into an internal bubble where the lines are straight, a (1, 0) state. If the bubble radius is, b and the initial plasma radius, a , then the final column radius is $r_1 = (a^2 + b^2)^{1/2}$. The column has abruptly expanded - on the fast MHD timescale - and, in an experiment, has pushed plasma into the walls or limiter. An $m = 3$ mode is also sketched in Fig. 12.

This view of the development of the external kink mode has been verified by numerical calculations. The linear growth rate of these modes is plotted in Fig. 13 and the final energy of the bubble state of a plasma cylinder inside a perfectly conducting outer cylinder, radius 1, is shown as a function of final plasma outer radius. The several curves are given for various values of nq/m and show when the bubble state is favored. The bubble state is accessible from the initial equilibrium when W_{bubble} decreases monotonically, otherwise a finite amplitude perturbation would be required to transfer from one equilibrium to the other.

The ideal MHD equations have been solved numerically for just such situations and a typical result is given in Fig. 14. Fortunately, these violent instabilities are easily eliminated by making the current non-uniform. This shears the field lines by making $q = q(r)$ and the bubble state is not topologically accessible. The external kink mode can only distort the external shape of the plasma column, as shown in Fig. 15. This simple model does not therefore explain the major disruptions of the plasma column which occur in Tokamak discharges at high current, and some non-ideal effects will have to be invoked to allow such modes to disrupt the magnetic field structure.

Magnetic Field Line Reconnection

The simple one fluid MHD model given in Table 1 included the effect of resistivity in Ohm's law. This is apparently a small term for a hot plasma but since $\eta J \equiv \eta c \nabla \times B / 4\pi$ it raises the order of the differential system and leads to boundary layer phenomena. These layers, of thickness $\eta^{1/3}$, can support instabilities with growth rates $O(\eta^{1/3})$ or some other fractional power of η , destroying the magnetic topology much faster than $O(\eta)$. Typical boundary layers are shown in Fig. 15 where the magnetic field B_x reverses direction across a current sheet J_z . Due to small but

finite resistivity the lines can cut and rejoin as indicated and the plasma-field configuration can tear apart. Examples of such behavior are to be found in Solar Flares, the tail of the earth's magnetosphere, and in Tokamaks.

The decay of a current layer is considered in more detail in the lower part of Fig. 15. If the current is perturbed by $\delta J \sim \cos kx$ then the first effect is to alter the magnetic topology in the layer, creating magnetic islands, of width $O(\sqrt{\delta J})$, as shown. These islands can help transport particles across the layer more rapidly than simple collisions and are thought to contribute to plasma diffusion in Tokamaks. Microinstabilities in low β plasmas hardly affect the field strength but can have a more damaging effect on the topology. Finite resistivity can allow these islands to tear apart, further amplifying the nonlinear impact of the instabilities.

Simulation of Sawtooth Oscillation in Tokamaks

Simple model calculations have been done on the nonlinear development of the internal kink mode in Tokamaks. The MHD equations have been modified by the inclusion of the energy equation shown on Slide 5 to include compression, ohmic heating and a model energy loss term. The normalized equations allow the resistivity to vary in the classical way with local electron temperature and an artificial viscosity is also included in the momentum equation. The plasma is assumed to be in a straight cylinder inside a square section chamber. The initial current is peaked in the center but the safety factor on axis is $q \approx 1.6$. The center of the plasma heats up, the current concentrates towards the center, and q falls till the kink condition, $q \leq 1$, is met over some central region. A small lateral displacement included in the initial conditions now begins to

grow and hot plasma is ejected from the plasma core. The cycle repeats itself and the resulting sawtooth oscillations of q and plasma pressure are shown in Fig. 17. To follow the plasma dynamics the results are transformed to the helical frame, $B r^* = B r$, $B^* = B - (2\pi r/L) B_z$. The magnetic surfaces are shown at various times in the first cycle of the sawtooth in Fig. 18. Initially, the helical field rotates clockwise around the magnetic surfaces. As the center heats up the field in the center rotates in the opposite direction. The shift of the plasma column causes another magnetic axis to form on the left and the finite resistivity cuts the field lines on the right at the shear layer where the helical field reverses. Subsequent frames show the hot core being expelled and cooled and the plasma reforming in a stable condition to repeat the cycle.

The classical resistive diffusion time for plasma loss should be, $30 < t_R = \sigma r^2 < 1000$, on the general scale of the plasma diffusion rate across the plasma. The MHD transit time in these normalized units is $O(1)$ and the observed loss time, due to the gross sawtooth oscillation, is $O(\frac{1}{4} t_R t_N)^{1/2} \sim 10$, consistent with theoretical expectations and experimental observation.

These calculations are qualitative but do illustrate the general understanding of the nonlinear consequences of the instabilities expected from linear theory.

Minimum Energy States in Toroidal Plasmas

Quiescent states have been observed in the final stages of very high current discharges in toroidal pinches with $q \ll 1$. The sort of violent instabilities described here are able to release much of the free energy in the plasma-field system and it is interesting to try and predict the

final minimum energy equilibrium state.

It was first shown by Moltjer that $K = \int \bar{A} \cdot \bar{B} \, dV$ is an invariant of the ideal MHD equations. It is straightforward to show (Slide 6) that, even with finite resistivity, K should only decay at a rate $O(\eta)$ and should be conserved by bubbles, kinks, and rapid resistive instabilities. At low β one can ignore the contributions of the plasma energy and try to minimize the magnetic energy $W = \int B^2 \, dV$ subject to the topological constraint, using a Lagrange multiplier, that K is constant. This gives the basic result $\bar{J} \propto \bar{B}$, or the force free statement, $\bar{J} \times \bar{B} = 0$. One can easily solve the the equation $\bar{\nabla} \times \bar{B} = \lambda \bar{B}$ in a straight cylinder in terms of Bessel functions. The parameter λ is an eigenvalue if the wall $r = a$ is perfectly conducting so that $B_r(a, \lambda) \equiv 0$, and determines the field profiles accordingly. It is easy to show that the minimum energy state is the minimum λ state and the value of K can be calculated directly, $\int \bar{A} \cdot \bar{B} \, dV$, and related to the number of volt seconds delivered to the discharge.

An elementary examination of the possible eigenvalues λ enables one to classify the possible quiescent states of a violent discharge without following the detailed nonlinear development of the plasma motions and instabilities: With low energy input the plasma current is azimuthally symmetric. As the voltage is raised the poloidal current flow, J_θ , is sufficient to reverse the toroidal magnetic field B_ϕ at the outside of the plasma. This is a very stable configuration for local MHD oscillations due to the high shear in the field. Finally, at higher voltages, the poloidal current is sufficient to introduce a helical twist to the plasma column. Further voltage increases merely increase the amplitude of the helicity without introducing higher poloidal mode structure, such as splitting the column into a double helix around the torus.

