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METHODS IN PROBABILITY AND
STATISTICAL INFERENCE

Progress Report
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ABSTRACT

I. Statistical Computing; Approximations to Statistical Tables and Functions.

David L. Wallace is developing efficient methods for approximating percentage points of the largest characteristic root of a Wishart matrix, and other statistical quantities of interest. He is also working on fitting non-additive models to two-way and higher-way tables, and the further development of the SNAP statistical computing systems.

II. Numerical Computation of Boundary-Crossing Probabilities for Brownian Motion and Related Stochastic Processes.

Michael J. Wichura has implemented numerical procedures for computing first passage distributions to a "moving" boundary for various diffusion processes, including Brownian motion and the Ornstein-Uhlenbeck process.

III. Probabilistic Methods in Statistical Mechanics.

Sandy Zabell and R. R. Bahadur have applied mathematical techniques from statistical mechanics to obtain a unified treatment of probabilities of large deviations of the sample mean, in the setting of locally convex topological vector spaces. Currently under investigation are large deviation results for dependent sequences and Lévy processes, and asymptotic expansions for large deviation probabilities of the Cramér type.

IV. Combining Independent Tests of Significance.

Michael D. Perlman and his co-author James A. Koziol are nearing completion of a comparative study of classical "omnibus" and Bayes procedures for combining several independent noncentral F-test statistics. Their earlier paper on the related problem of combining noncentral chi-square tests has been accepted for publication.

V. Small-Sample Efficiencies of Tests and Estimates.

R. R. Bahadur and David L. Wallace, together with James A. Koziol, are engaged in a numerical study of the small-sample powers of the Pearson chi-square and likelihood ratio tests for multinomial goodness-of-fit. They are also studying the relationship between asymptotic (large-sample) efficiency of test statistics, as measured by Bahadur's concept of exact slope, and actual small-sample efficiency. They have found that in the case of chi-square versus likelihood ratio tests, the actual small sample efficiency is better predicted by approximate slopes than by exact slopes.

VI. Improved Procedures for Simultaneous Estimation and Testing of Many Correlations.

Michael D. Perlman and a graduate student S. P. Lin have developed a promising new technique for the simultaneous estimation of all correlation coefficients in a multivariate population. Their method adapts the James-Stein "shrinking" estimator (for location parameters) to the estimating of correlations. Perlman and Kinley Larntz have obtained a simple statistic for testing equality of several correlation matrices. Their method requires no matrix inversions, may be carried out with a hand calculator, and is appropriate even for small sample sizes.

VII. Statistical Computing and Improved Regression Methods.

Ronald Thisted is studying alternatives to least squares regression when the data are ill-conditioned, with primary emphasis on ridge regression and empirical Bayes procedures. In addition, he is continuing his work on developing criteria for the evaluation of statistical software systems, and on the development and analysis of statistical algorithms.

VIII. Comparison of Several Populations.

James A. Koziol has developed techniques for making multiple comparisons among several populations differing by a scale parameter, such as exponential distributions with varying mean-lifetime. He has also developed a multiple comparison technique for comparing the means of several populations when the samples involved are subject to unequal patterns of censorship; this technique has application to the analysis of incomplete survival data. Further, Koziol has tabulated the null distribution of a certain nonparametric test for bivariate symmetry.

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A. INTRODUCTION

This report describes the research activities of the Department of Statistics, University of Chicago, carried out or in progress under the auspices of ERDA Contract No. EY-76-S-02-2751.*000, during the period 15 June 1976 to 14 June 1977. Eight research projects are described in Part B of this report. Part C contains a description of the statistical consultation activities of the Department which are of interest to ERDA. Attention is especially directed to the series of scientific interactions between the Department and the scientists at Argonne National Laboratories.

Approximate percentage efforts devoted to the research projects supported by this Contract are as follows: Bahadur 10%, Perlman 20%, ~~Thisted~~ 25%, Wallace 25%, Wichura 25%, Zabell 20%. These percentages are expected to continue through the duration of the current term. There has been complete compliance with all requirements of the Contract.

B. RESEARCH PROGRESS REPORT: 15 June 1976 - 14 June 1977

I. Statistical Computing; Approximations to Statistical Tables and Functions

Investigator: David L. Wallace

(a) Approximating tables of critical values, etc.

A number of tables arising in statistical theory take the form $Y(p,q)$ with two entry arguments (and often others, such as probability level). Additive (in p and q) fits either to Y or to a well chosen transform $Z = g(Y)$ sometimes result in approximations easy to understand and to implement. This traditional approach is close in spirit and technique to two-way fitting by least squares or by more resistant methods. Tukey's 1949 One degree of freedom for nonadditivity (1DOFNA) technique and its modern variants guide the choice of g .

If additivity is not good enough, mixed additive and multiplicative fits (Mandel-Gollob fits) are a useful extension, easily implementable on digital computers, and providing potentially high precision. Unweighted (or proportionally weighted) least squares over a rectangular grid leads to a closed form (singular-value decomposition) fit. Being nonlinear, the fit cannot use the 1DOFNA techniques to guide the choice of g . Especially with tables where the potential range of argument is unbounded, least squares has no great virtues and can give unsatisfactory fits.

