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January 11, 1977

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THE CHARGE TRANSPORT PROBLEM*

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Abstract

The net charge associated with a beam propagating in a conducting medium is calculated.

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Summary

In a recent report¹ Cooper and Neil demonstrate that the net charge transported by a beam pulse injected into a channel of finite conductivity equals the charge of the beam itself. The channel is taken to be infinite in the positive z direction, has finite radius and is terminated by a conducting ground plane at $z=0$. This result is not an obvious one, and it is restricted in its applicability by the special model assumed for the channel. It is my purpose in this report to explain the result of Cooper and Neil in more qualitative terms and to make similar calculations using several other channel models. It must be emphasized that these calculations are not concerned with the fate of the transported charge after the pulse has stopped, but rather with how much charge leaves the ground plane assuming the pulse does not stop.

It is found that the transported charge vanishes if the channel has infinite radius, while the result of Cooper and Neil holds for a bounded channel. However, if conductivity changes with time the transported charge may actually exceed that of the beam pulse.

Case 1 Conductivity (σ) is uniform and time independent everywhere above the $z = 0$ plane. The equation of charge continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \sigma \mathbf{E} + \nabla \cdot \mathbf{J}_b = 0, \quad (1)$$

where $\mathbf{J}_b$ is beam current density. For $z > 0$ we use Poisson's equation to eliminate $\mathbf{E}$ and find

$$\frac{\partial \rho}{\partial t} + 4\pi \sigma \rho = - \nabla \cdot \mathbf{J}_b. \quad (2)$$

There is also a charge layer at $z = 0$ not described by eq. (2), but this is not transported charge. The terms of eq. (2) are now integrated over the

entire volume ($z > 0$) to get

$$\frac{\partial Q}{\partial t} + 4 \pi \sigma Q = - \int d^3 r \nabla \cdot \mathbf{J}_b, \quad (3)$$

where Q is net charge

$$Q(t) = \int d^3 r \rho. \quad (4)$$

Let the beam clear the ground plane at time t_1 ; then for $t > t_1$ the rhs of eq. (3) may be converted to a surface integral which vanishes. Hence we have

$$Q = Q(t_1) e^{-4\pi \sigma t}, \quad (5)$$

so Q vanishes as t goes to infinity. Note that if σ is changing in time we have, instead of eq. (5),

$$Q = Q(t_1) e^{- \int_{t_1}^t dt' 4\pi \sigma(t')}, \quad (6)$$

which is non-zero for large t if σ decreases more rapidly than t^{-1} .

Case 2 The channel has finite radius but there is no ground plane -this problem preceeds discussion of the Cooper-Neil result. We consider here a case of somewhat greater generality than they, namely

$$\mathbf{J}_b = J_{b_z}(r, x) \hat{e}_z \quad (7)$$

$$\sigma = \sigma(r, x), \quad (8)$$

where

$$x = vt - z, \quad (9)$$

rather than restricting σ and J_{b_z} to be uniform inside radius a and σ independent of x . The equation of charge continuity is most conveniently written in the

form

$$\frac{\partial \rho}{\partial t} + \nabla_{\perp} \cdot \sigma \mathbf{E}_{\perp} + \frac{\partial J_z}{\partial z} = 0,$$

with subscript $\perp$ denoting components in the plane normal to $\tilde{\mathbf{e}}_z$. Integrating all terms over the entire normal plane one obtains

$$\frac{\partial q}{\partial t} + \frac{\partial I}{\partial z} = 0, \quad (11)$$

where

$$q = \int d^2r \rho = \text{charge per unit length}, \quad (12)$$

$$I = \int d^2r J_z = \text{current}. \quad (13)$$

The current is the sum of beam and plasma contributions:

$$I = I_b + I_p, \quad (14)$$

with
$$I_b = \int d^2r J_{bz}, \quad (15)$$

$$I_p = \int d^2r \sigma E_z. \quad (16)$$

q and I depend only on the variable x , so eq. (11) becomes

$$v \frac{\partial q}{\partial x} - \frac{\partial I}{\partial x} = 0. \quad (17)$$

We integrate once in x to obtain

$$q = \frac{I}{v} + \text{constant}. \quad (18)$$

I vanishes far ahead and behind the pulse but q may go to any constant value there.

We set the constant in eq. (18) equal to zero but this will not effect the

discussion of Cooper and Neil's result. Integrating the terms of eq. (18) over all x gives

$$Q = \int_{-\infty}^{+\infty} dx q = \int_{-\infty}^{+\infty} dx \frac{I_b + I_p}{v} . \quad (19)$$

The quantity

$$Q_b = \int_{-\infty}^{+\infty} dx \frac{I_b}{v} \quad (20)$$

is the beam charge. To calculate the plasma charge

$$Q_p = \int_{-\infty}^{+\infty} dx \frac{I_p}{v} = \int_{-\infty}^{+\infty} dx \int d^2 r_{\perp} \frac{\sigma E_z}{v} , \quad (21)$$

we use Maxwell's equations. These yield

$$\begin{aligned} -\nabla^2 \vec{E} + \nabla (\nabla \cdot \vec{E}) &= -\nabla^2 \vec{E} + \nabla (\nabla \cdot \vec{E}) \\ &= \nabla \times (\nabla \times \vec{E}) = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \vec{B}) \\ &= -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) . \end{aligned} \quad (22)$$

The z component of the electric field therefore satisfies

$$\nabla_{\perp}^2 E_z + \frac{\partial^2 E_z}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} = \frac{\partial}{\partial z} 4\pi \rho + \frac{\partial}{\partial t} \frac{4\pi J_z}{c^2} . \quad (23)$$