Summary

So far I have shown how variational principles describe the equilibrium of plasma in a torus on the fast MHD timescales. I have also shown how the equilibrium of plasma in a mirror machine can be described very elegantly on all timescales. The linear theory of plasma stability in real geometry determines which equilibria are stable and useful for magnetic fusion. Semi-quantitative models of the nonlinear development of these instabilities help in the understanding of experiments. It remains to describe the vital element of these configurations: the actual lifetimes for containment of particles and energy, which must be long enough to allow many fusion reactions in the plasma.

Diffusion in Magnetized Plasmas

This is a very complicated topic because it involves the interaction of very many processes in the case of present day tokamaks. The purpose of this discussion is merely to enumerate the principal effects and indicate their impact on plasma confinement.

The first point to make is that a plasma is composed of at least two species of charged particle, ions and electrons, which would diffuse at different rates. Collisions between like particles cause no net diffusion but collisions between ions and electrons do. The outward flow of ions from a plasma must balance the electron flow to keep the plasma approximately neutral and so an electric field always appears which equalizes the flows. Plasma diffusion is said to be ambipolar and this is true in Tokamaks and mirror machines.

The classical diffusion rate in a plasma cylinder is easily estimated as shown in Slide 7. The diffusion coefficient, $D = \eta n T_c^2 / B^2$ is very small compared with that in a plasma suffering from strong electrostatic ,

turbulence where $D = D_{\text{BCHM}}$. The scaling of this coefficient was derived in the first pair of lectures. Plasma diffusion in Tokamaks lies between these limiting cases. Like particle collisions do produce energy transport and, because ion-ion collisions are $(M/m)^{1/2}$ more frequent than ion-electron collisions at the same temperature because of the larger momentum exchange, the heat containment time is $(m/M)^{1/2}$ smaller than the particle containment time.

Even classical diffusion is stronger in a toroidal plasma because the electric field driving the plasma current gives an additional plasma motion of $cE_\phi \times B_0/B^2$. Pfirsch and Schluter derived the correction and found it to be $D_{\text{PS}} = q^2 D$, in a collisional plasma where the mean free path is much shorter than the distance around the torus. (Slide 8).

As the plasma temperature is increased the plasma essentially changes state by becoming collisionless and the transport is non-local. The diffusion is determined by the small class of particles trapped by the $1/R$ variation of the toroidal field and which execute a banana shaped drift motion. The well depth in the toroidal field is $O(a/R)$ and only traps particles with $v_{\perp}^2/v_{\perp}^2 \sim a/R$. The vertical drift velocity during a banana orbit gives a banana width much larger than a Larmor radius.

Since the disc in velocity space in which particles are trapped in banana orbits is small the rate at which it fills up due to collision is correspondingly large. The fraction of trapped particles is equally small and one can finally write down an estimate of the banana diffusion coefficient as $D_{\text{BANANA}} = D_{\text{PS}}(R/a)^{3/2}$. The ranges of validity of these expressions do not overlap and it has been found experimentally that the diffusion rate tends to form a plateau regime as shown in Slide 16. Diffusion in this regime is dominated by electrostatic microinstabilities in the plasma.

Tokamak plasma behavior in this regime is dominated by plasma turbulence, a very complex set of phenomena to describe. The fundamental result, derived in many different ways, is illustrated in Slide 9. A plasma instability will grow to the point where it has nearly eliminated the driving force or until wave-wave coupling can balance the full energy input with energy cascaded to other modes in the turbulent spectrum. In the case of an instability driven by a density gradient, the fluctuating density may reach an amplitude where it cancels the density gradient. Conservation of mass flow in a density fluctuation $\sim n e^{i k r + \gamma t + i \omega t}$ allows a crude estimate of the maximum plasma displacement, $\xi_{\max} \sim 1/k$. The average plasma flux gives an estimate of the diffusion due to a large amplitude wave, growth rate γ , as $D \sim \gamma/k^2$.

A typical microinstability is the current driven drift wave which gives one possible scaling law for $D(B, T_e, J, \text{etc.})$. Like a heated fluid layer which increases its heat and particle transport over the conduction and diffusion values by forming convection cells and eventually boiling, a Tokamak plasma can combine MHD kink activity in the center, turbulent plasma transport around this, and classical type transport; radiation, and interaction with neutral gas in the cool outer layers. The present models of these processes use a five or six regime model and include much of the practical physics and Tokamak geometry as summarized on Slide 10.

The final question is: "Does this model agree with experiments?" A comparison of theory and experiment is shown in Figure 13, and the answer is, "Not very well." The model of saturated plasma turbulence agrees qualitatively with particle code studies (cf. Methods in Computational Physics, Volumes 9, 16), but accurate predictions are not available for most real

instabilities. Measured fluctuation amplitudes in Tokamaks are 40-80% of the theoretical limit and the general character of the turbulent layers is in agreement with theory. The codes can be calibrated by comparison with experiments and the scaling laws seen correct in many cases. The U. S. Tokamak program is based on the experimental scaling laws discussed in the first lectures and the theory needs further development to describe plasma transport accurately.

Dimensional Analysis of Tokamak Scaling Laws

An immediate question is, "Where should one look to resolve such discrepancies?" A good indication is given by J. Connor and J. B. Taylor by using dimensional analysis: in some asymptotic limit, the plasma confinement time might become a power law relation between the parameter on which it depends--density n , temperature T , field strength B , and plasma size, a (cf. Slide 11). This dependence can be expressed in terms of dimensionless variables, β , a_i/a , collision frequency/cyclotron frequency, ν/Ω_c , and Debye length/ a_i , but this is not too useful. The important observation is that the power law relation is constrained by the transformation laws of any theoretical model one might use to describe the plasma. The constraints are listed in Slide 11 for the most plausible models of a hot, magnetized plasma. Two attempts to fit data from the ATC Tokamak and have a large number of Tokamaks to power law relations fail to fit any of these models. The conclusions are that (i) the models have omitted important physical effects like radiation, impurities, and neutral gas and correspondingly (ii) the computed fits did not include enough data on these effects to be reliable as scaling laws. If one designed a machine to eliminate these effects, the results might be very different from extrapolation of these computed fits. Fortunately,

we would expect the results to be better as the experimentalist got closer to the classical theory! In the meantime, this dimensional analysis shows that no matter how incredibly sophisticated we make our theory of plasma turbulence it will not resolve the differences between transport theory and experiments.