One class of situation arises often enough to be worth special study. A border of the table, say $Y(p,1)$ and $Y(1,q)$, may be known exactly in terms of conveniently simple functions. In that case, making an additive fit to match the border may be highly useful and worth the slight apparent degradation from an overall criterion. A second border (say, $Y(p,2)$ and $Y(2,q)$) is sometimes known or computable with relatively less effort and/or imprecision than for the bulk of the table. A convenient choice of fit -- for given g -- is:

$$\hat{Y}(p,q) = g^{-1}(\hat{Z}(p,q))$$

where

$$\hat{Z}(p,q) = Z(1,1) + U(p) + V(q) + \frac{W(p,2)W(2,q)}{W(2,2)}$$

$$Z(p,q) = g(Y(p,q))$$

$$U(p) = Z(p,1) - Z(1,1)$$

$$V(q) = Z(1,q) - Z(1,1)$$

$$W(p,q) = Z(p,q) - U(p) - V(q) - Z(1,1).$$

This algorithm trivializes the fitting for known g , but leaves the choice of g to systematic exploration and good intuition. The choice of criteria

remains. Extension to higher rank factor fits is obvious, but the applicability and gains are less clear.

An initial exploration has been carried out for tables of the expected value of the first root of a Wishart matrix (Mandel, Technometrics 1971) -- essentially the table needed for some of the theory of sampling error in fitting mixed additive factor models. For this table, the best transformation g of the form $(y+c)^\lambda$ turns out to be very nearly $c = 0$, $\lambda = 1/2$, perhaps the most natural a priori choice. The table is ideally suited to the fit because the first two (and only these) margins are known in closed form:

$$Y(q,1) = Y(1,q) = q$$

$$Y(q,2) = Y(2,q) = q + 2^{q-1} \frac{\Gamma^2\left(\frac{q+1}{2}\right)}{\Gamma(q)},$$

from which the fit is determined by the algorithm.

(b) Simplified probability plotting and displaying distribution approximations.

Probability plotting is a useful exploratory and diagnostic technique. Its value and general utility have often been limited by an insistence on a precise choice of plotting positions, by the need to estimate parameters, and a necessity for specially designed graph paper or else extensive tables or non-trivial computer programs. Such precision is inappropriate for exploratory graphical methods. In what follows, gamma (or chi-square) plotting is used as an important illustrative example of the development and exposition of approximate techniques.

The Wilson-Hilferty cube root transformation of chi-squares provides an approximate normalization that is at once moderately precise and convenient. Its superiority for moderate to large degrees of freedom is well known, and its convenience for graphing more than balances the superior precision of the Peizer-Pratt normalization. For graphing, the cube-root is quite tolerable even down to one degree of freedom. Figure 2 on page 5 shows the quality of the approximation. (Of course, for one degree of freedom, the "exact" half-normal plot is equally simple and preferable.)

If Y is purportedly distributed like $c\chi_v^2$, then

$$\sqrt[3]{Y} \sim \sqrt[3]{cv}\left(1 - \frac{2}{9v}\right) + \sqrt[3]{cv} \sqrt{\frac{2}{9v}} Z$$

where Z is unit Gaussian. Hence making an ordinary full normal (= rankit) plot of the cube-roots of a sample $Y_1, \dots, Y_n$, should produce nearly a straight line with intercept and slope simply related to c and v . Linearity of the plot is a check on Y being a multiple of chi-square, and graphical estimates of c and v are easily obtained. If c and v are known a priori, then the plot should resemble a specific line. Often v is known, but c is

unknown, and in this case the plot should correspond to a line with prescribed intercept $\left(\sqrt{\frac{9v}{2}} - \sqrt{\frac{2}{9v}}\right)$ on the rankit (i.e., z) scale. Figure 3 on page 5 is an example of a rankit plot of cube roots of a chi-square sample on one degree of freedom.

Sampling variation prevents even truly normal samples from yielding straight lines. For truly chi-square samples, the slight curvature introduced because of the inexactness of the cube root normalization will be small relative to sampling fluctuation except for very small degrees of freedom or very large samples, or in the (often uninteresting) lower tail of the sample. Several approaches to graphical and quantitative documentation and elaborations of these assertions are in progress.

The exploration of simplified probability plotting is an outgrowth of a study of ways to display distribution approximations. Parameter plots of the form

$$\{R(F(x)), R(\hat{F}(x))\},$$

such as Figure 2 on page 5, are useful in conveying overall impressions, but not for showing detailed behavior of the approximation. Plots of the form

$$\{R(F(x)), R(\hat{F}(x)) - R(F(x))\},$$

such as Figure 1 on page 5, show the detailed error structure compared to a horizontal reference line. In both Figure 1 and 2, the plotting transformation R was taken to be G^{-1} , the inverse cumulative of the unit Gaussian, while $F, \hat{F}$ denote the cumulative d.f. of $\sqrt[3]{Y}$ and its approximation. Varying the choices of R , and, in the second form, using a separate choice of R for the horizontal and vertical coordinates, provide for useful variants. Detailed exposition and comparisons of such displays is proposed.

(c) Statistical computing

Making the interactive graphic capabilities of the TEKTRONIX 4013 graphics terminal available to data analysts and other users was the chief development of the past year. With the library of FORTRAN subroutines comprising the TEKTRONIX software package at hand, the FORTRAN programmer has potentially full use of the terminal. But for practical data analysis, the graphing must be combined with an interactive program providing system control and full data analytic capabilities. The open-ended structure of the SNAP statistical system as developed and used here over the past eight years provided a convenient and direct implementation for computing under the TSO time sharing option on the IBM 370/168 computer. Several TEKTRONIX graphical commands were easily added to the SNAP system, and have met the basic needs of graphing. Specialized graphing operations, such as contour plotting, will be added as new commands available to all users. Any user can incorporate special graphical and analytic subprograms in the SNAP system for his/her own private use.

The SNAP system continued in heavy use in introductory and advanced courses in Statistics, and in increasingly varied use to construct flexible and specially tailored data analytic procedures for research investigations. Besides the addition of graphic operations, developments in SNAP during the year included improved interactive control, harmonic analysis commands, and commands facilitating passing data between the SNAP system and others (SPSS, BMD, or user FORTRAN programs), either to enable each system to be used where it is best, or to enable data to be moved into SNAP as the easiest route to graphing.