Substituting from eq. (9) for t and z we get

$$\nabla_{\perp}^2 E_z + \frac{\partial^2 E_z}{\partial x^2} - \frac{v^2}{c^2} \frac{\partial^2 E_z}{\partial x^2} = -\frac{\partial}{\partial x} 4\pi \rho + v \frac{\partial}{\partial x} \frac{4\pi J_z}{c^2} . \quad (24)$$

The terms of eq. (24) are now integrated over all x ; E_z , ρ and J_z all vanish at $x = \pm \infty$, so we are left with

$$\nabla_{\perp}^2 \int_{-\infty}^{+\infty} dx E_z = 0 . \quad (25)$$

The only well behaved solution is

$$\int_{-\infty}^{+\infty} dx E_z = 0 . \quad (26)$$

It is apparent from eq. (21) that if σ is independent of x then Q_p vanishes. In essence, the plasma charge induced by the increasing beam current of the pulse head is cancelled by an equal charge of opposite sign induced by falling beam current in the tail.

We consider the common situation where σ vanishes ahead of the pulse and is built up in the pulse by the beam-gas interaction. Then Q_p is non-zero in general and in fact is expected to have the sign of Q_b because E_z parallels the beam in the tail and beyond the pulse, where σ is large, while E_z opposes the beam in the head, where σ is small.

Case 3 Discussion of Cooper and Neils' result (that net transported charge is Q_b even in the presence of a ground plane.) The ground plane $z = 0$ is a perfect conductor and produces an image at $(-z, \chi)$ of any particle at $(+z, \chi)$ with reversed velocity and sign of charge. Hence the electromagnetic problem is extended to the entire volume $(-\infty \leq z \leq +\infty)$ with E_z , B_θ and J_z symmetric in z , and E_r and ρ antisymmetric. As the beam pulses emerge from the ground plane (one going in the $+Z$ direction, the other in the $-Z$ direction), a complicated pattern of fields and charges appears. However at time t_1 , when the pulses clear the ground plane, the entire pattern is resolved into two steady state solutions (case 2 described above) and a transient. That is, we have a solution of the type described by eqs. (7) - (26) plus its image plus some

left over fields. Note that each steady state solution has a "tail" which sticks through the ground plane. The sum of the steady state solutions satisfies the symmetry condition at $z = 0$ but it does not satisfy initial conditions at time t_1 . The transient (left over) field also satisfies Maxwell's equations and the symmetry condition. Furthermore, the transient contains no beam charge to act as an inhomogeneous source. Net transient charge is zero on $(-\infty < z < +\infty)$ and its density vanishes at large $|z|$. The system is diffusive so we expect the transient charge density to die out at large time - its positive and negative parts combining at $z = 0$. Thus only the steady state solutions persist, which establishes the advertised result. Recall that the steady state solutions were selected to have q vanish at $x = \pm \infty$. This does not affect the present result because any constant charge added to a steady state solution would be cancelled by its image.

Case 4 Transmission line model A useful model for the bulk treatment of current and charge is as follows. We assume the beam and plasma currents have uniform radial profile inside a channel of conductivity σ and radius a . Charge is taken to be located on a cylindrical shell at radius $R_c \geq a$, and displacement current is concentrated on another shell at very large radius $b \gg R_c$. Then one may derive four coupled equations relating I , I_p , q , and the mean field in the channel $\bar{E}_z$:

$$I = I_b + I_p , \quad (27)$$

$$\frac{\partial q}{\partial t} + \frac{\partial I}{\partial z} = 0 , \quad (28)$$

$$I_p = -\frac{\bar{E}_z}{R} , \quad (29)$$

$$\bar{E}_z = -\frac{\partial}{\partial t} (LI) - \frac{\partial}{\partial z} \left(\frac{q}{C} \right) \quad (30)$$

R, L, and C are respectively the resistance, inductance and capacitance per unit length and are given by the approximate formulas

$$R = (\pi a^2 \sigma)^{-1}, \quad (31)$$

$$C^{-1} = 2 \log \left(\frac{b}{R_c} \right), \quad (32)$$

$$L = \frac{1}{c^2} \left[2 \log \left(\frac{b}{a} \right) + \frac{1}{2} \right] \quad (33)$$

The radius b is selected to be of the same order magnitude as the length of the system considered - since it appears in the log function its magnitude need not be precisely defined. In general R, C and L are allowed to vary with t and z. If a conducting ground plane is placed at z = 0 then the method of images may be applied; q is antisymmetric while I, I_p and $\bar{E}_z$ are symmetric in z.

It was stated but not proved in case (3) that a transient charge vanishes at large time. This property is readily demonstrated with the transmission line model. We let R, L and C be constant for all z and examine the behavior of a mode of the appropriate form

$$q = \tilde{q} e^{-pt} \sin(kz), \quad (34)$$

$$\frac{\bar{E}_z}{R} = I_p = I = \tilde{I} e^{-pt} \cos(kz). \quad (35)$$

Then eqs. (28) - (30) yield

$$p = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{k^2}{LC}} \quad (36)$$

Wave number k must be greater than zero since the charge does not reach z = ∞ in a physical initial value problem. Hence the real part of p is positive and the mode decays exponentially with time.

Reference

1. V. K. Neil and R. K. Cooper, UCID 17346, "Relativistic Particle Beam in a Semi-infinite, Axially Symmetric Conducting Channel Extending from a Perfectly Conducting Plane, December 13, 1976.

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