Plasma Turbulence and Losses in Mirror Machines

The lifetime of ions in a mirror machine is determined ultimately by diffusion into the loss cone in velocity space. If the plasma were completely stable this diffusion would be controlled by classical collisions between the ions and electrons. I will summarize the essential physics of these processes and finish the lectures with the very interesting story of the effects of saturated plasma turbulence on the ion lifetimes in the present experiments.

Classical Collisional Losses

The energetics of mirror confined plasmas is dominated by the electrons because they are not effectively contained by the mirrors and cause a large energy drain from the plasma. The electrons scatter more quickly than the ions, find the loss cone, and leave until a large enough ambipolar potential ($\sim 4-5 T_e$) is built up to contain the electrons electrostatically. The ambipolar potential has two effects: the ion loss cone is expanded by the potential, $\epsilon = 1/2 m v_{ii}^2 + \mu B + e\phi$, increasing the ion loss rate (See Slide 12) and only the hottest electrons can escape over the potential barrier. The energy taken out by a typical electron is about $6 T_e$ due to this effect and, in present day experiments is $8-30 T_e$, depending on other conditions like ionizations of neutral gas, radiation, and secondary emission of cold electrons!! Due to the long range of the Coulomb potential and the effects of Debye screening on short range interactions between ions and electrons in a plasma, the principal scattering is by many small angle collisions. The diffusion in velocity space is given by a set of Fokker-Planck equations for each particle species (Slide 12). In the mirror machine ordering of timescales the ion-ion collision frequency is the lowest. The Fokker-Planck equation can therefore be averaged in space over the volume

of the plasma leaving just a 2D diffusion equation to be solved.

The basic result of many studies of these equations is that the energy amplification factor for a single mirror machine is around 1-1.5.

$$Q = \frac{\text{Nuclear Power Generated}}{\text{Neutral beam Power injected}} = \frac{1}{4} \frac{n \langle \sigma v \rangle}{I.E} W_{\text{NUC}} \sim 1.$$

This means that the efficiency of neutral injection and of energy recovery from charged particle losses out the end of a mirror machine have to be very high, ~ 80%, and very cheap to make economic power. Many ideas have been proposed to reduce mirror losses, reduce the electron energy drain, or stopper the ends but the most promising seems to be to use the ambipolar potential of a high temperature (1 Mev) mirror plasma at each end of a long central chamber to contain a fusion plasma (~ 20 keV) electrostatically. This triple mirror idea is being vigorously pursued in Japan, Novosibirsk, and at Livermore. A reactor could have $Q > 5$ for fairly reasonable engineering parameters and an overall length of 100 μ .

The Drift Cyclotron Loss Cone Instability

Another important effect of the loss cone is that the ion distribution function f is far from a Maxwellian thermal equilibrium. Electrostatic fluctuations, $\delta \vec{E} \sim \delta \vec{E} e^{i(\omega t + k y)}$, of the plasma can be unstable at frequencies $0(\omega_{ci})$ or higher since they destroy the invariance of μ and can use the free energy of the ion distribution function. The most serious of these instabilities in the 2XIIB machine was the "drift cyclotron loss cone" mode which couples the ion cyclotron oscillations to the azimuthal drift current arising from the electron density gradient. At this point I will introduce the Vlasov kinetic equation describing the behavior of a collisionless distribution function $F(\vec{R}, \vec{V}, t)$ of ions or electrons in an electrostatic field $\vec{E} = -\nabla \phi$ and a magnetic field $\vec{B}$:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{V} \cdot \frac{\partial F}{\partial \vec{R}} + \frac{e}{m} \left(\vec{\nabla} E + \frac{\vec{V} \times \vec{B}}{c} \right) \cdot \frac{\partial F}{\partial \vec{V}} = 0$$

The distribution function is split into a background equilibrium f and a perturbation δf due to the electric field. The linearized Vlasov equation can be written as

$$\frac{\partial \delta f}{\partial t} + \vec{V} \cdot \frac{\partial \delta f}{\partial \vec{R}} - \omega_c \frac{\partial \delta f}{\partial \phi} = - \frac{e}{m} \vec{\nabla} E \cdot \frac{\partial f}{\partial \vec{V}}$$

where $\omega_c = eB/mc$ and ϕ is the larmor angle on the cyclotron orbit. The equation can be formally solved by integration along the unperturbed particle orbits as

$$\delta f = - \frac{e}{m} \int_{-\infty}^t dt \delta E \cdot \frac{\partial f}{\partial \vec{V}}$$

The dispersion relation from the linear kinetic theory of a plasma slab in a uniform magnetic field $\vec{B} = B_0 \hat{z}$, is shown in Figure 19. The ions and electrons are coupled through Poisson's equation for the electric potential of charge fluctuations, $\nabla^2 \delta \phi = 4\pi \delta \rho$. There is not space to develop plasma kinetic theory here; suffice to observe the result of these calculations (Figure 29) which shows that such oscillations are unstable above a certain critical density which depends on the size of the loss cone (or mirror ratio), the growth rate is high, $\gamma/\omega_{ci} \sim 3$, the modes are not highly resonant $-5 \leq \omega/\omega_{ci} \leq 1$ and cannot therefore be stabilized by spoiling the resonance, and the perpendicular wavenumber is very large, $ka_i \sim 20$, and the modes are unaffected by the plasma geometry. The mode structure tends to be uniform along the magnetic field lines since the electrons move rapidly up and down the lines to short out any parallel electric fields. The modes could therefore be altered by boundary conditions but turn out to be insensitive to geometric effects in 2XII type plasmas. There are only two ways to stabilize or saturate this fierce

instability--by having enough particles in the loss cone to be closer to equilibrium or by building a machine large enough that the driving force from the radial density gradient is insignificant.

Quasilinear Model of Plasma Turbulence

Filling the loss cone was exactly the technique used in 2XIIB: a cold (100 ev) stream of plasma was trickled down the field lines into the hot central plasma to eliminate the DLC modes which otherwise cut the ion lifetime from 5 msec to 300 μ secs. Experimental and theoretical results were compared in the first set of lectures and I would now like to sketch the basis of the theory.