Other graphing developments in progress include the incorporation of TEKTRONIX graphing operations into the IDA (Interactive Data Analysis) BASIC system for the HP-2000 computers. During the forthcoming year, a Digital Equipment Corporation DEC-20 computer will become available for regular interactive computing at the University. The TEKTRONIX software was developed originally for the compatible DEC-10 computer, and several different routes for improved graphical techniques will become available. Planned development activity includes adapting the SNAP system for use on the DEC-20 by the local user community.

II. Numerical Computation of Boundary-Crossing Probabilities for Brownian Motion and Related Stochastic Processes.

Investigator: Michael J. Wichura

The "Anderssen-deHoog-Weiss" computational procedure for determining first passage time distributions and boundary crossing probabilities for Brownian motion and Brownian bridge (outlined in the May 1975 proposal) has been implemented and extended to apply to other diffusion processes for which the (not necessarily stationary) transition densities may be computed without undue expense. An example of such a process is the Ornstein-Uhlenbeck process, with or without drift. Even though this process may be transformed into Brownian motion by a change of time and state, it was found that the transformation typically converts a well-posed problem for the O.U. process into a nearly singular problem for the B.M. process; thus it is necessary to work directly with the Ornstein-Uhlenbeck process.

The program utilizes flexible input and output procedures which allow the user considerable freedom in setting up the problem and outputting the results of the computation. The user may select one of several "in-house" families of parameterized moving boundaries, or provide his own. Similarly, the user may select one of several "in-house" diffusions, or provide his own. The output routines permit the user to determine the amount of information printed and to store results for future analysis. A computer graphics routine allows the user to get immediate visual display of first passage densities and distribution functions, and boundaries. Higher resolution plots can be obtained using a Tektronix 4013 Graphics and Display Terminal.

The program is, of course, of wide applicability, and several members of the University community are either using it in their own research or have expressed interest in doing so. For example, Professor Luigi Ricciardi of the Department of Biophysics and Mathematical Biology, along with one of his graduate students, Charles Smith, have been using the program to study a problem in neuro-physiology which involves the first passage time of an Ornstein-Uhlenbeck process with drift to a moving boundary. No analytic solution is available for their problem, and a program such as this is the only feasible means of attack. The figure below is a low resolution plot of the first passage time densities to an exponential boundary for three Ornstein-Uhlenbeck processes having a common drift parameter but differing diffusion constants.

Present development of the program is being determined in part by requests of users for new and/or improved features. Calculation of various statistics (means, standard deviations, modes, percentiles), curve-fitting, and adaptive integration schemes are under study.

III. Probabalistic Methods in Statistical Mechanics

Investigators: R. R. Bahadur and Sandy Zabell

During the last year, R. R. Bahadur and S. Zabell have investigated the application of a technique in statistical mechanics (due to Lanford [3]) to the theory of large deviations. Lanford's theory is partially worked out in the case of a finite-dimensional vector space and bounded random vectors; Bahadur and Zabell have expanded the theory and extended it to the case of possibly infinite-dimensional vector spaces and possibly unbounded random vectors. Their conclusions are given in detail in Tech. Report No. 36 of the Department of Statistics, University of Chicago (January, 1977); the following is a brief outline.

Let $X_1, X_2, \dots$ be an i.i.d. sequence of random elements with values in a topological vector space V , and let $\bar{X}_n$ be the mean of the first n elements of the sequence. If A is a subset of V such that $n^{-1} \log P(\bar{X}_n \in A)$ tends to a limit as $n \rightarrow \infty$, this limit is called the entropy of A . It is shown that every open convex set has an entropy, and that the entropy of such a set A is the maximum over A of a function on V which is called the point entropy function. A number of general methods for finding the point entropy function are developed.

Bahadur and Zabell apply the general theory to several examples, including the following infinite dimensional ones. (i) Suppose that V is a separable Hilbert space and that the common distribution of the X_i is Gaussian. An explicit formula for the point entropy is obtained and used to compute the entropy of the complement of the unit ball. (ii) Suppose that the X_i are Brownian motions or Brownian bridges on the interval $[0, 1]$. It is shown that in these cases the point entropy of a path is minus the kinetic energy of the path; hence the entropy of an open convex set is minus the minimum energy of paths in the set. (iii) Suppose that $Y_1, Y_2, \dots$ is an i.i.d. sequence of real random variables, and for each n let F_n be the empirical distribution function based on $\{Y_1, \dots, Y_n\}$. The exponential rate at which the probability that F_n lies in a given set of distribution functions tends to zero was first studied by Sanov [4] and subsequently by many other authors. Bahadur and Zabell take X_i to be the probability distribution degenerate at Y_i , and they regard $X_1, X_2, \dots$ as an i.i.d. sequence in the vector space of functions of bounded variation, with $\bar{X}_n \equiv F_n$. They show that Sanov's conjecture is valid for every weakly open convex set of distribution functions, and that it is not valid for all weakly open sets.

A number of applications of the above conclusions to the theory of estimation are given in the Technical Report No. 36 mentioned earlier.

Some topics that Bahadur and Zabell intend to explore next are large deviation probabilities for dependent sequences (e.g., Markov chains), for Lévy processes,

and asymptotic expansions of the Cramér type [1], [2] for such probabilities.

References

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IV. Combining Independent Tests of Significance

Investigators: - James A. Koziol and Michael D. Perlman

The general problem under consideration is that of combining independent tests of significance when the distributions of the test statistics belong to a known parametric family. For example, it is often the case that the test statistics to be combined follow chi-square or F distributions, noncentral under the alternative. In such cases it is possible that combination procedures may be found which are more sensitive than "nonparametric" procedures such as Fisher's "omnibus" procedure based on the product $\prod P_i$ of the attained significance levels.

Perlman and Koziol have carried out an intensive investigation of the problem of combining independent chi-square or F tests. Their major theoretical contribution has been the construction of several families of "conjugate" prior distributions for noncentrality parameters, which families are wide enough to approximate any unimodal prior density fairly well, yet which easily may be inverted to obtain Bayes estimates and tests. They also have performed an extensive numerical study of these procedures and their "nonparametric" competitors such as Fisher's.

In this work they first developed usable approximations to the null distributions of the test statistics they derived, such as linear combinations or products of independent chi-square, F, or beta statistics. These results are of interest in their own right. Next, they computed the powers of the Bayes procedures and nonparametric competitors by exact numerical integration and Monte Carlo simulation. This program has been completed for the chi-square case, resulting in a paper [A] recently accepted for publication (after revision) in the Journal of the American Statistical Association. In this paper it is reported that the Bayes tests indeed present substantial improvement over the nonparametric tests when prior information is available, but that if no such information is available Fisher's test is usually preferable under a minimax criterion. (The exceptional case occurs when the chi-square statistics to be combined each have 1 degree of freedom, in which case the simple test based on the sum of the chi-square statistics is preferable on minimax grounds).

For the more complex problem of combining non-central F statistics, the theoretical and computational work has been completed. Perlman and Koziol are currently engaged in compiling these results in a second paper [B] to be submitted to JASA. This project should be completed by the end of summer, 1977. (Recently a paper by Monti and Sen [C] has appeared dealing with the problem of combining independent noncentral F statistics $F_1, \dots, F_n$. These authors show that the locally most powerful test statistic for alternatives restricted to a straight line emanating from the origin in the space of noncentrality parameters is a linear combination $\sum c_i U_i$ of the beta statistics $U_i = F_i / (1 + F_i)$, and advocate the use of this combination procedure. In our much more extensive study we have derived this statistic as a Bayes combination procedure and compared it with five other Bayes procedures and four nonparametric procedures. We find that the Bayes test based on the statistic $\prod (1 - U_i)^{a_i}$ is just as powerful as $\sum c_i U_i$ against alternatives most probable under unimodal prior distributions, while the former test is more powerful against less probable alternatives. That is, the former test is less sensitive to inaccuracies in the choice of prior distribution.)

References

- [A] Koziol, James A. and Perlman, Michael D. (1978). Combining independent noncentral chi-square tests. To appear in the Journal of the American Statistical Association.
- [B] Koziol, James A., Perlman, Michael D., and Rasmussen, U. Combining independent noncentral F tests. In preparation.
- [C] Monti, K., and Sen P.K. (1976). The locally optimal combination of independent test statistics. Journal of the American Statistical Association, Volume 71, 903-911.

V. Small-Sample Efficiencies of Tests and Estimates.

Investigators: R. R. Bahadur, James A. Koziol, and David L. Wallace

We study and compare the distribution of the exact significance level achieved by two tests -- the likelihood ratio and chi-square -- of a simple trinomial hypothesis, for various moderate sized samples. In this initial discussion, we consider a fixed null hypothesis and a fixed alternative. For any sample size N , and any prescribed Type II error, β , there is a unique randomized test, whose type I error will be denoted by $\alpha(N, \beta)$.

Empirical study demonstrates that the behavior of $\alpha(N, \beta)$ is most readily grasped when the errors α , β , are measured on a logit scale. Accordingly let y and u denote the logits of α and β , and corresponding to the relation $\alpha = \alpha(N, \beta)$, introduce $y = g(N, u)$.

Asymptotic theory of exact significance levels relates to the exponential rate of convergence of α to 0, where $-\ln \alpha$ and the negative logit y are essentially equivalent. That asymptotic theory asserts that, under appropriate conditions,

$$\lim_{N \rightarrow \infty} \frac{g(N, u)}{N} = c$$

where c is the "exact half slope" of the test and does not depend on the level u .

If the asymptotic limit were approached rapidly, then a comparison of tests could be based on a comparison of exact half-slopes. Inasmuch as the likelihood ratio has larger exact slopes than chi square for all but very special alternatives where the two slopes are the same, the likelihood ratio should be preferable. A chief finding of our examination of moderate sample performance is that the asymptotic comparison is seriously misleading, and the chi square is about as often superior as inferior, and that this comparison appears to hold over a wide range of sample sizes and interesting significance levels.

The functions $g(N, u)$ have been evaluated numerically for several null hypotheses, and a wider range of alternatives, and for type II errors of 10%, 20%, 50%, 75% (equispaced on a logit scale). For purposes of illustration, we present here the graphs of $g(N, u)$ versus N , $7 < N < 97$, for the aforementioned four type II errors, in the case that the null hypothesis is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the alternative is $(.52, .24, .24)$; Figure 1 on page 15 gives the graphs for the chi square test, while Figure 2 gives the graphs for the likelihood ratio test. The vertical scale corresponds to significance levels from .80 to .000050. The behavior is typical. Superposition of the two Figures shows chi square to be superior to likelihood ratio, and that the chi-square advantage does not disappear with increasing N . Figure 3 is a repeat of Figure 2 on log rather than logit scale. Figures 2 and 3 are almost identical except in the range of moderate to large α , where the logit plot is evidently preferable.

Within the range of N studied, the dominant effect is linear, but the slope is not well-approximated by the exact half-slope. Changing u (or β)

amounts nearly to a translation, with the amount nearly proportional to the change in u (logit scale again works well). The change in y is not quite constant, but increases slowly with N .

The local behavior is not smooth, and not even monotone in N ; this phenomenon arises from discreteness and the lattice structure of the underlying distribution. The detail of roughness is seemingly erratic, varying sharply with N , with u , with the test statistic, and with the alternative hypothesis.