The effect of the fluctuations predicted by the linearized theory is to alter the background distribution function f on a longer timescale than $0(\omega_c^{-1})$. The full Vlasov equation can be averaged on the fluctuation timescale to give the quasilinear equation

$$\frac{\partial f}{\partial t} + \vec{V} \cdot \frac{\partial f}{\partial \vec{R}} - \omega_c i \frac{\partial f}{\partial \phi} = - \frac{e}{m} \left\langle \delta E \cdot \frac{\partial \delta f}{\partial \vec{V}} \right\rangle$$

The energy in the fluctuating field is given in each mode by

$$\frac{\partial}{\partial t} (\delta E_k)^2 = 2\gamma_k (\delta E_k)^2 + \alpha_k = \frac{\partial \mathcal{E}_k}{\partial t}$$

where $\gamma_k(t)$ is the growth rate derived from the linearized dispersion relation and α_k a source of thermal fluctuation. The effect of the fluctuations is to cause ions to diffuse in velocity space towards the loss cone.

Of course, an adequate model of 2XIIB requires a description of the other processes involved--beam injection, cold plasma stream, cooling by

electrons, charge exchange losses, and the fact that an ion in the loss cone takes a finite transit time to leave the machine. The fact that the 2XIIB electrons are so cold (100 ev) means that hot ions (15kev) escape primarily by cooling down into the loss cone and not by scattering in angle (Figure 21). The final quasilinear equation for the ions is given in Slide 13. The electron temperature is strongly coupled to the ion turbulence and is severely limited by the energy losses (ηT_e) for each electron leaving the plasma. The set of equations is completed in Slide 14 by a simple approximate diffusion coefficient, the growth rate, and the effective width of the resonance coupling the fluctuations δE to the particles δf .

This sort of model has many adjustable parameters to describe the plasma volume, injection rate, stream input, etc. However, once calibrated, the model fit the experiment very well and provided the essential ruling laws for designing the next round of experiments.

CONCLUDING REMARKS

I have tried to show in these lectures how modern plasma theory is able to describe equilibria in real devices, the stability limits, both MHD and kinetic, of these equilibria, the nonlinear and quasilinear consequences of the instabilities, and the calculation of ion particle and energy confinement times. The details may only be comprehensible to someone working in the field, but it is at least evident that plasma theory is much more sophisticated than the back of an envelope calculation of the 50's and that there are many areas for further research before our models can be regarded as accurate by an engineer.

TABLE I. THE ONE FLUID MAGNETOHYDRODYNAMIC MODEL

CONSERVATION OF ION DENSITY

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$

MOMENTUM EQUATION

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = -\nabla p + \frac{\vec{J} \times \vec{B}}{c}$$

EQUATION OF STATE

$$\frac{d}{dt} (p/\rho) = 0$$

OHM'S LAW: IGNORE ELECTRON INERTIA

$$\vec{E} + \frac{\vec{V} \times \vec{B}}{c} = \eta \vec{J}$$

MAXWELL'S EQUATIONS

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} \qquad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

TABLE II. STATIC EQUILIBRIUM OF GUIDING CENTER PLASMA

EQUATIONS REDUCE TO:

$$c^{-1} \vec{j} \times \vec{B} = \nabla \cdot \vec{P} \quad \vec{P} = p_{\perp} \vec{I} + \left(\frac{p_{\parallel} - p_{\perp}}{B^2} \right) \vec{B} \vec{B}$$

$$\nabla \times \vec{B} = 4\pi c^{-1} \vec{j} \quad \nabla \cdot \vec{B} = 0$$

I. AXISYMMETRIC (TOKAMAK) SCALAR PRESSURE EQUILIBRIA

$$\vec{B} = F(\psi) \nabla \phi + \nabla \phi \times \nabla \psi \quad \vec{B} \cdot \nabla \psi \equiv 0$$

$$\nabla \cdot (R^{-2} \nabla \psi) = -S(\psi) \left(4\pi p'(\psi) + R^{-2} FF' \right) \quad (\rightarrow f(R)\psi)$$

TOROIDAL AND POLOIDAL FLUXES ARE

$$c = \int_{\psi \leq c} \nabla \cdot (\phi \vec{B}) dV \quad \chi(c) = \int_{\psi \leq c} \nabla \cdot (\psi \vec{B})$$

NUMERICAL METHOD TO CONTROL BIFURCATION

$$(\Delta_R + f(R) S(\psi_n) - \alpha_1) \psi^* = -(\Delta_z + \alpha_1) \psi_n$$

$$(\Delta_z + f(R) S(\psi^*) - \alpha_1) \psi_{n+1/3} = -(\Delta_R + \alpha_1) \psi^*$$

$$\psi_{n+1} = (1 - \alpha_2) \psi_n + 2 \alpha_2 \psi_{n+1/3} - \alpha_2 \psi_{n+2/3}$$

$$0 < \alpha_2 < \frac{1}{2} \quad \alpha_1 \sim 3f(R)_{\max}$$

Fig. 1. Numerically computed flux surfaces in a circular cross section Tokamak. The high β cases (c, d) are unstable with $q = .7, .5$ respectively. Callen and Dory (1972), Phys. Fluids 15, 1523-1528.

Fig. 2. Moderate $\beta_p \approx .25$ equilibria in Doublet II showing bifurcated solutions. Chu et. al. (1974) Phys. Fluids 17, 1183-1187.

Fig. 2a. A confined, blocked particle orbit, projected into the plane $\phi = 0$ in a high shear, $L = 3$, Stellarator. The position of the guiding center as the particle passes $\phi = 0$ is marked by X and numbers indicate the order of the transits. Note the size of the orbit relative to the Larmor radius. Gibson and Taylor (1972) Phys. Fluids 10, 2653-2659.

TABLE III.

MIRROR EQUILIBRIA

ION DISTRIBUTION FUNCTION

$$f = f(\epsilon, \mu, J)$$

$$\epsilon = \frac{1}{2} m v^2 \quad \mu = \frac{1}{2} \frac{m v_{\perp}^2}{B} \quad J = \oint v_{\parallel} dl$$

SYMMETRY PROPERTIES OF BASEBALL GEOMETRY

$$S_1: (r, \theta, z) \rightarrow (r, -\theta, z)$$

$$S_2: (r, \theta + \frac{\pi}{2}, z) \rightarrow (r, \frac{\pi}{2} - \theta, z)$$

$$S_3: (r, \frac{\pi}{4} + \theta, z) \rightarrow (r, \frac{\pi}{4} - \theta, -z)$$

OMNIGENOUS DRIFT SURFACES

$$\psi(r, \theta, z, v = \mu/\epsilon) = \psi_0(r, z, v) + \sum_1^{\infty} \psi_{2n}(r, z, v) \cos 2n \theta$$

$$\begin{aligned} &\approx \psi_0 + \psi_2 \cos 2 \theta \\ &r \rightarrow 0 \end{aligned}$$

$$\begin{aligned} S_3: \psi_2(-z) &= -\psi_2(z) \rightarrow 0 \\ &z \rightarrow 0 \end{aligned}$$

ION DISTRIBUTION FUNCTION $f = \omega(\psi(J)) F(\epsilon, \mu)$

PRESSURE TENSOR $\vec{P} = \omega(\psi) \vec{P}^*(B)$

EQUILIBRIA VALID ON ALL TIMESCALES

SOLVE FOR ψ B FROM MHD:

$$\vec{J} \times \vec{B} = \nabla \cdot \vec{P} \quad \nabla \times \vec{B} = 4\pi c^{-1} \vec{J} \quad \nabla \cdot \vec{B} = 0$$

Fig. 3. Constant J Surfaces for particles bouncing in Baseball II. Dotted line: Near the mirror peak; solid line: near the minimum of B.
Hall et. al. (1974) IAEA Tokyo Conference.