Smoothed versions of each function can be constructed that capture a major part of the behavior and facilitate comparison of tests. Doing this in a way that can build on an incomplete range of sample sizes and that can be extrapolated to larger N presents technical problems not fully solved.

Assessing the importance of differences between statistical procedures is often usefully done in terms of ratios of sample sizes required to meet some criterion. That approach is attractive here, once smoothed functional relations are obtained. Preliminary studies suggest that the ratio of sample sizes is nearly proportional to the rates not of exact slopes but of approximate slopes as conjectured by Bahadur (1967). Further explorations, especially toward this last conjecture, are being pursued.

Reference

R. R. Bahadur (1967). Rates of convergence of estimates and test statistics.
Ann. Math. Statistic 38 303-324.

NEG LOGIT ALPHA US N CHI-50

111: 2-40-36-7 N

Figure 1

NEG LOGIT ALPHA US N LIK-RATIO

111: 2-40-36-7 N

Figure 2

NEG LN ALPHA US N LIK-RATIO

111: 2-40-36-7 N

Figure 3

VI. Improved Procedures for Simultaneous Estimation and Testing of Many Correlations.

Investigator: Michael D. Perlman

The need for simple yet accurate procedures to test the equality of two or more population correlation matrices has long been recognized. Seal ([7], p.124) and Jennrich ([4], p.910) have noted that this testing problem arises in biological applications and factor analysis. However, due to the large number of correlations ($p(p-1)/2 \doteq p^2/2$) present in p-variate populations, and due to the great computational complexities arising in the calculation of maximum likelihood estimates and the LRT, no satisfactory procedures currently exist for small or moderate sample sizes. In 1967 Kullback [5] proposed a simple test statistic and claimed an asymptotic chi-square distribution under the null hypothesis, but two years later Aitken [1] showed this distribution theory to be incorrect, and that in fact the asymptotic null distribution of Kullback's statistic depends on the unknown parameters. In 1970 Jennrich [4] proposed a reasonable test statistic whose asymptotic null distribution is indeed chi-square. The accuracy of this approximation depends, however, on the approximate multivariate normality of the sample correlation matrix, and therefore requires rather large sample sizes. Furthermore, calculation of Jennrich's statistic requires the inversion of two $p \times p$ matrices.

Perlman and Kinley Larntz [6] have recently developed a simple statistic for testing equality of several correlation matrices which requires no matrix inversion, may be computed by means of a hand calculator, and whose asymptotic distribution is approached much more rapidly than that of Jennrich's statistic. In the two-sample case the statistic of Larntz and Perlman is based simply on

$T_{LP} \equiv \max\{|z_{ij}^{(1)} - z_{ij}^{(2)}|, 1 \leq i \leq j \leq p\}$, where $z_{ij}^{(\alpha)}$ is the z-transform of $r_{ij}^{(\alpha)}$, the ij -th sample correlation coefficient from the α -th population. In the past two years Larntz and Perlman carried out Monte Carlo studies comparing the performance of their statistic T_{LP} with that of Jennrich's statistic T_J (and two modifications) in the null and non-null cases. As mentioned before, for small sample sizes T_{LP} is clearly preferable to T_J . For large sample sizes T_J is more powerful than T_{LP} for some alternatives while T_{LP} dominates T_J for other alternatives. This work is nearing completion.

The problem of simultaneously estimating all correlations among the components of a multivariate population is extremely important, yet difficult if the dimension is not small and if accurate estimates are required. The reason for this is quite simple: if the multivariate observations are vectors of dimension p , then there are $p(p-1)/2 \doteq p^2/2$ correlations to be estimated (by contrast, there are only p means). Thus, much larger sample sizes are required to obtain accurate estimates of the population correlations. It is customary to warn students in introductory statistics courses about the expected inaccuracies of estimates of correlations, especially if the dimension p is large and the sample size n is not. In particular, estimates of correlation coefficients are often too large in magnitude.

Therefore, it would seem essential to develop improved techniques for obtaining simultaneous estimates of correlation coefficients. In the past twenty years, improved techniques for estimating multivariate location parameters (e.g., the population mean vector) have been developed and studied in detail. Notable contributions to this problem have been made by Charles Stein (cf. James and

Stein [3]), Efron and Morris [2], and others. It has occurred to Perlman and a doctoral student S.P. Lin that these techniques might be applied to the problem of simultaneous estimation of correlation coefficients. Their idea is to reduce the latter problem to the former by applying the Fisher z-transformation to each of the $p(p-1)/2$ sample correlation coefficients. This transforms the population correlation coefficients into location parameters for large sample sizes, and does so approximately for smaller sample sizes, so that the James-Stein and Efron-Morris estimation techniques may be applied to the transformed statistics. The transformation is then reversed, yielding estimates of the population correlations. Numerical studies indicate that this method yields estimates more accurate than the maximum likelihood estimates, both for large and small sample sizes. S.P. Lin is currently carrying out an extensive numerical investigation as part of his doctoral research. This work should be completed by June 1977.

The technique just described should be expected to lead to an improved estimate (relative to the maximum likelihood estimate) of the entire covariance matrix and in particular of its characteristic roots. Charles Stein, in the 1975 Rietz Lecture presented in Atlanta to the Institute of Mathematical Statistics, presented an alternate technique for obtaining improved estimates of these population roots. As part of their investigations, Lin and Perlman have compared these two techniques for estimating characteristic roots, and have laid out guidelines for their use.

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Other Research Projects of M. Perlman

During the past year (and continuing now) Perlman has also conducted research concerning the following problems, which are jointly supported by the National Science Foundation and ERDA:

Probability inequalities, Schur functions, and majorization with applications to tests of hypotheses and simultaneous confidence regions in multivariate analysis.