TABLE IV.

MAGNETOSTATICS OF MIRROR EQUILIBRIA

$$\begin{aligned}
 \vec{B} &= \vec{H} + 4\pi \vec{M}(\vec{B}) = \vec{H} + 4\pi (\vec{M}_\mu + \vec{M}_J) \\
 &= \vec{B}_{vac} - \nabla\Omega - \frac{4\pi}{B^2} (p_\perp - p_\parallel) \vec{B} + \xi \nabla\gamma \\
 \nabla \cdot \vec{B} &= 0 \qquad \nabla \times \vec{H} = 0 \\
 f &\neq f(\mu), \vec{M}_\mu \equiv 0 \qquad f \neq f(J), \vec{M}_J \equiv 0
 \end{aligned} \tag{1}$$

DRIFT MAGNETISATION POTENTIAL

$$\begin{aligned}
 \vec{B} \cdot \nabla \Gamma &= p_\parallel^* \qquad \bar{\Gamma} = \oint p_\parallel^* \frac{dl}{B} \\
 \xi &= \int_0^\omega \bar{\Gamma} d\omega \qquad \gamma = \Gamma/\bar{\Gamma} - \frac{1}{2}
 \end{aligned} \tag{2}$$

RADIAL PRESSURE PROFILE

$$\begin{aligned}
 \omega &= \omega(\bar{\Gamma}) \equiv \omega(x, y = z = 0) \\
 \vec{B} \cdot \nabla \omega &= 0
 \end{aligned} \tag{3}$$

MAGNETIC SCALAR POTENTIAL

$$\nabla \cdot \vec{B} = 0 = -\nabla^2 \Omega + 4\pi \nabla \cdot \vec{M} \tag{4}$$

Fig. 4. Isometric plot of a flux tube, $\omega = \omega(\vec{r})$ in 2XIIB shows fanning of flux tube. The drift surface cross section in the midplane is distorted from a circle by the quadrupole components of the vacuum field.

Fig. 5. 2XII Vacuum field $|B|$ in the plane $y = 0$ shows magnetic well. Contours are at 500 gauss with the highest at 12 kg.

Fig. 6. High β (~ 0.7) plasma in 2XII depresses central field by 2 kg and digs deeper well by plasma diamagnetism.

Fig. 7. Log plots of B_0/R_V^2 vs. actual mirror ratio. Mirror mode stability limits are at curves I, $n = .5, 1, 2$. Long thin approximation to pressure balance gives L curves, labelled with initial vacuum mirror ratio. Computed results from 3D McGus code are labelled M. Field curvature weakens at * points. Result of Fokker-Planck calculation and long thin approximation is labelled F. McNamara et. al. (1976) IAEA Berchtesgaden Conference.

Fig. 8. Mod. B contours in a weak magnetic well. Points are (a) coil cross section, (b) mirror point, (c) wedge cut in B contours by high β plasma, (d) small minimum B well region, (e) mirror point.

Fig. 9. Lower field line of each pair is a vacuum field line in well of Fig. 8. Upper field lines show reversal of stabilizing curvature by high β plasma.

SLIDE 3

ENERGY PRINCIPLES FOR MHD STABILITY

PERTURB AN EQUILIBRIUM:

$$\delta \vec{v} = \frac{d\vec{\xi}}{dt} \quad \vec{\xi}(\vec{x}) \sim \text{Plasma displacement}$$

$$\frac{d^2 \vec{\xi}}{dt^2} = F \cdot \vec{\xi} \quad F \sim \text{Linear, self adjoint operator}$$

PERTURBED ENERGY > 0 FOR STABILITY

$$\Delta W = (8\pi)^{-1} \int \delta W \, dV$$

$$\delta W = (\vec{H}_v/B)(\vec{Q} \times \hat{b})^2 + \partial \vec{H}_v / \partial B (\vec{Q} \cdot \hat{b})^2$$

Firehose Mirror

$$-4\pi (\vec{F} \cdot \Delta p_{\perp}) B^{-1} (2\vec{Q} \cdot \hat{b} + \vec{\xi} \cdot \nabla B) + 4\pi (\vec{\xi} \cdot \Delta p_{\parallel})(\hat{k} \cdot \vec{\xi})$$

Pressure Gradient Curvature

$$-4\pi c^{-1} H_v B^{-1} (\vec{J} \cdot \hat{b}) (\vec{Q} \cdot \vec{\xi} \times \hat{b})$$

Parallel Current

$$\text{where } \vec{Q} = \nabla \times (\vec{\xi} \times \vec{B}) \quad \vec{H}_v = \left(1 + 4\pi B^{-2} (p_{\perp} - p_{\parallel})\right) \vec{B}$$

$$\Delta = \nabla - \nabla B \, \partial / \partial B \quad \hat{k} = -\hat{b} \cdot \nabla \hat{b} \quad \hat{b} = \vec{B}/B$$

LOW FREQUENCY STABILITY IN MINIMUM-B

$$\epsilon = \frac{1}{2} m v^2 = \frac{1}{2} m v_{\parallel}^2 + \mu B$$

$$\text{Stability if } \frac{\partial f}{\partial \epsilon} < 0 \quad \nabla p \cdot \nabla B < 0$$

SLIDE 4. MHD STABILITY OF THE TOKAMAK

EXTREMISE LAGRANGIAN

$$\mathcal{L} = \omega^2 \int dV \frac{1}{2} \rho |\vec{\xi}|^2 - \Delta W \equiv \omega^2 K - \Delta W$$

GALERKIN PROCEDURE → MATRIX EIGENPROBLEM

$$\vec{\xi} = \sum_1^M a_n \phi_n$$

$$\sum_n \left[\omega_M^2 \langle \phi_n, |K| \phi_n \rangle - \langle \phi_n, |\Delta W| \phi_n \rangle \right] \cdot a_n^{(M)} = 0$$

PHYSICAL DECOMPOSITION OF $\vec{\xi}$

$$\vec{\xi} = \left(\frac{g \xi_\psi}{g R^2 B_0} \right) \nabla \theta \times \vec{B} + i \left(\frac{g \xi_s}{g R^2 B_0} \right) \vec{B} \times \nabla \psi + i \frac{\tau}{B_0} \vec{B}$$

$$\xi_\psi = \delta - 2\pi i \frac{\partial \zeta}{\partial \theta} \quad \xi_s = 2\pi \frac{\partial \zeta}{\partial \psi}$$

τ - sound branch ζ - shear Alfvén branch δ - compressibility allows fast Alfvén.

FOURIER DECOMPOSITION: $\phi \sim \sum_{l,n} \phi_{ln}(\psi) e^{(l\theta - n\phi)}$

EIGENVALUE METHODS:

FIND $A = K^{-1/2} \Delta W K^{-1/2}$

SOLVE $(A - \omega^2 I) \cdot \vec{\eta} = 0 \quad \vec{\eta} = K^{1/2} \cdot \vec{\xi}$

BY JACOBI ITERATION, QR ALGORITHM, BLOCK INVERSE ITERATION

Fig. 10. Flux and angle coordinate system (ψ, θ) for Tokamak stability calculations is shown with the first three eigenfunctions $\cos l\theta$.

R. Grimm, 1977, Trieste Lectures.

Fig. 11

Fig. 11. Unstable ballooning mode in TFTR is displayed as an arrow plot of ξ . The mode is generally active on the bad curvature side of the plasma. R. Grimm (1977), Trieste Lectures.

MAGNETIC BUBBLE FORMATION

TORUS

$$q = \frac{2\pi}{c}$$

$$= \frac{a}{R} \frac{B_\phi}{B_\theta}$$

Ignore Toroidal Effects and Consider a Cylinder

Bubble Formation → Plasma Expansion

$$r_1 = (a^2 + b^2)^{1/2}$$

$m = 3$ mode.

Fig. 13a. Linear kink modes. The square of the growth rate is plotted versus nq/m for $m=1, 2, 3$. The shaded region shows the range of q for which the bubble state is energetically favored ($a=0.5$).

Fig. 13b. Bubble state energy as a function of the plasma outer radius r_1 , $a=0.5$.

$$W_{\text{BUBBLE}} = \frac{\pi r_1^2 a^2}{2} - \frac{\pi}{\ln r_1} \left(\frac{1-r_1^2}{2} + \psi_{\text{HALL}}^2 \right) + \text{const.}$$

Fig. 14. The undamped time history of the plasma surface in a case for which the bubble state is highly favored. Here $a = 0.7$, $\psi_w = -0.06$, $nq/m = 1.0$.

Fig. 15. Minimal energy states for $m = 1$ in the case of a parabolic current profile. Compare with Fig. 5.

MAGNETIC FIELD LINE RECONNECTION

Examples: Solar Flares, Magnetospheric Tail, Tokamaks

Decay of Current Layers

Field Topology Changes With Current Perturbation.

Layer Tears Apart With Finite Resistivity

SLIDE 5. SIMULATION OF SAWTOOTH OSCILLATIONS IN TOKAMAKS

A. Sykes, J. A. Wesson, Phys. Rev. Lett. 37, 40, 1976

MODEL: MHD + ENERGY EQUATION:

$$\frac{1}{\gamma-1} \frac{\partial p}{\partial t} = \frac{1}{\gamma-1} \nabla \cdot (p \vec{V}) - p \nabla \cdot \vec{V} + \eta j^2 - \frac{1}{\gamma-1} \frac{p}{\tau(r)}$$

Compression Ohmic Energy
Heating loss

MODEL NORMALIZED TO $\rho_0, a, V_A = B_0/\rho_0^{1/2}$

Resistivity $\eta = .0005 \tau_e^{-3/2}$

Viscosity $\nu = .04$

Energy loss time $t = 5.2/r$

Initial Current $j_z = 1.22 \frac{B_0}{a} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a}, q_{axis} = 1.6$

RESULTS IN HELICAL FRAME:

$$B_r^* = B_r \quad B_\theta^* = B_\theta - (2\pi r/L) B_z$$

Resistive Loss time $30 < t_R = \frac{a^2}{\sigma} r^2 < 1000$

MHD Transit time $t_W = r/V_A^* \sim 1$

Observed loss time $t \sim \left(\frac{1}{4} t_R t_W \right)^{1/2} \sim 10$

The Tokamak Temperature Regulator:
 Fig. 17. The upper graph gives the pressure at the center of the minor cross section plotted against time. The lower graph shows the value of q on the magnetic axes (full lines) and that at the position of minimum q (dashed line).

Fig. 18. Plots of the transformed magnetic field line trajectories during the first cycle. The regions with $q < 1$ are shown shaded. The initially stable configuration becomes unstable as a result of current concentration. An island is formed (having $q > 1$) which grows and displaces the original unstable "island." The island with $q < 1$ decays away.

SLIDE 6. MINIMUM ENERGY STATES IN TOROIDAL PLASMAS

WOLTJER'S INVARIANT: $K = \int_{\text{volume of torus}} \vec{A} \cdot \vec{B} \, dV$ $\vec{B} = \nabla \times \vec{A}$

$$\frac{\partial \vec{A}}{\partial t} = \vec{E} + \nabla \phi$$

VARIES ONLY ON RESISTIVE TIMESCALE:

$$\begin{aligned} \frac{\partial K}{\partial t} &= \int dS [(\vec{A} \cdot \vec{V} - c\phi) \hat{n} \cdot \vec{B} + (c\vec{E} + \vec{V} \times \vec{B}) \cdot (\hat{n} \times \vec{A})] \\ &\quad - 2c \int dV \vec{E} \cdot \vec{B} \\ &= - 2c \int dV n_{||} \vec{J} \cdot \vec{B} \sim 0(n_{||}) \end{aligned}$$

CONSERVED BY BUBBLES, KINKS, RESISTIVE INSTABILITIES.

MINIMIZE MAGNETIC ENERGY IN TORUS WITH CONSTRAINT K

$$W_{\text{MAG}} = \int \frac{B^2}{8\pi} \, dV$$

GIVES (J. B. Taylor, Pulsed High- β Plasmas, Pergamon, 1976)

$$\nabla \times \vec{B} = \lambda \vec{B} \quad \text{Force Free}$$

MINIMUM ENERGY STATE IS MIN λ STATE:

$$W_1 - W_2 = (\lambda_1 - \lambda_2) K/8\pi.$$

$K = \psi_0 V_S$, GIVEN BY VOLT-SECONDS IN DISCHARGE.