Complete class theorems and admissibility results for non-exponential families of distributions arising in the study of invariant tests of multivariate hypotheses.

Schur-ordering of tail probabilities of sums of independent gamma variates with unequal scale parameters, with applications to monotonicity properties of the power functions of tests for covariance matrices and other problems.

Unbiasedness and monotonicity properties of the power functions of goodness-of-fit tests for the multinomial and other discrete distributions.

VII. Statistical computing and improved regression methods

Investigator: Ronald Thisted

(a) Evaluation of statistical software

Thisted has been actively involved in the evaluation of statistical software. In August 1976 a report [4] on the comparison of several statistical computer packages was presented as an invited paper at the American Statistical Association meetings. In that study, package users with varied backgrounds and experience compared three packages for usefulness and usability on several different problems in data analysis. In each case, the "best" package was a function both of the problem to be solved and of the individual user's perceptions of the best approach to the problem. Thus, no package could be considered "best" for all problems, or even for all users on a single problem. It was shown, however, that guidelines could be supplied to assist these users in selecting the package(s) best suited to their individual needs in solving problems. Work has continued on developing criteria which scientists and teachers can use to select statistical programs wisely and to utilize them efficiently. The emphasis is on "adaptive" criteria, which individual users can adapt to their own particular scientific needs, budgets, and available computing systems.

(b) Statistical computing

In work related to statistical computing, Thisted is compiling notes on methods for generation of random numbers on digital computers. He has developed a fast, memory-efficient algorithm for computing the distribution of a large class of two-sample rank tests, called "simple procedures" by Switzer [2]. In addition, work is in progress (with T. Sager) on developing algorithms for computing maximum likelihood estimates in isotonic modal regression problems, i.e., in problems where the modes of the populations are known to satisfy certain order restrictions. Thisted has been invited to speak on the teaching of statistical computing at the meetings of the American Statistical Association in Chicago in August 1977.

(c) Improved regression methods

The problem of improved regression methods, particularly those related to ridge regression [1] and to minimax empirical Bayes procedures [3], is under continued investigation. A report (with C. Morris) on several aspects of ridge regression estimators and their relationship to certain minimax estimators for the mean of a multivariate normal distribution is in progress. Thisted expects to continue his work on the behavior of stepwise regression procedures and on the analysis of Monte Carlo studies in statistical research.

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VIII. Comparison of several populations

Investigator: James A. Koziol

- (a) Multiple comparisons of K populations differing by scale parameters (with N. Reid).

A typical problem arising in life testing situations is that of determining whether K populations differ from one another by a shift in scale: for example, upon observing independent samples of exponentially distributed random variates from K populations, one might wish to detect multiplicative treatment effects. Given samples from two populations with underlying exponential distributions, the statistic introduced by Savage (1956) yields the locally most powerful rank test of the null hypothesis that the two populations are identically distributed, against the Lehmann alternative that they differ by a scale parameter. This two sample test may be extended to a test against differences in scale when $K > 2$ samples are given, using a general form of K sample rank tests described by Hájek and Sidák (1967, Section III.4.1).

When it is of less importance to test whether the K populations are identically distributed than to determine which populations differ from one another on an individual basis, a multiple comparisons procedure is preferable to a single global test. Koziol and Reid have developed two nonparametric methods of comparing K populations. Under the first procedure, all the observations in the combined K samples are jointly ranked, and populations π_i and $\pi_{i'}$ are compared with average rank scores as determined from the Hájek-Sidák K sample generalization of the Savage test. Under the second procedure, populations π_i and $\pi_{i'}$ are compared with rank scores from the pairwise ranking of their respective samples, this pairwise ranking excluding observations from the other $K - 2$ samples. It is shown that the two methods of comparing all pairs of populations have equivalent asymptotic properties, including the same asymptotic relative Pitman efficiency.

- (b) Multiple comparisons among K samples subject to unequal patterns of censorship.

A frequently arising problem in medical studies is the analysis of incomplete survival data. Suppose, for example, that a study is undertaken to assess the efficacies of various operative techniques of coronary bypass surgery, the parameter of interest being length of post-operative survival. At the conclusion of the study, there may be incomplete survival information on certain individuals: it may be known that a particular individual was alive a given length of time following surgery, but nothing is known concerning his survival status subsequent to that time. The complete survival information of that individual is lacking, for the length of his post-operative survival is unavailable, being censored at a given time point.

Among the statistical procedures for testing equality of the underlying survival distributions of two populations when observations are arbitrarily censored is Gehan's modifications of the Wilcoxon two sample rank test.

Breslow (1970) has proposed a generalization of Gehan's procedure for testing equality of K continuous survival distributions when observations in each of the $K \geq 2$ samples may be subject to arbitrary censorship. However, a single global test of whether the K populations are equivalent is somewhat uninformative; in the coronary bypass surgery study, for example, it would be of greater interest to select which, if any, operative techniques differ from one another in terms of enhanced survival. Koziol has addressed this problem, and derived a multiple comparisons procedure for arbitrarily censored populations. The procedure is illustrated with data from a recent coronary bypass surgery study.

(c) A test for bivariate symmetry based on the empirical distribution function.

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be a sequence of independent bivariate vectors distributed according to an arbitrary absolutely continuous distribution F , and consider the hypothesis of bivariate symmetry:

$$H_0 : F(x, y) = F(y, x), \text{ all } (x, y). \quad (1.1)$$

Hollander (1971) proposed a procedure for testing (1.1) based on the bivariate empirical distribution function F_n of the observations: the hypothesis of bivariate symmetry would be rejected for suitably large values of Hollander's statistic H_n^2 , with H_n^2 defined as

$$H_n^2 = n \iint [F_n(x, y) - F_n(y, x)]^2 dF_n(x, y). \quad (1.2)$$

Under (1.1), the statistic H_n^2 is not invariant under transformation of the coordinates by the same monotone function, hence is not distribution-free. In view of this Hollander determined the permutation distribution of H_n^2 conditional upon the observed sample, and thereupon proposed a conditional distribution-free test of bivariate symmetry.

Suppose that the null hypothesis (1.1) is restricted to

$$H_0 : F(x, y) = F^*(x) F^*(y), \text{ all } (x, y), \quad (1.3)$$

where F^* is an arbitrary absolutely continuous distribution function. (1.3) being a subhypothesis of (1.1), it follows from Hollander (1971) that H_n^2 is a consistent test statistic for (1.3). The asymptotic distribution of H_n^2 is distribution-free if (1.3) obtains, hence is of interest. Koziol has investigated this asymptotic distribution, and demonstrated that under (1.3), H_n^2 converges in distribution to

$$H^2 = \iint W^2(s, t) ds dt, \text{ where } W \text{ is a bivariate Gaussian process on the unit}$$

square. The eigenvalues and eigenfunctions of the integral operator associated with the covariance kernel of W are found, and the Karhunen-Loève expansion of the process W is given. By an argument analogous to that used in establishing the asymptotic distribution of the Cramér-von Mises statistic, it is proved that the distribution of H^2 is that of an infinite weighted sum of chi-square variates each with one degree of freedom, the weights being the eigenvalues of the covariance kernel of W . Critical values of H^2 are obtained by numerical inversion of its

characteristic function. A general formulation for the asymptotic power of H_n^2 in testing (1.3) against a sequence of local alternatives is given. A particular case, a version of the sphericity hypothesis [Anderson (1958), section 10.7] for bivariate normal observations, is examined in greater detail, with a numerical power comparison of H_n^2 with the best parametric test being made.

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C. CONSULTATION AND COOPERATIVE RESEARCH WITH OTHER SCIENTISTS AT
THE UNIVERSITY OF CHICAGO AND ARGONNE NATIONAL LABORATORIES

Through its consultation program the Department of Statistics participates in other research projects at the University of Chicago and Argonne National Laboratories which are of interest to ERDA. Alan Izenman has served as the coordinator of the consultation program during 1976-77. All members of the Department of Statistics, as well as its graduate students, participate in the consultation program to varying degrees. Some recent activities of the program are now described.

Consultation/Cooperative Research at the University of Chicago

Statistical Analysis of Nuclear Radiation Detector Counts. The following problem was brought to the consulting unit by Mr. Michael Neumann, a physicist at the Franklin McLean Cancer Research Institute (FMI) and Mr. John Shepherd, Chief Computer Engineer for the Institute for Computer Research (ICR). They were interested in testing the sensitivity of a new type of gamma-ray detector; such detectors are commonly used in medical diagnostic studies of the metabolic and physiologic functions of various organs, including the brain. Mr. Neumann had designed and constructed a detector which, on the basis of some preliminary studies, appeared to have extremely favorable performance characteristics when compared to similar detectors currently in use.

Several experimental runs were carried out in Mr. Neumann's laboratory using a single point source of radiation (germanium 68, typically preferred for brain imaging) and a detector of the kind constructed in that laboratory. These runs were of various durations, with the majority 100secs. Interest centered first on showing that for an experimental run of given duration, the intervals between successively recorded events followed an exponential distribution. A frequency distribution (or, histogram) of interval lengths for each run was computed by Mr. Shepherd. However, a limitation of the counting mechanism meant that no interval of length smaller than 20µsecs could be recorded; thus, even though events less than 20µsecs apart had been observed by the detector, the counter was not able to record such intervals, and so they did not appear in any of the histograms. Furthermore, the histograms that were produced did not use equal 'bins' or class-intervals. Instead, to bring in a very long tail, 20µsec-length bins were judiciously grouped into 'slices' which were then used as new class-intervals for the resulting histogram; these 'slices' were of differing lengths, depending on how many bins were combined to make the 'slice.' The frequency corresponding to a particular 'slice' was computed as the average of the frequencies of the bins comprising that 'slice.' These histograms, however, did not appear to be of exponential shape. Subsequent investigations by Izenman showed that incorrect readings were being taken due to a faulty setup of the equipment used in the experimental runs.

Some suggestions were then made by Izenman on how to analyze the mass of data that was potentially available. One was to design an experiment that would take into account the following possible sources of variation that could occur in a given experimental run: (1) total duration of the run (e.g. 50secs, 100secs, 200secs,...,1000secs); (2) voltage used (e.g. 0-3000 volts, with special emphasis on the range 1500-2500 volts); (3) position of detector relative to the source. A further suggestion was that a printout of all individual interval-lengths for each run in the order in which they occurred should be made and would be very useful as a diagnostic tool. If such details were readily available then modelling of the stream of events could be undertaken. This stream, in theory, should be a stationary Poisson process, and if details on the occurrence times of events for each run were available, then a spectrum analysis of the point process generated could be used to check the sensitivity of the detector.

Consultation/Cooperative Research with Argonne National Laboratories

Scientific interaction between the Department of Statistics and scientists at Argonne National Laboratories continued to be fruitful.