PREDICTS REVERSED FIELD AND HELICAL QUIESCENT STATES IN HIGH CURRENT

($q \gg 1$) DISCHARGES.

SLIDE 7. DIFFUSION IN MAGNETIZED PLASMA

1. DIFFUSION IS AMBIPOLAR

$$\frac{\sum e \Delta \vec{r}}{\sum |e|} = \frac{c}{B^2} \frac{\sum \Delta m \vec{V} \times \vec{B}}{\sum |e|} \equiv 0$$

2. CLASSICAL DIFFUSION

$$\vec{v}_p + \frac{\vec{J} \times \vec{B}}{c}, \quad n \vec{J} = \frac{\vec{v} \times \vec{B}}{c}$$

$$v_{\text{class}} = - \frac{nc^2}{B^2} \nabla p \equiv - \frac{nc^2 T}{B^2} \nabla n \equiv - \frac{p}{n} \nabla n$$

$$D = \eta n T \frac{c^2}{B^2} = v_{ei} a_e^2 \quad \sim 15 \text{ cm}^2/\text{sec}$$

$T = 10 \text{ keV} \quad \beta \sim 1$

$B = 40 \text{ kg}$

3. STRONG TURBULENCE, CONVECTIVE CELLS

$$D_{\text{BOHM}} = \frac{cT}{16 eB} \quad \sim 1.6 \cdot 10^6 \text{ cm}^2/\text{sec}$$

4. HEAT CONDUCTION

$$v_{ii} = \sqrt{\frac{M}{m}} v_{ie}$$

$$t_{\text{HEAT}} \sim t_{\text{PARTICLE}} \sqrt{\frac{m}{M}}$$

SLIDE 8. PLASMA TRANSPORT IN TOKAMAKS

Pfirsch Schlüter Toroidal Correction

$$v_{PS} \sim q^2 v_{CLASS} \quad \lambda_{MFP} < qR$$

Banana Diffusion $\lambda_{MFP} > qR(R/a)^{3/2}$

$$\frac{\Delta B_T}{B_T} \sim \frac{a}{R} \quad \text{Trapped Particles At Small } \frac{v_{\perp}^2}{v_z^2} \sim \frac{a}{R}$$

Vertical Drift Velocity $v_D \sim \frac{mc}{eB} \cdot \frac{mv_z^2}{R}$

Banana Width $\delta \sim v_D \frac{qR}{v_{\parallel}}$

Collision Rate $v_T \sim \frac{R}{a} v$ Trapped fraction = $n \frac{v_{\parallel}}{v}$

$$D_{BANANA} = \delta^2 v_T \frac{n_T}{n} = D_{PS} \left(\frac{R}{a} \right)^{3/2}$$

Fig 2

SLIDE 9. TURBULENT PLASMA DIFFUSION

$$\frac{\partial n}{\partial x} = \frac{\partial n_0}{\partial x} + \frac{\partial n_1}{\partial x} = 0$$

Incompressible, electrostatic mode:

$$\frac{\partial n_1}{\partial t} + v_x \frac{\partial n_0}{\partial x} = 0, \quad n_1 = -\xi \frac{\partial n_0}{\partial x}, \quad \xi_{\max} \sim \frac{1}{k}$$

Plasma Flux:

$$\Gamma = \langle n_1 v_x \rangle \approx - \left\langle \xi \frac{d\xi}{dt} \right\rangle \frac{dn_0}{dx} = -\gamma \xi^2 \frac{dn_0}{dx}$$

$$D \sim \gamma \xi^2 \sim \gamma / k^2 = \left(\frac{j_u}{nev_t} \right)^{3/2} \left(\frac{ckT_e}{eB} \right)^2 \frac{n}{c_s \phi_{\text{SHEAR}}} \left(\frac{dn}{dx} \right)$$

Current Driven Drift Waves

Fig. 19. Comparison of the radial profiles of particle density, n , electron temperature, T_e , ion temperature, T_i , for conditions applicable to a high density discharge in a small Tokamak ($R \approx 1.09$ m, $a = 0.14$ m, $n_0 = 1.65 \times 10^{19} \text{ m}^{-3}$, $I_p = 60$ kA, $B_0 = 3.7$ T) given at 35 msec by the code, ICARUS, using the neoclassical diffusion model (—), and the laboratory experiment (Dimock *et al.*, 1973) (---).

1. $D \sim \gamma/k_A^2$ Agrees with particle codes.
2. Saturation wave amplitudes not known accurately.
3. Fluctuation levels are 40-80% of quasilinear limit.
4. Fluctuations in Tokamaks have wide frequency spectra, but have the general characteristics of linear theory.

Fig. 19. H. L. Watkins *et. al.*, *Methods in Computational Physics*, Vol. 16, Academic Press.

PHYSICAL PROCESSES IN TOKAMAKS

1. PLASMA TRANSPORT: PARTICLES AND ENERGY.
 - MULTI-REGIME MICROINSTABILITY MODELS FOR PARTICLE AND ELECTRON THERMAL TRANSPORT.
 - NEOCLASSICAL ION THERMAL CONDUCTIVITY.
 - ENHANCED TRANSPORT DUE TO MHD MODES WHEN $q < 1$.
2. NEUTRAL GAS:
 - NEUTRAL INFLUX DUE TO RECYCLING OR PROGRAMMED GAS FEED.
 - MANY GENERATIONS OF CHARGE-EXCHANGE.
3. IMPURITIES:
 - LOW-Z IMPURITIES, E.G., OXYGEN.
 - HIGH-Z METALLIC IMPURITIES FROM WALL OR LIMITER.
 - NEOCLASSICAL INWARD IMPURITY TRANSPORT, TOGETHER WITH ANOMALOUS DIFFUSION.
 - CORONAL EQUILIBRIUM FOR $\langle z \rangle$ AND RADIATION.
4. NEUTRAL BEAM INJECTION:
 - SELF-CONSISTENT HEATING BY SLOWING-DOWN OF BEAM IONS.
 - SELF-CONSISTENT BEAM-ION RADIAL DEPOSITION PROFILE, INCLUDING IMPURITY EFFECTS.
 - ADIABATIC COMPRESSION OF BEAM IONS AND BACKGROUND PLASMA.

SCHEMATIC ILLUSTRATION OF THE MULTIREGIME MODEL FOR THE DEPENDENCE OF DIFFUSION AND THERMAL CONDUCTIVITY COEFFICIENTS ON COLLISION FREQUENCY.

SLIDE 11. DIMENSIONAL ANALYSIS OF TOKAMAK SCALING LAWS
(Connor and Taylor)

ASSUME

$$B\tau = n^p T^q B^r a^s$$

$$\equiv m/e \, G(\beta, a_i/a, v/\Omega_c, \lambda_D/a)$$

Transformation Laws of each model impose constraints.

MODEL	$B\tau$	CONSTRAINTS
Collisionless Vlasov, High β	$F\left(\frac{T}{a^2 B^2}, \frac{na^2}{B^4 a^5}\right)$	$3p + 2q + s = 0$ $4p + 2q + r = 0$
Collisional Vlasov, High β	$F(na^2, Ta^{1/2}, Ba^{5/4})$	$0 = 2p + \frac{q}{2} + \frac{5r}{4} - s$
Ohmic Heating: T not independent $\tau^{5/2} \sim \beta^2 \tau / na^2$	$F(na^2, Ba^{5/4})$	$C = 8p + 5r - 4s$ $= 0$

EXPERIMENTS: Computed fit to results

$$\text{ATC Tokamak} \quad p = 1 \quad r = \frac{1}{4} \quad s = 3/4$$

$$C = 6 \, 1/4!!$$

Hugill and Sheffield Survey (Culham)

$$p = .61 \pm .08 \quad r = 1.99 \pm .13 \quad s = 1.57 \pm .17$$

$$C = 8.05 \pm 1.97$$

Models omit Radiation, Wall effects, Impurities, Neutrals.

SLIDE 12. COLLISIONAL LOSSES IN MIRRORS FOKKER PLANCK EQUATIONS

$$\frac{\partial f}{\partial t} + \vec{V} \cdot \frac{\partial f}{\partial \vec{R}} + \frac{\vec{F} \cdot \vec{C}}{m} \cdot \frac{\partial f}{\partial \vec{V}} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} + \text{Source} + \text{Loss}$$

$$\frac{1}{\Gamma} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = - \frac{\partial}{\partial V_i} \left(f \frac{\partial h}{\partial V_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial V_i \partial V_j} \left(f \frac{\partial^2 g}{\partial V_i \partial V_j} \right)$$

$$g_a = \sum_b \left(\frac{Z_b}{Z_a} \right)^2 \ln \Lambda_{ab} \int f_b(\vec{V}') |\vec{V} - \vec{V}'| d\vec{V}'$$

$$h_a = \sum_b \frac{m_a + m_b}{m_b} \left(\frac{Z_b}{Z_a} \right)^2 \ln \Lambda_{ab} \int f_b(\vec{V}') |\vec{V} - \vec{V}'|^{-1} d\vec{V}'$$

$$\Gamma = 4\pi Z^4 e^4 / m^2$$

MIRROR MACHINE ORDERING

$$\omega_p \gg \Omega_c \gg \Omega_{\text{bounce}} \gg \Omega_{\text{DRIFT}} \gg v_{ii}$$

$$v_{ee} \gg v_{ie} \gtrsim v_{ii}$$

THE DRIFT CYCLOTRON LOSS CONE MODE

Electrostatic Fluctuations

$$\phi = \phi_0 e^{i(\omega t + ky)}$$

Dispersion Relation: Linear Theory

Density Gradient

$$1 + \frac{\omega_{pe}^2}{\Omega_e^2} = \frac{\omega_{pe}^2}{\omega \Omega_e} \frac{1}{kn} \frac{dn}{dx}$$

Electron Inertia

$$+ \frac{8\pi^2 n_e^2 \omega}{k^2 M} \int dv_{\perp} dv_{\parallel}^2 \frac{\partial f}{\partial v_{\perp}^2} \sum_{-\infty}^{\infty} \frac{j_n^2(b_0)}{n^2 + n_i^2}$$

Ion Loss Cone

$$\omega_p^2 = \frac{4\pi n e^2}{m} \quad \Omega = \frac{eB}{mc}$$

Fig. 20. Maximum growth rate of drift cone mode versus density, real frequency, and $b_0 = ka_i$ for various mirror ratios, $R_p = 25 v_{th,i}/\omega_{c1}$, $\beta = 0, 0.1$. Numbers on each contour refer to the mirror ratio.

Slide 13. Quasilinear Diffusion equations for Turbulent Mirror Plasma. D. E.
Baldwin et. al. (1976), Present Status of Mirror Stability Theory,
UCID 17038.

I. EVOLUTION OF DISTRIBUTION FUNCTION

$$\begin{aligned} \frac{\partial F}{\partial t} = & \frac{\partial}{\partial v_{\perp}^2} D(v_{\perp}^2) \frac{\partial F}{\partial v_{\perp}^2} + \nu_{\text{DRAG}} \frac{\partial}{\partial v_{\perp}^2} (v_{\perp}^2 F) - \nu_{\text{TRANSIT}} (v_{\perp}^2) F \\ & + n S_{\text{beam}} (v_{\perp}^2) - F \int dv_{\perp}^2 [1 - \alpha_{\text{ion}} (v_{\perp}^2) S_{\text{beam}} (v_{\perp}^2)] + S_{\text{strm}} (v_{\perp}^2) - \nu_{\text{CHX}} F \end{aligned}$$

II. EVOLUTION OF ELECTRON TEMPERATURE

$$\frac{\partial}{\partial t} \left(\frac{3}{2} n T_e \right) = n \nu_{\text{DRAG}} T_i - \eta \frac{\partial n_L}{\partial t} T_e$$

TURBULENT DIFFUSION COEFFICIENT

$$D(V_{\perp}^2) = 4 \sum_k m^2 \frac{\omega_{ci} \bar{V}^2 \sigma_k}{\Delta \omega_k} J_m^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_{ci}} \right) \mathcal{E}_k$$

$$\frac{\partial \mathcal{E}_k}{\partial t} = 2 \gamma_k \mathcal{E}_k + \alpha_k$$

$$\frac{\gamma_k}{\omega_{ci}} = \sigma_k \bar{V}^2 \omega_{ci} \int_0^{\infty} dv_{\perp}^2 \frac{\partial F}{\partial V_{\perp}^2} \frac{J_m^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_{ci}} \right)}{\Delta \omega_k}$$

$$\Delta \omega_k = \omega_{ci} \left(\frac{m \langle V_{\parallel} \rangle}{\omega_{ci} L} \right)^{1/3} + k^2 D / \omega_{ci}^2 V_{\perp}^2$$

Slide 14. The Turbulent Diffusion Coefficient, Electrical Energy equation,
and growth rate. D. E. Baldwin et. al. (1976).