Dr. George Sacher of Argonne recently sent to the Department of Statistics a proposed project entitled "A Statistical Evaluation of Alternatives in Initiating a Long-Term Experiment Using Dogs," written by S. A. Tyler, G. A. Sacher, and W. P. Norris. He asked that it be reviewed from the point of view of statistical design, etc. Comments on the paper were obtained from Professor Paul Meier, Shelby Haberman, and Joan Chmiel. The underlying problem was that of assessing the long-range effects of low-level radiation in man extrapolating from the effects of low and moderate-level radiation in lesser animals. The proposal contained two key elements, namely (a) the identification of break-points in the regression of log death-rate on log dose, and (b) the effective allocation of animals in the assessment of the regression slope (with a highly restricted choice set for 3 doses, i.e., 80, 40, and 40; or 40, 80, and 40; or 40, 40, and 80). Meier wondered whether the two-slope model of point (a) above was really satisfactory, and whether some continuous-slope model might be comparably informative and more tractable. He also suggested an alternative approach via the Cox regression model [D. R. Cox, J. Royal Statistical Soc. Series B, (1972), p. 187] in which an unknown underlying life-distribution, or equivalently, death-rate function, is assumed, and the effect of radiation is modelled as a multiplier of that function. He was also concerned about how the two segments of the two-slope model (each fitted by least squares) are to be separated. He suggested a literature search of papers on regression involving breakpoints. Concerning point (b), Meier thought that a reasonable thing to do would be to seek a performance measure which reflected the desired discriminatory power, and thereby determine its value for each of the three alternatives. Haberman agreed with Meier that Cox-style work was appropriate to the problem. He and Chmiel also noted that although a control group was mentioned in the proposed study, no discussion of methods of assignment (randomly or otherwise) to treatments or controls was indicated. Chmiel questioned the use of Bartlett's test and of the one-sided limits given the paper. She also felt that the analysis was unclear at various places and often seemed arbitrary. She also agreed that a Cox-style analysis would be more appropriate. Sacher replied that on the basis of these comments, the paper had been revised.

There are plans for a sequence of time-series seminars at Argonne this coming Spring with several groups participating, including a group of faculty and graduate students from the Department of Statistics.

We consider the interaction between the Department of Statistics and scientists at Argonne to be rewarding for both groups, and desire to continue and expand this activity in subsequent years.

D. TECHNICAL REPORTS PREPARED OR IN PREPARATION UNDER FULL OR

PARTIAL ERDA SUPPORT: 15 JUNE 1976 -- 14 JUNE 1977

CURRENT FACULTY

R. R. Bahadur

"Large deviations of the sample mean in general vector spaces" (With S. L. Zabell). Technical Report No. 36.

"Large deviation probabilities concerning Markov chains." In preparation.

"Asymptotic efficiency of estimates of the parameters of a Markov chain." In preparation.

"Large deviation probabilities of randomly stopped means." In preparation.

"The asymptotic variance of functions of the transition counts for a Markov chain." In preparation.

"Small-sample efficiencies of the Pearson chi-square and likelihood ratio tests for multinomial goodness-of-fit." (With James A. Koziol and David L. Wallace). In preparation.

Michael D. Perlman

"Combining independent chi-square tests." (With James A. Koziol). Accepted for publication in the Journal of the American Statistical Association.

"Combining independent non-central F-tests." (With James A. Koziol and U. Rasmussen). In preparation.

"A simple procedure for testing equality of correlation matrices." (With Kinley Larntz). In preparation.

"Invariant tests for means in the presence of covariates." (With John Marden). In preparation.

"A note on the matrix-variate F distribution." Accepted for publication in Sankhya.

"Reflection groups, generalized Schur functions, and the geometry of majorization" (With M. L. Eaton). Accepted for publication in the Annals of Probability.

"Generating $\theta(n)$ by reflections." (With M. L. Eaton). Submitted to Pacific Journal of Mathematics.

Ronald Thisted

"User documentation and control language: Evaluation and comparison of statistical computer packages." Technical Report No. 35.

"Ridge regression: an overview." (With Carl Morris). In preparation.

"Risk estimation for stepwise regression procedures." In preparation.

"An efficient algorithm for computing distributions for a class of two-sample rank statistics." In preparation.

"Maximum likelihood estimation of isotonic model regression." (With T. Sager). In preparation.

"User documentation and control language II: Evaluation criteria." In preparation.

"Teaching statistical computing" A computer packages approach." In preparation.

David L. Wallace

"Pearson curves as asymptotic approximations." Technical Report No. 42.

"Analytic approximations to distributions of largest eigenvalues."

"Small-sample efficiencies of the Pearson chi-square and likelihood ratio tests for multinomial goodness-of-fit." (With R. R. Bahadur and James A. Koziol). In preparation.

"Gamma probability plotting can be simple." In preparation.

Michael J. Wichura

"Convergence rates for laws of the iterated logarithm." Special invited paper delivered to Institute of Mathematical Statistics; to be submitted to Annals of Probability.

"Boundary-crossing probabilities associated with Chung's law of the iterated logarithm." In preparation.

"Asymptotic boundary-crossing probabilities for Bessel diffusions." In preparation.

"Another general form of the so-called law of the iterated logarithm." In preparation.

Sandy Zabell

"Large deviations of the sample mean in general vector spaces." (With R. R. Bahadur) Technical Report No. 36.

"Large deviations of Lévy processes." In preparation.

"On an inversion theorem for conditional expectations." In preparation.

"Limit theorems for conditional expectations. I. The large deviations case." In preparation.

"Limit theorems for conditional expectations. II. The general case." In preparation.

DOCTORAL DISSERTATIONS

S. P. Lin (Research Advisor: Michael D. Perlman)

"Improved procedures for the estimation of correlation matrices."
(Completion expected June, 1977).

J. Marden (Research Advisor: Michael D. Perlman)

"Complete class theorems for invariant tests in multivariate analysis."
(Completion expected August, 1977).

S. G. Machado (Research Advisor: David L. Wallace)

"On transformations of multivariate data."
(Completed April, 1976).

A. G. de Wet (Research Advisor: David L. Wallace)

"Linear functional and structural relationships."
(Completion expected May, 1977).